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## On the Spin Wave Field Theory and its Application to the Microwave Resonance

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The exact operator system describing the spin system such as ferro-, ferri- and antiferromagnetic ones was obtained in which spin  $S$  needs not to be equal to one half. The field operators adopted here are well-known annihilation and creation operators usually used for the Bose field, with subsidiary conditions limiting the infinite space of numbers to a finite value of  $2S+1$ . Exact solutions of these Hamiltonians, of course, can not be obtained because of not bi-linear form but they are solved approximately in the case of low temperatures. The resonance conditions obtained by this method for ferro-, ferri- and antiferromagnetic cases are the same as those already obtained from the Heisenberg's equation of motions.

### § 1. Introduction

To simplify the problem of magnetic behaviors such as susceptibility and specific heat or consideration about microwave resonance or relaxation phenomena etc. in antiferromagnetic or ferrimagnetic substances, it is very convenient to describe the Hamiltonian of spin system with the operators which are usually used in the field theories<sup>1)</sup>. The theories hitherto proposed contained more or less unreasonable operators for the system containing spins of small value in which  $1/S$  should not be neglected. In Bloch's well known exchange problem,<sup>2)</sup> however, the exact operator system for spin of  $1/2$  was used. With respect to the spin wave system, in general, it may be worth while to give an exact operator system as above without the restriction of  $1/2$  spin, and for this reason we shall describe the method below. Unfortunately it is also unable to solve the definite problems by this method as in Bloch's case since the Hamiltonian thus transformed is made up of not bi-linear form of the operators which create or annihilate the number of spin wave quanta "spinon" by unity. However this treatment has some convenience to obtain the useful bi-linear forms of Hamiltonian which are usually used at low temperatures. Although the simplification of Hamiltonian for the ferromagnetic or antiferromagnetic problems have been used also by many authors, the method discussed below shows a more straightforward way to construct the Hamiltonian for similar problems which will be seen in the following sections with some examples. Moreover, this method makes it easy to compare the relation between the theories hitherto proposed.

## § 2. Spin wave field

Now we introduce the operators satisfying the following commutation relations,

$$[a_j, a_k^+] = \delta_{jk}, \quad [b_j, b_k^+] = \delta_{jk}, \quad (j, k, = \dots N), \quad (2.1)$$

otherwise zero. Denoting the quantum numbers with  $\bar{N}_k$  and  $\bar{N}_k$  (all of them can be assumed to have integral non-negative values), the matrix elements of these operators are

$$\begin{cases} (a_k) \dots; \bar{N}_k, \bar{N}_k'; \dots = (a_k^+) \dots; \bar{N}_k', \bar{N}_k; \dots = \sqrt{\bar{N}_k'} \delta_{\bar{N}_k, \bar{N}_k'-1}, \\ (b_k) \dots; \bar{N}_k, \bar{N}_k'; \dots = (b_k^+) \dots; \bar{N}_k', \bar{N}_k; \dots = \sqrt{\bar{N}_k'} \delta_{\bar{N}_k, \bar{N}_k'-1}, \end{cases} \quad (2.2)$$

regarding all other quantum numbers  $a_k$ ,  $a_k^+$ ,  $b_k$  and  $b_k^+$  etc. act as unit matrices. Letting the spin eigen function  $\Psi(\dots; m_{sk}; \dots)$  be  $\Psi(\dots; \bar{N}_k, \bar{N}_k; \dots)$ , the spin operators  $S_k^+ = S_{kx} + iS_{ky}$ ,  $S_k^- = S_{kx} - iS_{ky}$  and  $S_{kz}$  can be represented\* by the above operators, we have:

$$S_k^+ = a_k^+ b_k, \quad S_k^- = b_k^+ a_k \quad \text{and} \quad S_{kz} = (1/2)(a_k^+ a_k - b_k^+ b_k) \quad (2.3)$$

with subsidiary conditions

$$(a_k^+ a_k + b_k^+ b_k) \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) = (\bar{N}_k + \bar{N}_k) \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots), \quad (2.4)$$

$$\bar{N}_k + \bar{N}_k = 2S_k, \quad (2.5)$$

and

$$\begin{cases} \bar{N}_k \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) = (S_k - m_{sk}) \Psi(\dots; m_{sk}; \dots), \\ \bar{N}_k \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) = (S_k + m_{sk}) \Psi(\dots; m_{sk}; \dots). \end{cases} \quad (2.6)$$

To prove the relations (2.3) it is sufficient to show the next relations hold to exactly:

$$\begin{aligned} S_k^+ \Psi(\dots; m_{sk}; \dots) &= a_k^+ b_k \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) = \sqrt{(\bar{N}_k + 1) \bar{N}_k} \Psi(\dots; \bar{N}_k + 1, \bar{N}_k - 1; \dots) \\ &= \sqrt{(S_k + m_{sk})(S_k - m_{sk} + 1)} \Psi(\dots; m_{sk} + 1; \dots), \end{aligned} \quad (2.7.1)$$

$$\begin{aligned} S_k^- \Psi(\dots; m_{sk}; \dots) &= b_k^+ a_k \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) = \sqrt{\bar{N}_k (\bar{N}_k + 1)} \Psi(\dots; \bar{N}_k - 1, \bar{N}_k + 1; \dots) \\ &= \sqrt{(S_k - m_{sk})(S_k + m_{sk} + 1)} \Psi(\dots; m_{sk} - 1; \dots), \end{aligned} \quad (2.7.2)$$

$$\begin{aligned} S_{kz} \Psi(\dots; m_{sk}; \dots) &= (1/2)(a_k^+ a_k - b_k^+ b_k) \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) \\ &= (1/2)(\bar{N}_k - \bar{N}_k) \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) \\ &= m_{sk} \Psi(\dots; m_{sk}; \dots), \end{aligned} \quad (2.7.3)$$

and

\*) After this work it called the authors attention that J. Schwinger had obtained these representations, in "On Angular Momentum" (1952).



$$\begin{aligned}
 S_k^2 \Psi(\dots; m_{sk}; \dots) &= \{(1/2)(S_k^+ S_k^- + S_k^- S_k^+) + S_{zk}^2\} \Psi(\dots; m_{sk}; \dots) \\
 &= \{(1/2)(a_k^+ b_k b_k^+ a_k) + (1/2)(a_k^+ b_k b_k^+ a_k) + (1/4)(a_k^+ a_k - b_k^+ b_k)^2\} \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) \\
 &= (1/4)(a_k^+ a_k + b_k^+ b_k)(a_k^+ a_k + b_k^+ b_k + 2) \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) \\
 &= S_k(S_k + 1) \Psi(\dots; m_{sk}; \dots),
 \end{aligned} \tag{2.7.4}$$

with the help of relations (2.1), (2.2), (2.4), (2.5) and (2.6).

The Hamiltonian of spin system  $\mathcal{H}_{spin}$  now in question is assumed to be constructed by the Zeeman energy term, exchange energy term, dipole-dipole interaction term and the anisotropy term, which is in this case, for simplicity, assumed to be uniaxial anisotropy ;

$$\begin{aligned}
 \mathcal{H}_{spin} &= g\beta H \sum_j S_{zj} + \sum_{j>k} J_{jk} (S_j S_k) + \sum_{j>k} K_{jk} S_{zj} S_{zk} \\
 &\quad + \sum_{j>k} D_{jk} [(S_j S_k) - 3r_{jk}^{-2} (S_j r_{jk}) (S_k r_{jk})].
 \end{aligned} \tag{2.8}$$

In the above expression the letters have usual meanings, i.e.:  $g$  is the Landé factor,  $\beta$  the Bohr magneton,  $H$  the external magnetic field the direction of which lies in  $z$ -direction,  $J_{jk}$  the exchange integral between  $j$ -th and  $k$ -th magnetic ions,  $K_{jk}$  the anisotropy constant, and  $D_{jk}$  the coefficient of dipolar or pseudodipolar interaction between the above mentioned ions. In pure dipolar case  $D_{jk}$  is expressed by  $g^2 \beta^2 r_{jk}^{-3}$ .

Using the relation (2.3) the above expression of Hamiltonian is easily transformed into

$$\begin{aligned}
 \mathcal{H}_{spin} &= (1/2)g\beta H \sum_j (a_j^+ a_j - b_j^+ b_j) + \sum_{j>k} \tilde{J}_{jk} [S_j S_k - (1/2)(a_j^+ b_k^+ - a_k^+ b_j^+)(a_j b_k - a_k b_j)] \\
 &\quad + (1/4) \sum_{j>k} K_{jk} (a_j^+ a_j - b_j^+ b_j)(a_k^+ a_k - b_k^+ b_k) \\
 &\quad - (3/4) \sum_{j>k} D_{jk} \{\gamma_{kj}^2 (a_j^+ a_j - b_j^+ b_j)(a_k^+ a_k - b_k^+ b_k) \\
 &\quad + \gamma_{jk}^+ \gamma_{jk}^- (a_j^+ a_k b_k^+ b_j + a_k^+ a_j b_j^+ b_k) \\
 &\quad + \gamma_{jk} \gamma_{jk}^- [(a_k^+ a_k - b_k^+ b_k) a_j^+ b_j + (a_j^+ a_j - b_j^+ b_j) a_k^+ b_k] \\
 &\quad + \gamma_{jk} \gamma_{jk}^+ [(a_k^+ a_k - b_k^+ b_k) b_j^+ a_j + (a_j^+ a_j - b_j^+ b_j) b_k^+ a_k] \\
 &\quad + (\gamma_{jk}^-)^2 a_j^+ a_k^+ b_j b_k + (\gamma_{jk}^+)^2 b_j^+ b_k^+ a_j a_k\},
 \end{aligned} \tag{2.9}$$

where,  $\tilde{J}_{jk}$  is  $J_{jk} + D_{jk}$ ,  $\gamma_{jk}$  the direction cosine of  $z_{jk}$ ,  $\gamma_{jk}^\pm$  is equal to  $\alpha_{jk} \pm i\beta_{jk}$  and  $\alpha_{jk}$  and  $\beta_{jk}$  are also direction cosines of  $x_{jk}$  and  $y_{jk}$ . For the case in which all  $S_k$ 's are equal to one half, Bloch obtained already an expression similar to the above.

As stated in the introduction the direct application of this theory for ferromagnet and so forth is mathematically impossible. The spin wave nature of the Hamiltonian (2.9), however, is easily understood to some extent. In the ground state configuration we know all  $a_k^+ a_k$  or  $b_k^+ b_k$  are nearly equal to zero or  $2S_k$ , hence we consider that, for example,  $a_k^+ a_k$  is either zero or unity. In such a case the matrix elements for  $b_k$  etc. are nearly constant with respect to  $N_k$ , so that we can conclude that Hamiltonian (2.9) can be considered to be of nearly bi-linear form of the spin wave operators  $a_k^+$  and

$b_j$ , etc. Changing the representation from configuration space to momentum space, it is supposed that the Hamiltonian (2.9) can be approximated by that of an assembly of harmonic oscillators with definite wave numbers  $\mathbf{k}_\lambda$  by the well known theorem of diagonalization of bi-linear form. We shall show explicitly the Hamiltonians thus obtained for the cases of ferromagnetic, antiferromagnetic and ferrimagnetic ground states and also the procedures of diagonalizations in the corresponding cases in the following section.

### § 3. Approximate forms of Hamiltonian in special cases

We shall discuss here the simple approximate forms of the Hamiltonians above mentioned for the cases of ferromagnetic, ferrimagnetic and antiferromagnetic spin arrangements at low temperatures. At first we consider the case of ferromagnetic spin ordering.

#### a) Ferromagnetic case

If one assumes

$$b_j \sim b_j^+ \sim \sqrt{2S_j} \quad (3.1)$$

and

$$b_j^+ b_j \Psi(\dots; \bar{N}_j, \bar{N}_j; \dots) = (2S_j - a_j^+ a_j) \Psi(\dots; \bar{N}_j; \dots), \quad (3.2)$$

then the Hamiltonian (2.9) is found to be the one deduced by Holstein and Primakoff<sup>(3)</sup> with anisotropy terms, and this corresponds to the "ferromagnetic" case :

$$\begin{aligned} \mathcal{H}_{spin} = & g\beta H \sum_j (a_j^+ a_j - S_j) - \sum_{j>k} \tilde{J}_{jk} \{ (\sqrt{S_j} a_k^+ - \sqrt{S_k} a_j^+) (\sqrt{S_j} a_k - \sqrt{S_k} a_j^+) \\ & - a_j^+ a_j a_k^+ a_k - S_j S_k \} \\ & + \sum_{j>k} K_{jk} (a_j^+ a_j - S_j) (a_k^+ a_k - S_k) + 3 \sum_{j>k} D_{jk} [\gamma_{jk}^2 (S_j - a_j^+ a_j) (S_k - a_k^+ a_k) \\ & + (\sqrt{S_j S_k}/2) \{ \gamma_{jkl}^+ \gamma_{jk}^- (a_j^+ a_k + a_k^+ a_j) + (\gamma_{jk}^-)^2 a_j^+ a_k^+ + (\gamma_{jk}^+)^2 a_j a_k \} \\ & + \gamma_{jk} \gamma_{jk}^- \{ \sqrt{S_j/2} a_j^+ (a_k^+ a_k - S_k) + \sqrt{S_k/2} a_k^+ (a_j^+ a_j - S_j) \} \\ & + \gamma_{jkl} \gamma_{jk}^+ \{ \sqrt{S_j/2} a_j (a_k^+ a_k - S_k) + \sqrt{S_k/2} a_k (a_j^+ a_j - S_j) \} ]. \end{aligned} \quad (3.3)$$

Now we consider the case in which the lattice containing magnetic ions can be divided into two interpenetrating sublattices I and II, and all spins situated on the sublattice I are  $S_I$  and those on II are  $S_{II}$ . We shall now define new operators  $a_{\lambda 1}^+$  and  $a_{\lambda 2}$ , etc. by the following relations :

$$\left\{ \begin{array}{l} a_k = \sum_{\lambda} \sqrt{1/N_I} \exp [i(\mathbf{k}_\lambda \cdot \mathbf{r}_k)] a_{\lambda 1}, \\ a_k^+ = \sum_{\lambda} \sqrt{1/N_I} \exp [-i(\mathbf{k}_\lambda \cdot \mathbf{r}_k)] a_{\lambda 1}^+, \end{array} \right\} \quad \left\{ \begin{array}{l} a_j = \sum_{\mu} \sqrt{1/N_{II}} \exp [i(\mathbf{k}_\mu \cdot \mathbf{r}_j)] a_{\mu 2}, \\ a_j^+ = \sum_{\mu'} \sqrt{1/N_{II}} \exp [-i(\mathbf{k}_{\mu'} \cdot \mathbf{r}_j)] a_{\mu' 2}^+. \end{array} \right. \quad (3.4)$$

Here the summations are taken over the whole sublattice points of I and II, so that  $N_I$  and  $N_{II}$  are both equal to  $N/2$ . The commutation rules are now found to be :



$$[a_{\lambda 1}, a_{\lambda' 1}^+] = \delta_{\lambda \lambda'}, \quad [a_{\mu 2}, a_{\mu' 2}^+] = \delta_{\mu \mu'}, \quad (3.5)$$

while all other commutators vanish. Using these operators the main (bi-linear) part of the exchange Hamiltonian is transformed into

$$\begin{aligned} \mathcal{H}_{spin} &= - \sum_{j>k} \tilde{J}_{jk} (\sqrt{S_{II}} a_j^+ - \sqrt{S_I} a_j^+) (\sqrt{S_{II}} a_k - \sqrt{S_I} a_k) \\ &= -1/2 \sum_{\lambda} \{ S_{II} J(\rho, 0) a_{\lambda 1}^+ a_{\lambda 1} + S_{II} J(\rho, 0) a_{\lambda 2}^+ a_{\lambda 2} - \sqrt{S_I S_{II}} J(\rho, \lambda) (a_{\lambda 1}^+ a_{\lambda 2} + a_{\lambda 2}^+ a_{\lambda 1}) \}, \end{aligned}$$

where

$$J(\rho, \lambda) = \sum_{\text{neighbors}} \tilde{J}_{ik} \exp [i(\mathbf{k}_{\lambda} \cdot \rho_n)]. \quad (3.7)$$

Using further transformations

$$\begin{cases} a_{\lambda 1} = A_{\lambda} \cos \theta_{\lambda} + B_{\lambda} \sin \theta_{\lambda}, \\ a_{\lambda 1}^+ = A_{\lambda}^+ \cos \theta_{\lambda} + B_{\lambda}^+ \sin \theta_{\lambda}, \end{cases} \quad \begin{cases} a_{\lambda 2} = -A_{\lambda} \sin \theta_{\lambda} + B_{\lambda} \cos \theta_{\lambda}, \\ a_{\lambda 2}^+ = -A_{\lambda}^+ \sin \theta_{\lambda} + B_{\lambda}^+ \cos \theta_{\lambda}, \end{cases} \quad (3.8)$$

with the commutation relations

$$[A_{\lambda}, A_{\lambda'}^+] = [B_{\lambda}, B_{\lambda'}^+] = \delta_{\lambda \lambda'}, \quad \text{otherwise zero}, \quad (3.9)$$

the exchange Hamiltonian (3.6) takes the form

$$\mathcal{H}_{spin} = \sum_{\lambda} (C_{\lambda} A_{\lambda}^+ A_{\lambda} + D_{\lambda} B_{\lambda}^+ B_{\lambda}), \quad (3.10)$$

where

$$\begin{cases} C_{\lambda} = (1/2) \{ S_I \sin^2 \theta_{\lambda} + S_{II} \cos^2 \theta_{\lambda} \} J(\rho, 0) + \sqrt{S_I S_{II}} J(\rho, \lambda) \sin \theta_{\lambda} \cos \theta_{\lambda}, \\ D_{\lambda} = (1/2) \{ S_I \sin^2 \theta_{\lambda} + S_{II} \cos^2 \theta_{\lambda} \} J(\rho, 0) - \sqrt{S_I S_{II}} J(\rho, \lambda) \sin \theta_{\lambda} \cos \theta_{\lambda}, \end{cases} \quad (3.11)$$

and  $\theta_{\lambda}$  in the above expression is determined by the equality

$$\tan 2\theta_{\lambda} = 2 \sqrt{S_I S_{II}} J(\rho, \lambda) / (S_I - S_{II}) J(\rho, 0). \quad (3.12)$$

Using this value of  $\theta_{\lambda}$ , (3.11) is rewritten down as

$$\begin{cases} C_{\lambda} = (1/4) \{ J(\rho, 0) (S_I + S_{II}) + \sqrt{(S_I - S_{II})^2 J^2(\rho, 0) + 4 S_I S_{II} J^2(\rho, \lambda)} \}, \\ D_{\lambda} = (1/4) \{ J(\rho, 0) (S_I + S_{II}) - \sqrt{(S_I - S_{II})^2 J^2(\rho, 0) + 4 S_I S_{II} J^2(\rho, \lambda)} \}. \end{cases} \quad (3.13)$$

If  $S_I$  and  $S_{II}$  are equal with each other, then (3.10) turns back to the well-known expression already proposed by Holstein and Primakoff:

$$\mathcal{H}_{spin} = -S \sum_{\lambda} \{ J(\rho, 0) - J(\rho, \lambda) \} a_{\lambda}^+ a_{\lambda}. \quad (3.14)$$

Alternatively if we use the truncated bi-linear Hamiltonian including dipolar, anisotropy and Zeeman energies as well as exchange interaction energy, then the Hamiltonian can also be diagonalized. The result is as follows:

$$\mathcal{H}_{spin} = \sum (\mathcal{Q}_{\lambda+} A_{\lambda+}^+ A_{\lambda+} + \mathcal{Q}_{\lambda-} A_{\lambda-}^+ A_{\lambda-}), \quad (3.15)$$

where

$$\begin{aligned} \mathcal{Q}_{\lambda\pm} = & \left[ \frac{S_I + S_{II}}{2} \{J(\rho, 0) + K(\rho, 0) + D_z(\rho, 0)\} + g\beta H \pm \left[ \left( \frac{S_I + S_{II}}{2} \right)^2 \{J(\rho, 0) \right. \right. \\ & \left. \left. + K(\rho, 0) + D_z(\rho, 0)\}^2 + S_I S_{II} \{J(\rho, \lambda) + D_x(\rho, \lambda)\}^2 \right]^{1/2} \right] \\ & \times \left[ \frac{S_I + S_{II}}{2} \{J(\rho, 0) + K(\rho, 0) + D_z(\rho, 0)\} + g\beta H \pm \left[ \left( \frac{S_I - S_{II}}{2} \right)^2 \{J(\rho, 0) \right. \right. \\ & \left. \left. + K(\rho, 0) + D_z(\rho, 0)\}^2 + S_I S_{II} \{J(\rho, \lambda) + D_y(\rho, \lambda)\}^2 \right]^{1/2} \right] \end{aligned} \quad (3.16)$$

with

$$D_z(\rho, 0) = 3 \sum_{\text{neighbors}} D_{jk} \gamma_{jk}^2, \quad (3.17)$$

$$D_x(\rho, \lambda) = 3 \sum_{\text{neighbors}} D_{kj} \exp[i(\mathbf{k}_\lambda \cdot \rho_n)] \alpha_{jk}^2, \quad (3.18)$$

and

$$D_y(\rho, \lambda) = 3 \sum_{\text{neighbors}} D_{jk} \exp[i(\mathbf{k}_\lambda \cdot \rho_n)] \beta_{jk}^2. \quad (3.19)$$

In the above calculation we neglected the long range nature of dipolar force, because if we introduce this effect into consideration the final result becomes complicated one. However it can be surely supposed that eqs. (3.15) or (3.16) is nearly correct if we approximate (3.17) to (3.19) by the well known demagnetizing factors.

### b) Antiferromagnetic case

Alternative simple application of the theory can be developed in the case of antiferromagnetism near the absolute zero of temperature. In this case we also divide the crystal lattice now in consideration into two sublattices as before. In this case we put

$$b_j \sim b_j^+ \sim \sqrt{2S_j} b_j^+ \Psi(\dots; \bar{N}_j, \bar{N}_j; \dots) = (2S_j - a_j^+ a_j) \Psi(\dots; \bar{N}_j; \dots), \quad (3.20)$$

for the sublattice I, and

$$a_k \sim a_k^+ \sim \sqrt{2S_k} a_k^+ \Psi(\dots; \bar{N}_k, \bar{N}_k; \dots) = (2S_k - b_k^+ b_k) \Psi(\dots; \bar{N}_k; \dots), \quad (3.21)$$

for the sublattice II. Neglecting the dipolar terms in the Hamiltonian (2.9), we obtain

$$\begin{aligned} \mathcal{H}_{spin} = & g_I \beta H \sum_j (a_j^+ a_j - S) - g_{II} \beta H \sum_k (b_k^+ b_k - S) \\ & + \sum_{j>k} J_{jk} [(a_j^+ a_j + b_k^+ b_k + a_j^+ b_k^+ + a_j b_k) S - a_j^+ a_j b_k^+ b_k - S^2] \\ & - \sum_{j>k} K_{jk} [(a_j^+ a_j + b_k^+ b_k) S - a_k^+ a_j b_k^+ b_k - S^2]. \end{aligned} \quad (3.22)$$

Here we assumed that all  $S_k$  are equal to one another and to  $S$  as before. This expression is almost the same as those already obtained by Anderson<sup>4)</sup>, Kubo<sup>5)</sup>, Nakamura<sup>6)</sup> and



Ziman<sup>7)</sup>. So we shall not be concerned about this problem any more.

c) *Ferrimagnetic case*

If the spin of the sublattice I is  $S_I$  and that of II is  $S_{II}$  and when  $S_I$  is not equal to  $S_{II}$ , or the number of the elementary magnet is not same with each other, this case corresponds to the so-called "Ferrimagnetic" case. The spin wave nature of this magnetism was discussed by Kaplan<sup>8)</sup> very briefly basing upon the Heisenberg scheme. On the contrary we shall try to calculate the same Hamiltonian by means of adding the anisotropy energy using the above mentioned method of canonical transformation.

In this case we use the same approximations of the spin wave field operators as previously cited (3.20) and (3.21). The Hamiltonian takes the form

$$\begin{aligned} \mathcal{H}_{spin} = & g_I \beta H \sum_j (a_j^+ a_j - S_I) - g_{II} \beta H \sum_k (b_k^+ b_k - S_{II}) \\ & + \sum_{j>k} J_{jk} [(S_{II} a_j^+ a_j + S_I b_k^+ b_k + \sqrt{S_I S_{II}} (a_j^+ b_k^+ + b_k a_j) - a_j^+ a_j b_k^+ b_k - S_I S_{II})] \\ & + \sum_{j>k} K_{jk} [S_I S_{II} - (S_I b_k^+ b_k + S_{II} a_j^+ a_j) + a_j^+ a_j b_k^+ b_k]. \end{aligned} \quad (3.23)$$

By the canonical Fourier transformation defined by

$$\begin{cases} a_j = \sqrt{1/N_{II}} \sum_{\lambda} \exp[i(\mathbf{k}_{\lambda} \cdot \mathbf{r}_j)] a_{\lambda}, \\ a_j^+ = \sqrt{1/N_{II}} \sum_{\lambda'} \exp[-i(\mathbf{k}_{\lambda'} \cdot \mathbf{r}_j)] a_{\lambda'}^+, \end{cases} \quad \begin{cases} b_k = \sqrt{1/N_I} \sum_{\mu} \exp[i(\mathbf{k}_{\mu} \cdot \mathbf{r}_k)] b_{\mu}, \\ b_k^+ = \sqrt{1/N_I} \sum_{\mu'} \exp[-i(\mathbf{k}_{\mu'} \cdot \mathbf{r}_k)] b_{\mu'}^+, \end{cases}$$

$$[a_{\lambda} a_{\lambda'}^+] = [b_{\lambda} b_{\lambda'}^+] = \delta_{\lambda\lambda'}, \quad \text{otherwise zero}, \quad (3.24)$$

main truncated parts of (3.23) are transformed into:

$$\mathcal{H}_{spin} = \sum_{\lambda} \{A_{\lambda} a_{\lambda}^+ a_{\lambda} + B_{\lambda} b_{\lambda}^+ b_{\lambda} + C_{\lambda} (a_{\lambda}^+ b_{-\lambda}^+ + a_{\lambda} b_{-\lambda})\}, \quad (3.25)$$

with

$$\begin{aligned} A_{\lambda} &= g_I \beta H + \{J(\rho, 0) - K(\rho, 0)\} S_{II}, \\ B_{\lambda} &= -g_{II} \beta H + \{J(\rho, 0) - K(\rho, 0)\} S_I, \end{aligned} \quad (3.26)$$

and

$$C_{\lambda} = \sqrt{S_I S_{II}} J(\rho, \lambda).$$

Using the following successive canonical transformations

$$\left. \begin{aligned} a_{\lambda} &= \frac{1}{\sqrt{2}} (q_{1\lambda} + i p_{1\lambda}), \\ a_{\lambda}^+ &= \frac{1}{\sqrt{2}} (q_{1\lambda} - i p_{1\lambda}), \end{aligned} \right\} \left. \begin{aligned} b_{-\lambda} &= \frac{1}{\sqrt{2}} (q_{2\lambda} + i p_{2\lambda}), \\ b_{-\lambda}^+ &= \frac{1}{\sqrt{2}} (q_{2\lambda} - i p_{2\lambda}), \end{aligned} \right\} [q_{\sigma\lambda} p_{\sigma'\lambda'}] = i \delta_{\sigma\sigma'} \delta_{\lambda\lambda'}, \quad (3.27)$$

and

$$\left. \begin{aligned} p_{1\lambda} &= p_{x\lambda} \cos \theta_{\lambda} + p_{y\lambda} \sin \theta_{\lambda}, \\ p_{2\lambda} &= -p_{x\lambda} \sin \theta_{\lambda} + p_{y\lambda} \cos \theta_{\lambda}, \end{aligned} \right\} \left. \begin{aligned} q_{1\lambda} &= x_{\lambda} \cos \theta_{\lambda} + y_{\lambda} \sin \theta_{\lambda}, \\ q_{2\lambda} &= -x_{\lambda} \sin \theta_{\lambda} + y_{\lambda} \cos \theta_{\lambda}, \end{aligned} \right\} \quad (3.28)$$

$$\text{where} \quad \tan 2\theta_{\lambda} = 2C_{\lambda} / (A_{\lambda} - B_{\lambda}), \quad (3.29)$$

and further

$$\begin{cases} x_\lambda = \{A_\lambda + B_{-\lambda} + \sqrt{(A_\lambda - B_{-\lambda})^2 + 4C_\lambda^2}\}^{1/4} \{A_\lambda + B_{-\lambda} - \sqrt{(A_\lambda - B_{-\lambda})^2 + 4C_\lambda^2}\}^{-1/4} X_\lambda, \\ y_\lambda = \{A_\lambda + B_{-\lambda} - \sqrt{(A_\lambda - B_{-\lambda})^2 + 4C_\lambda^2}\}^{1/4} \{A_\lambda + B_{-\lambda} + \sqrt{(A_\lambda - B_{-\lambda})^2 + 4C_\lambda^2}\}^{-1/4} Y_\lambda, \end{cases}$$

and

$$\begin{cases} X_\lambda = \xi_\lambda \cos \vartheta_\lambda + \eta_\lambda \sin \vartheta_\lambda, \\ Y_\lambda = -\xi_\lambda \sin \vartheta_\lambda + \eta_\lambda \cos \vartheta_\lambda, \end{cases} \quad \begin{cases} P_{X\lambda} = p_{\xi\lambda} \cos \vartheta_\lambda + p_{\eta\lambda} \sin \vartheta_\lambda, \\ P_{Y\lambda} = -p_{\xi\lambda} \sin \vartheta_\lambda + p_{\eta\lambda} \cos \vartheta_\lambda, \end{cases} \quad (3.31)$$

with

$$\tan 2\vartheta_\lambda = 8C_\lambda \sqrt{A_\lambda B_{-\lambda} - C_\lambda^2} (A_\lambda^2 - B_{-\lambda}^2)^{-1}, \quad (3.32)$$

and finally

$$\begin{cases} p_{\xi\lambda} = -i(A_\lambda B_{-\lambda} - C_\lambda^2)^{-1/4} \mathcal{Q}_{\lambda+}^{1/2} 2^{-1/2} (\alpha_\lambda - \alpha_\lambda^+), & \xi_\lambda = (A_\lambda B_{-\lambda} - C_\lambda^2)^{1/4} \mathcal{Q}_{\lambda+}^{-1/2} 2^{-1/2} (\alpha_\lambda + \alpha_\lambda^+), \\ p_{\eta\lambda} = -i(A_\lambda B_{-\lambda} - C_\lambda^2)^{-1/4} \mathcal{Q}_{\lambda-}^{1/2} 2^{-1/2} (\beta_\lambda - \beta_\lambda^+), & \eta_\lambda = (A_\lambda B_{-\lambda} - C_\lambda^2)^{1/4} \mathcal{Q}_{\lambda-}^{-1/2} 2^{-1/2} (\beta_\lambda + \beta_\lambda^+), \end{cases} \quad (3.33)$$

in which  $\mathcal{Q}_{\lambda+}$  and  $\mathcal{Q}_{\lambda-}$  are given by

$$\mathcal{Q}_{\pm\lambda} = (1/\sqrt{2}) \{A_\lambda^2 + B_{-\lambda}^2 - 2C_\lambda^2 \pm (A_\lambda - B_{-\lambda}) \sqrt{(A_\lambda + B_{-\lambda})^2 - 4C_\lambda^2}\}^{1/2},$$

we are able to obtain the final expression of Hamiltonian as follows:

$$\mathcal{H}_{spin} = \sum_\lambda \{ \mathcal{Q}_{\lambda+} \alpha_\lambda^+ \alpha_\lambda + \mathcal{Q}_{\lambda-} \beta_\lambda^+ \beta_\lambda + (A_\lambda + B_{-\lambda} - \mathcal{Q}_{\lambda+} - \mathcal{Q}_{\lambda-})/2 \}. \quad (3.35)$$

Inserting (3.26) into this Hamiltonian we can obtain the energy  $E_\lambda$  of the lower branch as a function of wave number vector.

$$\begin{aligned} E_\lambda = & \frac{g_I S_I - g_{II} S_{II}}{S_I - S_{II}} \beta H + \frac{2 S_I S_{II}}{S_I - S_{II}} [K(\rho, 0) + \{J(\rho, 0) - J(\rho, \lambda)\}] \\ & + \{K(\rho, 0)/J(\rho, 0)\} \{S_I S_{II}/(S_I - S_{II}) - 4 S_I^2 S_{II}^2/(S_I - S_{II})^2\} + \dots \end{aligned} \quad (3.36)$$

From the above relation, spin wave nature is found easily, i.e. the curved bracket in the right hand side of eq. (3.36) is proportional to the square of wave number vector.

#### 4. Microwave resonance conditions

In usual ferromagnetic resonance phenomena it is well known that the resonance condition is

$$h\nu = g\beta H, \quad (4.1)$$

where  $\nu$  is the frequency of the microwave used. This condition is easily deduced<sup>9)</sup> if the  $g$  factor of each spin is the same with other. On the other hand the resonance condition in such ferromagnetic materials, for example "Hypomanganite",<sup>10,11)</sup> in which there are two types of magnetic ions with  $g_I$  and  $g_{II}$ , is also obtained by adding the corresponding Zeeman energies to the Hamiltonian (3.6), the result in the first approximation is given by

$$h\nu = (S_I g_I + S_{II} g_{II}) (S_I + S_{II})^{-1} \beta H. \quad (4.2)$$

The resonance condition for the ferrimagnetic case is also given by (3.36) putting  $\mathbf{k}_\lambda = 0$ . If there are no anisotropy energy the relation similar to the above is expressed by

$$h\nu = (S_I g_I - S_{II} g_{II}) (S_I - S_{II})^{-1} \beta H. \quad (4.3)$$



These two relations (4.2) and (4.3) were already obtained by the present writer<sup>12)</sup> solving the Heisenberg's equations of motion. The resonance conditions involving the demagnetizing effect as well as the anisotropy effect corresponding to these cases were obtained by Wangsness.<sup>13)</sup> Putting  $g_I = g_{II}$  one can find from (3.36) the resonance formula proposed by Kittel<sup>14)</sup> for the case of ferrimagnetic resonance. If we use (3.26), in which  $g_I = g_{II}$  and  $S_I = S_{II}$ , antiferromagnetic resonance condition can be found directly from (3.34) and the result is the same with those given by Nagamiya,<sup>15)</sup> Kittel,<sup>14)</sup> Nakamura<sup>6)</sup> and Ziman.<sup>7)</sup>

The intensity of absorption line may also be derived from the relations (3.8) and (3.27) to (3.33) with corresponding microwave quanta, as we are not interested at present, in this problem as well as in the statistical behaviour of these systems, so we shall not discuss them furthermore.

## § 5. Conclusions

In this article the exact representation of Hamiltonians for the spin wave field was obtained using the well known Bose field operators with subsidiary conditions. The approximate solutions were obtained for ferro-, ferri- as well as antiferromagnetic cases in which two types of the spins and Landé factors of magnetic ions exist. In special cases these results of course agree with those obtained already by many authors. The microwave resonance conditions for two sublattice systems corresponding to the above mentioned cases were also discussed. These were the same ones as those obtained by the present authors and by Wangsness from the classical equation of motion.

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# Super-stationary Variational Method\*

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The stationary character in the usual variational method is extended by using a trial function with stationary character. A successive method for the improvement of the stationary character is also indicated. We have applied this method both for bound state problems and for continuous spectra (scattering problems).

## § 1. Introduction

Consider the eigenvalue equation associated with real eigenvalues

$$A\psi_k = \lambda_k B \psi_k, \quad (1)$$

where  $A$  and  $B$  are self-adjoint operators, and the normalization of  $\psi_k$  is taken to be  $(\psi_k B \psi_{k'}) = \pm \delta_{kk'}$ . Let a normalized set of trial functions be  $\phi_k$ ,  $(\phi_k B \phi_k) = \pm 1$ , which are close to the correct eigenfunctions  $\psi_k$  so that

$$\phi_k = (1 + \Delta_k) \psi_k, \quad (2)$$

where  $\Delta$  is "small". Expanding  $\phi_k$  by the complete set\*\*  $\psi_{k'}$ ,

$$\phi_k = a_{kk} \psi_k + \sum_{k' \neq k} a_{kk'} \psi_{k'}, \quad a_{kk} = 1 + O(\Delta^2), \quad a_{kk'} = O(\Delta); \quad k \neq k',$$

and defining  $\omega_k \equiv (\phi_k A \phi_k) / (\phi_k B \phi_k) = \lambda_k + O(\Delta^2)$ , it follows directly that

$$(\phi_k (A - \omega_{k'}) B \phi_{k'}) = (\omega_k - \omega_{k'}) a_{k'k} + O(\Delta^2) \equiv (\omega_k - \omega_{k'}) \alpha_{k'k}, \quad \alpha_{k'k} = a_{k'k} + O(\Delta^2).$$

for  $k \neq k'$ , Hence

\* Recently a paper entitled "A variation principle for eigenfunctions" by L. C. Biedenharn and J. M. Blatt has been published in Phys. Rev. **93** (1954) 230, which has almost the same contents as in §1 of my article appeared in *Soryushiron-kenkyu* (mimeographed circular in Japanese), **5**, 1014, (July, 1953). No application to actual problem is discussed in B-B's paper.

\*\* This is in the sense of mean convergence for  $B$  or  $A$ , namely, for any almost continuous functions  $\Psi$  and  $\Phi$  which are restricted by suitable conditions (for example; if  $B$  is positive definite,  $(\Phi B \Phi) < \infty$ ,  $(\Psi B \Psi) < \infty$ ), the following relations hold.

$$(\Phi B \Psi) = \sum_k a_k (\Phi B \psi_k) = \sum_{k'} b_{k'} (\psi_{k'} B \Psi), \quad a_k = (\psi_k B \Psi) / (\psi_k B \psi_k),$$

$$(\Phi A \Psi) = \sum_k \lambda_k a_k (\Phi B \psi_k) = \sum_{k'} \lambda_{k'} b_{k'} (\psi_{k'} B \Psi). \quad b_{k'} = (\Phi B \psi_{k'}) / (\psi_{k'} B \psi_{k'}).$$

If  $B$  is of "short range", (3) is valid for the trial functions  $\phi_k$  which are restricted only by (2) only in the region where  $B$  is appreciable, ( $B \sim O(1)$ )

$$\phi_k \equiv \phi_k - \sum_{k' \neq k} \alpha_{kk'} \phi_{k'} = \phi_k + O(\Delta^2). \quad (3)^*$$

Using  $\phi_k$  as the trial function, we have

$$\omega_k \equiv (\phi_k | A | \phi_k) / (\phi_k | B | \phi_k) = \lambda_k + O(\Delta^4). \quad (4)$$

When this  $\omega_k$  is considered as a functional of  $\phi_{k'}$ , the first, second and third functional derivatives of  $\omega_k$  vanish by dint of (1) for all  $\phi_{k'}$ . We shall call it the super-stationary character of  $\omega_k$ . Let  $\phi_k$  be constructed from the first trial functions  $\phi_{k'}$ , and take these  $\phi_k$  as a new set of trial functions. Then it is possible to calculate the approximate values of  $\lambda_k$ , which has errors of the order only  $O(\Delta^8)$ . This process can be performed endlessly as one pleases. In Sec. 2 we examine the accuracy of  $\omega_k$  with two examples, namely the problem in which the depth of a potential is to be determined when a neutron and a proton interact through the exponential potential with zero binding energy. For trial functions we take a) the correct solutions for Hulthén potential and b) the correct solutions for a square well potential. For continuous spectra the quantity to be solved is not the eigenvalues  $\lambda_k$  but, for example, the phase shift or the scattering amplitude. In Sec. 3 the present method is extended to such problems. Two examples in Sec. 3 are: 1) To solve the scattering length in  $n$ - $p$  collision. 2) To solve the two phase shifts  $\delta^{(1)}$ ,  $\delta^{(2)}$  and the mixing ratio  $\varepsilon$  in the scattering of neutron-proton with tensor forces. The results obtained both for bound state and scattering problems are good as expected by our theory.

## § 2. Bound state problem

Problem: A neutron and a proton constitute a ground state with zero binding energy by the exponential type potential. Find the depth of this potential.

In this case  $A$  and  $B$  are with the conditions  $\phi_k(0)=0$ ,  $\phi_k(r) \rightarrow \text{const}$ ;  $r \rightarrow \infty$ ,

$$A = -d^2/dr^2, \quad B = e^{-r}.$$

### a). Hulthén type trial functions

As the set of trial functions we take the correct solutions for the Hulthén potential;  $B = e^{-r}/(1 - e^{-r})$ :

$$\phi_n = c_n x d^{n+1}/dx^{n+1} [x^n (1-x)^{n+1}], \quad x \equiv 1 - e^{-r}, \quad n \geq 0.$$

The behaviors of the exponential and the Hulthén potentials differ considerably at the origin so that one might think that such trial functions will not give a good approximation. Nevertheless the high accuracy of the order of 0.02% is reached as shown in the following calculations. The correct eigenvalue corresponding to the ground state is

$$\lambda_0 = 1.4458(0).$$

\*) If the normalization of  $\phi_k$  is not specified,  $a_{kk'}$  is given by

$$a_{kk'} = (\phi_{k'} | (A - \omega_k B) \phi_k) / (\phi_{k'} | (A - \omega_k B) \phi_{k'}) = (A_{k'k} B_{kk} - A_{kk} B_{k'k}) / (A_{k'k} B_{kk} - A_{kk} B_{k'k'}),$$

$$A_{kk'} \equiv (\phi_k | A | \phi_{k'}), \quad B_{kk'} \equiv (\phi_k | B | \phi_{k'}).$$

From the usual variational method we obtain

$$\omega_0 = 1.5, \quad \Delta\omega_0/\lambda_0 \sim 0.04 = (20\%)^2.$$

It is inferred that these Hulthén type trial functions have errors of about 20%. Calculating  $\alpha_{k0}$  we get

$$\alpha_{01} = \sqrt{10}/32, \quad \alpha_{0n} = 0: \quad n \geq 2.$$

Thus

$$\phi_0 = \text{const} \times (22x + 15x^2).$$

Putting it into (4) we have

$$\omega_0 = 1611/1114 = 1.44614, \quad \Delta\omega_0/\lambda_0 \sim 0.0002.$$

This result is better than our prediction which is about  $(20\%)^4 = 0.0016$ .

*b). Square well type trial functions*

Since the intrinsic range for  $e^{-r}$  is  $3.54^{11)}$ , we shall take the solutions for a square well whose range is 4 for simplicity.

$$\phi_n = c_n \sin \{(2n+1)\pi r/8\}, \quad r < 4, \quad n \geq 0.$$

This set of trial functions does not constitute a complete set for true solutions in a strict sense, but this set is shown to be good enough for approximate calculations. To do this we cut off the potential as

$$B = e^{-r}; \quad r < 4, \quad = 0; \quad r > 4.$$

This gives

$$\omega_0 = 1.75, \quad \Delta\omega_0/\lambda_0 \sim (35\%)^2.$$

In fact, calculating  $\alpha_{0k}$  we get

$$\alpha_{01} = 0.227, \quad \alpha_{02} = 0.026, \quad \alpha_{03} = 0.007, \quad \alpha_{04} = 0.002.$$

From both examples *a)* and *b)* it is seen that the convergence of  $\alpha_{nk}$  series is good. Since we may neglect the quantities of order  $J^2$ , it is convenient to set  $\alpha_{02} = \alpha_{03} = \alpha_{04} = \dots = 0$ . Thus we have

$$\phi_0 = c \{ \sin(\pi r/8) + 0.095 \sin(3\pi r/8) \}; \quad r < 4, \quad = 0.905c; \quad r > 4.$$

From (4) using this  $\phi_0$  without cutting off the potential, we get

$$\omega_0 = 1.464, \quad \Delta\omega_0/\lambda_0 \sim 0.012.$$

The error of  $\omega_0$  is thus of the same order of  $(35\%)^4 = 0.015$ . This result may be regarded rather good in considering that further approximation such as the cutting off of the potential is made.

As seen in the preceding examples the merit of the present method may be found in the following three points. 1). The computation is simple. 2). The measure for errors can always be inferred. In this connection we notice that  $\Delta\omega_k \sim \omega_k - \omega_k \sim (J^2)$ , or  $\text{Max } |\alpha_{kk'}| \sim (J)$ ,  $(k \neq k')$ . 3). The convergence property of  $\alpha_{kk'}$  is good and we can (and should) neglect consistently  $\alpha_{kk'}$  which are of the order of  $J^2$ .



### § 3. Scattering problem

In this case the quantity to be solved is not the eigenvalues  $\lambda_k$  but, for example, the phase shift. We shall extend this method to the phase shift  $\delta$ , where  $\lambda$  is given. If the phase shift is given and the corresponding depth  $\lambda$  is to be found, however, the problem is just the same in Sec. 1 and Sec. 2.

For simplicity, first consider the radial wave equation for  $S$ -scattering with a central potential  $V(r)$ .

$$L\psi \equiv (A - \lambda B)\psi = -(\partial^2/\partial r^2 + \lambda V(r))\psi = 0, \quad \psi(0) = 0.$$

According to Kato<sup>2)</sup>, we normalize a trial function  $\phi$  as

$$\phi(r) \rightarrow \cos(kr + \theta) + A_t \sin(kr + \theta); \quad r \rightarrow \infty, \quad \phi(0) = 0,$$

and also normalize  $\psi$  in the same way as  $\phi$  where  $\theta$  is a fixed constant and  $A$  is an adjustable parameter depending on the asymptotic form of  $\phi$ . The usual variational principle for the scattering problem is derived from

$$\begin{aligned} k \cot(\delta - \theta) &= kA_t + (\phi | L \phi) - (A \phi | L A \phi), \quad \Delta\phi = \phi - \psi = O(\Delta), \\ kA &\equiv kA_t + (\phi | L \phi), \quad \Delta A_t \equiv \cot(\delta - \theta) - A = O(\Delta), \quad \Delta A = O(\Delta^2). \end{aligned} \quad (5)$$

Once having found  $kA$  which has only an error of the order of  $\Delta^2$ , we choose a set of trial functions  $\phi_k$  (since  $\phi$  corresponding to  $\lambda$  is one of  $\phi_k$ ,  $\phi$  will be written as  $\phi_0$ ) which give this  $kA$  asymptotically. By imposing this boundary condition on the system, there exist the eigenvalues  $\lambda_k$  and the corresponding eigenfunctions  $\phi_k$  whose asymptotic forms have the same  $kA$  determined as above. The trial functions  $\phi_k$  we have chosen should be close to such eigenfunctions. Then it is possible to construct the new trial functions  $\phi_k$ , as described in the preceding paragraph, which involve errors of only  $O(\Delta^2)$ . Applying this  $\phi_0$  to (5) we get an accurate  $kA = kA + (\phi_0 | L \phi_0)$ ,  $k \cot(\delta - \theta) + O(\Delta^4)$ . Thus  $kA$  has the superstationary character.

One of the methods by which  $kA$  is found is to put

$$(\phi_0 | L \phi_0) \equiv (\phi_0 | (A - \lambda B) \phi_0) = 0, \quad (6)$$

by assuming only one adjustable parameter related to the phase shift. (6) is the Hulthén's variational principle, and is analogous to the relation in Sec. 1 as

$$\lambda = (\phi_0 | A \phi_0) / (\phi_0 | B \phi_0).$$

$\lambda$  is only an approximate quantity for the boundary condition imposed by taking  $\phi_0$  as above. Hence  $\lambda$  corresponds to  $\omega_0$  in Sec. 1. ( $\lambda \equiv \omega_0$ ). The situation is indeed the same as in the Rayleigh-Ritz method excepting that the point of view is reverse. Therefore the usual variational methods may be called en bloc the Rayleigh-Ritz-Hulthén method.

(Example 1)

Solve the scattering length for  $S$ -wave neutron-proton collisions in the exponential potential.

We shall take the trial functions analogously as in Sec. 4,

$$\phi_n = \{-x/(n+1)!\} d^{n+1}/dx^{n+1} \cdot [x^n(1-x)^{n+1}] + (-1)^{n-1} \cdot \alpha r, \quad x \equiv 1 - e^{-r}, \quad (7)$$

where  $\alpha$  is an adjustable parameter which corresponds to the reciprocal of scattering length. If  $\alpha$  is determined from (6), we have

$$\alpha = (9\lambda - 6 - \sqrt{36 + 36\lambda - 15\lambda^2}) / (24\lambda), \quad -0.76 < \lambda < 3.16.$$

Using  $\phi_n$  in (7) with the above  $\alpha$ , we get

$$\phi_k = c_k (\phi_k - \sum_{k' \neq k} \alpha_{kk'} \phi_{k'}), \quad \phi_k \rightarrow (-1)^{k-1} (1 - \alpha r); \quad r \rightarrow \infty,$$

where  $\alpha_{kk'}$  are calculated as in the footnote in Sec. 1. Thus

$$-a = -\alpha + (\phi_0 | L | \phi_0).$$

First we shall consider the case:  $\lambda \equiv \omega_0 = 2$ .  $\alpha_{ok}$  are

$$\alpha_{o1} = 0.122, \quad \alpha_{o2} = 0.028, \quad \alpha_{o3} = -0.013, \dots$$

Since  $\alpha_{on} \sim O(\lambda^2)$ , ( $n \geq 2$ ), these  $\alpha_{on}$  ( $n \geq 2$ ) should be neglected. The values of  $a$  calculated by neglecting  $\alpha_{on}$  ( $n \geq 2$ ) are tabulated in Table 1. This example concerns zero energy scattering. At all energies the square well type trial functions may be of use as in the example *b*) of Sec. 2.

$\alpha$ $\lambda$	$a$ (Hulthén)	$a$ (Super-stationary)	numerical solution
1	-0.190	-0.166	(-0.136)
3/2	0.0000	0.0146	0.0144
2	0.1057	0.1142	0.1149
3	0.2500	0.2546	(0.2528)

Table 1. The reciprocal values of scattering length for various  $\lambda$ . Assuming the trial functions as (7), we get  $a$  by Hulthén's variational method and  $a$  by the present method. In the column of numerical solution we put the values obtained from the interpolation formula in table 5 of reference of which validity is restricted by  $-0.05 < a < 0.15$  as mentioned in that reference. Hence for  $\lambda=1$  and  $\lambda=3$  the values of  $a$  are to be superior to those values.

### (Example 2)

The super-stationary method is extended to the tensor force scattering in neutron-proton collision, as an example for complicated problems.

When we notice  $S-D$  mixed scattering in the spin triplet state, the equations to be solved are with the boundary conditions  $u(0) = w(0) = 0$ ,

$$(d^2/dr^2 + k^2 + \mu V(r))u(r) + \sqrt{8} \nu W(r)w(r) = 0,$$

$$(d^2/dr^2 + k^2 + \mu V(r) - 2\nu W(r) - 6/r^2)w(r) + \sqrt{8} \nu W(r)u(r) = 0,$$

or with an abbreviation, (7) is rewritten as

$$-L\phi=0, \quad \phi=\begin{bmatrix} u(r) \\ \tau v(r) \end{bmatrix}.$$

For such coupled equations there are two independent solutions. We choose these two trial functions by setting  $\theta=0$  as follows<sup>3)</sup>

$$\phi^{(1)}=\begin{cases} u^{(1)}(r)\rightarrow\cos \varepsilon^{(1)}(\cos kr+\cot \delta^{(1)}\sin kr), \\ \tau v^{(1)}(r)\rightarrow\sin \varepsilon^{(1)}(n_2(kr)+\cot \delta^{(1)}j_2(kr)), \quad (r\rightarrow\infty). \end{cases} \quad (8)$$

$$\phi^{(2)}=\begin{cases} u^{(2)}(r)\rightarrow-k^2\sin \varepsilon^{(2)}(\cos kr+\cot \delta^{(2)}\sin kr), \\ \tau v^{(2)}(r)\rightarrow k^2\cos \varepsilon^{(2)}(n_2(kr)+\cot \delta^{(2)}j_2(kr)). \end{cases}$$

$$n_2(kr)=\sqrt{\pi kr/2} J_{-5/2}(kr)\rightarrow-\cos kr,$$

$$j_2(kr)=\sqrt{\pi kr/2} J_{5/2}(kr)\rightarrow-\sin kr, \quad J_l(x); \quad l\text{-th order Bessel function,}$$

where  $\delta^{(1)}$ ,  $\delta^{(2)}$  are the phase shifts for  $S$ -predominant,  $D$ -predominant waves  $\phi^{(1)}$ ,  $\phi^{(2)}$ , and  $\varepsilon$  is a quantity concerning the mixing of  $S$  and  $D$ -waves. For correct solutions  $\phi^{(i)}$ , the relation:  $\varepsilon^{(1)}=\varepsilon^{(2)}=\varepsilon$  holds. We have first to get the stationary expressions for  $\delta^{(1)}$ ,  $\delta^{(2)}$  and  $\varepsilon$ . To do this it is convenient to use the following exact relations, from which one can easily find the variational formulae.

$$k^{4i-3}\cot \delta^{(i)}\cos(\varepsilon-\varepsilon^{(i)})=k^{4i-3}\cot \delta^{(i)}\cos(\varepsilon-\varepsilon^{(i)})+(\phi^{(i)}L\phi^{(i)})-(\Delta\phi^{(i)}L\Delta\phi^{(i)}), \quad (i=1,2)$$

$$k^3\sin(\varepsilon-\varepsilon^{(2)}) (\cot \delta^{(2)}-\cot \delta^{(1)})=-(\phi^{(1)}, L\phi^{(2)})+(\Delta\phi^{(1)}L\Delta\phi), \quad (9)$$

$$k^3\sin(\varepsilon-\varepsilon^{(1)}) (\cot \delta^{(2)}-\cot \delta^{(1)})=-(\phi^{(2)}, L\phi^{(1)})+(\Delta\phi^{(2)}L\Delta\phi^{(1)}).$$

where  $\Delta\phi^{(i)}\equiv\phi^{(i)}-\phi^{(i)}=O(\mathcal{A})$ , and  $(\quad)$  denotes both radial integration and inner product of  $2-2$  matrix related to  $u(r)$  and  $\tau v(r)$ . Since the quantities of order  $\mathcal{A}^2$  may be dropped in the variational expressions, the stationary expressions<sup>4)</sup> for  $\delta^{(1)}$ ,  $\delta^{(2)}$  will be derived by noticing that

$$\cos(\varepsilon-\varepsilon^{(i)})=1+O(\mathcal{A}^2), \quad (\Delta\phi^{(i)}L\Delta\phi^{(i)})=O(\mathcal{A}^2),$$

and similarly we have

$$\sin \varepsilon=\sin \varepsilon^{(i)}+\mathcal{A}^{(i)}\cos \varepsilon^{(i)}+O(\mathcal{A}^2), \quad (i=1,2)$$

$$\sin(\varepsilon-(\varepsilon^{(1)}+\varepsilon^{(2)})/2)=(\mathcal{A}^{(1)}+\mathcal{A}^{(2)})/2+O(\mathcal{A}^2), \text{ etc,}$$

$$\mathcal{A}\equiv(\phi^{(i)}, L\phi^{(j)})/\{k^3(\cot \delta^{(i)}-\cot \delta^{(j)})\}, \quad (i\neq j)$$

and also one's corresponding to  $\cos \varepsilon$  or  $\tan \varepsilon$ . For applying the present method to this problem, we first take the trial functions  $\phi^{(1)}$  and  $\phi^{(2)}$  which involve the four adjustable parameters  $\delta^{(i)}$ ,  $\varepsilon^{(i)}$ , ( $i=1,2$ ) as in (8). These parameters are determined, for example, with  $L\equiv H_0-\mu H_1-\nu H_2$ , by

$$(\phi^{(i)}, L\phi^{(j)})=0. \quad (i=1,2, j=1,2).$$

$\delta^{(i)}$  and  $\varepsilon^{(i)}$  thus determined have stationary character as seen from the above arguments.

Noticing the  $S$ -predominant solutions, then there exist eigenfunctions  $\psi_k^{(1)}$  which give above  $\delta^{(1r)}$  and  $\varepsilon^{(1r)}$  asymptotically, and corresponding eigen-values  $\mu_k^{(1)}$  and  $\nu_k^{(1)}$ . For these eigenvalues there exist also  $D$ -predominant solutions  $\psi_k^{(2)}$  in which  $S$  and  $D$ -waves have the same phase shift as in (8). Let  $\phi_k^{(1)}$  and  $\phi_k^{(2)}$  be the trial functions which are close to  $\psi_k^{(1)}$  and  $\psi_k^{(2)}$ .  $\phi_k^{(1)}$  have to give above  $\delta^{(1r)}$  and  $\varepsilon^{(1r)}$  asymptotically.  $\sigma_k^{(1)}$ ,  $\rho_k^{(1)}$  are defined as

$$(\phi_k^{(i)}, (H_0 - \sigma_k^{(i)} H_1 - \rho_k^{(i)} H_2) \phi_k^{(i)}) = 0, \quad (i=1,2)$$

which are an extension of  $(\phi_k(A - \omega_k B) \phi_k) \equiv 0$  in Sec. 1. And

$$\sigma_k^{(1)} = \mu_k^{(1)} + O(\mathcal{A}^2), \quad \rho_k^{(1)} = \nu_k^{(1)} + O(\mathcal{A}^2); \quad \phi_k^{(2)} - \psi_k^{(2)} = O(\mathcal{A}),$$

are verified from (9). Then we have

$$\begin{aligned} \phi_k^{(1)} &= \phi_k^{(1)} - \sum_{k' \neq k} \alpha_{kk'}^{(1)} \phi_{k'}^{(1)} + O(\mathcal{A}^2), \quad \psi_k^{(1)} = \psi_k^{(1)} + O(\mathcal{A}^2), \\ \alpha_{kk'}^{(1)} &\equiv (\phi_{k'}^{(1)} (H_0 - \sigma_k^{(1)} H_1 - \rho_k^{(1)} H_2) \phi_k^{(1)}) / (\phi_{k'}^{(1)} (H_0 - \sigma_k^{(1)} H_1 - \rho_k^{(1)} H_2) \phi_{k'}^{(1)}). \end{aligned}$$

For the  $D$ -predominant solutions the arguments are given in the similar way. Thus it is possible to find  $\delta^{(1)}$ ,  $\delta^{(2)}$  and  $\varepsilon$  with super-stationary character. To more general problems the argument may be extended in a similar way.

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## On Static Solutions of Einstein's Generalized Theory of Gravitation, I

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In Einstein's generalized theory of gravitation, we clarify the expressions of the basic quantity of the electromagnetic field and define "field strength" so as to introduce "four-vector potential". When the distribution of matter and charge is static and spherically symmetric, we can choose, by using the electrostatic potential, a suitable boundary condition independently of the coordinate system used and can overcome the difficulty of Wyman and others. After that the asymptotic behaviour of Wyman's solution is studied and the resulting forms are found to be satisfactory. The mutual effect of matter and charge is also discussed and the results are compared with those of general relativity for a point charge.

## § 1. Introduction

Since Einstein proposed a generalized theory of gravitation,<sup>1)</sup> many authors have studied static spherically symmetric solutions of the field equations. In these investigations it has been assumed that the non-symmetric fundamental tensor  $g_{\lambda\mu}$  is spherically symmetric, that is, form-invariant under the group of "spatial" rotations. In the static case, such a tensor has the form

$$g_{\lambda\mu} = \begin{pmatrix} -A(r), & 0, & 0, & f(r) \\ 0, & -B(r), & h(r) \sin \theta, & 0 \\ 0, & -h(r) \sin \theta, & -B(r) \sin^2 \theta, & 0 \\ -f(r), & 0, & 0, & C(r) \end{pmatrix}, \quad (1.1)$$

in "spherical polar coordinates"  $x^\lambda = (r, \theta, \varphi, ct)$ . At first, two solutions were given by Papapetrou satisfying the conditions  $h=0, f \neq 0$  (case I) and  $f=0, h \neq 0$  (case II) respectively.<sup>2)\*</sup> Afterwards Takeno et al. obtained a solution such as  $f/h \neq 0$ , which reduces to that of the case I or II of Papapetrou by putting  $h=0$  or  $f=0$  respectively.<sup>4)</sup> The solution of the case I of Papapetrou is the most general one under the condition  $h=0$ , while that of the case II is quite particular. The general solution for the latter case (i.e., the case  $f=0$ ) was obtained by Wyman by a skilful method.<sup>5)</sup> In the above researches

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\*) Papapetrou has solved Schrödinger's final affine field equations, which contain a constant  $\lambda$ .<sup>3)</sup> His solutions reduce to those of Einstein's new theory by putting  $\lambda=0$ . Throughout this paper "Papapetrou's solutions" are to be understood in this sense.

the components of  $g_{\lambda\mu}$  are real quantities. More general solutions in which the components are allowed to be imaginary were derived by Bonnor.<sup>6)</sup>

The general solution of the case  $h=0$  has the following properties: the symmetric part of the fundamental tensor  $g_{\lambda\mu}$  goes over asymptotically for  $r \rightarrow \infty$  to Schwarzschild's exterior solution of general relativity; the antisymmetric part  $g_{\lambda\mu}$  describes the field of a magnetic pole, if one identifies the dual of  $g_{\lambda\mu}$  with the electromagnetic field strength. In Einstein's new theory it is considered that the charge-current density is given by

$$j^\lambda = \epsilon^{\lambda\alpha\beta\gamma} g_{\alpha\beta,\gamma} / 3!, \quad (1.2)$$

where  $\epsilon^{\lambda\mu\nu\sigma}$  is the Levi-Civita tensor density. Then it follows that no charge and current can exist anywhere in the case  $h=0$ , because the right-hand member of (1.2) vanishes identically. We must therefore discuss solutions such as  $h \neq 0$ , when we wish to deal with the electromagnetic field produced by sources.

In the case  $h \neq 0$ ,  $f=0$ , the works of the above authors are not sufficient in the following points.

(1) There is some difficulty in choosing suitable boundary conditions, which state that matter and charge have no effect at a large distance. We shall explain this in some detail. As the expression of the boundary condition, the above authors have assumed that  $g_{\lambda\mu}$  approaches to its Galilean values as  $r \rightarrow \infty$ . At first sight this assumption seems natural, but it is not invariant for the coordinate transformations and is significant only when the coordinate system is fixed. For instance, if we impose this assumption on Wyman's solution in spherical polar coordinates,  $h$  vanishes identically and the solution degenerates into that for the pure gravitational field. However, in certain other coordinates, the same assumption implies that  $h$  does not necessarily vanish and there can exist some solutions containing both the gravitational and electromagnetic fields. (See § 4.) This has been pointed out by Wyman himself. We see similar situations also in Bonnor's solutions and he has chosen a certain advantageous coordinate system in putting boundary conditions.

(2) There are no profound and convincing investigations concerning the connection between the obtained solutions and those of the corresponding classical problems. As is well known, Maxwell's electromagnetic theory gives satisfactory results to a great extent. Also, in general relativity, the spherically symmetric solution of the field equations for empty space explains the observed facts successfully. Consequently it is desirable that, as the field becomes weak for  $r \rightarrow \infty$ , some of the spherically symmetric solutions of Einstein's new theory may approach to the corresponding solutions of classical Einstein's and Maxwell's equations (Schwarzschild's exterior solution, the Coulomb field of a point charge, etc.). There have been some attempts to clarify this point, but they have not yet arrived at satisfactory conclusions. For example, Bonnor concluded that the electric part of his solutions tends to the field of a point charge as  $r \rightarrow \infty$ . But he made only a little discussion on the asymptotic form of the gravitational part.

(1) and (2) are the least requirements to be satisfied in order that the solutions of the field equations have physical meanings. In more detail, (1) is a criterion to select

solutions of physical meanings out of those obtained by purely mathematical computations. Further, if one finds a solution meeting these requirements satisfactorily, one may consider it a basis on which one can conclude that Einstein's new theory is a good generalization of general relativity.

In this paper we shall first define the quantity which corresponds to the strength of the Maxwell field. When a new theory is proposed in physics, one must first make clear in what manner the basic quantity of the theory is related to the classical one or, in other words, what expression the physical quantity assumes in the theory. Without this it is almost impossible to interpret the resulting solutions physically. Nevertheless, it seems that the above authors have been eager only to solve the field equations and have paid little attention to the study of the expression of a quantity corresponding to the field strength of Maxwell's theory. We begin with the clarification of this point. We shall first impose no restriction on the distribution of matter and charge and shall search for the mathematical expression of "electromagnetic field strength" so as to be able to introduce "four-vector potential" as in the ordinary theory (§ 2). Then we shall clarify the connection between the electrostatic potential and the fundamental tensor, when the magnetic field is absent and the electric one is static. Further it will be shown that, if the distribution of matter and charge is spherically symmetric, the fundamental tensor must be spherically symmetric, that is, assumes the form (1.1) in a suitable coordinate system (§ 3).

By using the results of our discussion concerning the electromagnetic field quantity, it will be shown that the above requirements (1) and (2) can be satisfied. That is, the boundary condition is no more dependent on the coordinate system used, if we impose the condition on the electrostatic potential instead of  $g_{\lambda\mu}$  (§ 4). In this way it will also be shown that the general solution of the case  $f=0$  has the following asymptotic form for  $r \rightarrow \infty$ :  $g_{\lambda\mu}$  goes over to Schwarzschild's exterior solution and the electrostatic potential to that of a point charge (§ 5). We shall further study the mutual effect of matter and charge and shall compare the results with that of general relativity for the corresponding problem (§ 6).

## § 2. Introduction of four-vector potential

In Einstein's generalized theory of gravitation the fundamental quantity is a nonsymmetric tensor  $g_{\lambda\mu}$ , whose symmetric part  $g_{\lambda\mu}$  and antisymmetric part  $g_{\lambda\mu}^*$  refer to the field of gravitation and to that of electromagnetism respectively. If we put  $g_{\lambda\mu}^*=0$ , the field equations reduce to those of gravitation in general relativity. Accordingly  $g_{\lambda\mu}^*$  may be considered to represent the gravitational potential in the same way as the metric tensor does in general relativity.\*

On the other hand, there are some difficulties for the case of the electromagnetic field. That is, even when the gravitational field vanishes and  $g_{\lambda\mu}$  is assumed to be the metric of the Minkowski space-time (for example,  $g_{11}=g_{22}=g_{33}=-g_{44}=-1$ , other  $g_{\lambda\mu}$

\*) There are other possible interpretations. For example, see Reference 7).

$=0$ ), the resulting field equations are not linear with respect to  $g_{\lambda\mu}$  and do not coincide with those of Maxwell. Similar situations hold for the weak field. For instance, the equations for linear approximation are weaker than those of Maxwell.<sup>9)</sup> Also, for the second approximation, Schrödinger has drawn some conclusions which are entirely different from those obtained by Maxwell's theory.<sup>10)</sup>

These show the fact that it is difficult to find a classical analogue of the electromagnetic quantities in the new theory. So one may have the following questions concerning the electromagnetic field. What is the basic quantity which is worth being interpreted as the field strength? In what relation is  $g_{\lambda\mu}$  to that basic quantity? It seems to the present author that the ambiguous attitude towards these problems has brought about the difficulties concerning the boundary value problems. (See § 1.) Therefore, in this section, we shall define the "strength" of the electromagnetic field  $F_{\lambda\mu}$  so as to introduce the "four-vector potential"  $\phi_\lambda$  and then shall express  $F_{\lambda\mu}$  in terms of  $g_{\lambda\mu}$ .

In Maxwell's electromagnetic theory the field strength satisfies

$$F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} + F_{\nu\mu,\lambda} = 0, \quad (2.1)$$

where the comma denotes the ordinary differentiation with respect to the  $x$ 's. This equation implies the existence of four-vector potential  $\phi_\lambda$ , of which  $F_{\lambda\mu}$  is the curl, that is,

$$F_{\lambda\mu} = \phi_{\lambda,\mu} - \phi_{\mu,\lambda}. \quad (2.2)$$

By the use of this  $\phi_\lambda$  many problems become very simple. Accordingly, in the generalized theory of gravitation, we shall introduce "four-vector potential"  $\phi_\lambda$  in like manner. For this purpose we must define "field strength"  $F_{\lambda\mu}$  which satisfies (2.1) identically. However,  $g_{\lambda\mu}$  or its dual tensor is not suitable for being identified with  $F_{\lambda\mu}$ , because they do not necessarily satisfy this condition. In order to obtain from  $g_{\lambda\mu}$  an antisymmetric tensor of the second order satisfying (2.1), we have only to define

$$F_{\lambda\mu} \equiv \epsilon_{\lambda\mu\alpha\beta} g^{\alpha\beta} / 2!, \quad (2.3)$$

where  $g^{\lambda\mu}$  is given by

$$g^{\lambda\mu} = g^{\lambda\mu} \sqrt{-g}, \quad g_{\lambda\mu} g^{\mu\alpha} = \delta_\lambda^\alpha, \quad g = \det. g_{\lambda\mu} (\neq 0). \quad (2.4)$$

$F_{\lambda\mu}$  thus defined satisfies (2.1) on account of one of the field equations:

$$g^{\lambda\mu}_{;\nu} = 0. \quad (2.5)$$

Thus we can introduce the four-vector potential by the defining equation (2.2).

If  $g_{\lambda\mu} = 0$ , we have  $F_{\lambda\mu} = 0$  from the definition of  $F_{\lambda\mu}$ . Conversely when  $F_{\lambda\mu} = 0$ , we have  $g_{\lambda\mu} = 0$ , because it follows from (2.3) that

$$g^{\lambda\mu} = \epsilon^{\lambda\mu\alpha\beta} F_{\alpha\beta} / 2!. \quad (2.6)$$

Consequently, instead of  $g_{\lambda\mu}$ , we may adopt  $F_{\lambda\mu}$  or  $\phi_\lambda$  as the basic quantity of the electromagnetic field.

As the field equations are not linear, it is impossible to superpose the fields and the



introduction of  $\phi_\lambda$  does not make the problem so easy as might be expected from Maxwell's theory. However, it will be made clear that this introduction is not only very advantageous in choosing boundary conditions (§ 4), but also useful in comparing the results of the new theory with those of Maxwell's (§ 5).

We shall now express  $F_{\lambda\mu}$  in terms of

$$h_{\lambda\mu} = g_{\lambda\mu} \quad \text{and} \quad f_{\lambda\mu} = g_{\lambda\mu}, \quad (2.7)$$

for later uses. In the following the indices of  $f_{\lambda\mu}$  are raised and lowered by means of  $h^{\lambda\mu}$  and  $h_{\lambda\mu}$ , where  $h^{\lambda\mu}$  is the conjugate of  $h_{\lambda\mu}$ . We have from (2.4)

$$g_S^{\lambda\mu} = g^{\lambda\mu} \{ h^{\lambda\mu} (1 + f_{\alpha\beta} f^{\alpha\beta} / 2) - f^{\lambda\alpha} f^{\mu}_{\alpha} + f^{\lambda\mu} + i \epsilon^{\lambda\mu\alpha\beta} f_{\alpha\beta} / 2 \},$$

where we have put

$$\rho = \epsilon^{\alpha\beta\gamma\delta} f_{\alpha\beta} f_{\gamma\delta} / 8, \quad g = \det. h_{\lambda\mu}. \quad (2.8)$$

Substituting this result into (2.3), we obtain

$$F_{\lambda\mu} = -(\rho f_{\lambda\mu} + \epsilon_{\lambda\mu\alpha\beta} f^{\alpha\beta} g / 2) / \sqrt{-g}, \quad (2.9)$$

which is the desired expression for  $F_{\lambda\mu}$ .

When Einstein derived the field equations by the variational method, he has already required that  $g_{\lambda\mu}$  be derived from a "vector potential", that is,<sup>10)</sup>

$$g_{\lambda\mu} = g^{\lambda\mu\alpha}, \quad (2.10)$$

where  $g^{\lambda\mu\alpha}$  is a tensor density which is antisymmetric with respect to all indices. If we define  $g^{\lambda\mu\alpha}$  in terms of the above  $\phi_\lambda$  by

$$g^{\lambda\mu\alpha} = \epsilon^{\lambda\mu\alpha\beta} \phi_\beta, \quad (2.11)$$

(2.10) holds good for this  $g^{\lambda\mu\alpha}$ . Einstein has used the vector potential for convenience<sup>\*</sup> sake in deriving the field equations, while we have succeeded in taking it in the theory as an entity more physical.

### § 3. The electrostatic field

Hereafter we shall assume that the field arises from a static distribution of matter and charge and, for the time being, that the distribution has not necessarily any spatial symmetry in general. In this case it is natural to assume that in a suitable coordinate system the four-vector potential is given by

$$\phi_\lambda = (0, 0, 0, \phi). \quad (3.1)$$

where  $\phi$  is a function of "spatial coordinates"  $x^i = (x^1, x^2, x^3)$ .<sup>\*</sup> We have then from (2.9)

<sup>\*</sup>) Latin indices take the values 1, 2, 3.

$$\phi_{,i} = (\rho f_{i4} + \epsilon_{i4\alpha\beta} f^{\alpha\beta} \underline{g}/2) / \sqrt{-g}, \quad (3.2)$$

$$\rho f_{ij} + \epsilon_{ij\alpha\beta} f^{\alpha\beta} \underline{g}/2 = 0. \quad (3.3)$$

The gravitational field is also static from the assumption and in the above coordinate system we may represent this field by

$$h_{i4} = 0, \quad \dot{h}_{\lambda\mu} = 0, \quad (3.4)$$

where the dot denotes the differentiation with respect to "time coordinate"  $x^5$ . This condition for the staticness is usually assumed in general relativity. Then we obtain from (3.3)

$$\rho = 0, \quad f_{i4} = 0,^* \quad (3.5)$$

hence from (3.2)

$$\phi_{,i} = \epsilon_{i4\alpha\beta} f^{\alpha\beta} \underline{g}/2 \sqrt{-g},$$

or

$$\phi_{,i} = f^{jk} \underline{g} / \sqrt{-g}, \quad (i, j, k = \text{cycl. } (1, 2, 3)), \quad (3.6)$$

where the symbol  $(i, j, k = \text{cycl. } (1, 2, 3))$  shows that the equation holds good for any cyclic permutation  $(i, j, k)$  of  $(1, 2, 3)$ .

Now the charge-current density is represented by (1.2), i.e.  $j^\lambda = \epsilon^{\lambda\alpha\beta\gamma} g_{\alpha\beta\gamma}^r / 3!$ , which is based on the fact that  $j^\lambda$  satisfies the continuity equation  $j^\alpha_{;\alpha} = 0$ . We note further that, for linear approximation,  $j^\lambda = j^\lambda / \sqrt{-g}$  is identical with  $H^{\lambda\alpha}_{;\alpha}$ . When  $g_{\lambda\mu}$  (i.e.,  $h_{\lambda\mu}$ ) is of the form (3.4), it follows for the electrostatic field that

$$j^i = (f_{jk,i} + f_{k4,j} + f_{ij,k}) / 3 = 0,^{**} \quad (i, j, k = \text{cycl. } (1, 2, 3)) \quad (3.7)$$

$$j^4 = (f_{12,3} + f_{23,1} + f_{31,2}) / 3 = \text{function of } x^i \text{ alone.} \quad (3.8)$$

This justifies the definition (3.1) of the electrostatic field.

So far the distribution of matter and charge has been considered to be of no spatial symmetry in general. We shall hereafter assume the distribution to be spherically symmetric. Then the above formulae will be simplified further. That is, the fields being spherically symmetric, we may assume that in spherical polar coordinates

$$g_{\lambda\mu} = \begin{pmatrix} -A(r), & 0, & 0, & 0 \\ 0, & -B(r), & 0, & 0 \\ 0, & 0, & -B(r) \sin^2 \theta, & 0 \\ 0, & 0, & 0, & C(r) \end{pmatrix} \quad (3.9)$$

$$\phi = \phi(r) \quad (3.10)$$

\*) In order to derive (3.5), only the first equation of (3.4) is sufficient. For its derivation see Appendix.

\*\*) The proof of  $f_{ij,i} = 0$  is given in the Appendix.

In this case it follows from (3.6) that

$$\begin{aligned} f_{12} &= f_{23} = 0, \\ \phi' &= -[AC/\{B^2 \sin^2 \theta + (f_{23})^2\}]^{1/2} f_{23}, \end{aligned} \quad (3.11)$$

where the prime denotes the differentiation with respect to  $r$ . Solving (3.11) in terms of  $f_{23}$  we have

$$f_{23} = \pm B \phi' \sin \theta / \sqrt{\{AC - (\phi')^2\}} = \sin \theta \times (\text{function of } r).$$

Therefore, when the fields arise from a static spherically symmetric distribution of matter and charge,  $g_{\lambda\mu}$  must have the form (1.1) with  $f=0$ , i.e.,

$$g_{\lambda\mu} = \begin{pmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & h(r) \sin \theta, & 0 \\ 0, & -h(r) \sin \theta, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix}, \quad (3.12)$$

which is usually derived from the invariancy property of  $g_{\lambda\mu}$  under the group of spatial rotations. As for the electrostatic potential and the charge-current density we have from (3.11) and (3.8)

$$\phi = - \int \sqrt{\{AC/(B^2 + h^2)\}} h dr + \text{const.}, \quad (3.13)$$

$$j^4 = h' \sin \theta / 3. \quad (3.14)$$

In the following sections, the above arguments will be applied to examine the static spherically symmetric solution obtained by Wyman.

#### § 4. The problem of boundary conditions

It follows, from the results of the preceding sections, that we can solve the field equations on the assumptions (3.9) and (3.12), when the distribution of matter and charge is static and spherically symmetric. The resulting solutions will contain some integral constants, whose values must be determined so as to satisfy a suitable boundary condition. This condition usually states that the field vanishes at large distance from matter and charge.

In formulating this condition mathematically, Wyman met with the following difficulty: On passing from the spherical polar to the "cartesian coordinates"  $x^\lambda = (x, y, z, ct)$  by

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta, \quad t = t, \quad (4.1)$$

(3.12) reduces to the form

$$g_{\lambda\mu} = \begin{pmatrix} 0 & vz/r, & -vy/r, & 0 \\ -vz/r, & 0, & vx/r, & 0 \\ vy/r, & -vx/r, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{pmatrix}, \quad v = h/r^2. \quad (4.2)$$

He considered that matter and charge distribute at finite distance from the "spatial origin", and assumed as the boundary condition that :

$$\text{The field approaches to zero as } r \rightarrow \infty. \quad (4.3)$$

For the electromagnetic field this was expressed mathematically by

$$g_{\lambda\mu} \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \quad (4.4)$$

At first sight this expression seems natural, but it has different meanings according to the coordinate system chosen. For example, in the spherical polar and the cartesian coordinates, (4.4) has the form  $1/r \rightarrow 0$  and  $1/r^2 \rightarrow 0$  respectively. The former is obviously stronger than the latter and both are not the same. Thus the mathematical expression (4.4) of the boundary condition (4.3) depends on the coordinate system used and has no definite meanings. This is the difficulty stated in § 1. It seems to the present author that this difficulty may have been caused by the ambiguity for the basic quantity of the electromagnetic field. In fact, if we use the results of the preceding sections, we can easily overcome the difficulty. In the following we shall show this in detail.

Before carrying out this plan, we shall reflect on the behaviour of (4.3) with respect to the coordinate transformations and shall clarify the intrinsic nature of the above difficulty. It is to be noted that the condition (4.3) itself is not necessarily invariant and depends on the system of coordinates used. As an example, let us imagine a charged matter located at the spatial origin in some coordinate system. If we use another system in which the matter is far away from the origin, then the field does not vanish any more for  $r \rightarrow \infty$ . Thus (4.3) itself has no definite significance till a class of coordinate systems is suitably chosen. However, if one defines by  $x^4 = \text{const.}$  a space-like surface on which matter and charge exist at some moment and further defines spatial infinity on that surface by  $r \rightarrow \infty$ , then (4.3) has certainly a definite meaning. *The mathematical expressions of (4.3) must therefore be invariant at least for the two classes of transformations, such as*

$$S = \{\text{transformations of the spatial coordinates } x^i \text{ which leave infinity invariant}\},$$

$$T = \{\text{linear transformations of the time coordinate } x^4\}^*.$$

The works of Wyman and others have ambiguities in this point, that is, the mathematical expression of (4.3) is changed by a transformation of  $S$  such as (4.1).

Now, as was already shown in § 2, it is not  $g_{\lambda\mu}$  but  $F_{\lambda\mu}$  that is worth being interpreted as the field strength. Consequently, for the electrostatic field, it is natural to express the boundary condition (4.3) in the form

$$\phi \rightarrow \text{const.} (\neq \infty) \quad \text{as} \quad r \rightarrow \infty, \quad (4.5)$$

<sup>1)</sup> By a nonlinear transformation of  $x^4$  the staticness of the field is apparently lost, since some components with the index 4 of a world tensor will be multiplied by a function of  $x^4$  by such a transformation. A similar situation holds for the pure gravitational field and Takeno has discriminated the staticness of the space-time and that of the metric. See Reference 11).

instead of (4.4). This is obviously invariant for  $T$  and also for  $S$ , since  $\phi$  is an invariant scalar for the transformations of  $x^i$ . Thus we have obtained an expression of the boundary condition (4.3), which has an intrinsic meaning for  $S$  and  $T$ . The condition (4.5) is of course not equivalent to (4.4). Even when (4.5) is satisfied and the field strength  $F_{\lambda\mu}$  approaches to zero,  $g_{\lambda\mu}$  does not necessarily go to zero.

As for the gravitational field, we may express (4.3) in the form

$$g_{\lambda\mu} \rightarrow \text{Galilean values} \quad \text{as} \quad r \rightarrow \infty, \quad (4.6)$$

as in general relativity. This takes the form

$$A \rightarrow 1, \quad B \rightarrow r^2, \quad C \rightarrow 1 \quad \text{as} \quad r \rightarrow \infty, \quad (4.7)$$

in the spherical polar coordinates. If we make a transformation:  $t = \tau' + \text{const.}$ , where  $\tau'$  is constant, then  $C \rightarrow 1$  reduces to  $'C \rightarrow \tau'^2$  (=the Galilean value of  $g_{44}$  in the new coordinates). In this sense  $C \rightarrow 1$  is invariant for  $T$ . It is also invariant for  $S$ . The condition  $B \rightarrow r^2$  will be used in the form  $B \rightarrow \infty$  in the actual computations of § 5. The latter condition shows that the area on a sphere  $r = \text{const.}$  subtending a definite solid angle increases infinitely as  $r \rightarrow \infty$ . It can therefore be considered that  $B \rightarrow r^2$  has an invariant meaning for  $S$  and  $T$ . As will be seen in § 5, the condition  $A \rightarrow 1$  will determine not the values of integral constants but the asymptotic form of  $C$ . In other words, it is not a restriction on the solution itself but a condition for the choice of the radial coordinate  $r$ . Thus we can conclude that the expression (4.6) of the boundary condition is invariant for  $S$  and  $T$ .

Thus the requirement (1) of § 1 is satisfied. It is worth noticing that this is not achieved until we introduce the electrostatic potential in the new theory.

## § 5. The asymptotic form of a spherically symmetric solution

Throughout the remainder of this paper, we shall deal with the static spherically symmetric solution such that  $f = 0$  and  $h$  is real. The general solution in this case was derived by Wyman. His solution is given by that  $C$  is an arbitrary function of  $r$ , and  $A$ ,  $B$  and  $h$  satisfy

$$h + iB = 4m^2 p / [(e^a C^{1/2} \sqrt{V_p} + e^{-a} C^{-1/2} \sqrt{V_p})^2 C(q + i)], \quad (5.1)$$

$$A = C'^2 (B^2 + h^2) / 4m^2 C. \quad (5.2)$$

In these equations,  $m (\neq 0)$  and  $q$  are arbitrary real constants,  $a$  is an arbitrary complex constant, and  $p = 1 + i(3e/m^2)$ , where  $e$  is an arbitrary real constant\*.

In this section, using the boundary conditions (4.5) and (4.7), we shall determine these constants and shall derive the asymptotic form of the solution.

\* We do not treat of the Wyman's solutions for which  $m = 0$ . By a similar way as in this section, we can see that one of them corresponds to a charge distribution without mass and the other to the trivial case without matter and charge.



By substituting (4.7) into (5.1), we find

$$e^{2\alpha} = -1, \quad (5.3)$$

and then (5.1) becomes

$$h + iB = -\frac{4m^2\sqrt{p}}{q+i} \cdot \frac{d}{dX} \left\{ \frac{1}{(1-X)^{1/p} - 1} \right\}, \quad (5.4)$$

where we have put

$$C = 1 - X. \quad (5.5)$$

If we expand  $\{(1-X)^{1/p} - 1\}^{-1}$  in terms of  $X$  and then differentiate, the imaginary and real parts of (5.4) reduce to

$$B = \frac{4m^2}{(q^2+1)X^2} + \frac{eq}{q^2+1} + \frac{eqX}{q^2+1} + O(X^2), \quad (5.6)$$

and

$$h = -\frac{4m^4q}{(q^2+1)X^2} + \frac{e}{q^2+1} + \frac{eX}{q^2+1} + O(X^2) \quad (5.7)$$

respectively, where  $O(X^2)$  denotes terms of the second and higher orders in  $X$ .

On account of (5.2), (3.13) reduces to

$$\phi = -(1/2m) \{ C' h dr + \text{const.} = (1/2m) \{ h dX + \text{const.},$$

hence from (5.7),

$$\phi = \frac{1}{2m} \left[ -\frac{4m^4q}{(q^2+1)X} + \frac{eX}{q^2+1} + O(X^2) \right] + \text{const.} \quad (5.8)$$

As  $r \rightarrow \infty$ ,  $X \rightarrow 0$ . Accordingly, making use of the condition (4.5), we have from (5.8)

$$q = 0. \quad (5.9)$$

If we substitute (5.6) and (5.7) into (5.2), we have

$$A = (X'/X^2)^2 \{1 + X + O(X^2)\} \{4m^2 + O(X^3)\}. \quad (5.10)$$

Then it follows from (4.7) that  $X' = \pm X^2/2m$  for  $X \ll 1$ , and hence

$$X = \mp 2m/r, \quad (X \ll 1), \quad (5.11)$$

by a suitable choice of the integral constant. This shows that the boundary condition  $A \rightarrow 1$  is a restriction on the asymptotic form of  $C$  and not the condition on the solution itself. (See the end of § 4.)

By means of (5.9) and (5.11),  $B$ ,  $C$  and  $\phi$  have asymptotically the expressions

$$\left. \begin{aligned} B &= r^2 \{1 + O(1/r)\}, \\ C &= 1 - (2m/r) \{1 + O(1/r)\}, \\ \phi &= (e/r) \{1 + O(1/r)\} + \text{const.}, \end{aligned} \right\} \quad (5.12)$$

where we have adopted the positive sign of  $X$  on the analogy of general relativity. The

second term of  $C$  has the same form as in general relativity, which is used to interpret  $m$  as the mass producing the field. The first term of  $\phi$  shows that *the electrostatic field tends to that of Coulomb for a point charge as  $r \rightarrow \infty$ .*\* We can therefore consider the remaining constants of integration as referring to the mass and charge from which the fields arise, that is,

$$m \cdots \text{mass}, \quad e \cdots \text{charge}. \quad (5.13)$$

Thus it is possible to conclude that the above solution of the field equations is a significant one, and that we are given assurance for our discussion in the preceding sections.

As a further result of our boundary conditions, we shall prove that  $g_{\lambda\mu}$  tends to the metric of the Schwarzschild space-time asymptotically for  $r \rightarrow \infty$ . In Wyman's solution  $C$  is given arbitrarily with the only restriction (5.12). Let us now write

$$X = (2m/r) \{1 + k/r + O(1/r^2)\}, \quad (5.14)$$

where  $k$  is an arbitrary constant. Then we have

$$X'/X^2 = -(1/2m) \{1 + O(1/r^2)\},$$

and hence from (5.10)

$$A = 1 + (2m/r) \{1 + O(1/r)\}, \quad (5.15)$$

whatever the value of  $k$  may be. Thus it is shown that  $A$ ,  $B$  and  $C$  given by (5.12) and (5.15) reduce to those of the Schwarzschild space-time for large  $r$ .

From these results it may be concluded that Einstein's new theory is a good generalization of the theory of gravitation. And it is to be noted that *the asymptotic form of the solution thus obtained is a consequence of the boundary condition (4.5) and (4.7) only, and any further condition has not been assumed.*

## § 6. The mutual effect of matter and charge

Since it is likely that the mutual effect of matter and charge does not appear at a great distance, the solution is expected to approach to that derived from the classical theory for each free field. The result of the last section actually presents this feature. In this section, going a step further, we shall study what effect they have with each other. For that purpose we expand the solution asymptotically to the higher order.

Assuming that  $X$  has the expansion formula (5.14), it follows that

$$\begin{aligned} B &= r^2 \{1 - 2k/r + O(1/r^2)\}, \\ \phi &= (e/r) \{1 + (m+k)/r + O(1/r^2)\} + \text{const.}, \end{aligned} \quad (6.1)$$

in the same way as in the last section. In order to obtain the term of the order  $1/r^2$  in the expansion of  $A$ , we must consider the term of the order  $1/r^3$  in  $C$  as well as that

\*) This does not mean that the field arises from a point charge. Rather it may possibly be produced by a dispersed distribution of charge, as is seen from the form of the charge density (3.14). See Reference 6).

of the order  $1/r^2$ , as is easily seen by a direct computation. Accordingly, since  $C$  can be arbitrarily chosen with only one restriction (5.12), we shall assume for convenience' sake that the term of the order  $1/r^3$  in  $C$  vanishes, that is,

$$C = 1 - (2m/r) \{1 + k/r + O(1/r^3)\}, \quad (6.2)$$

which is always possible by a transformation of the form  $r \rightarrow r + O(1/r)$ . Then from (5.10) we have

$$A = 1 + 2m/r + 2(2m^2 + mk - k^2)/r^2 + O(1/r^3). \quad (6.3)$$

When the spherically symmetric space-time is treated in general relativity, there is ambiguity in choosing the radial coordinate  $r$ . A similar situation appears in Wyman's solution and presents itself in that  $C$  is an arbitrary function of  $r$ . It seems to be natural to determine the radial coordinate so that

$${}'B = {}'r^2 \{1 + O(1/{}'r^2)\}.*$$

For that purpose we may use a transformation

$$r = {}'r + k.$$

Then, by using the transformation law of a tensor, the solution has the following expression in the new coordinates:

$$\begin{aligned} 1/A &= 1 - (2m/r) \{1 - k^2/mr + O(1/r^2)\}, \\ B &= r^2 \{1 + O(1/r^2)\}, \\ C &= 1 - (2m/r) \{1 + O(1/r^2)\}, \\ \phi &= (c/r) \{1 + m/r + O(1/r^2)\} + \text{const.}, \end{aligned} \quad (6.4)$$

where we have omitted the primes which show that the new system of coordinates is used. We note that in the new coordinate system the term of the order  $1/r^2$  in  $C$  vanishes, while that of the order  $1/r^3$  appears.

The constant  $k$  in (6.4) is arbitrary. However, if we assume the above solution (6.4) to coincide with that of Schwarzschild:

$$1/A = C = 1 - 2m/r, \quad B = r^2$$

by putting  $g_{\nu\nu} = 0$ ,  $k$  must be a constant which depends on  $c$  and reduces to zero by putting  $c = 0$ . In the following argument  $k$  will be considered as such a constant.

By a suitable transformation of the form  $r \rightarrow r \{1 + O(1/r)\}$ , we can take  $k$  to be zero. Hence there is a coordinate system in which  $g_{\lambda\mu}$  is of the same form as Schwarzschild's exterior solution to within terms of the second and higher orders.\*\*

\*) It is more natural to take  $'B = {}'r^2$ , but in this case the transformation equation of  $r$  is much complicated and the treatment of the problem becomes difficult.

\*\*) The order in the asymptotic expansion of the solution is considered with respect to  $1/r$ . It is further assumed that, for the 0-th order, the solutions reduce to those of the corresponding classical problems.



To within terms of the same orders, the deviation of  $\phi$  from the Coulomb potential is given by  $me/r^2$ , which is considered as representing the mutual effect of matter and charge. It is noted that, *even if the terms of the second and higher orders are neglected, there is no coordinate system for which  $\phi$  assumes exactly the form of the Coulomb potential.* Otherwise, as is seen from (6.1), we have  $m+k=0$  in such a coordinate system. This is obviously contradictory to the above dependence of  $k$  on  $e$ .

In general relativity the solution for a point charge is given by

$$1/A = C = 1 - 2m/r + 4\pi e^2/r^2, \quad B = r^2, \quad \phi = e/r. \quad (6.5)$$

Comparing (6.5) with (6.4) we can easily see that there appears, even in the first order, a difference between the solution of general relativity and that of Einstein's new theory\*.

The term  $me/r^2$  of  $\phi$  does not vanish in any coordinate system, for which  $B$  has the form (6.4). We can therefore interpret this as the "intrinsic" mutual effect. On the other hand, the term  $2k^2/r^2$  of  $1/A$  in (6.4) depends on the radial coordinate used and can be put to zero by a suitable transformation of  $r$ . Therefore this term can be considered as representing the "apparent" effect.

### Concluding remarks

In Einstein's generalized theory of gravitation there are various antisymmetric tensors of the second order which may be interpreted as the strength of the electromagnetic field. Of these quantities we have chosen a certain one so as to introduce the four-vector potential. By using this potential, when the fields arise from a static spherically symmetric distribution of matter and charge, we have overcome the difficulty concerning the boundary value problems. Further we have derived the asymptotic form of Wyman's solution, which coincides with those of the corresponding classical problems.

Our results are sufficiently satisfactory in the sense that it satisfies the two requirements of § 1. From our standpoint we can discuss the case  $f \rightarrow 0$ ,  $h=0$ , too. Studies for this case will be published in a subsequent paper.

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### Appendix

#### *Proof of (3.5)*

We have only to prove (3.5) at an arbitrary point  $P$  of the space-time. We choose the coordinates so that the components of  $h_{\lambda\mu}$  at  $P$  are

$$h_{11} = h_{22} = h_{33} = -h_{44} = -1, \quad \text{other } h_{\lambda\mu} = 0.$$

Since  $h_{i4}=0$  holds identically, the transformation to such coordinates leaves the assumption

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\*) For example, the terms  $4\pi e^2/r^2$  in (6.5) and  $2k^2/r^2$  in (6.4).

(3.1) unaltered, and we shall prove using these coordinates. Then (3.3) becomes

$$\rho f_{ij} + f_{ki} = 0, \quad (i, j, k = \text{cycl. } (1, 2, 3)).$$

Multiplying by  $f_{ki}$  and summing for  $k$ , we have by means of (2.8)

$$\rho^2 + \sum_k (f_{ki})^2 = 0.$$

From this we can easily obtain (3.5), since  $f_{ii}$  and  $\rho$  are all real quantities.

*Proof of  $f_{ij,i} = 0$  in (3.7)*

We have from (3.6)

$$f^{ij} \underline{g} = \epsilon^{ij\alpha\beta} \phi_{,\alpha} \sqrt{-g}, \quad (\text{A.1})$$

and hence

$$f_{ab} f^{ab} \underline{g}^2 = -g g_{ab} g_{cd} \epsilon^{acp4} \epsilon^{bdq4} \phi_{,p} \phi_{,q}, \quad (\text{A.2})$$

On account of (3.5)  $g$  takes the form

$$g = \underline{g} (1 + f_{ab} f^{ab} / 2).$$

Substituting this into (A.2),  $f_{ab} f^{ab}$  is found to be a function of  $h_{\lambda\mu}$  and  $\phi_{,i}$ . Since  $h_{\lambda\mu,i} = 0$  and  $\phi_{,i} = 0$ , we have  $(f_{ab} f^{ab})_{,i} = 0$ , and accordingly  $g_{,i} = 0$ . Thus we can see the staticness of  $f_{ij}$  by means of (A.1).

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## The Classical Equations of a Point Particle in a Symmetric Meson Field

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The equations of motion of a point particle interacting with charge symmetrical mesons are deduced on the action-at-a-distance theory from a variation principle which is analogous to that of Fokker in electrodynamics. In the first article an action is first defined with the help of a Riesz function in terms of a parameter  $\alpha$  to be determined by the method of analytical continuation. It is then shown that by expressing the mesic charge-current and dipole densities with the help of a Riesz function which for  $\alpha \rightarrow 2$  behaves like Dirac's  $\delta$ -function, the adjunct fields, which are introduced on purely formal grounds, satisfy the fundamental equations of meson field in the usual field theory for  $\alpha \rightarrow 2$ . The variation of the action function for  $\alpha \rightarrow 2$  finally gives the required equations of motion. In the second article the adjunct fields, which are singular on the world line of the particle, are evaluated by Riesz method of analytical continuation for  $\alpha \rightarrow 2$ . The scattering of mesons by nucleons are calculated in the last article.

### Introduction

The derivation of the equation of motion of a point particle involves an intimate knowledge of the behaviour of the fundamental particles and the nature of the interaction between them. The interaction of the particles is customarily described with the help of the field concept: the particle creates a field around itself which then acts on the other particle located in the field. But the idea of the self-field involves the question of the action of the particle upon itself. The conception of the self-force, *i.e.* the force experienced by a moving particle due to its own field has given rise to serious divergence difficulties in the formulation of the theory of fundamental particles and in the construction of their equations of motion. It has been observed that the equations of motion containing the radiative reaction terms are obtained only when the field produced by the particle is taken to be the retarded field. It is, in fact, the retarded field of the particle upon itself that provides the radiative reaction. And as this field is singular at the position of the particle it has to be evaluated by a limiting process as suggested for instance, by Dirac<sup>1)</sup> and Riesz,<sup>2)</sup> to obtain the divergence free equations of motion. In the case of the symmetric field,



defined as the sum of the retarded and advanced fields, the radiative reaction terms are completely absent. Wheeler and Feynman<sup>(3)</sup> have recently revived the alternative theory of action at a distance previously studied by Schwarzschild,<sup>(4)</sup> Tetrode<sup>(5)</sup> and Fokker.<sup>(6)</sup> In this description of direct interparticle action no direct use is made of the concept of the field. Each particle moves in compliance with the principle of stationary action formulated by Fokker. It has been shown by Wheeler and Feynman that the theory of direct interparticle action may be considered to be equivalent to what they call an adjunct field theory in which the fields acting on a given particle arise only from the particles other than the given one and are given by the symmetric fields. They have thereby shown that in the limiting case of universe in which all electromagnetic disturbances are ultimately absorbed, the symmetric fields acting on a given particle are made up of (i) the sum of the retarded fields generated by every particle other than the given one, and (ii) the radiation field, which is defined as half the difference of the retarded and advanced fields produced by the given particle. Wheeler and Feynman thus succeeded in obtaining the equation of motion of a particle interacting with the electromagnetic field which correctly described the radiative reaction. The theory of Wheeler and Feynman has been extended recently by Kanazawa<sup>(7)</sup> and Havas<sup>(8)</sup> to obtain the equations of motion of a nucleon interacting with the neutral meson field characterised by mesic charge  $g$ . The present authors<sup>(9)</sup> generalised the theory further to obtain the equations of motion of a spinning particle in a neutral meson field characterised by a mesic dipole moment  $f$ . It is noteworthy that while the adjunct field theory leads to identical equations of motion for a point particle interacting with the electromagnetic field, it gives equations which differ significantly in the case of a point particle interacting with the meson field with regard to the terms involved in the integrals; the integrations are to be extended over the past and future motion of the particle.

The present paper is an extension of the previous works on the action at a distance theory and deals with the formulation of the equations of motion of a point nucleon in charge symmetrical meson theory in which the combination of positive, negative and neutral mesons are involved in the description of interactions. The classical theory of the interaction of point particles with charged fields was first given by Fierz.<sup>(10)</sup> The corresponding equations of motion of a point particle in charge symmetric scalar and vector meson field in the field theory had been already given by Vachaspati and one of the present authors.<sup>(11)</sup> Le Couteur<sup>(12)</sup> has also established the similar equations in vector meson field. In the first article of the present paper we derive the equations of motion of a point particle for symmetric interactions with mesons from a variation principle which is similar to that of Fokker in electrodynamics. We have thereby defined an action function with the help of a new Riesz function which is an analogue of the usual Green's function and expressed it in terms of a parameter  $\alpha$  which is to be determined by the method of analytical continuation. Further the charge-current and the dipole densities are expressed with the help of a Riesz function which for analytical continuation of the parameter  $\alpha \rightarrow 2$  behaves like Dirac's  $\delta$ -function. It is then shown that the adjunct fields, which are introduced in the theory on purely formal grounds, satisfy the fundamental equations of meson field for  $\alpha \rightarrow 2$ . In the second article the adjunct Riesz fields have been first defined in terms of

an integral which is an analytical function of a parameter  $\alpha$ , assumed large enough for the integrals to be convergent. The fields are then calculated on the world line of the particle by analytical continuation with regard to the parameters  $\alpha$  to 2 for which they satisfy the meson equations. The third article deals with the scattering of mesons by nucleons on the basis of charge symmetrical theory.

### § 1. Variation principle and the equations of motion

We shall derive in this section the equations of motion of a point particle interacting with the symmetric scalar or vector mesons. We shall adopt the symmetrical meson theory of Kemmer<sup>(13)</sup> in which the field quantities are regarded as three component vectors in isotopic spin space (or charge space). Thus

$$\varphi_\mu = (\varphi_{\mu 1}, \varphi_{\mu 2}, \varphi_{\mu 3}) = \alpha_1 \varphi_{\mu 1} + \alpha_2 \varphi_{\mu 2} + \alpha_3 \varphi_{\mu 3}, \quad (1)$$

where  $\alpha_1, \alpha_2, \alpha_3$  are three orthogonal unit vectors in the isotopic spin space;  $\alpha_1$  and  $\alpha_2$  components together describe the charged mesons and  $\alpha_3$  refers to the neutral mesons. If  $\tau_1, \tau_2$  and  $\tau_3$  are the isotopic spin components in this space we have

$$\tau = \alpha_1 \tau_1 + \alpha_2 \tau_2 + \alpha_3 \tau_3. \quad (2)$$

We take the metric tensor  $g_{\mu\nu} = 0$  if  $\mu \neq \nu$  and

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1 \quad (3)$$

and the velocity of light is put equal to unity ( $c=1$ ). We shall denote the coordinates of the  $i$ -th particle by  $x_{i\mu}$  which are functions of the proper time  $\tau_i$ , measured from an arbitrary point on the world line of the  $i$ -th particle. A dot over a symbol will mean differentiation with respect to the proper time and the velocity  $\dot{x}_{i\mu}$  of the  $i$ -th particle will be denoted by  $v_{i\mu}$ . The spins of the particle will be described by tensors  $S_{i\mu}$  and  $S_{4\mu\nu}$  for the scalar and vector mesons respectively and are considered to be a function of  $\tau_i$ .

Following Fokker and incorporating in it the idea of Riesz we now deduce the equations of motion from a variation principle in which the action  $J^{(\alpha)}$ , which is defined for a parameter  $\alpha$  to be determined by the method of analytical continuation, is an extremum. We define the action function as

$$J^{(\alpha)} = J_0^{(\alpha)} + J_{\text{int}}^{(\alpha)}. \quad (4)$$

Here  $J_0^{(\alpha)}$  is the action for the free particles and  $J_{\text{int}}^{(\alpha)}$  represents the action for the interaction of the particles. We write\*

$$J_0^{(\alpha)} = \sum_i \int m_i v_i d\tau_i + \sum_i \int T_i d\tau_i + \sum_i \int L_i d\tau_i, \quad (5)$$

\* The more general equations of motion as obtained by Le Couteur may be easily deduced from the equation (5) by replacing the mass ' $m$ ' by

$$m - \frac{1}{4} L \left( \frac{d\tau}{d\tau} \right)^2 + \frac{1}{4} K \frac{dS_{\sigma\rho}}{d\tau} \frac{dS^{\sigma\rho}}{d\tau},$$

where  $L$  and  $K$  are constants as defined by him. This general case, however, has little practical significance and we are not giving the results here.

where

$$v_i^2 = v_{i\mu} v_i^\mu = 1. \quad (6)$$

The first term stands for the translational motion, the second term for the rotational motion of the spin and the third term for the rotation of the isotopic spin of the point particle.

We define the variation of  $T_i$  as follows

$$\delta T_i = \frac{1}{2} I_i S_{i\mu\nu} \delta \omega_i^{\mu\nu} \quad (7a)$$

for the vector meson, and

$$\delta T_i = \frac{1}{2} I_i \epsilon_{\mu\nu\rho\sigma} \delta \omega_i^{\mu\nu} v_i^\rho S_i^\sigma \quad (7b)$$

for the scalar mesons, where  $\omega_{i\mu\nu}$  is the angular velocity of the  $i$ -th particle, defined by  $\omega_{i\mu\nu} = d\Omega_{i\mu\nu}/d\tau_i$ ,  $\Omega_{i\mu\nu}$  being the angular coordinate.  $\epsilon_{\mu\nu\rho\sigma}$  is a tensor antisymmetric in all indices with  $\epsilon_{0123} = -1$ .

We further define for the case of isotopic spin

$$\delta L_i = \lambda_i \tau_i \cdot \delta \hat{\theta}_i, \quad (8)$$

where  $\theta_i$  denotes the angular coordinate of the isotopic spin of the  $i$ -th particle in charge space and  $\hat{\theta}_i = d\theta_i/d\tau_i$  and  $\lambda_i$  is a constant having the dimension of an angular momentum.

We define

$$\begin{aligned} J_{\text{int}}^{(\alpha)} = & \sum_{i < k} \iint \tau_i \cdot \tau_k \left[ g_i v_{i\nu} + \frac{g_i}{\chi^2} v_{i\rho} \frac{\partial^2}{\partial z_i^\nu \partial z_{i\rho}} - f_i S_{i\mu\nu} \frac{\partial}{\partial z_{i\mu}} \right] \\ & \times \left[ g_k v_{k\nu} + \frac{g_k}{\chi^2} v_{k\sigma} \frac{\partial^2}{\partial z_{k\nu} \partial z_{k\sigma}} - f_k S_{k\lambda} \frac{\partial}{\partial z_{k\lambda}} \right] G^{(\alpha)}(z_i - z_k) d\tau_i d\tau_k \end{aligned} \quad (9a)$$

for the vector meson, and

$$J_{\text{int}}^{(\alpha)} = \sum_{i < k} \iint \tau_i \cdot \tau_k \left[ g_i' v_i - f_i' S_{i\nu} \frac{\partial}{\partial z_{i\nu}} \right] \left[ g_k' v_k - f_k' S_{k\mu} \frac{\partial}{\partial z_{k\mu}} \right] G^{(\alpha)}(z_i - z_k) d\tau_i d\tau_k \quad (9b)$$

for the scalar meson, where  $\chi$  as usual is a characteristic constant for meson having the dimension of a reciprocal length, and  $g, g'$  and  $f, f'$ , are mesic charges and dipole moments for the vector and scalar mesons respectively.  $G^{(\alpha)}(s_k)$  stands for the Riesz-function given by

$$G^{(\alpha)}(s_k) = \frac{\chi^{1-\alpha}}{2^{\alpha-2/2} \Gamma(\alpha/2)} (\chi s_k)^{\alpha-1/2} J_{\alpha-1/2}(\chi s_k), \quad (10)$$

which, as can be easily verified, satisfies the equation

$$(\partial^2/\partial x_\mu \partial x^\mu + \chi^2) G^{(\alpha)}(s_k) = G^{(\alpha-2)}(s_k), \quad (11)$$

where

$$s_k^2 = s_{k\mu} s_k^\mu, \quad s_{k\mu} = x_\mu - z_{k\mu}. \quad (12)$$



We now define

$$\varphi_{k\nu}^{(\alpha)}(x) = \int_{-\infty}^{\infty} \tau_k \left[ g_k \tau'_{k\nu} + \frac{g_k}{\chi^2} \tau_{k\sigma} \frac{\partial^2}{\partial \mathcal{Z}_k^\nu \partial \mathcal{Z}_{k\sigma}} - f_k S_{k\lambda\nu} \frac{\partial}{\partial \mathcal{Z}_{k\lambda}} \right] G^{(\alpha)}(s_k) d\tau_k \quad (13a)$$

$$\begin{aligned} &= g_k \int_{-\infty}^{\infty} \tau_k v_{k\nu} G^{(\alpha)}(s_k) d\tau_k + \frac{g_k}{\chi^2} \frac{\partial^2}{\partial x^\nu \partial x_\sigma} \int_{-\infty}^{\infty} \tau_k \tau'_{k\sigma} G^{(\alpha)}(s_k) d\tau_k \\ &\quad + f_k \frac{\partial}{\partial x_\lambda} \int_{-\infty}^{\infty} \tau_k S_{k\lambda\nu} G^{(\alpha)}(s_k) d\tau_k, \end{aligned} \quad (14a)$$

and

$$G_{k\mu\nu}^{(\alpha)}(x) = \partial \varphi_{k\nu}^{(\alpha)}(x) / \partial x^\mu - \partial \varphi_{k\mu}^{(\alpha)}(x) / \partial x^\nu \quad (15a)$$

for the vector meson, and

$$U_k^{(\alpha)}(x) = \int_{-\infty}^{\infty} \tau_k \left[ g_k' v_k - f_k' S_{k\mu} \frac{\partial}{\partial \mathcal{Z}_{k\mu}} \right] G^{(\alpha)}(s_k) d\tau_k \quad (13b)$$

$$= g_k' \int_{-\infty}^{\infty} \tau_k v_k G^{(\alpha)}(s_k) d\tau_k + f_k' \frac{\partial}{\partial x_\mu} \int_{-\infty}^{\infty} \tau_k S_{k\mu} G^{(\alpha)}(s_k) d\tau_k, \quad (14b)$$

and

$$E_{k\mu}^{(\alpha)}(x) = \partial U_k^{(\alpha)}(x) / \partial x^\mu \quad (15b)$$

for the scalar meson.

It is easily found that by operating  $\tau_k [g_k \tau'_{k\nu} + g_k / \chi^2 \cdot \tau_{k\sigma} \partial^2 / \partial \mathcal{Z}_k^\nu \partial \mathcal{Z}_{k\sigma} - f_k S_{k\lambda\nu} \partial / \partial \mathcal{Z}_{k\lambda}]$  on both sides of the equation (11) and integrating with respect to  $\tau_k$  between the limits  $-\infty$  and  $+\infty$ , we get for vector mesons

$$\left( \frac{\partial^2}{\partial x_\mu \partial x^\mu} + \chi^2 \right) \varphi_{k\nu}^{(\alpha)}(x) = 4\pi J_{k\nu}^{(\alpha)}(x) + \frac{4\pi}{\chi^2} \frac{\partial^2 J_{k\sigma}^{(\alpha)}(x)}{\partial x^\nu \partial x_\sigma} + 4\pi \frac{\partial \Sigma_{k\lambda\nu}^{(\alpha)}(x)}{\partial x_\lambda}, \quad (16a)$$

where

$$J_{k\nu}^{(\alpha)}(x) = \frac{g_k}{4\pi} \int \tau_k v_{k\nu} G^{(\alpha-2)}(s_k) d\tau_k, \quad (17a)$$

$$\Sigma_{k\rho\nu}^{(\alpha)}(x) = \frac{f_k}{4\pi} \int \tau_k S_{k\rho\nu} G^{(\alpha-2)}(s_k) d\tau_k. \quad (18a)$$

It can be further shown that

$$\partial \varphi_{k\mu}^{(\alpha)}(x) / \partial x_\mu = 4\pi / \chi^2 \cdot \partial J_{k\mu}^{(\alpha)}(x) / \partial x_\mu, \quad (19)$$

and

$$\partial G_{k\mu\nu}^{(\alpha)}(x) / \partial x_\mu + \chi^2 \varphi_{k\nu}^{(\alpha)}(x) = 4\pi J_{k\nu}^{(\alpha)}(x) + 4\pi \partial \Sigma_{k\lambda\nu}^{(\alpha)}(x) / \partial x_\lambda. \quad (20a)$$

Similarly for the scalar meson, we obtain

$$\left( \frac{\partial^2}{\partial x_\mu \partial x^\mu} + \chi^2 \right) U_k^{(\alpha)}(x) = 4\pi J_k^{(\alpha)}(x) + 4\pi \frac{\partial \Sigma_{k\nu}^{(\alpha)}(x)}{\partial x_\nu}, \quad (16b)$$

$$\frac{\partial F_{k\mu}^{(\alpha)}(x)}{\partial x_\mu} + \chi^2 U_k^{(\alpha)}(x) = 4\pi J_k^{(\alpha)}(x) + 4\pi \frac{\partial \Sigma_{k\nu}^{(\alpha)}(x)}{\partial x_\nu}, \quad (20b)$$

where

$$J_k^{(\alpha)}(x) = \frac{g'_k}{4\pi} \int \tau_k G^{(\alpha-2)}(s_k) d\tau_k, \quad (17b)$$

$$\Sigma_{k\nu}^{(\alpha)}(x) = \frac{f'_k}{4\pi} \int \tau_k S_{k\nu} G^{(\alpha-2)}(s_k) d\tau_k. \quad (18b)$$

It is to be noted that for  $\alpha=2$ ,

$$G^{(\alpha-2)}(s_k) = 0 \quad (21)$$

for a point outside the world line of the  $k$ -th particle, and hence  $J_k^{(2)}$  and  $\Sigma_{k\nu}^{(2)}$  for the vector meson, and  $J_k^{(2)}$  and  $\Sigma_{k\nu}^{(2)}$  for the scalar meson also vanish outside the world line. Further following Fremberg it can be shown by analytical continuation to  $\alpha=2$  that at any point on the world line of the  $k$ -th particle

$$J_{k\nu}^{(2)} = g_k / 4\pi \cdot \tau_k v_{k\nu}(\tau_k), \quad (22a)$$

$$\Sigma_{k\rho\nu}^{(2)} = f_k / 4\pi \cdot \tau_k S_{k\rho\nu}(\tau_k) \quad (23a)$$

for the vector meson, and

$$J_k^{(2)} = g'_k / 4\pi \cdot \tau_k, \quad (22b)$$

$$\Sigma_{k\nu}^{(2)} = f'_k / 4\pi \cdot \tau_k S_{k\nu}(\tau_k) \quad (23b)$$

for the scalar meson.

It further follows from (19) and (21) that for  $\alpha=2$ ,

$$\sum_{k \neq \alpha} \partial \varphi_{k\mu}^{(\alpha=2)} / \partial x_{\alpha\mu} = 0. \quad (24)$$

It should be noted that the quantities defined by (14) and (15) satisfy the fundamental equations for the meson field when  $\alpha=2$  and may therefore be considered equivalent to the potentials and fields adjunct generated by the  $k$ -th particle at the point  $x_\mu$ . It is to be further observed that the potentials as defined above are distinguished from all other solutions of meson equations by being symmetric, i.e. equal to half the sum of the retarded and advanced potentials. Similarly the adjunct fields are also symmetric fields:

We can now write the action function representing the interactions

$$\begin{aligned} J_{\text{int}}^{(\alpha)} = & \sum_{i < k} g_i \int \tau_i \cdot v_{i\nu} \varphi_k^{(\alpha)\nu}(s_{ik}) d\tau_i + \sum_{i < k} \frac{g_i}{\chi^2} \int \tau_i \cdot v_{ip} \frac{\partial^2 \varphi_k^{(\alpha)\nu}(s_{ik})}{\partial x_i^\nu \partial x_{ip}} d\tau_i \\ & - \sum_{i < k} f_i \int \tau_i \cdot S_{i\mu\nu} \frac{\partial \varphi_k^{(\alpha)\nu}(s_{ik})}{\partial x_{i\mu}} d\tau_i, \end{aligned} \quad (25a)$$

and

$$J_{\text{int}}^{(\alpha)} = \sum_{i < k} g'_i \int \tau_i \cdot v_i U_k^{(\alpha)}(s_{ik}) d\tau_i - \sum_{i < k} f'_i \int \tau_i \cdot S_{i\nu} \frac{\partial U_k^{(\alpha)}(s_{ik})}{\partial x_{i\nu}} d\tau_i, \quad (25b)$$

for vector and scalar mesons respectively and where

$$S_{ik}^{\alpha} = (z_{i\mu} - z_{k\mu}) (z_i^{\mu} - z_k^{\mu}). \quad (26)$$

To determine the equations of motion, we shall have to perform the variation of the action functions given by (4), (5) and (25) subject to the conditions

$$v^2 = v_{\mu} v^{\mu} = 1, \quad S_{\mu\nu} v^{\nu} = 0, \quad S_{\mu} v^{\mu} = 0. \quad (27)$$

We therefore consider the variation of

$$P^{(\alpha)} = J_0^{(\alpha)} + J_{\text{int}}^{(\alpha)} + \sum_i \{ \xi_i (v_{i\mu} v_i^{\mu} - 1) d\tau_i + \sum_j \{ \eta_j^{\mu} S_{i\mu\nu} v_i^{\nu} d\tau_i, \quad (28a)$$

$$P^{(\alpha)} = J_0^{(\alpha)} + J_{\text{int}}^{(\alpha)} + \sum_i \{ \xi_i (v_{i\mu} v_i^{\mu} - 1) d\tau_i + \sum_j \{ \eta_j S_{i\nu} v_i^{\nu} d\tau_i \quad (28b)$$

for vector and scalar mesons respectively, where  $\xi_i$ ,  $\eta_i^{\mu}$  and  $\xi_i$ ,  $\eta_i$  are Lagrange's multipliers to be determined later, and then pass to the limit  $\alpha \rightarrow 2$  by the process of analytical continuation.

*The translational equations of motion:*

Now by altering the world line of the particle 'a' from  $z_a^{\mu}(\tau_a)$  to  $z_a^{\mu}(\tau_a) + \delta z_a^{\mu}(\tau_a)$  and requiring that the variations in  $P$  in (28) are zero, we obtain the translational equation of motion of the  $a$ -th particle for  $\alpha \rightarrow 2$  with the help of the relation (24)

$$m_a \dot{v}_{a\mu} + \frac{d}{d\tau_a} \{ 2\xi_a v_{a\mu} + \eta_a^{\lambda} S_{a\lambda\mu} \} = \sum_{k \neq a} g_a \{ \tau_a \cdot G_{k\mu\nu}^{(\alpha=2)} v_a^{\nu} - \tau_a \cdot \varphi_{k\mu}^{(\alpha=2)} \} \\ - \frac{1}{2} \sum_{k \neq a} f_a \tau_a \cdot S_{a\lambda\nu} \frac{\partial G_k^{(\alpha=2)\lambda\nu}}{\partial z_a^{\mu}}, \quad (29a)$$

and

$$m_a \dot{v}_{a\mu} + \frac{d}{d\tau_a} \{ 2\xi_a v_{a\mu} + \eta_a S_{a\mu} \} = \sum_{k \neq a} g_a' \left\{ \tau_a \cdot E_{k\mu}^{(\alpha=2)} - \frac{d}{d\tau_a} (\tau_a \cdot v_{a\mu} U_k^{(\alpha=2)}) \right\} \\ - \sum_{k \neq a} f_a' \tau_a \cdot S_{a\nu} \frac{\partial E_k^{(\alpha=2)\nu}}{\partial z_a^{\mu}} \quad (29b)$$

for vector and scalar mesons respectively.

*The rotational equations of motion:*

The equations of motion for the rotation of the spin of the point particle are easily obtained by the variation of the angular co-ordinate of the particle 'a' and using the relation (7). Thus we obtain by passing to the limit  $\alpha \rightarrow 2$ ,

$$I_a \dot{S}_{a\mu\nu} + \eta_a^{\rho} \{ S_{a\rho\nu} v_{a\mu} - S_{a\rho\mu} v_{a\nu} \} = - \sum_{k \neq a} f_a \tau_a \cdot (G_{k\mu}^{(\alpha=2)\rho} S_{a\rho\nu} - G_{k\nu}^{(\alpha=2)\rho} S_{a\rho\mu}) \quad (30a)$$

for the vector meson and

$$\frac{d}{d\tau_a} (I_a \epsilon_{\mu\nu\rho\sigma} v_a^{\rho} S_a^{\sigma}) - \eta_a (v_{a\mu} S_{a\nu} - v_{a\nu} S_{a\mu}) \\ = - \sum_{k \neq a} f_a' \tau_a \cdot (E_{k\mu}^{(\alpha=2)} S_{a\nu} - E_{k\nu}^{(\alpha=2)} S_{a\mu}) \quad (30b)$$



for the scalar meson.

*The equations of motion for isotopic spin :*

The equations of motion for the isotopic spin vector  $\tau_a$  of the particle 'a' are obtained from the variation principle by considering a small rotational displacement  $\delta\theta_a$  of  $\tau_a$  in the charge space and using the relation (8) and passing to the limit  $\alpha \rightarrow 2$ . We thus obtain

$$\dot{\tau}_a = \frac{1}{\lambda_a} \sum_{k \neq a} \left[ \tau_a \times \left( g_a v_{a\nu} \varphi_k^{(\alpha=2)\nu} - \frac{1}{2} f_a S_{a\mu\nu} \mathbf{G}_k^{(\alpha=2)\mu\nu} \right) \right] \quad (31a)$$

for vector mesons and

$$\dot{\tau}_a = \frac{1}{\lambda_a} \sum \left[ \tau_a \times \left( g_a' U_k^{(\alpha=2)} - f_a' S_{a\mu} \mathbf{F}_k^{(\alpha=2)\mu} \right) \right] \quad (31b)$$

for scalar mesons.

It is to be mentioned that in deriving the equations of motion we have freely used the following relations ;

$$\partial S_{a\mu\nu} = (\partial Q_{a\mu\rho} S_{a\nu}^\rho - \partial Q_{a\nu\rho} S_{a\mu}^\rho), \quad (32a)$$

$$\partial S_{a\mu} = \partial Q_{a\mu\nu} S_a^\nu \quad (32b)$$

and

$$\partial \tau_a = [\partial \theta_a \times \tau_a]. \quad (33)$$

We now determine the Lagrangian multipliers from (29a) and (29b) by multiplying them by  $\tau_a^\mu$ , and  $\tau_a^\rho$  and  $\tau_a$  from (30a) and (30b) by multiplying them by  $\tau_a^\nu$ . We thus obtain after some simplifications

$$2\hat{z}_a = - \sum_{k \neq a} \frac{1}{2} f_a \tau_a \cdot S_{a\lambda\nu} \mathbf{G}_k^{(\alpha=2)\lambda\nu}, \quad (34a)$$

$$\tau_a^\rho = I_a \dot{\tau}_a^\rho - \sum_{k \neq a} f_a \tau_a \cdot \tau_a^\nu \mathbf{G}_{k\nu}^{(\alpha=2)\rho}$$

for vector mesons and

$$2\hat{z}_a = - \sum_{k \neq a} f_a' \tau_a \cdot \mathbf{F}_k^{(\alpha=2)\nu} S_{a\nu}, \quad (34b)$$

$$\tau_a = \sum_{k \neq a} f_a' \tau_a \cdot \mathbf{F}_{k\nu}^{(\alpha=2)} \tau_a^\nu$$

for scalar mesons.

Thus substituting (34) in (29) and (30) the translational and rotational equations of motion finally reduce to

$$\begin{aligned} m_a \dot{\tau}_{a\mu} + \frac{d}{d\tau_a} \left[ I_a \dot{S}_{a\mu\nu} \tau_a^\nu - \sum_{k \neq a} f_a \tau_a \cdot \left( \frac{1}{2} \tau_{a\mu} S_{a\lambda\nu} \mathbf{G}_{k\lambda\nu}^{(\alpha=2)} + S_{a\mu}^\rho \mathbf{G}_{k\rho\nu}^{(\alpha=2)} \tau_a^\nu \right) \right] \\ = \sum_{k \neq a} \left[ \tau_a \cdot \left( g_a \mathbf{G}_{k\mu\nu}^{(\alpha=2)} \tau_a^\nu - \frac{1}{2} f_a S_{a\lambda\nu} \frac{\partial \mathbf{G}_{k\lambda\nu}^{(\alpha=2)}}{\partial z_a^\mu} \right) - g_a \dot{\tau}_a \cdot \varphi_{k\mu}^{(\alpha=2)} \right], \end{aligned} \quad (35a)$$

$$I_a (\dot{S}_{a\mu\nu} + v_{a\mu} \dot{S}_{a\nu\rho} v_a^\rho - v_{a\nu} \dot{S}_{a\mu\rho} v_a^\rho) = - \sum_{k \neq a} f_a \tau_a \cdot [S_{a\mu}^\rho \mathbf{D}_{k\nu\rho}] , \quad (36a)$$

where

$$\mathbf{D}_{k\nu\rho} = \mathbf{G}_{k\nu\rho}^{(\alpha=2)} - \mathbf{G}_{k\lambda\rho} \tau_a^\lambda \tau_{a\nu} \quad (37a)$$

for vector mesons and

$$m_a \dot{v}_{a\mu} + \frac{d}{d\tau_a} \left[ \sum_{k \neq a} \tau_a \cdot (g_a' v_{a\mu} \mathbf{U}_k^{(\alpha=2)} - f_a' S_{a\rho} \mathbf{F}_k^{(\alpha=2)\rho} v_{a\mu} + f_a' \mathbf{F}_{k\rho}^{(\alpha=2)} v_a^\rho S_{a\mu}) \right. \\ \left. - I_a \epsilon^{\mu\nu\rho\sigma} \dot{v}_a^\nu \dot{v}_a^\rho S_a^\sigma \right] = \sum_{k \neq a} \tau_a \cdot \left( g_a' \mathbf{F}_{k\mu}^{(\alpha=2)} - f_a' S_a^\nu \frac{\partial \mathbf{F}_{k\nu}^{(\alpha=2)}}{\partial \dot{v}_a^\mu} \right), \quad (35b)$$

$$I_a \epsilon_{\mu\nu\rho\sigma} v_a^\nu \dot{S}_a^\rho = - \sum_{k \neq a} f_a' \tau_a \cdot [\mathbf{D}_{k\mu} S_{a\nu}]_-, \quad (36b)$$

where

$$\mathbf{D}_{k\mu} = \mathbf{F}_{k\mu}^{(\alpha=2)} - \mathbf{F}_{k\rho}^{(\alpha=2)} v_a^\rho \tau_{a\mu} \quad (37b)$$

for the scalar meson. The negative sign as subscript after the square bracket [ ]<sub>-</sub> indicates that the same terms as in the bracket, but with  $\mu$  and  $\nu$  interchanged, are to be subtracted. From (30) and (28) it is easy to show that

$$S_a^{\mu\nu} S_{a\mu\nu} = \text{constant}, \quad S_a^\mu S_{a\mu} = \text{constant}, \quad (38)$$

for the vector and scalar mesons respectively; and from (31) it follows that

$$\tau_a^2 = \text{constant}. \quad (38')$$

It will be noted that the fields entering in the equations of motion of a given point particle interacting with vector or scalar mesons as described above contain symmetric fields which are produced by all the particles other than the given one. Now following Wheeler Feynman, we write

$$\sum_{k \neq a} Q_k^{\text{sym}} = \sum_{k \neq a} Q_k^{\text{ret}} + 1/2 \cdot (Q_a^{\text{ret}} - Q_a^{\text{adv}}) - 1/2 \cdot \sum_{\text{all } k} (Q_k^{\text{ret}} - Q_k^{\text{adv}}), \quad (39)$$

where  $Q$  denotes either potential or field strength. The condition of 'complete absorption' postulated by Wheeler and Feynman gives

$$1/2 \cdot \sum_{\text{all } k} (Q_k^{\text{ret}} - Q_k^{\text{adv}}) = 0. \quad (40)$$

The application of this very significant condition, so to say, reduces the fields involved in the equations of motion of a given particle ' $\alpha$ ' to those of the retarded fields of all particles other than the particle ' $\alpha$ ' together with the radiation field of the particle ' $\alpha$ ' itself which provides the radiative reaction. The fields of the theory of action at a distance therefore differ from the retarded self-field of the usual field theory in the terms which involve integrals over past and future of the particle.

## § 2. Evaluation of the adjunct fields on the world line of the particle

For the purpose of investigating the scattering of charge symmetric mesons by nucleons from the equations of motion established in the previous article it will be necessary to

evaluate the potentials and the fields on the world line of the nucleons. The calculations are carried out by Riesz method of analytical continuation as already discussed in a previous paper by two of the present authors<sup>1,2</sup> in connection with the neutral meson fields. We shall give the results only.

(a) *Charged symmetric vector mesons :*

The potential on the world line may be obtained from (14a)

$$\begin{aligned}
 \varphi_v^{(\alpha=2)\text{rad}}(\text{on the world line}) = & -\frac{1}{2} \dot{\mathbf{T}}_v - \left[ v_v \dot{v}^\mu \dot{\mathbf{T}}_\mu + \frac{1}{2} v_v \dot{v}^\mu \mathbf{T}_\mu + \frac{1}{2} \dot{v}_v \dot{v}^\mu \mathbf{T}_\mu \right] \\
 & + \frac{1}{\chi^2} \left[ \frac{1}{3} \ddot{\mathbf{T}}_v + \frac{1}{3} \dot{\mathbf{T}}_v \dot{\tau}^2 + \frac{1}{3} \mathbf{T}_v (\dot{\tau} \ddot{\tau}) - \left\{ 2 \dot{v}_v \dot{v}^\mu \mathbf{T}_\mu (\dot{\tau} \ddot{\tau}) + 2 \dot{v}_v \dot{v}^\mu \dot{\mathbf{T}}_\mu \dot{\tau}^2 + \dot{v}_v \dot{v}^\mu \mathbf{T}_\mu \dot{\tau}^2 \right. \right. \\
 & + \dot{v}_v \dot{v}^\mu \mathbf{T}_\mu \dot{\tau}^2 + \frac{4}{3} \dot{v}_v \dot{v}^\mu \ddot{\mathbf{T}}_\mu + 2 \dot{v}_v \dot{v}^\mu \ddot{\mathbf{T}} + \frac{4}{3} \dot{v}_v \dot{v}^\mu \dot{\mathbf{T}}_\mu + \frac{1}{3} \dot{v}_v \ddot{\tau}^\mu \mathbf{T}_\mu + 2 \dot{v}_v \dot{v}^\mu \ddot{\mathbf{T}}_\mu \\
 & + 2 \dot{v}_v \dot{v}^\mu \dot{\mathbf{T}}_\mu + \frac{2}{3} \dot{v}_v \ddot{\tau}^\mu \mathbf{T}_\mu + \frac{4}{3} \ddot{v}_v \dot{v}^\mu \dot{\mathbf{T}}_\mu + \frac{2}{3} \ddot{v}_v \dot{v}^\mu \mathbf{T}_\mu + \frac{1}{3} \ddot{\tau}^\mu \dot{v}^\nu \mathbf{T}_\mu \left. \right\} \Big] \\
 & - \frac{\chi}{2} \left\{ \int_{-\infty}^{\tau} \mathbf{T}_v \frac{J_1(\chi S)}{S} d\tau' - \int_{\tau}^{\infty} \mathbf{T}_v \frac{J_1(\chi S)}{S} d\tau' \right\} + \frac{1}{2} \left\{ \int_{-\infty}^{\tau} \mathbf{T}_v \frac{J_2(\chi S)}{S^2} d\tau' - \int_{\tau}^{\infty} \mathbf{T}_v \frac{J_2(\chi S)}{S^2} d\tau' \right\} \\
 & - \frac{\chi}{2} \left\{ \int_{-\infty}^{\tau} s_v s^\mu \mathbf{T}_\mu \frac{J_3(\chi S)}{S^3} d\tau' - \int_{\tau}^{\infty} s_v s^\mu \mathbf{T}_\mu \frac{J_3(\chi S)}{S^3} d\tau' \right\} + \frac{2}{3} \mathbf{T}_{\mu\nu} \ddot{\tau}^\mu + \dot{\mathbf{T}}_{\mu\nu} \dot{\tau}^\mu \\
 & + \frac{\chi^2}{2} \left\{ \int_{-\infty}^{\tau} s^\mu \mathbf{T}_{\mu\nu} \frac{J_2(\chi S)}{S^2} d\tau' - \int_{\tau}^{\infty} s^\mu \mathbf{T}_{\mu\nu} \frac{J_2(\chi S)}{S^2} d\tau' \right\}. \tag{41a}
 \end{aligned}$$

Similarly the field on the world line is obtained from (15a)

$$\begin{aligned}
 G_{\mu\nu}^{(\alpha=2)\text{rad}}(\text{on the world line}) = & \frac{2}{3} \mathbf{T}_{\mu\nu} (\dot{v} \ddot{v}) + \frac{2}{3} \mathbf{T}_{\mu\nu} \dot{v}^2 + \frac{2}{3} \ddot{\mathbf{T}}_{\mu\nu} \\
 & + \left[ \frac{4}{3} v_\mu \ddot{\mathbf{T}}_{\nu\sigma} v^\sigma + 2 v_\mu \dot{\mathbf{T}}_{\nu\sigma} v^\sigma \dot{v}^2 + \frac{2}{3} \ddot{v}_\mu \mathbf{T}_{\nu\sigma} \dot{v}^\sigma + \frac{4}{3} \ddot{v}_\mu \dot{\mathbf{T}}_{\nu\sigma} v^\sigma + \frac{4}{3} v_\mu \dot{\mathbf{T}}_{\nu\sigma} \ddot{v}^\sigma \right. \\
 & + \frac{2}{3} \dot{v}_\mu \mathbf{T}_{\nu\sigma} \ddot{v}^\sigma + \frac{1}{3} v_\mu \mathbf{T}_{\nu\sigma} \ddot{v}^\sigma + 2 v_\mu \ddot{\mathbf{T}}_{\nu\sigma} \dot{v}^\sigma + 2 \dot{v}_\mu \ddot{\mathbf{T}}_{\nu\sigma} v^\sigma + 2 \dot{v}_\mu \ddot{\mathbf{T}}_{\nu\sigma} \dot{v}^\sigma + v_\mu \mathbf{T}_{\nu\sigma} \dot{v}^\sigma \dot{v}^2 \left. \right] \\
 & + \chi^2 \left\{ \dot{\mathbf{T}}_{\mu\nu} + \frac{1}{2} [\dot{v}_\mu \dot{\mathbf{T}}_{\nu\sigma} \dot{v}^\sigma]_- \right\} + \chi^2 \left\{ \int_{-\infty}^{\tau} \mathbf{T}_{\mu\nu} \frac{J_2(\chi S)}{S^2} d\tau' - \int_{\tau}^{\infty} \mathbf{T}_{\mu\nu} \frac{J_2(\chi S)}{S^2} d\tau' \right\} \\
 & + \frac{\chi^2}{2} \left\{ \int_{-\infty}^{\tau} [s_\mu \mathbf{T}_{\nu\sigma} s^\sigma]_- \frac{J_3(\chi S)}{S^3} d\tau' - \int_{\tau}^{\infty} [s_\mu \mathbf{T}_{\nu\sigma} s^\sigma]_- \frac{J_3(\chi S)}{S^3} d\tau' \right\} \\
 & - \frac{1}{3} \dot{v}^2 (v_\mu \mathbf{T}_\nu - v_\nu \mathbf{T}_\mu) - (v_\mu \ddot{\mathbf{T}}_\nu - v_\nu \ddot{\mathbf{T}}_\mu) + (\dot{v}_\mu \dot{\mathbf{T}}_\nu - \dot{v}_\nu \dot{\mathbf{T}}_\mu) - \frac{1}{3} (\ddot{v}_\mu \mathbf{T}_\nu - \ddot{v}_\nu \mathbf{T}_\mu) \\
 & + \frac{\chi^2}{2} \left\{ \int_{-\infty}^{\tau} (s_\mu \mathbf{T}_\nu - s_\nu \mathbf{T}_\mu) \frac{J_2(\chi S)}{S^2} d\tau' - \int_{\tau}^{\infty} (s_\mu \mathbf{T}_\nu - s_\nu \mathbf{T}_\mu) \frac{J_2(\chi S)}{S^2} d\tau' \right\}, \tag{42a}
 \end{aligned}$$



where

$$\mathbf{T}_\nu = g \boldsymbol{\tau} v_\nu, \quad \mathbf{T}_{\mu\nu} = f \boldsymbol{\tau} S_{\mu\nu}. \quad (43)$$

(b) *Charged symmetric scalar meson:*

The potential and the field are given by (14b) and (15b)

$$\begin{aligned} U_{(\alpha=2)}^{\text{rad}} = & -\dot{\mathbf{Z}} - \frac{1}{2} \chi \left\{ \int_{-\infty}^{\tau} \mathbf{Z}(\tau') \frac{J_1(\chi s)}{s} d\tau' - \int_{\tau}^{\infty} \mathbf{Z}(\tau') \frac{J_1(\chi s)}{s} d\tau' \right\} \\ & - \left( \ddot{\mathbf{Z}}_\mu \dot{\tau}^\mu + \dot{\mathbf{Z}}_\mu \ddot{\tau}^\mu + \frac{1}{3} \mathbf{Z}_\mu \ddot{\tau}^\mu \right) \\ & + \frac{\chi^2}{2} \left\{ \int_{-\infty}^{\tau} s^\mu \mathbf{Z}_\mu \frac{J_2(\chi s)}{s^2} d\tau' - \int_{\tau}^{\infty} s^\mu \mathbf{Z}_\mu \frac{J_2(\chi s)}{s^2} d\tau' \right\}, \end{aligned} \quad (41b)$$

$$\begin{aligned} \mathbf{H}_\mu^{\text{rad}(\alpha=2)} = & -\frac{1}{3} \mathbf{Z}(\dot{\tau}_\mu \dot{\tau}^{\mu 2} + \ddot{\tau}_\mu) - \dot{\mathbf{Z}} \dot{\tau}_\mu - \ddot{\mathbf{Z}} v_\mu - \frac{1}{2} \chi^2 \mathbf{Z} \dot{\tau}_\mu + \frac{1}{2} \chi^2 \left\{ \int_{-\infty}^{\tau} s_\mu \mathbf{Z}(\tau') \frac{J_2(\chi s)}{s^2} d\tau' \right. \\ & - \left. \int_{\tau}^{\infty} s_\mu \mathbf{Z}(\tau') \frac{J_2(\chi s)}{s^2} d\tau' \right\} + \frac{1}{3} \mathbf{Z}_\mu (\dot{v} \ddot{v}) + \frac{1}{3} \dot{\mathbf{Z}}_\mu \dot{v}^2 + \frac{1}{3} \ddot{\mathbf{Z}}_\mu \\ & - \left[ 2v_\mu \dot{\mathbf{Z}}_\sigma v^\sigma \dot{v}^2 + v_\mu \mathbf{Z}_\sigma \dot{v}^\sigma \dot{v}^2 + \frac{4}{3} v_\mu \ddot{\mathbf{Z}}_\sigma v^\sigma + 2v_\mu \ddot{\mathbf{Z}}_\sigma \dot{v}^\sigma + \frac{4}{3} v_\mu \dot{\mathbf{Z}}_\sigma \ddot{v}^\sigma \right. \\ & + \left. \frac{1}{3} v_\mu \mathbf{Z}_\sigma \ddot{v}^\sigma + 2\dot{v}_\mu \ddot{\mathbf{Z}}_\sigma v^\sigma + 2\dot{v}_\mu \dot{\mathbf{Z}}_\sigma \dot{v}^\sigma + \frac{2}{3} \dot{v}_\mu \mathbf{Z}_\sigma \ddot{v}^\sigma + \frac{4}{3} \ddot{v}_\mu \dot{\mathbf{Z}}_\sigma v^\sigma + \frac{3}{2} \ddot{v}_\mu \mathbf{Z}_\sigma \dot{v}^\sigma \right] \\ & + \frac{1}{2} \chi^2 (\dot{\mathbf{Z}}_\mu - \dot{\tau}_\mu \dot{\mathbf{Z}}_\sigma \tau^\sigma) + \frac{\chi^2}{2} \left\{ \int_{-\infty}^{\tau} \mathbf{Z}_\mu(\tau') \frac{J_2(\chi s)}{s^2} d\tau' - \int_{\tau}^{\infty} \mathbf{Z}_\mu(\tau') \frac{J_2(\chi s)}{s^2} d\tau' \right\} \\ & - \frac{\chi^2}{3} \left\{ \int_{-\infty}^{\tau} s_\mu s^\sigma \mathbf{Z}_\sigma(\tau') \frac{J_3(\chi s)}{s^3} d\tau' - \int_{\tau}^{\infty} s_\mu s^\sigma \mathbf{Z}_\sigma(\tau') \frac{J_3(\chi s)}{s^3} d\tau' \right\}, \end{aligned} \quad (42b)$$

where

$$\mathbf{Z} = g' \boldsymbol{\tau}, \quad \mathbf{Z}_\mu = f' \boldsymbol{\tau} S_\mu. \quad (44)$$

For our calculations of the scattering cross sections we shall need the potentials and the fields generated by the nucleons in the rest system, *i.e.*, for  $v_\mu = (1, 0, 0, 0)$ . To this approximation we may replace  $d\tau'$  by  $dt'$  and  $s = t - t'$ . Thus we obtain for *vector meson*

$$\begin{aligned} \varphi_\nu^{\text{rad}(\alpha=2)} = & -\frac{3}{2} g \dot{\tau} v_\nu - \frac{g}{\chi^2} \ddot{\tau} v_\nu + \frac{g}{2} \left\{ \int_{-\infty}^t \tau v_\nu \frac{J_2(\chi s)}{s^2} dt' - \int_t^\infty \tau v_\nu \frac{J_2(\chi s)}{s^2} dt' \right\} \\ & - \frac{1}{2} g \chi \left\{ \int_{-\infty}^t \tau v_\nu \frac{J_1(\chi s)}{s} dt' - \int_t^\infty \tau v_\nu \frac{J_1(\chi s)}{s} dt' \right\} \\ & - \frac{1}{2} g \chi \left\{ \int_{-\infty}^t \tau s_\nu s_\mu v_\mu \frac{J_3(\chi s)}{s^3} dt' - \int_t^\infty \tau s_\nu s_\mu v_\mu \frac{J_3(\chi s)}{s^3} dt' \right\} \end{aligned} \quad (45a)$$

and

$$\mathbf{G}_{\mu\nu}^{\text{rad}(\alpha=2)} = \frac{2}{3} f \{ \ddot{\tau} S_{\mu\nu} + 3\dot{\tau} \dot{S}_{\mu\nu} + 3\dot{\tau} \ddot{S}_{\mu\nu} + \tau \ddot{\ddot{S}}_{\mu\nu} \} + \chi^2 f (\dot{\tau} S_{\mu\nu} + \tau \dot{S}_{\mu\nu}) \\ + \chi^2 f \left\{ \int_{-\infty}^t \tau S_{\mu\nu} \frac{J_2(\chi s)}{s^2} dt' - \int_t^{\infty} \tau S_{\mu\nu} \frac{J_2(\chi s)}{s^2} dt' \right\}. \quad (46a)$$

for scalar meson

$$\mathbf{U}^{\text{rad}(\alpha=2)} = -g' \dot{\tau} - \frac{1}{2} g' \chi \left\{ \int_{-\infty}^t \tau \frac{J_1(\chi s)}{s} dt' - \int_t^{\infty} \tau \frac{J_1(\chi s)}{s} dt' \right\}, \quad (45b)$$

and

$$\mathbf{F}_{\mu}^{\text{rad}(\alpha=2)} = -g' \ddot{\tau} - \frac{\chi^2}{2} g' \tau \dot{\tau}_{,\mu} + \frac{1}{2} g' \chi^2 \left[ \int_{-\infty}^t \tau S_{\mu} \frac{J_2(\chi s)}{s^2} dt' - \int_t^{\infty} \tau S_{\mu} \frac{J_2(\chi s)}{s^2} dt' \right] \\ + \frac{1}{3} f' (\ddot{\tau} S_{\mu} + 3\dot{\tau} \dot{S}_{\mu} + 3\dot{\tau} \ddot{S}_{\mu} + \tau \ddot{\ddot{S}}_{\mu}) + \frac{1}{2} \chi^2 f' (\dot{\tau} S_{\mu} + \tau \dot{S}_{\mu}) \\ + \frac{1}{2} f' \chi^2 \left\{ \int_{-\infty}^t \tau S_{\mu} \frac{J_2(\chi s)}{s^2} dt' - \int_t^{\infty} \tau S_{\mu} \frac{J_2(\chi s)}{s^2} dt' \right\}. \quad (46b)$$

The time-symmetric equations of motion of the action-at-a-distance theory for the mesic charge ( $g \neq 0, f=0$ ;  $g' \neq 0, f'=0$ ) of the nucleon have been published by Mehl and Havas<sup>(16)</sup> while the present paper was under preparation and will not be repeated here. Their corresponding equations of motion for the rotation of the spin of the nucleon ( $g=0, f \neq 0$ ;  $g'=0, f' \neq 0$ ) follows immediately from the equations (36) and (46). We write thereby

$$S_{23}, S_{31}, S_{12} = M_1, M_2, M_3; \mathbf{G}_{23}, \mathbf{G}_{31}, \mathbf{G}_{12} = \mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3, \quad (47)$$

$$\text{and } 1/2 \cdot S_{kl} S^{kl} = M_l M^l = 1$$

and drop the suffix 'a' which denotes the 'a'-th particle.

We get the equations of motion

$$I \frac{dM}{dt} = [M, f \tau \cdot \mathbf{H}^{\text{in}}] + f^2 \left[ M, \frac{2}{3} \frac{d^3 M}{dt^3} + 2 \frac{dM}{dt} \tau \frac{d^2 \tau}{dt^2} + \chi^2 \frac{dM}{dt} \right. \\ \left. + \chi^2 \tau \cdot \left\{ \int_{-\infty}^t \tau M \frac{J_2(\chi s)}{s^2} dt' - \int_t^{\infty} \tau M \frac{J_2(\chi s)}{s^2} dt' \right\} \right] \quad (48a)$$

for the vector meson and

$$I \frac{dS}{dt} = [S, f' \tau \cdot \mathbf{H}^{\text{in}}] + f'^2 \left[ S, \frac{1}{3} \frac{d^3 S}{dt^3} + \tau \cdot \frac{d^2 \tau}{dt^2} \frac{dS}{dt} + \frac{1}{2} \chi^2 \frac{dS}{dt} \right. \\ \left. + \frac{1}{2} \chi^2 \tau \cdot \left\{ \int_{-\infty}^t \tau S \frac{J_2(\chi s)}{s^2} dt' - \int_t^{\infty} \tau S \frac{J_2(\chi s)}{s^2} dt' \right\} \right] \quad (48b)$$

for the scalar meson. Here  $M$  denotes the components  $M_1, M_2, M_3$  and  $S$  the com-

ponents  $S_1, S_2, S_3$ . Similarly  $H$  stands for the components  $H_1, H_2, H_3$  and  $F$  for  $F_1, F_2, F_3$ .

Similarly the equations of motion on the action-at-a-distance theory for the rotation of the isotopic spin of the nucleon are obtained from the equations (31) together with the help of the potentials and the fields given by (45) and (46) :

$$\begin{aligned} \lambda \frac{d\tau}{dt} = & [\tau \times \{g \varphi^{0im} - f(MH^{im})\}] - \left[ \tau \times g^2 \left\{ \frac{3}{2} \frac{d\tau}{dt} + \frac{1}{\chi_2} \frac{d^3\tau}{dt^3} \right. \right. \\ & - \frac{1}{2} \left( \int_{-\infty}^t \frac{d\tau}{dt'} \frac{J_2(\chi s)}{s} dt' - \int_t^{\infty} \frac{d\tau}{dt'} \frac{J_2(\chi s)}{s} dt' \right) \\ & + \frac{1}{2} \chi \left( \int_{-\infty}^t \tau \frac{J_1(\chi s)}{s} dt' - \int_t^{\infty} \tau \frac{J_1(\chi s)}{s} dt' \right) \left. \right] \\ & - \left[ \tau \times f^2 \left\{ 2 \left( M \frac{d^2 M}{dt^2} \right) \frac{d\tau}{dt} + \frac{2}{3} \frac{d^3\tau}{dt^3} + \chi^2 \frac{d\tau}{dt} \right. \right. \\ & + M_i \chi^2 \left( \int_{-\infty}^t M_i \tau \frac{J_2(\chi s)}{s^2} dt' - \int_t^{\infty} M_i \tau \frac{J_2(\chi s)}{s^2} dt' \right) \left. \right] \end{aligned} \quad (49a)$$

for the vector mesons and

$$\begin{aligned} \lambda \frac{d\tau}{dt} = & [\tau \times \{g' U^{im} - f'(S E^{im})\}] \\ & - \left[ \tau \times g'^2 \left\{ \frac{d\tau}{dt} + \frac{1}{2} \chi \left( \int_{-\infty}^t \tau \frac{J_1(\chi s)}{s} dt' - \int_t^{\infty} \tau \frac{J_1(\chi s)}{s} dt' \right) \right\} \right] \\ & - \left[ \tau \times f'^2 \left\{ \left( S \frac{d^2 S}{dt^2} \right) \frac{d\tau}{dt} + \frac{1}{3} \frac{d^3\tau}{dt^3} + \frac{1}{2} \chi^2 \frac{d\tau}{dt} \right. \right. \\ & + \frac{1}{2} \chi^2 S_i \left( \int_{-\infty}^t \tau S_i \frac{J_2(\chi s)}{s^2} dt' - \int_t^{\infty} \tau S_i \frac{J_2(\chi s)}{s^2} dt' \right) \left. \right] \end{aligned} \quad (49b)$$

for the scalar meson.

The equations (48) and (49) can be easily solved for small oscillations performed by  $M, S$  and  $\tau$  about their initial directions. If we neglect terms which are of second order in the displacements, the equations for  $d\tau/dt$  and  $dM/dt$  for vector mesons as well as for  $d\tau/dt$  and  $dS/dt$  for scalar mesons separate, and the equations for  $dM/dt$  and  $dS/dt$  become identical except for the terms within the integrals with the corresponding equations of motion of dipoles obtained by Bhabha and Harish Chandra in neutral meson field.

### § 3. The scattering of mesons by nucleons

The cross-sections of the scattering of neutral mesons by mesic charge and dipole have been already evaluated by Bhabha,<sup>16)</sup> Harish Chandra<sup>17)</sup> and Pauli<sup>18)</sup> and of the charged mesons by Fierz,<sup>10)</sup> Majumdar and Vachaspati<sup>11)</sup> and Le Couteur<sup>12)</sup> on the basis of the



retarded field theory. We shall in this section give results for the corresponding scattering cross sections on the basis of Wheeler-Feynman theory of action-at-a-distance developed in earlier sections. The calculations of cross-sections are practically the same as employed by these authors and we shall give only the essential results. For comparison we shall side by side write down the scattering cross-sections obtained from the retarded field theory. The results of the scattering of mesons by mesic charge are already published<sup>15)</sup> as mentioned before and we omit them from our discussions.

### 1) *Scattering by the spin of the nucleon:*

The scattering cross-sections as calculated below will refer to the scattering which correspond to that of a transversely polarised neutral mesons by the rotation of the spin of the neutron or proton. As already discussed by Bhabha that a dipole free to rotate will not scatter longitudinal meson waves at all. The scattering of longitudinally polarised mesons is due to the translational motion of the dipole and is practically negligible in the approximations used in our calculations.

Let us now suppose that a periodic force  $H$  with a frequency  $\omega_0/2\pi$  acts on the dipole

$$H = H_0 \cos \omega_0 t \quad (50)$$

where  $H_0$  lies along the  $x$ -axis. We consider the scattering for weak fields, so that the oscillation of the dipole is small and we may write

$$M(t) = M_0 + M_1 \sin \omega_0 t, \quad (M_0 \cdot M_1) = 0. \quad (51)$$

$M_0$  is the initial direction of the dipole and  $M_0^2 = 1$ . We will neglect terms quadratic in  $M$ . Introducing (50) and (51) in (48a) and writing for brevity

$$\alpha = 3/2 \cdot I/f^2, \quad \beta = 3/2 \cdot |H_0|/f, \quad \xi = (\omega_0^2 - x^2)/\omega_0 \quad (52)$$

we obtain

$$3/2f \cdot [M_0 \times H_0] = \alpha M_1 \omega_0 + [M_0 \times M_1] \omega_0 \xi. \quad (53)$$

To solve this equation we denote by  $\theta$  the angle between  $M_0$  and  $I I_0$ , and  $\eta$  the angle between vectors  $[M_0 \times I I_0]$  and  $M_1$ ,  $M_1$  lying in such a direction that the angle between the planes  $[M_0 \times M_1]$  and  $[M_0 \times H_0]$  is  $(\pi/2) - \eta$ . It then follows that

$$|M_1| = \frac{\beta}{\omega_0 (\alpha^2 + \xi^2)^{1/2}}, \quad \tan \eta = -\xi/\alpha \quad (54)$$

and the total energy radiated by the dipole is

$$\frac{3}{4} H_0^2 \sin^2 \theta \frac{\xi}{\alpha^2 + \xi^2}. \quad (55)$$

The scattering cross-section, *i.e.*, energy scattered divided by the energy flow due to the incident wave, which is  $H_0^2 \omega_0 (\omega_0^2 - x^2)^{-1/2} / 8\pi$ , is, when averaged over all initial orientations of the dipole,

$$\sigma_v = \frac{16\pi}{9} \left( \frac{f^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^2}{\omega_0^2} \cdot \frac{1}{1 + \frac{4}{9} \left( \frac{f^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2}}. \quad (56a)$$

Exactly in the same way the scattering cross-section for scalar mesons is obtained from the equation (48b) :

$$\sigma_s = \frac{8\pi}{9} \left( \frac{f'^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^2}{\omega_0^2} \cdot \frac{1}{1 + \frac{1}{9} \left( \frac{f'^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2}}. \quad (56b)$$

The corresponding scattering cross-sections in the retarded field theory are

$$\sigma_v = \frac{16\pi}{9} \left( \frac{f^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^2}{\omega_0^2} \frac{1 + \frac{4}{9} \left( \frac{f^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2} + \frac{4}{9} \left( \frac{f^2}{I} \right)^2 \frac{\chi^6}{\omega_0^2}}{\left[ 1 + \frac{4}{9} \left( \frac{f^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2} + \frac{4}{9} \left( \frac{f^2}{I} \right)^2 \frac{\chi^6}{\omega_0^2} \right]^2 - \frac{16}{9} \left( \frac{f^2}{I} \right)^2 \frac{\chi^6}{\omega_0^2}} \quad (57a)$$

for vector mesons and

$$\sigma_s = \frac{8\pi}{9} \left( \frac{f'^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^2}{\omega_0^2} \frac{1 + \frac{1}{9} \left( \frac{f'^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2} + \frac{1}{9} \left( \frac{f'^2}{I} \right)^2 \frac{\chi^6}{\omega_0^2}}{\left[ 1 + \frac{1}{9} \left( \frac{f'^2}{I} \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2} + \frac{1}{9} \left( \frac{f'^2}{I} \right)^2 \frac{\chi^6}{\omega_0^2} \right]^2 - \frac{4}{9} \left( \frac{f'^2}{I} \right)^2 \frac{\chi^6}{\omega_0^2}} \quad (57b)$$

for scalar meson.

## II) Scattering by the isotopic spin of the nucleon :

We now solve the equations (49a) and (49b) under the action of a weak, simply periodic incoming field of frequency  $\omega_0/2\pi$ . We put

$$\tau = \alpha_1 \epsilon_1 \sin(\omega_0 t + \sigma_1) + \alpha_2 \epsilon_2 \sin(\omega_0 t + \sigma_2) + \alpha_3, \quad (58)$$

where  $\alpha = \pm 1$  according as the particle is initially a neutron or a proton and  $\epsilon_1$  and  $\epsilon_2$  are small quantities such that their higher powers are neglected.

*Vector meson :* We now substitute (58) in (49a) and equate the coefficients of  $\alpha_1$  and  $\alpha_2$  on both sides of the equation, whence we obtain

$$-\frac{\lambda\omega_0}{a} \epsilon_1 \cos(\omega_0 t + \sigma_1) = \partial \sin\left(\omega_0 t \pm \frac{\pi}{2}\right) + \frac{1}{\chi^2} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right) (\omega_0^2 - \chi^2)^{3/2} \epsilon_2 \cos(\omega_0 t + \sigma_2) \quad (59)$$

and

$$\frac{\lambda\omega_0}{a} \epsilon_2 \cos(\omega_0 t + \sigma_2) = \partial \sin \omega_0 t + \frac{1}{\chi^2} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right) (\omega_0^2 - \chi^2)^{3/2} \epsilon_1 \cos(\omega_0 t + \sigma_1), \quad (60)$$

where we have put

$$\begin{aligned}\{g\varphi^{0in}-f(MH^{in})\} \alpha_2 &= \delta \sin(\omega_0 t \pm \pi/2), \\ \{g\varphi^{0in}-f(MH^{in})\} \alpha_1 &= \delta \sin \omega_0 t,\end{aligned}\quad (61)$$

where  $\pm$  signs are associated with positive or negative mesons. It immediately follows from (59) and (60) that

$$\sigma^2 = \sigma_1 \pm \pi/2 \quad \text{and} \quad \epsilon_1 = \epsilon_2 = \epsilon \quad (62)$$

and hence

$$\epsilon^2 = \frac{\delta^2}{\lambda^2 \omega_0^2} \frac{1}{1 + \frac{1}{\lambda^2 \chi^4} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2}}. \quad (63)$$

Before we evaluate the scattering cross-section it is to be noticed that the term  $\tau \cdot H^{in}$  in (48a) is of the second order of smallness and hence the forced oscillation of  $M$  does not contribute to the scattering of charged mesons. Then it can be easily verified that the variation of  $fM \cdot d\tau/dt$  due to the forced oscillation of  $\tau$  induces a purely transverse retarded field given by

$$\begin{aligned}\varphi_k^{\text{ret}} &= \epsilon f (\omega_0^2 - \chi^2)^{1/2} \{ \alpha_1 \cos(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_1) \\ &\quad + \alpha_2 \cos(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_2) \} \frac{[rM]}{r^2}\end{aligned}\quad (64)$$

and the variation of  $g \cdot d\tau/dt$  induces a longitudinal field given by

$$\begin{aligned}\varphi^0_{\text{ret}} &= -\frac{\epsilon g}{r \chi^2} (\omega_0^2 - \chi^2) \{ \alpha_1 \sin(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_1) + \alpha_2 \sin(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_2) \} \\ \varphi_k^{\text{ret}} &= -\frac{\epsilon g \omega_0 \sqrt{\omega_0^2 - \chi^2}}{r \chi^2} \{ \alpha_1 \sin(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_1) + \alpha_2 \sin(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_2) \}.\end{aligned}\quad (65)$$

It is to be noted that the charge of the meson is conserved in the scattering process. The total energy radiated in transverse waves is  $1/2 \cdot \epsilon^2 f^2 \omega_0 (\omega_0^2 - \chi^2)^{3/2} \sin^2 \phi$  and in longitudinal waves  $1/2 \chi^2 \cdot \epsilon^2 \omega_0 (\omega_0^2 - \chi^2)^{3/2}$  where  $\phi$  is the angle between the direction  $r$  in which the meson radiation is observed and the spin  $M$ . The energy flow associated with incident transverse and longitudinal waves on the other hand are given by,  $\frac{1}{8\pi} \omega_0 (\omega_0^2 - \chi^2)^{-1/2} \frac{\delta^2}{f^2 \cos^2 \theta}$  and  $\frac{\chi^2}{8\pi} \omega_0 (\omega_0^2 - \chi^2)^{-1/2} \frac{\delta^2}{g^2}$  respectively, where  $\theta$  is the angle between  $M$  and  $H^{in}$ . Averaging over all inclinations of  $r$  relative to  $M$  the total radiated energy is

$$\frac{1}{2} \frac{\epsilon^2}{\chi^2} \omega_0 (\omega_0^2 - \chi^2)^{3/2} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right).$$

Dividing this by the inflowing energy, averaged over all orientations of  $M$ , we finally obtain for the effective cross-section for the scattering of transverse mesons

$$\sigma_r = \frac{4\pi}{3} \frac{(\omega_0^2 - \chi^2)^2}{\omega_0^2} \frac{f^2}{\lambda^2 \chi^2} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right) \frac{1}{1 + \frac{1}{\lambda^2 \chi^4} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2}} \quad (66)$$

and of longitudinal mesons

$$\sigma_r^t = \frac{4\pi}{\chi^3} \frac{(\omega_0^2 - \chi^2)^2}{\omega_0^2} \frac{g^2}{\chi^2} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right) \frac{1}{1 + \frac{1}{\lambda^2 \chi^3} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2}}. \quad (67)$$

*Scalar meson:* Substituting again (58) in (49b) and equating the coefficient of  $\alpha_1$  and  $\alpha^2$  we obtain

$$-\frac{\lambda\omega_0}{\alpha} \epsilon_1 \cos(\omega_0 t + \sigma_1) = \{g' U^{\text{in}} - f'(M H^{\text{in}})\} \alpha_2 + Q \epsilon_2 \cos(\omega_0 t + \sigma_2) \quad (68)$$

and

$$\frac{\lambda\omega_0}{\alpha} \epsilon_2 \cos(\omega_0 t + \sigma_2) = \{g' U^{\text{in}} - f'(M H^{\text{in}})\} \alpha_1 + Q \epsilon_1 \cos(\omega_0 t + \sigma_1) \quad (69)$$

where

$$Q = -(\omega_0^2 - \chi^2)^{1/2} \left\{ g'^2 - \frac{1}{3} f'^2 (\omega_0^2 - \chi^2) \right\}. \quad (70)$$

The variation of  $f' M \cdot d\tau/dt$  and  $g' \cdot d\tau/dt$  due to the forced oscillations of  $\tau$  generate the potentials at a very distant point  $r$  from the source which are respectively given by

$$\begin{aligned} U^{\text{ret}} &= \epsilon f' (\omega_0^2 - \chi^2)^{1/2} \{ \alpha_1 \sin(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_1) \\ &\quad + \alpha_2 \sin(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_2) \} \frac{(r \cdot M)}{r^2}, \\ U^{\text{ret}} &= \frac{\epsilon g'}{r} \{ \alpha_1 \sin(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_1) \\ &\quad + \alpha_2 \sin(\omega_0 t - r \sqrt{\omega_0^2 - \chi^2} + \sigma_2) \}. \end{aligned} \quad (71)$$

Now the similar procedure as in the vector meson case gives the total scattering cross section as follows

$$\sigma_s = \frac{\lambda^2 \omega_0^2}{4\pi} \frac{g'^3 + \frac{2}{3} g'^2 f'^2 (\omega_0^2 - \chi^2) + \frac{1}{9} f'^3 (\omega_0^2 - \chi^2)^2}{1 + \frac{1}{\lambda^2} \frac{(\omega_0^2 - \chi^2)}{\omega_0^2} \left\{ f'^3 - \frac{2}{3} g'^2 f'^2 (\omega_0^2 - \chi^2) + \frac{1}{9} f'^3 (\omega_0^2 - \chi^2)^2 \right\}}. \quad (72)$$

The corresponding cross sections for the scattering of transverse and longitudinal mesons in retarded field theory are respectively given by

$$\begin{aligned} \sigma_r^t &= \frac{4\pi}{3} \frac{(\omega_0^2 - \chi^2)^2}{\omega_0^2} \frac{f^2}{\lambda^2 \chi^2} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right) \\ &\quad \times \frac{1}{\left[ 1 \pm \frac{\alpha \chi}{\lambda \omega_0} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right) \right]^2 + \frac{1}{\lambda^2 \chi^4} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2}} \end{aligned} \quad (66')$$

and

$$\sigma_r^l = 4\pi \frac{(\omega_0^2 - \chi^2)^2}{\omega_0^2} \frac{g^2}{\lambda^2 \chi^4} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right)$$



$$\times \frac{1}{\left[1 \pm \frac{a\chi}{\lambda\omega_0} \left(g'^2 + \frac{2}{3} f'^2 \chi^2\right)\right]^2 + \frac{1}{\lambda^2 \chi^4} \left(g'^2 + \frac{2}{3} f'^2 \chi^2\right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2}}. \quad (67')$$

Similarly the scattering cross section for scalar meson in retarded field theory is

$$\sigma_s = \frac{4\pi}{\lambda^2 \omega_0^2} \times \frac{g'^4 + \frac{2}{3} g'^2 f'^2 (\omega_0^2 - \chi^2) + \frac{1}{9} f'^4 (\omega_0^2 - \chi^2)^2}{\left[1 \pm \frac{a\chi}{\lambda\omega_0} \left(g'^2 + \frac{1}{3} f'^2 \chi^2\right)\right]^2 + \frac{1}{\lambda^2} \frac{(\omega_0^2 - \chi^2)}{\omega_0^2} \left\{ g'^4 - \frac{2}{3} g'^2 f'^2 (\omega_0^2 - \chi^2) + \frac{1}{9} f'^4 (\omega_0^2 - \chi^2)^2 \right\}}. \quad (72')$$

An examination of the above expressions for the scattering cross-sections shows that all the terms which are due to the inertial reactions of the meson field resulting in an apparent change of the mass, spin and the isotopic spin of the nucleon in the retarded field theory are completely absent in action at a distance theory. The terms which, on the other hand, are due to the reactions of the field resulting in an irreversible damping on the motion of the nucleon remain the same in both the theories.

The implications of the classical scattering cross-sections given by (56), (66), (67) and (72) above and their experimental verifications will be discussed in the light of quantum mechanics in a subsequent paper.

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# On a Treatment of Many-particle Systems in Quantum Field Theory

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Using the distribution of the virtual particles in the true vacuum and around a real particle, a complete set of states are defined, each of which may be interpreted as representing a prescribed number of real particles. The representation spanned by this set of states is non-orthogonal and the Schrödinger equation transformed into this representation includes the non-Hermitian interaction Hamiltonian. However, the usual definition of the transition probability can be used also in this representation. The equation has the advantage that the energy of the system appears as the difference from the vacuum energy and that the vacuum or the single particle self-energy process no longer occurs. An alternative Hermitian equation is also derived, which, applied to proton-neutron system, gives the non-adiabatic Dancoff potential and the finite single nucleon self-energy term to the second order in the coupling constant. To this order, the divergent vacuum self-energy term reappears, but it is shown to be cancelled by including the contributions of the higher order terms. The simple relations between the true vacuum state, the real one-particle states and the other unbound stationary states are also deduced. All these formulations may be used to treat the many-particle systems on the knowledge of the solutions of the one-particle systems.

## § 1. Introduction and summary

Contrary to the complete success of the quantum electrodynamics, the meson theory based upon the conventional formulation of quantum field theory seems to have failed so far in giving the quantitative explanation of the nuclear phenomena. Although the failure might be due finally to the fundamental defect of the meson theory itself, there still remain several points to be clarified within the framework of the current theory, before we can draw any definite conclusion about the validity of our meson theory. Among all, there seems to remain a possibility of obtaining the better results by taking into account to a sufficient extent the higher order effects (we do not necessarily mean the use of the perturbation theory), which are believed to play a decisive role in meson theory. Indeed, in the weak coupling approximation, the fourth order calculations often proved to change the results of the second order ones considerably. However, using the perturbation theory, the higher order calculations beyond the fourth are very complicated and difficult. Moreover, the meaning of such calculation itself is in doubt in view of the questions raised against the convergence of the perturbation expansions<sup>1)</sup>.

In these circumstances, it will be of some value to formulate a method that enables one to treat the many-particle systems like pion-nucleon and nucleon-nucleon systems from the knowledge of the distribution of the virtual cloud around a nucleon, namely, the one-

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particle solution. If this is possible, then the difficulties of evaluating the higher order effects, which have been met in every problem, could be concentrated into the determination of the virtual cloud of a nucleon. The virtual cloud of a nucleon may be determined, for instance, half-phenomenologically from the experimental data like the anomalous magnetic moments of nucleons and the electron-neutron interaction. If the underlying theory were correct, this would correspond to taking account of the higher order corrections. This procedure, which is by no means a complete solution of the difficulty, will, however, serve to see how far the current meson theory can describe various types of nuclear phenomena without contradiction.

The present paper will be devoted to the formulation of the methods that connect the solutions of one- and many-particle systems. It includes two such formulations. One is a remark about the simple relations between the stationary states representing the true vacuum, the real one-particle and the scattering of any number of particles. Indeed, it will be shown that the exact knowledge of the distribution of the virtual particles in the true vacuum or around a real particle immediately leads to the complete determination of the other stationary scattering states except those including the bound systems. The applicability of the relations to practical problems along the above mentioned line will depend on how good one-particle solution we can obtain phenomenologically. The other is concerned with the construction of a new representation in the quantum field theory which is suited for the above mentioned program and is applicable for the bound as well as the unbound systems.

The usual representation, employed in the perturbation theory as well as in Tamm-Dancoff's method, is one which diagonalizes the free Hamiltonian, so that each basic vector represents a state of a prescribed number of bare particles. To exploit the one-particle solution explicitly, we define, corresponding to each eigenstate of the free Hamiltonian, a set of states each of which may be interpreted as representing a prescribed number of real particles accompanied by their characteristic virtual clouds. Namely, the states are defined as such in which the individual virtual clouds of the real particles involved, defined in the absence of the other particles, co-exist independently without being deformed by the presence of the other. Since the states thus defined form a complete set, they can be used as the basic vectors of the new representation. The interaction Hamiltonian appearing in the Schrödinger equation in this representation can be interpreted as the interaction between the real particles and the matrix-elements of it can be calculated once we know the one-particle solutions. The true vacuum state and the real one-particle states are represented by the eigenstates of the free Hamiltonian in this representation.

Unfortunately, however, the basic vectors do not form an orthogonal set, which introduces some complications in the formulation. This manifests itself in the fact that the interaction Hamiltonian in this representation is not Hermitian. Discussions will be made on how to solve the equations and how to derive the observable quantities from the solutions of the Schrödinger equation. It will be shown that for scattering problems the usual formula of the transition probability can be used also in this representation, in spite of the fact that the definition of the expectation value must be changed in general in this

representation. This is confirmed by calculating the nuclear potential to the second order in the coupling constant using the Schrödinger equation in this representation. It turns out to be the same as the usual adiabatic potential. In the eigenvalue problems for bound systems, the non-Hermitian interaction Hamiltonian may well lead to the complex eigenvalue, the imaginary part of which has no physical meaning and can be dropped if it is small compared to the real part of the eigenvalue. An alternative equation is also derived, which contains only Hermitian operators, so that the real property of the eigenvalue is guaranteed.

The orthogonality of the base could have been recovered if we had taken a different approach by employing an infinite series of successive unitary transformations which remove the virtual processes step by step as a generalization of Bloch-Nordsieck transformation. This will, however, be useless for our purpose, since every step of the successive unitary transformations will correspond already to solving the one-particle problem by perturbation.

This representation has one more advantage that the energy of the system appears in the Schrödinger equation in the form of the difference from the vacuum energy\*, which is just the observed quantity and that the interaction Hamiltonian never gives rise to the vacuum or the single particle self-energy processes. Using the Hermitianized equation, such self-energy terms reappear if we employ the non-adiabatic treatment. However, it is shown by the example of nuclear potential, that the single particle self-energy term is finite without subtraction, and is equal to what Dyson<sup>11)</sup> identified as the physically meaningful finite part of the diverging expression resulting from the use of the old Tamm-Dancoff method. The vacuum self-energy term is divergent in the second order, but in this case the contributions from every order must be considered also, which altogether turn out to be finite.

## § 2. Relation between the stationary states

In this section we shall define the real vacuum and the real one-particle states for the later purpose and also derive the relations that connect these states with those that describe the scattering processes.

We shall take, throughout this paper, the case of the nucleon (say, proton) field  $\psi$  interacting with the neutral pseudo-scalar meson field  $\phi$  through the pseudo-scalar coupling. We choose the units  $\hbar=c=1$ . Quantities containing the time-variable explicitly are to refer generally to the interaction representation, while those with no time-variable will refer to the time-origin, where the Heisenberg, Schrödinger and interaction representations are to coincide. To which representation a quantity shall refer will be evident in each case if not mentioned explicitly. First we shall summarize some notations.

$H_0$ : free Hamiltonian (including the mass renormalization terms)

$H_i$ : interaction Hamiltonian (with mass renormalization terms subtracted)

$H=H_0+H_i$ : total Hamiltonian

$\Phi_\alpha$ : eigenstates of the free Hamiltonian

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\* The same result was obtained by Dyson<sup>12)</sup> in the different representation. This seems to be a common feature of the representations that employ the true vacuum state instead of the free one as the fundamental basic vector, from which other basic vectors are constructed.



$\Psi_a$ : eigenstates of the total Hamiltonian.

If necessary, we shall write  $\Phi_a$  as  $\Phi(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{q}_1, \dots, \mathbf{q}_m; \mathbf{k}_1, \dots, \mathbf{k}_l)$ , where the arguments represents  $n$  protons with momenta  $\mathbf{p}_1, \dots, \mathbf{p}_n$ ,  $m$  anti-protons with momenta  $\mathbf{q}_1, \dots, \mathbf{q}_m$  and  $l$  mesons with momenta  $\mathbf{k}_1, \dots, \mathbf{k}_l$ . The same notation will be used also for  $\Psi_a$ . Especially the free and true vacuum states will be denoted simply as  $\Phi_0$  and  $\Psi_0$ , respectively. We introduce as usual the destruction (creation) operators  $a_p, (a_p^*)$ ,  $b_p, (b_p^*)$  and  $c_p, (c_p^*)$  of a bare proton, anti-proton and meson, respectively, with the momentum (and spin we shall not write the spin variable explicitly—)  $\mathbf{p}$ , by the Fourier expansions of the field operators in the interaction representation,

$$\begin{aligned}\psi(x) &= \sum_p \psi^{(+)}(\mathbf{p}) e^{i\mathbf{p}x} + \sum_p \psi^{(-)}(\mathbf{p}) e^{-i\mathbf{p}x} \\ &= \sum_p a_p u_+(\mathbf{p}) e^{i\mathbf{p}x - i\varepsilon_p t} + \sum_p b_p^* u_-(-\mathbf{p}) e^{-i\mathbf{p}x + i\varepsilon_p t}\end{aligned}$$

and

$$\begin{aligned}\phi(x) &= \sum_k \phi^{(+)}(\mathbf{k}) e^{i\mathbf{k}x} + \sum_k \phi^{(-)}(\mathbf{k}) e^{-i\mathbf{k}x} \\ &= \sum_k 1/\sqrt{2\omega_k} (c_k e^{i\mathbf{k}x - i\omega_k t} + c_k^* e^{-i\mathbf{k}x + i\omega_k t}),\end{aligned}$$

with

$$\varepsilon_p = \sqrt{\mathbf{p}^2 + m^2} \quad \text{and} \quad \omega_k = \sqrt{\mathbf{k}^2 + \mu^2}.$$

$m$  and  $\mu$  are the renormalized masses of the proton and the meson.  $\psi^{(\pm)}$  and  $\phi^{(\pm)}$  are the usual positive and negative frequency part of  $\psi$  and  $\phi$ . The thin letter argument  $p$ ,  $q$  and  $k$  denote the four momenta, while the bold ones the three dimensional momenta.  $u_{\pm}(\mathbf{p})$  is a c-number Dirac spinor for the positive and negative energy state.

We introduce the adiabatic transformation function<sup>\*)</sup> in the interaction representation, say,  $U(t, -\infty)$ , which is defined by

$$i\partial U(t, -\infty)/\partial t = H_I(t)U(t, -\infty); \quad U(-\infty, -\infty) = 1.$$

Then, as is well known<sup>(6) 7) 8)</sup>, the states defined by the equation

$$\Psi_a = U(0, -\infty)\Phi_a; \quad H_0\Phi_a = \varepsilon_a\Phi_a, \quad (1)$$

are the unbound eigenstates of the total Hamiltonian  $H$ , and we have

$$H\Psi_a = (\varepsilon_0 + \varepsilon_a)\Psi_a. \quad (2)$$

That the eigenvalue is given by  $\varepsilon_0 + \varepsilon_a$ , with  $\varepsilon_0$  denoting the energy eigenvalue of the true vacuum, will be shown in the Appendix. Then, the true vacuum state  $\Psi_0$  defined from the free vacuum state  $\Phi_0$  by the equation<sup>\*\*)</sup>

\* We suppose that  $U(0, -\infty)$  is obtained from that transformation function which is defined by the interaction Hamiltonian  $H_I(t) \exp(-a|t|)$  by taking the limit  $a \rightarrow 0$ . Then the normalization of  $\Psi_a$  is guaranteed, since the due consideration of  $a$  makes  $U(0, -\infty)$  unitary<sup>9)</sup>. This can be illustrated, for instance, exactly by the example discussed by Van Hove<sup>7)</sup> (contrary to his result) and to the second order in the coupling constant in the case considered here.

\*\* The apparent difference between the equation (3) and that given by Gell-Mann and Low is simply an indefinite constant phase factor, which we suppose included in  $\Psi_0$ . (See the equation (8). The imaginary part of  $A$  is just this phase factor.)

$$\Psi_0 = U(0, -\infty) \Phi_0, \quad (3)$$

will satisfy

$$H \Psi_0 = \varepsilon_0 \Psi_0, \quad (4)$$

and the one-particle states defined by

$$\left. \begin{aligned} \Psi(p; 0; 0) &= U(0; -\infty) \Phi(p; 0; 0); (H_0 - \varepsilon_p) \Phi(p; 0; 0) = 0, \\ \Psi(0; q; 0) &= U(0, -\infty) \Phi(0; q; 0); (H_0 - \varepsilon_q) \Phi(0; q; 0) = 0, \\ \Psi(0; 0; k) &= U(0, -\infty) \Phi(0; 0; k); (H_0 - \omega_k) \Phi(0; 0; k) = 0, \end{aligned} \right\} \quad (5)$$

satisfy

$$\left. \begin{aligned} H \Psi(p; 0; 0) &= (\varepsilon_0 + \varepsilon_p) \Psi(p; 0; 0), \\ H \Psi(0; q; 0) &= (\varepsilon_0 + \varepsilon_q) \Psi(0; q; 0) \\ H \Psi(0; 0; k) &= (\varepsilon_0 + \omega_k) \Psi(0; 0; k). \end{aligned} \right\} \quad (6)$$

Now, expanding  $U(0, -\infty)$  into the power series of the coupling constant, decomposing each term into its normal constituents and rearranging the resulting terms, we get

$$U(0, -\infty) = D_0^* (1 + \sum_p A_p'^* a_p + \sum_q B_q'^* b_q + \sum_k C_k'^* c_k + \dots). \quad (7)$$

Here,  $D_0^*$ ,  $A_p'^*$ ,  $B_q'^*$  and  $C_k'^*$  are some infinite sums of products of the free creation operators and the dotted part represents the terms with more than one destruction operators, which describe the scattering processes. The true vacuum state defined by (3) will then be written as

$$\Psi_0 = D_0^* \Phi_0 = e^{\Delta} e^{d_0^*} \Phi_0. \quad (8)$$

The last expression of  $D_0^*$  results from the multiple occurrence of the same processes in the vacuum<sup>(8)</sup>.  $\Delta$  is a constant which comes from the contributions of the single connected "bubble" diagrams in  $U(0, -\infty)$  and in general a complex number.  $d_0^*$  is a sum of products of creation operators and corresponds to the diagrams as shown in Fig. 1. Each graph contributes a term in  $d_0^*$  and each end of a free line corresponds to a creation operator in the term. The real one-particle states defined by (5) can be written from (7) as

$$\left. \begin{aligned} \Psi(p; 0; 0) &= A_p^* D_0^* \Phi_0 = A_p^* \Psi_0; \quad A_p^* = a_p^* + A_p'^*, \\ \Psi(0; q; 0) &= B_q^* D_0^* \Phi_0 = B_q^* \Psi_0; \quad B_q^* = b_q^* + B_q'^*, \\ \Psi(0; 0; k) &= C_k^* D_0^* \Phi_0 = C_k^* \Psi_0; \quad C_k^* = c_k^* + C_k'^*, \end{aligned} \right\} \quad (9)$$

Where we have used the following relations.

$$\begin{aligned} [D_0^*, A_p^*] &= [D_0^*, B_q^*] = [D_0^*, C_k^*] = [A_p^*, C_k^*] = [B_q^*, C_k^*] = 0, \\ [A_p^*, B_q^*]_{-} &= [A_p^*, A_{p'}^*]_{+} = [B_q^*, B_{q'}^*]_{-} = 0. \end{aligned} \quad (10)$$

This follows from the fact that the terms of  $D_0^*$  and  $C_k^*$  contain an even number of,

while those of  $A_p^*$  and  $B_q^*$  an odd number of free fermion creation operators. The terms of  $A_p^*$  correspond to the diagrams shown in Fig. 2. Here, the incident line must be represented by the c-number spinor  $u_+(\mathbf{p})$ . It is to be noted that the terms of  $A_p^*$  contain at least two creation operators, since the terms, due to the self-energy processes are cancelled by the effects of the mass renormalization terms. The same is true for  $B_q^*$  and  $C_k^*$ . The terms of  $C_k^*$  correspond to the diagrams shown in Fig. 3.

Comparing the diagrams of Fig. 2 with those of Fig. 1, we can observe a kind of correspondence between them except the diagram 2<sup>o</sup>. For instance, the diagram 2a can be obtained from the diagram 1a by removing the free anti-proton line and adding the incident proton line. The contributions of the diagram 1a to  $d_0^*$  and 2a to  $A_p^*$  are

$$-ig \sum_{q, p', k} \bar{\psi}^{(-)}(p') \gamma_5 \psi^{(-)}(q) \\ \times \phi^{(-)}(k) \delta(\mathbf{p}' + \mathbf{q} + \mathbf{k}) / (\varepsilon_{p'} \\ + \varepsilon_q + \omega_k)$$

and

$$-ig \sum_{p', k} \bar{\psi}^{(-)}(p') \gamma_5 u_+(\mathbf{p}) \phi^{(-)}(k) \\ \times \delta(\mathbf{p}' - \mathbf{p} + \mathbf{k}) / (\varepsilon_{p'} - \varepsilon_p + \omega_k),$$

respectively. The factor  $\delta(\mathbf{p}' + \mathbf{q} + \mathbf{k}) / (\varepsilon_{p'} + \varepsilon_q + \omega_k)$  is characteristic of  $U(0, -\infty)$  and replaces the four dimensional  $\delta$ -function of S-matrix,  $U(\infty, -\infty)$ . The difference between the above two expressions are: (1) anti-proton operator  $\psi^{(-)}(-\mathbf{p})$  in  $d_0^*$  is replaced by the c-number Dirac spinor  $u_+(\mathbf{p})$  with the prescribed momentum  $\mathbf{p}$  and (2) the sign of  $\varepsilon_p$  is changed. The same relation holds for every couple of the corresponding diagrams of  $d_0^*$  and  $A_p^*$ . In fact, a general term of  $d_0^*$ , which represents the creation of  $n$  proton pairs and  $l$  mesons, would assume a form

$$\text{const.} \sum_{p_1 \dots p_n, q_1 \dots q_l, k_1 \dots k_l} \bar{\psi}_{\alpha_1}^{(-)}(p_1) \dots \bar{\psi}_{\alpha_n}^{(-)}(p_n) A_{\alpha_1 \dots \alpha_n; \beta_1 \dots \beta_n}(p_1 \dots p_n; q_1 \dots q_n; k_1 \dots k_l) \\ \times \psi_{\beta_1}^{(-)}(q_1) \dots \psi_{\beta_n}^{(-)}(q_n) \phi^{(-)}(k_1) \dots \phi^{(-)}(k_l) \cdot \delta(\mathbf{p}_1 + \dots + \mathbf{q}_1 + \dots$$

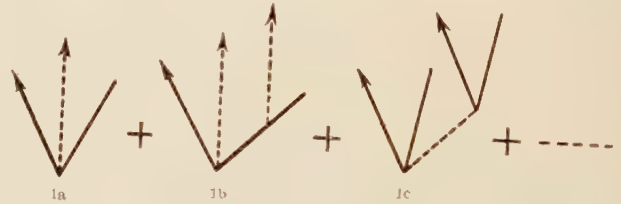


Fig. 1

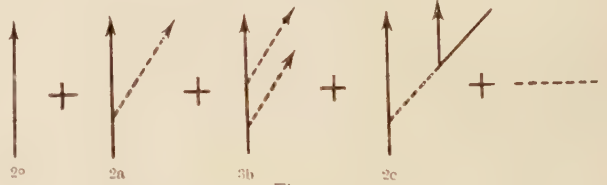


Fig. 2

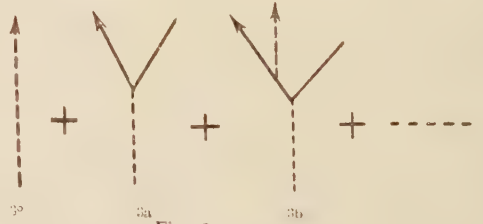


Fig. 3

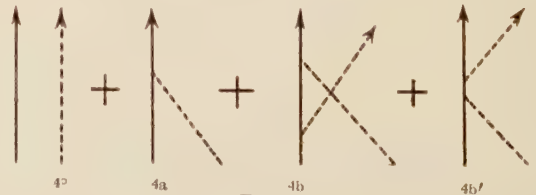


Fig. 4

$$+ \mathbf{k}_1 + \dots + \mathbf{k}_l) / (\mathbf{p}_{10} + \dots + \mathbf{q}_{10} + \dots + \mathbf{k}_{10} + \dots + \mathbf{k}_{l0}), \quad (11)$$

after the integrations on the intermediate state momenta have been performed. Since the negative frequency parts of the free operator are taken, any  $\mathbf{p}_{i0}$ ,  $\mathbf{q}_{i0}$  and  $\mathbf{k}_{i0}$  must be negative. The corresponding term of  $A_p^*$  would have the same form except that one  $\psi^{(-)}$ -operator must now be replaced by the positive frequency part  $\psi^{(+)}(\mathbf{p})$  with  $\mathbf{p}_0 > 0$ . Thus, we can infer the equation

$$A_p^* = a_p^* + \delta d_0^* / \delta \psi^{(-)}(-\mathbf{p}) \big|_{\varepsilon_p \rightarrow -\varepsilon_p} u_+(\mathbf{p}), \quad (12)$$

or from (8) and (9),

$$\Psi(\mathbf{p}; 0; 0) = \Phi(\mathbf{p}; 0; 0) + \delta \Psi_0 / \delta \psi^{(-)}(-\mathbf{p}) \big|_{\varepsilon_p \rightarrow -\varepsilon_p} u_+(\mathbf{p}). \quad (12')$$

In the same way, we reach the equation

$$\Psi(0; 0; \mathbf{k}) = \Phi(0; 0; \mathbf{k}) + \delta \Psi_0 / \delta \phi^{(-)}(-\mathbf{k}) \big|_{\omega_k \rightarrow -\omega_k} / \sqrt{2\omega_k}. \quad (13)$$

Of course, in these relations it is presumed that  $d_0^*$  is expressed by the terms of the form like (11) and especially that  $A$  is a well-defined function of the three dimensional momenta  $\mathbf{p}'$ 's,  $\mathbf{q}'$ 's and  $\mathbf{k}'$ 's as well as the energies  $\varepsilon_p$ 's,  $\varepsilon_q$ 's and  $\omega_k$ 's, since otherwise the replacement  $\varepsilon_p \rightarrow -\varepsilon_p$  would become ambiguous. In this respect, it is to be noted that  $A$  is not necessarily an invariant function of the four-momenta of the created particles, since  $U(0, -\infty)$  is not a Lorentz-invariant operator.\* Therefore, the spatial and temporal components of the particle four-momenta would appear in  $A$  in a complicated way, especially for the terms which contain the radiative effects. However, it may be remarked that in  $A$ , considered as a function of the particle four-momenta, the special components  $\mathbf{p}$ ,  $\mathbf{q}$  and  $\mathbf{k}$  should not appear in quadratic forms  $\mathbf{p}^2$ ,  $\mathbf{q}^2$  and  $\mathbf{k}^2$ , but only in the form of  $(\mathbf{p}, \mathbf{p}')$ ,  $(\mathbf{p}, \mathbf{k})$  etc. This would be the sufficient condition to make the replacement  $\varepsilon_p \rightarrow -\varepsilon_p$  unambiguous. To see this, consider instead of  $U(0, -\infty)$  an invariant operator  $U(\sigma, -\infty)$ ,  $\sigma$  being a space-like surface defined by  $\varepsilon_\mu x_\mu = 0$ , with an arbitrary time-like vector  $\varepsilon_\mu$ . Then the terms of  $U(\sigma, -\infty)$  must be the invariant functions of the four-momenta  $\mathbf{p}$ ,  $\mathbf{q}$  and  $\mathbf{k}$  as well as of  $\varepsilon_\mu$ . The quadratic terms  $\mathbf{p}^2$ ,  $\mathbf{q}^2$  and  $\mathbf{k}^2$  can be replaced by  $-m^2$  and  $-\mu^2$  and no longer appear. Then, taking  $\varepsilon_\mu = (0, 0, 0, i)$  the above statement is established.

The other stationary states that describe the scattering processes can also be expressed in analogous manner. For example, consider the scattering of a meson by a proton, which is described by the terms of  $U(0, -\infty)$  corresponding to the diagrams shown in Fig. 4. Comparing these diagrams with those of Fig. 2 and Fig. 1, we obtain

$$\Psi(\mathbf{p}; 0; \mathbf{k}) = \Phi(\mathbf{p}; 0; \mathbf{k}) + \delta^2 \Psi_0 / \delta \psi^{(-)}(-\mathbf{p}) \delta \phi^{(-)}(-\mathbf{k}) \big|_{\substack{\varepsilon_p \rightarrow -\varepsilon_p \\ \omega_k \rightarrow -\omega_k}} u_+(\mathbf{p}) / \sqrt{2\omega_k}$$

\* As is well known, if we replace the factor  $\delta(\mathbf{p}_1 + \dots + \mathbf{k}_l) / (\mathbf{p}_{10} + \dots + \mathbf{k}_{l0})$  that appears in  $U(0, -\infty)$  by  $-2\pi i \delta^{(4)}(\mathbf{p}_1 + \dots + \mathbf{k}_l)$ , we obtain the S-matrix. The invariant property of S-matrix might at first sight suggest the invariance of the remaining factor  $A$ . But this is not true, since it can contain terms of the form

$$\text{non-invariant function} \times (\mathbf{p}_{10} + \dots + \mathbf{k}_{l0})$$

which drop in S-matrix by multiplying the four dimensional  $\delta$ -function.



$$= \Phi(p; 0; k) + \delta \Psi(p; 0; 0) / \delta \phi^{(-)}(-k) \Big|_{\omega_k \rightarrow -\omega_k} / \sqrt{2\omega_k}. \quad (14)$$

Analogously, for the scattering of two protons, we have

$$\Psi(pp'; 0; 0) = \Phi(pp'; 0; 0) + \delta \Psi(p; 0; 0) / \delta \phi^{(-)}(-p') \Big|_{\varepsilon_{p'} \rightarrow -\varepsilon_{p'}} u_+(p'). \quad (15)$$

Thus, the complete knowledge of the distribution of the virtual particles in the vacuum, namely,  $\mathcal{Q}_0^*$ , would determine the other stationary states except those containing bound systems. Also, the exact knowledge of the virtual clouds around a nucleon,  $A_p^*$ , would be sufficient to determine the scattering processes involving a nucleon. This suggests a possibility of applying these relations for practical problems. For instance, we may determine  $A_p^*$  phenomenologically so as to fix the magnetic moments of nucleon and electron-neutron potential, assuming that  $A_p^*$  consists of the three terms like 2<sup>c</sup>, 2a and 2b and assigning to each of them some adequate function  $\mathcal{A}$  to take into account the higher order radiative corrections. If the two-meson component 2b contributes appreciably to the magnetic moments as suggested by Sachs<sup>9)</sup>, then the functional form of  $\mathcal{A}$  could be determined to some extent. Then, using (14), it can be used to determine the one-meson components 4b and 4b' of  $\Psi(p; 0; k)$  which determine the meson-nucleon scattering cross section. The inverse procedure, namely, to determine the two-meson components of  $A_p^*$  from the meson-nucleon scattering data may also be applicable. However, in these procedures it is probable that the large arbitrariness of the form of  $\mathcal{A}$  as a function of particle momenta and energies, as discussed above, may decrease the practical value of this method.

### § 3. Construction of the real-particle representation

Using the virtual clouds of the vacuum and real particles,  $A_p^*$ ,  $B_q^*$ ,  $C_k^*$  and  $D_0^*$ , we now proceed to define the basic states of the new representation introduced in § 1. We shall call this the real-particle representation or  $\mathcal{Q}$ -representation and denote its basic vectors as  $\mathcal{Q}_a$ . The true vacuum state and the real one-particle states must be included in the set  $\{\mathcal{Q}_a\}$ ;

$$\mathcal{Q}_0 = \Psi_0, \quad \mathcal{Q}(p; 0; 0) = \Psi(p; 0; 0), \quad \text{etc.}$$

We shall define the state  $\mathcal{Q}(pp'; 0; 0)$  which represents the two real protons with momenta  $p$  and  $p'$  by

$$\mathcal{Q}(pp'; 0; 0) = A_p^* A_{p'}^* D_0^* \Phi_0 = A_p^* A_{p'}^* \mathcal{Q}_0,$$

in which the individual virtual clouds of the vacuum and the two particles,  $D_0^*$ ,  $A_p^*$  and  $A_{p'}^*$ , co-exist except the terms forbidden by Pauli-principle, which automatically drop. In the same way, to every eigenstate  $\Phi(p_1, \dots, p_n; q_1, \dots, q_m; k_1, \dots, k_l)$  of the free Hamiltonian, we define the corresponding real particle state

$$\mathcal{Q}(p_1, \dots, p_n; q_1, \dots, q_m; k_1, \dots, k_l) = A_{p_1}^* \dots A_{p_n}^* B_{q_1}^* \dots B_{q_m}^* C_{k_1}^* \dots C_{k_l}^* \mathcal{Q}_0. \quad (16)$$

According to (10), the position of the operators  $C_k^*$  (and  $D_0^*$  included in  $\mathcal{Q}_0$ ) can be arbitrarily moved, while the interchange of the two adjacent operators among  $A_p^*$ 's and

$B_q^*$ 's causes the change of the sign of the expression. From (7) and (9) it is clear that we can also express  $\mathcal{Q}_a$  as

$$\mathcal{Q}_a = U(0, -\infty)^{disc.} \Phi_a. \quad (17)$$

$U(0, -\infty)^{disc.}$  means that only the contributions from those diagrams that consist of the separate disconnected parts, each associated with one incident particle, should be taken. This equation will be useful for calculating  $\mathcal{Q}_a$  by perturbation.

The basic vectors  $\mathcal{Q}_a$  thus defined do not form an orthogonal set, as is readily seen by calculating  $\mathcal{Q}$ 's to the first few orders in the coupling constant. Therefore, it would be convenient to define the "reciprocal lattice"  $\{\bar{\mathcal{Q}}_a\}$  that satisfies the relation

$$(\bar{\mathcal{Q}}_a, \mathcal{Q}_b) = \delta_{ab}. \quad (18)$$

We shall illustrate the method of constructing the set  $\{\bar{\mathcal{Q}}_a\}$  by an example. Since the states with different total charges and different total momenta are orthogonal, we shall limit ourselves to the states of the total charge  $e$  and the total momentum  $p$ . Such space is spanned by a set of states

$$A_p^* \mathcal{Q}_0, A_{p-k}^* C_k^* \mathcal{Q}_0, A_{p-k-k'}^* C_k^* C_{k'}^* \mathcal{Q}_0, \dots$$

Remark the fact that all the terms of  $A_p^*$ ,  $B_q^*$  and  $C_k^*$  contain at least one creation operator,\* that is, they contain no c-number term. This means that any term of  $A_{p-k}^* C_k^*$  contains at least two, and any term of  $A_{p-k-k'}^* C_k^* C_{k'}^*$  at least three creation operators. More precisely

$$\begin{aligned} \mathcal{Q}(p_1, \dots, p_n; q_1, \dots, q_m; k_1, \dots, k_l) = e^\Delta [\Phi(p_1, \dots, p_n; q_1, \dots, q_m; k_1, \dots, k_l) \\ + \sum_{(n'+m'+l') > (n+m+l)} f(p'_1, \dots; q'_1, \dots; k'_1, \dots) \Phi(p'_1, \dots, p'_{n'}; q'_1, \dots, q'_{m'}; k'_1, \dots, k'_{l'})]. \end{aligned} \quad (19)$$

( $n'$  and  $m'$  are not less than  $n$  and  $m$ , respectively, but  $l'$  can be smaller than  $l$ , since mesons can be transformed into proton pairs.) Therefore,  $\Phi(p; 0; 0)$  is orthogonal to every  $\mathcal{Q}_a$  except to  $\mathcal{Q}(p; 0; 0)$ , and we can take

$$\bar{\mathcal{Q}}(p; 0; 0) = e^{-\bar{\Delta}} \Phi(p; 0; 0).$$

By the same reason,  $\Phi(p-k; 0; k)$  is orthogonal to every  $\mathcal{Q}_a$  except to  $\mathcal{Q}(p-k; 0; k)$  and  $\mathcal{Q}(p; 0; 0)$ . Therefore, we assume

$$\bar{\mathcal{Q}}(p-k; 0; k) = \alpha \Phi(p-k; 0; k) + \beta \bar{\mathcal{Q}}(p; 0; 0).$$

The coefficient  $\alpha$  is fixed from the normalization condition (18) and is simply  $e^{-\bar{\Delta}}$  from

\* This is trivial for  $A_p^*$  and  $B_q^*$  but not for  $C_k^*$  ( $k=0$ ), which could contain the c-number terms corresponding to such graphs as Fig. 5. The contributions of such graphs vanish for pseudo-scalar mesons but not for scalar mesons, for which the method will need some modifications.



Fig. 5

(19).  $\beta$  is then determined from the requirement that  $\mathcal{O}(\boldsymbol{p}-\boldsymbol{k}; 0; \boldsymbol{k})$  shall be orthogonal to  $\mathcal{Q}(\boldsymbol{p}; 0; 0)$  and we obtain

$$\beta = -e^{-\bar{\Delta}}(\mathcal{Q}(\boldsymbol{p}; 0; 0), \Phi(\boldsymbol{p}-\boldsymbol{k}; 0; \boldsymbol{k})).$$

In this way, we can construct the set  $\{\mathcal{O}_\alpha\}$  successively. From the method of construction, it is evident that in general  $\mathcal{O}_\alpha$  has the following form.

$$\begin{aligned} \mathcal{O}(\boldsymbol{p}_1, \dots, \boldsymbol{p}_n; \boldsymbol{q}_1, \dots, \boldsymbol{q}_m; \boldsymbol{k}_1, \dots, \boldsymbol{k}_l) = & e^{-\bar{\Delta}} [\Phi(\boldsymbol{p}_1, \dots, \boldsymbol{p}_n; \boldsymbol{q}_1, \dots, \boldsymbol{q}_m; \boldsymbol{k}_1, \dots, \boldsymbol{k}_l) \\ & + \sum_{(n'+m'+l') < (n+m+l)} g(\boldsymbol{p}'_1, \dots, \boldsymbol{p}'_{n'}; \boldsymbol{q}'_1, \dots, \boldsymbol{q}'_{m'}; \boldsymbol{k}'_1, \dots, \boldsymbol{k}'_{l'}) \Phi(\boldsymbol{p}'_1, \dots, \boldsymbol{p}'_{n'}; \boldsymbol{q}'_1, \dots, \boldsymbol{q}'_{m'}; \boldsymbol{k}'_1, \dots, \boldsymbol{k}'_{l'})]. \end{aligned} \quad (20)$$

(Again  $n'$  and  $m'$  are not greater than  $n$  and  $m$ , but  $l'$  can be greater than  $l$ .)

In order that the set  $\{\mathcal{Q}_\alpha\}$  can be used as the base of a representation, it must be shown to form a complete set.  $\mathcal{Q}_\alpha$  coincides with the corresponding  $\Phi_\alpha$  in the zeroth order of the coupling constant. Hence, two different  $\mathcal{Q}_\alpha$ 's never correspond to the same  $\Phi_\alpha$  and the correspondence is one-to-one. However, this may not be sufficient to infer the completeness of the set  $\{\mathcal{Q}_\alpha\}$ . The exact proof of the completeness of the set  $\{\mathcal{Q}_\alpha\}$  seems hardly accessible and here we shall be content with showing by a constructive method that any  $\Phi_\alpha$  can be expressed by an infinite series of  $\mathcal{Q}$ 's, assuming the convergence of it.

Take, as an example a free proton state  $\Phi(\boldsymbol{p}; 0; 0)$ . From (19), the only  $\mathcal{Q}_\alpha$  in the space of the total charge  $e$  that contains the term  $\Phi(\boldsymbol{p}; 0; 0)$  is  $\mathcal{Q}(\boldsymbol{p}; 0; 0)$ . So, as a first approximation, we represent  $\Phi(\boldsymbol{p}; 0; 0)$  by  $A_p^* \mathcal{Q}_0$ . Then, the error produced would be the remaining terms of  $A_p^*$  (and  $I_0^*$ ) such as Fig. 2a, 2b, etc., and are at least first order in the coupling constant. Next, to cancel, for instance, the term 2a, which is proportional to  $\Phi(\boldsymbol{p}-\boldsymbol{k}; 0; \boldsymbol{k})$  we subtract  $A_{p-k}^* C_k^* \mathcal{Q}_0$  with an appropriate constant factor. (In reality, this cancels not only the term 2a, but a class of infinite terms including such as 2b and 2c.) The error here produced, would be at least second order in the coupling constant. In this way, we can cancel the redundant terms successively by adding some  $\mathcal{Q}_\alpha$ 's and the order of the error in the coupling constant becomes higher and higher, although the terms to be cancelled increase infinitely as we proceed.

Having defined the mutually reciprocal bases  $\{\mathcal{Q}_\alpha\}$  and  $\{\mathcal{O}_\alpha\}$ , we shall now transform into the representations defined by these bases, say  $\mathcal{Q}$ - and  $\mathcal{O}$ -representations. In this section, the well-known formal aspects of the non-orthogonal representations will be summarized. Define the transformation functions  $W$  and  $V$  by

$$\left. \begin{aligned} W = \sum_\alpha \mathcal{Q}_\alpha \langle \Phi_\alpha \quad \text{or} \quad \mathcal{Q}_\alpha = W \Phi_\alpha \\ V = \sum_\alpha \mathcal{O}_\alpha \langle \Phi_\alpha \quad \text{or} \quad \mathcal{O}_\alpha = V \Phi_\alpha \end{aligned} \right\} \quad (21)$$

From (18), we have

$$V^\dagger W = W V^\dagger = 1. \quad (22)$$

Transform a state-vector  $\Psi$  in the usual representation (call it hereafter  $\Phi$ -representation) into  $X$  and  $Y$  in  $\mathcal{Q}$ - and  $\mathcal{O}$ -representations according to

$$\Psi = W X \quad \text{and} \quad \Psi = V Y. \quad (23)$$

Then, from (21) and (22),

$$\left. \begin{aligned} \langle a | X \rangle &= (\Phi_a, X) = (\mathcal{G}_a, \Psi) \\ \langle a | Y \rangle &= (\Phi_a, Y) = (\mathcal{Q}_a, \Psi). \end{aligned} \right\} \quad (24)$$

They are the so-called the covariant and contravariant components\* of  $\Psi$  in the  $\mathcal{Q}$ -representation;

$$\Psi = \sum_a \langle a | X \rangle \mathcal{Q}_a = \sum_a \langle a | Y \rangle \mathcal{G}_a.$$

They are connected by the equations

$$\langle a | Y \rangle = \sum_b (\mathcal{Q}_a, \mathcal{Q}_b) \langle b | X \rangle \quad \text{and} \quad \langle a | X \rangle = \sum_b (\mathcal{G}_a, \mathcal{G}_b) \langle b | Y \rangle,$$

or simply

$$Y = G X, \quad X = G Y, \quad (25)$$

with

$$G = W^\dagger W; \quad \langle b | G | a \rangle = (\mathcal{Q}_b, \mathcal{Q}_a) \quad (26)$$

and

$$G^{-1} = V^\dagger V; \quad \langle b | G^{-1} | a \rangle = (\mathcal{G}_b, \mathcal{G}_a).$$

The scalar product of two vectors is given in these representations by

$$(\Psi, \Psi') = (X, G X') = (Y, G^{-1} Y') = (Y, X'). \quad (27)$$

#### § 4. Schrödinger equation in the real-particle representation

We shall transform the Schrödinger equation

$$(E - H_0) \Psi = 0 \quad (28)$$

into the  $\mathcal{Q}$ - and  $\mathcal{G}$ -representations. From (23) and (22) we readily obtain

$$(E - \tilde{H}) X = 0 \quad \text{and} \quad (E - \tilde{H}^\dagger) Y = 0, \quad (29)$$

where the transformed Hamiltonian

$$\tilde{H} = V^\dagger H W \quad (30)$$

has the matrix elements, according to (21),

$$\langle b | \tilde{H} | a \rangle = (\mathcal{G}_b, H \mathcal{Q}_a). \quad (31)$$

To calculate the matrix element of the total Hamiltonian  $H$  we shall evaluate  $H \mathcal{Q}_a$  for an arbitrary  $\mathcal{Q}_a$ .  $H_0$  can be written as

$$H_0 = \sum_p a_p^* a_p \epsilon_p + \sum_q b_q^* b_q \epsilon_q + \sum_k c_k^* c_k \omega_k$$

\* The concept of co- and contra-variant components was used first by Nishijima<sup>10</sup> in quantum field theory.



and  $H_i$ ,

$$H_i = \int_{x_0=0} [ig \bar{\psi}(x) \gamma_5 \psi(x) \phi(x) - \delta m \bar{\psi}(x) \psi(x) - \frac{1}{2} \partial \mu^2 \phi(x)^2] d^3 x.$$

Denote by  $H^c$  the creation part of  $H$ , namely,

$$H^c = H_i^c = \int_{x_0=0} [ig \bar{\psi}^{(-)} \gamma_5 \psi^{(-)} \phi^{(-)} - \delta m \bar{\psi}^{(-)} \psi^{(-)} - \frac{1}{2} \partial \mu^2 \phi^{(-)} \phi^{(-)}] d^3 x.$$

We shall introduce a kind of contraction symbol  $H^{\rightarrow} A_p^{**}$  which means that all the annihilation operators involved in  $H$  shall contract with the creation operators in  $A_p^{**}$  in every possible way. Therefore, is not a c-number but still contains an infinite sum of products of creation operators. The symbol  $H^{\rightarrow} A_p^{**} A_{p'}^{**}$  will be used to represent those terms that arise from the part of  $H$  with more than one annihilation operators contracting simultaneously with the creation operators in both  $A_p^{**}$  and  $A_{p'}^{**}$  in all possible ways. In this case, therefore,  $H_0$  makes no contribution.  $H^{\rightarrow} A_p^{**} A_{p'}^{**} A_{p''}^{**}$  shall have the analogous meaning. Using these notations, for arbitrary operators  $A$  and  $B$  which are the sums of products of creation operators, there holds a relation

$$H^{\rightarrow} (AB)^* = H^{\rightarrow} A^* B + H^{\rightarrow} A B^* + H^{\rightarrow} A^* B^*. \quad (32)$$

Now, the total Hamiltonian operated on  $\mathcal{Q}_0$  gives

$$H \mathcal{Q}_0 = H D_0^* \Phi_0 = (H^c D_0^* + H^{\rightarrow} D_0^{**}) \Phi_0.$$

Comparing this equation with (4), we get

$$H^c D_0^* + H^{\rightarrow} D_0^{**} = \varepsilon_0 D_0^*. \quad (33)$$

Analogously, operating  $H$  on one-particle states (9), and remarking the equations (6), we obtain

$$\left. \begin{aligned} H^c A_p^* D_0^* + H^{\rightarrow} (A_p^* D_0^*)^* &= (\varepsilon_p + \varepsilon_0) A_p^* D_0^*, \\ H^c B_q^* D_0^* + H^{\rightarrow} (B_q^* D_0^*)^* &= (\varepsilon_q + \varepsilon_0) B_q^* D_0^*, \\ H^c C_k^* D_0^* + H^{\rightarrow} (C_k^* D_0^*)^* &= (\omega_k + \varepsilon_0) C_k^* D_0^*. \end{aligned} \right\} \quad (34)$$

Operating  $H$  on the two-particle state  $\mathcal{Q}(pp'; 0; 0)$ , using (32) and rearranging the resulting terms, we have

$$\begin{aligned} H \mathcal{Q}(pp'; 0; 0) &= H A_p^* A_{p'}^* D_0^* \Phi_0 \\ &= \{H^c A_p^* D_0^* + H^{\rightarrow} (A_p^* D_0^*)^*\} A_{p'}^* \Phi_0 + A_p^* \{H^c A_{p'}^* D_0^* + H^{\rightarrow} (A_{p'}^* D_0^*)^*\} \Phi_0 \\ &\quad - A_p^* A_{p'}^* (H^c D_0^* + H^{\rightarrow} D_0^{**}) \Phi_0 + (H_i^{\rightarrow} A_p^{**} A_{p'}^{**} D_0^* + H_i^{\rightarrow} A_p^{**} A_{p'}^{**} D_0^{**}) \Phi_0. \end{aligned}$$

Here we have used the relation (10) and the fact that  $H^c$  commutes with  $D_0^*$ ,  $A_p^*$ ,  $B_q^*$  and  $C_k^*$ . From (33) and (34), we get

$$\begin{aligned} H \mathcal{Q}(pp'; 0; 0) &= (\varepsilon_p + \varepsilon_{p'} + \varepsilon_0) \mathcal{Q}(pp'; 0; 0) \\ &\quad + (H_i^{\rightarrow} A_p^{**} A_{p'}^{**} D_0^* + H_i^{\rightarrow} A_p^{**} A_{p'}^{**} D_0^{**}) \Phi_0. \end{aligned}$$

The first term represents the energy of the real particles plus the vacuum energy. The second can be interpreted as the interaction between the two real particles, since it connects the  $A_p^*$  and  $A_{p'}^*$ , and deforms the virtual clouds of the two particles. This equation can be readily generalized to

$$H\mathcal{Q}_a = (\varepsilon_0 + \varepsilon_a)\mathcal{Q}_a + H_i^{\leftrightarrow}\mathcal{Q}_a^*, \quad (35)$$

where the abbreviation  $H_i^{\leftrightarrow}\mathcal{Q}_a^*$  means

$$\begin{aligned} H_i^{\leftrightarrow}\mathcal{Q}^* &\equiv H_i^{\leftrightarrow}(A^*B^*C^*\dots D_0^*)^*\Phi_0 \\ &\equiv H_i^{\leftrightarrow}(A^{**}B^{**}C^{**}\dots + A^{**}B^*C^{**}\dots + A^*B^{**}C^{**}\dots + \dots)(D_0^* + D_0^{**})\Phi_0 \\ &\quad + H_i^{\leftrightarrow}(A^{**}B^{**}C^{**}\dots + \dots)D_0^*\Phi_0. \end{aligned} \quad (36)$$

The matrix elements of the transformed Hamiltonian  $\tilde{H}$ , (31), can now be written down using (35),

$$\langle b|\tilde{H}|a\rangle = (\varepsilon_0 + \varepsilon_a)\delta_{ba} + (\mathcal{G}_b, H_i^{\leftrightarrow}\mathcal{Q}_a).$$

Therefore, introducing the effective interaction Hamiltonian  $K$  defined by

$$\langle b|K|a\rangle = (\mathcal{G}_b, H_i^{\leftrightarrow}\mathcal{Q}_a), \quad (37)$$

$\tilde{H}$  decomposes into three parts,

$$\tilde{H} = \varepsilon_0 + H_0 + K, \quad (38)$$

and we can rewrite the equations (29) as

$$(\varepsilon - H_0)\mathbf{X} = K\mathbf{X}; \quad \varepsilon = E - \varepsilon_0, \quad (I)$$

and

$$(\varepsilon - H_0)\mathbf{Y} = K^\dagger\mathbf{Y}. \quad (I')$$

These are the fundamental Schrödinger equations in our new representations. The eigenstates of the free Hamiltonian in  $\mathcal{Q}$ -representation,  $\mathbf{X} = \mathcal{Q}_a$ , corresponds to the real-particle state  $\mathcal{Q}_a$  in the usual representation. Especially, the true vacuum state  $\mathcal{Q}_0$  and the real one-particle states such as  $\mathcal{Q}(\mathbf{p}; 0; 0)$  are represented by the free eigenstates  $\Phi_0$  and  $\Phi(\mathbf{p}; 0; 0)$  in  $\mathcal{Q}$ -representation;  $K$  operations on these states vanishes. On the other hand,  $\mathbf{Y} = \mathcal{Q}_a$  corresponds to  $\mathcal{G}_a$ , which permits no direct physical interpretation as  $\mathcal{Q}_a$ . Hence, we shall consider  $\mathcal{Q}$ -representation as the fundamental and  $\mathcal{G}$ -representation as the supplementary representation.

One advantage of equation (I) is that the energy of the system appears as the difference from the vacuum energy,  $\varepsilon = E - \varepsilon_0$ , which is just the observed quantity. Further,  $K$  gives rise neither to the vacuum self-energy nor to the single particle self-energy processes, from its definition (37) and (36). The matrix-elements of the interaction Hamiltonian  $K$  are not expressed in a simple form, but can be calculated once we know the one-particle solutions, namely,  $D_0^*$ ,  $A_p^*$ ,  $B_q^*$  and  $C_k^*$ . However,  $K$  is not Hermitian, reflecting the fact that the base  $\mathcal{Q}_a$  forms a non-orthogonal set. This may introduce some complications in applying these equations. Postponing the discussions on this point at the end of this

section, here we shall derive an alternative Hermitian equation instead of (I). To derive this, transform the equation (28) by  $W$  according to (23) as before and multiply from left by  $W^\dagger$  instead by  $V^\dagger$ . We obtain

$$(EG - W^\dagger H W)X = 0,$$

where  $G$  is defined by (26). From equation (35) we readily get

$$\begin{aligned} \langle b|W^\dagger H W|a\rangle &= (\mathcal{Q}_b, H\mathcal{Q}_a) \\ &= \left(\varepsilon_0 + \frac{\varepsilon_a + \varepsilon_b}{2}\right)(\mathcal{Q}_b, \mathcal{Q}_a) + \frac{1}{2}[(\mathcal{Q}_b, H_i^\dagger \mathcal{Q}_a^*) + (H_i^\dagger \mathcal{Q}_b^*, \mathcal{Q}_a)]. \end{aligned}$$

Hence, introducing the Hermitian operator  $I(\varepsilon)$  defined by

$$\begin{aligned} \langle b|I(\varepsilon)|a\rangle &= \frac{1}{2}(\varepsilon_a + \varepsilon_b - 2\varepsilon)[(\mathcal{Q}_b, \mathcal{Q}_a) - \langle b|a\rangle] \\ &\quad + \frac{1}{2}[(\mathcal{Q}_b, H_i^\dagger \mathcal{Q}_a^*) + (H_i^\dagger \mathcal{Q}_b^*, \mathcal{Q}_a)], \end{aligned} \quad (39)$$

we get

$$(\varepsilon - H_0)X = I(\varepsilon)X. \quad (\text{II})$$

Although the definition of  $I(\varepsilon)$  is rather complicated, and permits no simple physical interpretation, it could be used for the energy eigenvalue problems, since the real property of the eigenvalue is guaranteed. Further, in this equation, no explicit knowledge of the reciprocal base  $\{\mathcal{Q}_a\}$  is necessary.

Turning back to equation (I), we shall examine what follows from the non-Hermitian interaction Hamiltonian. We shall begin with the scattering problems. Although the equation (I) retains the usual form of the Schrödinger equation and permits formally the conventional method of finding the transition probabilities, it must be examined whether this could be justified or not, since the definition of scalar products of two vectors is changed in  $\mathcal{Q}$ -representation as shown in (27). The usual steps to find the transition probability applied to equation (I) would be: given the incident wave  $\Phi_a$ , solve (I) under the boundary condition of outgoing wave as

$$X_a = \Phi_a + \frac{1}{\varepsilon_a - H_0 + i\alpha} K' X_a \quad (\alpha \rightarrow 0); \quad (H_0 - \varepsilon_a)\Phi_a = 0, \quad (40)$$

and obtain the transition probability from the formula

$$\sigma_{ba} = \frac{2\pi}{\hbar} |\langle b|\tilde{R}|a\rangle|^2 \delta(\varepsilon_b - \varepsilon_a),$$

where the matrix  $\tilde{R}$  is given by

$$\langle b|\tilde{R}|a\rangle = (\Phi_b, KX_a). \quad (41)$$

Or, what is equivalent, define the S-matrix in  $\mathcal{Q}$ -representation, say,  $\tilde{S}$ , as usual by

$$\tilde{S} = \sum_{n=0}^{\infty} \left( \frac{1}{i} \right)^n \frac{1}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P[K(t_1), K(t_2), \dots, K(t_n)] dt_1 dt_2 \cdots dt_n \quad (42)$$

with

$$K(t) = e^{iH_0 t} K e^{-iH_0 t}$$

and calculate its matrix-elements, which are related to  $\tilde{R}$  by the equation

$$\langle b | \tilde{S} | a \rangle = \langle b | a \rangle - 2\pi i \delta(\varepsilon_b - \varepsilon_a) \langle b | \tilde{R} | a \rangle. \quad (43)$$

We can show that these formal procedures are indeed justified, by proving that the matrix-elements of  $\tilde{S}$  are equal to those of the S-matrix in the usual representation, say  $S$ , (see Appendix II)

$$\langle b | \tilde{S} | a \rangle = \langle b | S | a \rangle = \langle b | U(\infty, -\infty) | a \rangle. \quad (44)$$

This implies from (43) that

$$\langle b | \tilde{R} | a \rangle = \langle b | R | a \rangle = (\Phi_b, H_i \Psi_a) \quad \text{for } \varepsilon_a = \varepsilon_b, \quad (45)$$

where  $\Psi_a$  is the eigenstate of  $H$ , define by (1) and satisfies

$$\Psi_a = \Phi_a + \frac{1}{\varepsilon_a - H_0 + i\alpha} (H_i - \varepsilon_0) \Psi_a.$$

Thus, we see that the non-Hermitian interaction Hamiltonian introduces no complication in scattering problems. These results may be considered as the consequences of the fact that the transformation function  $U$  (or  $\mathcal{Q}_a$ ) contains no term representing the real processes.

Next we shall consider the energy eigenvalue problem of bound systems. The exact solution of equation (I) must have the real eigenvalue  $\varepsilon$ , since it is a transform of the usual Schrödinger equation. But the approximations that must be done in defining  $\mathcal{Q}_a$  and hence the matrix-elements of  $K$  as well as the approximate treatment of equation (I) such as Tamm-Dancoff's may lead to the complex eigenvalue  $\varepsilon$ . Denote the Hermitian parts of  $K$  as  $K_h$  and  $K_a$ , respectively. The necessary and sufficient condition for the eigenvalue  $\varepsilon$  to be real for a solution  $\mathbf{X}_n$ , is

$$(\mathbf{X}_n, K_a \mathbf{X}_n) = 0. \quad (46)$$

Suppose that for the approximate  $K$ , we have solved (I) and obtained the eigenvector  $\mathbf{X}_n$  and eigenvalue  $\varepsilon_n$ . Then, the condition (46) will not be satisfied in general, and the eigenvalue  $\varepsilon_n$  will have the complex value given by

$$\begin{aligned} \text{Re } \varepsilon_n &= (\mathbf{X}_n, (H_0 + K_h) \mathbf{X}_n), \\ \text{Im } \varepsilon_n &= (1/i) \cdot (\mathbf{X}_n, K_a \mathbf{X}_n). \end{aligned} \quad (47)$$

Since for the exact  $K_a$ ,  $\text{Im } \varepsilon_n$  must vanish from (46), it is expected that for the properly defined  $K$ , that

$$\text{Im } \varepsilon_n \ll \text{Re } \varepsilon_n, \quad (48)$$

and we may discard  $\text{Im } \varepsilon_n$  as physically meaningless. Or inversely, if solving (I) the



obtained eigenvalue  $\varepsilon$  should not satisfy the condition (48), it must be concluded that the  $K$  used has not been good. The relation (48) would mean that we could treat  $K_a$  as perturbation. We could then replace the original equation (I) by

$$(\varepsilon' - H_0 - K_h)X' = 0. \quad (49)$$

if the obtained eigenvector  $X'_n$  and eigenvalue  $\varepsilon'_n$  satisfy the condition

$$1/i \cdot (X'_n, K_a X'_n) = \varepsilon'_n,$$

since the error thus produced would be, in general, not greater than the imaginary part of the eigenvalue obtained from (I). (second order in  $K_a$ )

For the practical treatment of equation (I), we may employ the so-called Tamm-Dancoff's method.<sup>11)</sup> Representing equation (I) by the eigenstate  $\Phi_a$  of the free Hamiltonian, we obtain an infinite set of simultaneous equations for the components  $\langle a|X \rangle$ . Eliminating all the components other than the specified one that represents a prescribed number of particles, for instance, two protons, we would obtain an equation of the form

$$\langle pq|X \rangle = \sum_{p'q'} \langle pq|V|p'q' \rangle \langle p'q'|X \rangle,$$

where the potential  $V$  would be given by

$$\begin{aligned} \langle pq|V|p'q' \rangle &= \left\langle pq \left| K \left( 1 - \frac{1}{\varepsilon - H_0} K \right)^{-1} \right| p'q' \right\rangle \\ &= \langle pq|K|p'q' \rangle + \left\langle pq \left| K \frac{1}{\varepsilon - H_0} K \right| p'q' \right\rangle + \dots. \end{aligned} \quad (50)$$

It should be remarked that the intermediate states should not include the two-proton states. The so-called Tamm-Dancoff's method consists in restricting the intermediate states to a small number of specified states (two-proton plus one-meson states, for instance), while the adiabatic potential could be obtained by calculating the first few terms of (50) and replacing  $\varepsilon$  by  $\varepsilon_{p'} + \varepsilon_{q'}$ . Both non-adiabatic and adiabatic potentials would be expected to be non-Hermitian, and the foregoing general argument for the complex eigenvalue would hold here too.

## § 5. Example

In order to realize the structure of our fundamental equations (I) and (II), and also to check the arguments of the foregoing section, we shall take as example the neutron-proton system and solve these equations to the second order in the coupling constant  $g$ . We take the neutral pseudoscalar meson as before.  $p$ ,  $q$  and  $k$  will now be used to denote the momenta of proton, neutron and meson, respectively, and  $p$ ,  $q$ , the momenta of the anti-particles. The nuclear potential  $V$  obtained by solving (I) is given by (50).

We shall calculate the matrix-elements of  $K$  by perturbation using the expression (19) for  $\mathcal{Q}_a$ .  $\mathcal{O}_a$  can be obtained by the method described in § 3. The results are:

$$\mathcal{Q}(p, q) = i \left[ \Phi(p, q) - ig \sum_k \frac{(\bar{u}_+(\mathbf{p}-\mathbf{k}) \gamma_5 u_+(\mathbf{p}))}{(\varepsilon_{p-k} + \omega_k - \varepsilon_p)} \Phi(p-k, q, k) \right]$$

$$-ig \sum_{p'k} \frac{(\bar{u}_+(\mathbf{p}'-\mathbf{k})\gamma_5 u_-(\mathbf{p}'))}{(\varepsilon_{p'-k} + \omega_k + \varepsilon_{p'})} \Phi(\mathbf{p}, \mathbf{p}'-\mathbf{k}, -\bar{\mathbf{p}}', \mathbf{q}, \mathbf{k}) + N.T. + O(g^2) \Big].$$

The first line comes from the contribution of the diagrams 2<sup>o</sup> and 2a. The third term represents the contribution of the diagram 1a. *N.T.* (Neutron term) means the similar expressions as the second and third terms and are connected with neutron.  $O(g^2)$  denotes the terms at least second order in  $g$ .

$$\mathcal{G}(\mathbf{p}, \mathbf{q}) = e^{-\bar{\Delta}} \Phi(\mathbf{p}, \mathbf{q})$$

$$\mathcal{G}'(\mathbf{p}, \mathbf{q}, \mathbf{k}) = e^{-\bar{\Delta}} \left[ \Phi(\mathbf{p}, \mathbf{q}, \mathbf{k}) + \left\{ ig \frac{(\bar{u}_+(\mathbf{p}+\mathbf{k})\gamma_5 u_+(\mathbf{p}))}{(\varepsilon_p + \omega_k - \varepsilon_{p+k})} \sqrt{2\omega_k} + O(g^2) \right\} \Phi(\mathbf{p}+\mathbf{k}, \mathbf{q}) + N.T. \right].$$

The matrix-elements of  $K$  are given by

$$\begin{aligned} \langle \mathbf{p}\mathbf{q} | K | \mathbf{p}'\mathbf{q}' \rangle &= \mathcal{G}(\mathbf{p}, \mathbf{q}), H_i^{\pi} \mathcal{G}'(\mathbf{p}', \mathbf{q}') \\ &= g^2 (\bar{u}_+(\mathbf{p})\gamma_5 u_+(\mathbf{p}')) (\bar{v}_+(\mathbf{q})\gamma_5 v_+(\mathbf{q}')) \left[ \frac{1}{\varepsilon_p + \omega_{p-p'} - \varepsilon_{p'}} + \frac{1}{\varepsilon_q + \omega_{p-p'} - \varepsilon_{q'}} \right] \times \\ &\quad \times \frac{1}{2\omega_{p-p'}} \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') + O(g^4), \\ \langle \mathbf{p}\mathbf{q} | K | \mathbf{p}''\mathbf{q}''\mathbf{k}'' \rangle &= ig (\bar{u}_+(\mathbf{p})\gamma_5 u_+(\mathbf{p}'')) \frac{1}{\sqrt{2\omega_{k''}}} \delta(\mathbf{p} - \mathbf{p}'' - \mathbf{k}'') \delta(\mathbf{q} - \mathbf{q}'') \\ &\quad + N.T. + O(g^3), \end{aligned}$$

$$\text{other } \langle \mathbf{p}\mathbf{q} | K | \alpha \rangle = 0,$$

$$\langle \mathbf{p}''\mathbf{q}''\mathbf{k}'' | K | \mathbf{p}\mathbf{q} \rangle = O(g^3).$$

Thus, to the order  $g^2$ , only the first term of the expansion (50) makes non-vanishing contribution and we have

$$\begin{aligned} \langle \mathbf{p}\mathbf{q} | V | \mathbf{p}'\mathbf{q}' \rangle &= g^2 (\bar{u}_+(\mathbf{p})\gamma_5 u_+(\mathbf{p}')) (\bar{v}_+(\mathbf{q})\gamma_5 v_+(\mathbf{q}')) \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \times \\ &\quad \times \left[ \frac{1}{\varepsilon_p + \omega_{p-p'} - \varepsilon_{q'}} + \frac{1}{\varepsilon_q + \omega_{p-p'} - \varepsilon_{q'}} \right] \frac{1}{2\omega_{p-p'}}, \end{aligned} \quad (51)$$

where  $v_{\pm}(\mathbf{q})$  denotes the Dirac spinor for neutron. This is the usual second order adiabatic potential. The usual non-adiabatic potential cannot be obtained from equation (I) to this order. It will be shown in the following that it is obtained by solving the Hermitian equation (II). The potential resulting from equation (II) would be given again by (50) replacing  $K$  by  $I(\varepsilon)$ .

$$\langle \mathbf{p}\mathbf{q} | V | \mathbf{p}'\mathbf{q}' \rangle = \langle \mathbf{p}\mathbf{q} | I(\varepsilon) | \mathbf{p}'\mathbf{q}' \rangle + \sum_{\alpha} \langle \mathbf{p}\mathbf{q} | I(\varepsilon) | \alpha \rangle \frac{1}{\varepsilon - \varepsilon_{\alpha}} \langle \alpha | I(\varepsilon) | \mathbf{p}'\mathbf{q}' \rangle + \dots \quad (52)$$

To the second order in  $g$ , the intermediate states are restricted to two-nucleon plus one-

meson states and two-nucleon plus one nucleon-pair plus one-meson states. From the definition of  $I(\varepsilon)$ , (39), and the expression (19) for  $\mathcal{Q}_a$  we can readily calculate the matrix-elements of  $I(\varepsilon)$ . The results are:

$$\begin{aligned} \langle p q | I | p' q' \rangle &= g^2 (\bar{u}_+(\mathbf{p}) \gamma_5 u_+(\mathbf{p}')) (\bar{v}_+(q) \gamma_5 v_+(q')) \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \times \\ &\times \frac{1}{2\omega_{p-p'}} \left[ \frac{\varepsilon_q + \varepsilon_{p'} - \omega_{p-p'} - \varepsilon}{(\varepsilon_p + \omega_{p-p'} - \varepsilon_{p'}) (\varepsilon_q - \omega_{p-p'} - \varepsilon_{q'})} + \left( \begin{smallmatrix} p \rightarrow q' \\ p' \rightarrow q' \end{smallmatrix} \right) \right] + O(g^4), \quad (53) \\ \sum_{p'', q'', k} \langle p q | I | p'' q'' k \rangle &= \frac{1}{\varepsilon - \varepsilon_{p'} - \varepsilon_{q'} - \omega_k} \langle p'' q'' k | I | p' q' \rangle \\ &= g^2 (\bar{u}_+(\mathbf{p}) \gamma_5 u_+(\mathbf{p}')) (\bar{v}_+(q) \gamma_5 v_+(q')) \times \\ &\times \frac{1}{2\omega_{p-p'}} \left[ \frac{(\varepsilon - \varepsilon_p - \varepsilon_q) (\varepsilon - \varepsilon_{p'} - \varepsilon_{q'}) \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}')}{(\varepsilon_p + \omega_{p-p'} - \varepsilon_{p'}) (\varepsilon_q - \omega_{p-p'} - \varepsilon_{q'}) (\varepsilon - \varepsilon_{q'} - \varepsilon_p - \omega_{p-p'})} + \left( \begin{smallmatrix} p \rightarrow q' \\ p' \rightarrow q' \end{smallmatrix} \right) \right] + \\ &+ \text{nucleon self-energy term } (g^2) + O(g^4) \quad (54) \end{aligned}$$

and

$$\begin{aligned} \sum_{p'', q'', t, t', k} \langle p q | I | p'' q'' t t' k \rangle &= \frac{1}{\varepsilon - \varepsilon_{p'} - \varepsilon_{q'} - \varepsilon_t - \varepsilon_{t'} - \omega_k} \langle p'' q'' t t' k | I | p' q' \rangle \\ &= \text{nucleon self-energy term } (g^2) + \text{vacuum self-energy term } (g^2) + O(g^4). \quad (55) \end{aligned}$$

Adding (53) and (54), we get the second order non-adiabatic potential.

$$\begin{aligned} \langle p q | I^2 | p' q' \rangle_{\text{N.A.}}^{(2)} &= -g^2 (\bar{u}_+(\mathbf{p}) \gamma_5 u_+(\mathbf{p}')) (\bar{v}_+(q) \gamma_5 v_+(q')) \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \times \\ &\times \frac{1}{2\omega_{p-p'}} \left[ \frac{1}{\varepsilon - \varepsilon_{q'} - \omega_{p-p'} - \varepsilon_p} + \frac{1}{\varepsilon - \varepsilon_{p'} - \omega_{p-p'} - \varepsilon_q} \right]. \quad (56) \end{aligned}$$

The vacuum and nucleon self-energy processes reappear in the symmetrized equation (II), contrary to the equation (I). The second order nucleon self-energy contribution from (54) and (55) are, however, convergent;

$$\begin{aligned} (\varepsilon - \varepsilon_p - \varepsilon_q)^2 N &\equiv -g^2 (\varepsilon - \varepsilon_p - \varepsilon_q)^2 \delta(\mathbf{p} - \mathbf{p}') \delta(\mathbf{q} - \mathbf{q}') \sum_k (1/2\omega_k) \times \\ &\times \left[ (\bar{u}_+(\mathbf{p}) \gamma_5 A^{(+)}(\mathbf{p} - \mathbf{k}) \gamma_5 u_+(\mathbf{p})) / \{(\varepsilon_{p-k} + \omega_k - \varepsilon_p)^2 (\varepsilon - \varepsilon_{p-k} - \omega_k - \varepsilon_q)\} \right. \\ &- (\bar{u}_+(\mathbf{p}) \gamma_5 A^{(-)}(\mathbf{p} - \mathbf{k}) \gamma_5 u_+(\mathbf{p})) / \{(\varepsilon_{p-k} + \omega_k + \varepsilon_p)^2 (\varepsilon - \varepsilon_{p-k} - \omega_k - 2\varepsilon_p - \varepsilon_q)\} \left. \right] \\ &+ N.T. \quad (57) \end{aligned}$$

Here  $A^{\pm}$  means the usual projection operator for  $\pm$  energy states. The first term comes from (54) and the second from (55). This expression is the same as that Dyson<sup>(1)</sup> identified as the physically significant part of the diverging expression which results from the old Tamm-Dancoff method. It will be the advantage of the equation (II) that this finite part can be obtained without subtraction. On the other hand, the vacuum self-energy process gives rise to a badly divergent term,

$$(\varepsilon - \varepsilon_p - \varepsilon_q)^2 M \equiv -g^2 \sum_{k, t, t'} Tr(\gamma_5 A^{(+)}(\mathbf{t}) \gamma_5 A^{(-)}(-\mathbf{t}')) \delta(\mathbf{t} + \mathbf{t}' + \mathbf{k})^2 \delta(\mathbf{p} - \mathbf{p}') \delta(\mathbf{q} - \mathbf{q}')$$

$$\times \frac{(\varepsilon - \varepsilon_p - \varepsilon_q)^2}{(\varepsilon_t + \varepsilon_{t'} + \omega_k)^2} \frac{1}{2\omega_k} \frac{1}{(\varepsilon - \varepsilon_p - \varepsilon_q - \varepsilon_t - \varepsilon_{t'} - \omega_k)}. \quad (58)$$

It may appear strange that the equations (I) and (II) for the same state vector  $\mathbf{X}$  lead to different results in the same order  $g^2$ . This is, of course, related to the expandability of  $\mathbf{X}$  in powers of  $g$ . Indeed, if we employ the Born approximation, the above vacuum and nucleon self-energy terms vanish on account of the factor  $(\varepsilon - \varepsilon_p - \varepsilon_q)^2$  and the two equations (I) and (II) give the same result. For bound states,  $\mathbf{X}$  can not be expanded in powers of  $g$  and as for the diverging quantity, it may well be expected that the exact evaluation of the higher order terms may change the diverging aspect drastically. In fact we can show that there are contributions from the higher order terms of the vacuum self-energy type, which added to (58) give the finite results altogether. Namely, we shall consider the following vacuum self-energy terms from (52).

$$\begin{aligned} & \sum_{\pi} \langle pq | I | pq\pi \rangle' \frac{1}{\varepsilon' - \varepsilon_{\pi}} \langle pq\pi | I | pq \rangle' \\ & + \sum_{\pi\pi'} \langle pq | I | pq\pi \rangle' \frac{1}{\varepsilon' - \varepsilon_{\pi}} \langle pq\pi | I | pq\pi' \rangle' \frac{1}{\varepsilon' + \varepsilon_{\pi'}} \langle pq\pi' | I | pq \rangle' \\ & + \dots \end{aligned}$$

where  $\pi$  stands for  $p' p'' k$  such as  $p' + p'' + k = 0$  (or  $q' q'' k$  such as  $q' + q'' + k = 0$ )  $\varepsilon_{\pi} = \varepsilon_{p'} + \varepsilon_{p''} + \omega_k$ ,  $\varepsilon' = \varepsilon - \varepsilon_p - \varepsilon_q$ . The prime on the matrix-elements means that only the vacuum self-energy processes independent of the original nucleon  $p$  and  $q$  should be considered. It will be shown in the appendix III, that the above series contains the following terms

$$\varepsilon'^2 M - \varepsilon'^3 M^2 + \varepsilon'^4 M^3 - \dots = \varepsilon'^2 M \frac{1}{1 + \varepsilon' M} = \varepsilon', \quad (59)$$

since  $M$ , (58), is the diverging quantity. The equation for the two-nucleon components  $\langle pq | \mathbf{X} \rangle$  may now be written as

$$(\varepsilon - \varepsilon_p - \varepsilon_q)^2 N \langle pq | \mathbf{X} \rangle = \sum_{p'q'} \langle pq | V | p'q' \rangle_{\text{A.N.}}^{(2)} \langle p'q' | \mathbf{X} \rangle.$$

However, the grouping of the diverging series as done in (59) may be of course unallowable. The only point that must be stressed here is that as for the vacuum divergence the perturbation calculation is completely false.

In conclusion the author wishes to express his sincere thanks to Professor K. Sakuma for his kind encouragements throughout the course of this work, and to Mr. Hiida for his valuable discussions.

## Appendix I

We want to prove that the states defined by the equation

$$\Psi_{\alpha} = U(0, -\infty; \alpha) \Phi_{\alpha} \quad (\alpha \rightarrow 0); \quad H_0 \Phi_{\alpha} = \varepsilon_{\alpha} \Phi_{\alpha}, \quad (\text{A.1})$$



are the eigenstates of the total Hamiltonian with the eigenvalue  $\varepsilon_0 + \varepsilon_a$ ,

$$H\Psi_a = (\varepsilon_0 + \varepsilon_a)\Psi_a. \quad (\text{A} \cdot 2)$$

The adiabatic transformation function is defined by the power series expansion

$$U(0, -\infty; \alpha) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^0 \cdots \int_{-\infty}^0 P[H_i'(t_1), \dots, H_i'(t_n)] dt_1 \cdots dt_n, \quad (\text{A} \cdot 3)$$

where

$$H_i'(t) = H_i(t)e^{-\alpha t}.$$

On performing the integration, we have<sup>5)6)</sup>

$$U(0, -\infty; \alpha)|a\rangle = \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{\varepsilon_a - H_0 + ni\alpha} H_i \cdots \frac{1}{\varepsilon_a - H_0 + i\alpha} H_i \right] |a\rangle. \quad (\text{A} \cdot 4)$$

Using this expression, we find

$$(\varepsilon_a - H_0)U|a\rangle = H_i U|a\rangle - i\alpha g \partial U / \partial g |a\rangle,$$

or, since the state  $|a\rangle$  is arbitrary,

$$U^\dagger (H_0 + H_i) U = U^\dagger H U = H_0 + i\alpha g U^\dagger \partial U / \partial g. \quad (\text{A} \cdot 5)$$

Hence, in order that the equation (A·2) holds, it is necessary and sufficient to prove that

$$Q \equiv \lim_{\alpha \rightarrow 0} i\alpha g U^\dagger \partial U / \partial g = \varepsilon_0 I, \quad (\text{A} \cdot 6)$$

$I$  being the unit matrix. Again using the expression (A·4), we get

$$\langle b|Q|a\rangle = \lim_{\alpha \rightarrow 0} \sum_{n=1}^{\infty} \frac{i\alpha}{\varepsilon_a - \varepsilon_b + ni\alpha} \langle b|(U^\dagger H_i U)^{(n)}|a\rangle, \quad (\text{A} \cdot 7)$$

where  $(U^\dagger H_i U)^{(n)}$  means the  $n$ -th order term of  $U^\dagger H U$ . From the series expansion (A·3), singling out the factors that form the connected graphs containing  $H_i$  as a vertex, we obtain

$$(U^\dagger H_i U)^{(n)} = \sum_{m=0}^{n-1} (U^\dagger H_i U)_{\text{connec.}}^{(n-m)} (U^\dagger U)^{(m)},$$

where it should be understood that any pair of operators from each of the two factors  $(U^\dagger H_i U)^{(n-m)}$  and  $(U^\dagger U)^{(m)}$ , shall not contract and  $(\ )_{\text{connec.}}$  means that only the single connected graphs should be taken. But

$$U^\dagger U = 1$$

and only the term  $m=0$  remains

$$(U^\dagger H_i U) = (U^\dagger H_i U)_{\text{connec.}}^{(n)}. \quad (\text{A} \cdot 8)$$

Then the diagonal matrix-element of  $Q$  is given by

$$\langle a|Q|a\rangle = \lim_{\alpha \rightarrow 0} \langle a| \sum_{n=0}^{\infty} \frac{1}{n} (U^\dagger H_i U)_{\text{connec.}}^{(n)} |a\rangle.$$

It is easy to show from (A·4) that this gives in the limit  $\alpha \rightarrow 0$ ,

$$\langle a|Q|a\rangle = \langle a|\sum_{n=1}^{\infty}\left[\left(H_t P \frac{1}{\varepsilon_a - H_0}\right)^{n-1} H_t\right]_{\text{connec.}}|a\rangle, \quad (\text{A}\cdot 9)$$

which is simply the expression that gives the energy shift. It consists of the two parts, one the self-energy of the vacuum and the other the single nucleon self-energies. But we have subtracted the mass renormalization terms just to cancel the latter. Therefore, we have

$$\langle a|Q|a\rangle = \varepsilon_0. \quad (\text{A}\cdot 10)$$

It only remains to show that the non-diagonal matrix-elements of  $Q$  vanish. On account of the factor  $i\alpha(\varepsilon_a - \varepsilon_b + ni\alpha)^{-1}$  in (A·7),  $\langle b|Q|a\rangle$  vanishes unless  $\varepsilon_b = \varepsilon_a$ . Therefore,  $Q$  conserves the energy as well as momentum (spin and parity). This automatically excludes the non-diagonal elements for  $\Phi_0$ ,  $\Phi(\mathbf{p}; 0; 0)$ ,  $\Phi(0; \mathbf{q}; 0)$  and  $\Phi(0; 0; \mathbf{k})$ . However, this is not the case for the other states with more than one particle. In this case, (A·5) divides into two parts, one which is associated with the scattering of the particles involved and the other that represents the vacuum and the single nucleon self-energy processes. In the former, the final states  $b$  form a continuous set of states and this means as usual that the factor  $(\varepsilon_a - \varepsilon_b + ni\alpha)^{-1}$  must be understood as  $2\pi i \delta_+(\varepsilon_a - \varepsilon_b)$ . Hence, the additional factor  $i\alpha$  makes such matrix-elements vanish. On the other hand, the latter gives just  $\varepsilon_0$  as before. Thus, we can generally infer the equation (A·2).

## Appendix II

We shall prove the relation (45).  $\tilde{S}$  defined by (42) can also be given by

$$\tilde{S} = \lim_{t, -t_0 \rightarrow \infty} e^{iH_0 t} e^{-i(H_0 + K)(t - t_0)} e^{-iH_0 t_0}.$$

Hence,

$$\begin{aligned} \langle b|\tilde{S}|a\rangle &= \lim_{t, -t_0 \rightarrow \infty} (\Phi_b, e^{-i(\tilde{H} - \varepsilon_0)(t - t_0)} \Phi_a) e^{i\varepsilon_b t - i\varepsilon_a t_0} \\ &= \lim_{t, -t_0 \rightarrow \infty} (\tilde{\Phi}_b, e^{-i(H - \varepsilon_0)(t - t_0)} \mathcal{Q}_a) e^{i\varepsilon_b t - i\varepsilon_a t_0}, \end{aligned}$$

where we have used the equations (38), (30), (22), and (21). But

$$U(t, t_0) \equiv e^{iH_0 t} e^{-i(H - \varepsilon_0)(t - t_0)} e^{-iH_0 t_0}$$

is the transformation function in the interaction representation defined by the interaction Hamiltonian  $H_i - \varepsilon_0$ . (This is obtained from the usual interaction representation by a trivial unitary transformation.) Therefore,

$$\langle b|\tilde{S}|a\rangle \lim_{t, -t_0 \rightarrow \infty} = (\tilde{\Phi}_b, e^{-iH_0 t} U(t, t_0) e^{iH_0 t_0} \mathcal{Q}_a) e^{i\varepsilon_b t - i\varepsilon_a t_0}. \quad (\text{A}\cdot 11)$$

Expand  $\mathcal{Q}_a$  and  $\tilde{\Phi}_b$  according to (19) and (20), namely,

$$\left. \begin{aligned} \mathcal{Q}_a &= e^{\Delta} \Phi_a + \sum_{a'} f_{a'} \Phi_{a'}, \\ \tilde{\Phi}_b &= e^{-\bar{\Delta}} \Phi_b + \sum_{b'} g_{b'} \Phi_{b'}, \end{aligned} \right\} \quad (\text{A}\cdot 12)$$

where it should be remarked that the summations are restricted to those states that satisfy

$$\varepsilon_{a'} > \varepsilon_a, \quad \varepsilon_{b'} < \varepsilon_b. \quad (\text{A} \cdot 13)$$

(No term representing the real processes is contained in  $\mathcal{Q}_a$ ). Consider a typical term thus obtained.

$$(\Phi_{b'}, U(t, t_0) \Phi_{a'}) e^{i(\varepsilon_b - \varepsilon_{b'})t - i(\varepsilon_a - \varepsilon_{a'})t_0}.$$

The first factor gives the matrix-elements of the usual S-matrix in the limit of  $t \rightarrow \infty$  and  $t_0 \rightarrow -\infty$  and contains no oscillatory factor. (If we had not subtracted  $\varepsilon_0$  from  $H_i$ ,  $U(t, t_0)$  would have contained the oscillatory factor due to the vacuum closed loop graphs. The above mentioned trivial unitary transformation corresponds to the Dyson's rule to drop the vacuum graphs from  $S$ .) Therefore, the term vanishes unless  $\varepsilon_{b'} = \varepsilon_b$  and  $\varepsilon_{a'} = \varepsilon_a$  on account of the oscillating second and third factors. This means from (A·13) that only the first terms of  $\mathcal{Q}_a$  and  $\mathcal{Q}_b$  in (A·12) contribute to (A·11). Thus we get

$$\langle b | \bar{S} | a \rangle = \langle b | U(\infty, -\infty) | a \rangle \equiv \langle b | S | a \rangle.$$

We shall also mention the following relation which follows from the analogous argument as above.

$$\mathbf{X}_a = e^{\Delta} V^\dagger \Psi_a,$$

where  $\mathbf{X}_a$  is the solution of (40) and  $\Psi_a$  is that of

$$\Psi_a = \Phi_a + \frac{1}{\varepsilon_a - H_0 + i\alpha} (H_i - \varepsilon_0) \Psi_a, \quad (\alpha \rightarrow 0).$$

### Appendix III

We shall prove

$$\begin{aligned} & \sum_{\pi} \langle pq | I | pq\pi \rangle' \frac{1}{\varepsilon' - \varepsilon_{\pi}} \langle pq\pi | I | pq \rangle' + \sum_{\pi\pi'} \langle pq | I | pq\pi \rangle' \frac{1}{\varepsilon' - \varepsilon_{\pi}} \langle pq\pi | I | pq\pi' \rangle' \times \\ & \quad \times \frac{1}{\varepsilon' - \varepsilon_{\pi'}} \langle pq\pi' | I | pq \rangle' + \dots \\ & = \sum_{n=1}^{\infty} (-\varepsilon')^{n+1} \left[ \sum_{\pi} (\mathcal{Q}_0, \mathcal{Q}_{\pi}) \frac{1}{\varepsilon' - \varepsilon_{\pi}} (\mathcal{Q}_{\pi}, \mathcal{Q}_0) \right]^n \\ & = \sum_{n=1}^{\infty} (-\varepsilon')^{n+1} \left[ \sum_{\pi} \langle pq | I | pq\pi \rangle \frac{1}{\varepsilon' - \varepsilon_{\pi}} \langle pq\pi | I | pq \rangle \right]^n. \end{aligned} \quad (\text{A} \cdot 14)$$

From (39), we have

$$\begin{aligned} \langle pq | I | pq\pi \rangle' & = (-\varepsilon' + \frac{1}{2} \varepsilon_{\pi}) (\mathcal{Q}_{pq}, \mathcal{Q}_{pq\pi}) \\ & \quad + \frac{1}{2} [(H_i^{\leftarrow} \mathcal{Q}_{pq}^*, \mathcal{Q}_{pq\pi}) + (\mathcal{Q}_{pq}, H_i^{\rightarrow} \mathcal{Q}_{pq\pi}^*)]. \end{aligned}$$

Since only the vacuum processes are considered, we may put

$$\begin{aligned}(\mathcal{Q}_{pq}, \mathcal{Q}_{pq\pi}) &= (\mathcal{Q}_0, \mathcal{Q}_\pi), \quad (H_i^{\rightarrow} \mathcal{Q}_{pq}, \mathcal{Q}_{pq\pi}) = 0, \\(\mathcal{Q}_{pq}, H_i^{\rightarrow} \mathcal{Q}_{pq\pi}) &= (\mathcal{Q}_0, H_i^{\rightarrow} \mathcal{Q}_\pi) = (\mathcal{Q}_0, (H - \varepsilon_0 - \varepsilon_\pi) \mathcal{Q}_\pi) \\&= -\varepsilon_\pi (\mathcal{Q}_0, \mathcal{Q}_\pi).\end{aligned}$$

Therefore, we get

$$\langle pq | I | pq\pi \rangle' = -\varepsilon' (\mathcal{Q}_0, \mathcal{Q}_\pi).$$

In the same way, to evaluate

$$\begin{aligned}\langle pq\pi | I | pq\pi' \rangle' &= (-\varepsilon' + \frac{1}{2} \varepsilon_\pi + \frac{1}{2} \varepsilon_{\pi'}) (\mathcal{Q}_{pq\pi}, \mathcal{Q}_{pq\pi'}) \\&+ \frac{1}{2} [(H_i^{\rightarrow} \mathcal{Q}_{pq\pi}, \mathcal{Q}_{pq\pi'}) + (\mathcal{Q}_{pq\pi}, H_i^{\rightarrow} \mathcal{Q}_{pq\pi'})],\end{aligned}$$

we may put

$$\begin{aligned}(\mathcal{Q}_{pq\pi}, \mathcal{Q}_{pq\pi'}) &= (\mathcal{Q}_\pi, \mathcal{Q}_0) (\mathcal{Q}_0, \mathcal{Q}_{\pi'}), \\(\mathcal{Q}_{pq\pi}, H_i^{\rightarrow} \mathcal{Q}_{pq\pi'}) &= (\mathcal{Q}_\pi, \mathcal{Q}_0) (\mathcal{Q}_0, H_i^{\rightarrow} \mathcal{Q}_{\pi'}) = -\varepsilon_{\pi'} (\mathcal{Q}_\pi, \mathcal{Q}_0) (\mathcal{Q}_0, \mathcal{Q}_{\pi'}).\end{aligned}$$

Hence, we have

$$\langle pq\pi | I | pq\pi' \rangle' = -\varepsilon' (\mathcal{Q}_\pi, \mathcal{Q}_0) (\mathcal{Q}_0, \mathcal{Q}_{\pi'}).$$

Thus we get the relation (A·14). Since to the order  $g^2$ ,

$$\sum_{\pi} \langle pq | I | pq\pi \rangle' \frac{1}{\varepsilon - \varepsilon_{\pi}} \langle pq\pi | I | pq \rangle' = \varepsilon'^2 M,$$

(A·14) contains the series (59). From the above argument, it is clear that the series (52) include the following more general vacuum terms

$$\sum_{n=1}^{\infty} (-\varepsilon')^{n-1} \left[ \sum_a (\mathcal{Q}_0, \mathcal{Q}_a) \frac{1}{\varepsilon' - \varepsilon_a} (\mathcal{Q}_a, \mathcal{Q}_0) \right]^n \delta(p-p') \delta(q-q'). \quad (\text{A} \cdot 15)$$

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## Some Remarks on the Charge Conjugation

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The charge conjugation is studied, and it is represented by a reflection in a some space, i.e., the charge space. The discontinuity of this transformation gives rise to some arbitrariness and so several transformation types. Regarding these transformation types as the intrinsic property of the individual elementary particles, we shall investigate the possibility that we could introduce the qualitative difference between elementary particles and the conservation law of the heavy particles, more generally, transition rule between elementary particles would be supported by such a superselection rule that recently Wigner *et al.* proposed in association with the time reversal.

## § 1. Introduction

The charge conjugation is usually considered as a following transformation.

$$(I) \quad \left. \begin{array}{l} \psi_a \longrightarrow \psi_{\bar{a}} \\ \psi_{\bar{a}} \longrightarrow \psi_a \end{array} \right\} \text{ for a spinor field,} \quad (1.1)$$

where  $a$  and  $\bar{a}$  describe some particle and antiparticle, respectively.

$$(II) \quad \left. \begin{array}{l} \phi_\alpha \longrightarrow \phi_\alpha^\dagger \\ \phi_\alpha^\dagger \longrightarrow \phi_\alpha \end{array} \right\} \text{ for a boson field,} \quad (1.2)^*)$$

where  $\alpha$  describes a tensor suffix.

(III) The coupling constant of the interaction of  $(\bar{\psi}_a \psi_b \phi)$  type between the fermion and boson is transformed as

$$\begin{aligned} \mathcal{G}_{S, P\phi, P\psi} &\longrightarrow \mathcal{G}_{S, P\phi, P\psi} \\ \mathcal{G}_{V, T, (I\psi)} &\longrightarrow -\mathcal{G}_{V, T, (I\psi)}, \end{aligned} \quad (1.3)^{**})$$

where  $\mathcal{G}_{S, P\phi, P\psi}$  denotes the coupling constant for the scalar, pseudoscalar and pseudovector coupling and  $\mathcal{G}_{V, T, (I\psi)}$  for the vector, tensor and pseudotensor coupling, respectively. It should be noted here that this transformation may be interpreted as a reflection in some space called a charge space as we shall show in a following section and that the relation between the representation  $\psi_a, \psi_{\bar{a}}$  of  $a$  and  $\bar{a}$  particles:

$$\psi_{\bar{a}} = \rho K \bar{\psi}_a \quad (1.4)$$

\*) The dagger describes the hermitian conjugate.

\*\*) This transformation of the coupling constant is unique, when the boson field is neutral and the spinor particles  $a$  and  $b$  are identical, because otherwise the interaction term vanishes.

is not unique and involves an arbitrary factor  $\rho = \pm 1, \pm i$ ,<sup>1)</sup> where  $K$  is some unitary operator consisted of  $E_\mu$ -matrices.\*<sup>1)</sup> Further, the discontinuity associated with a reflection and the appearance of arbitrary factor are closely connected with each other.

In this paper we shall treat the charge conjugation from a general point of view which involves the conventional one above mentioned as a special case.

## § 2. The charge conjugation

As a simple example of the charge conjugation of the spinor field, we shall treat the electron field, whose interaction with the electromagnetic field is invariant under this transformation. However, the following discussion may be independent from the electric charge and holds for such a field as the neutron which does not interact with the electromagnetic field in a vector type.

The interaction Hamiltonian density between the electron and the electromagnetic field has a following form.

$$H' = ie/\cdot 2 \{ \bar{\psi}_e E_\mu \psi_e - \bar{\psi}_e^- E_\mu \psi_e^- \} A_\mu, \quad (2.1)$$

where  $\psi_e$  and  $\psi_e^-$  describe the electron and positron fields, respectively, and whose relation is considered as fixed by a specific value  $\rho_s$  of the arbitrary factor in (1.4). Now, let us consider  $\psi_e$  and  $\psi_e^-$  as two components of the representation of the electron-positron field by a specified reference system in a certain two dimensional space which we shall hereafter generally  $P$ - $A$  space and denote this set as

$$\phi = \begin{pmatrix} \psi_e \\ \psi_e^- \end{pmatrix}. \quad (2.2)$$

Using  $\phi$  and Pauli's isotopic spin matrix  $\tau_i$  whose operand is the above mentioned  $P$ - $A$  space, we can rewrite (2.1) in a following simple form,

$$H' = ie^{(3)}/2 \cdot \{ \phi E_\mu \tau_3 \phi \} A_\mu^{(3)}, \quad (2.3)$$

where  $e^{(3)}$  and  $A_\mu^{(3)}$  are defined by

$$\begin{aligned} e^{(3)} &= e \\ A_\mu^{(3)} &= A_\mu. \end{aligned} \quad (2.4)$$

From the expression of (2.3), we can interpret  $H'$  as a quantity associated with the third

\*) In this paper, we use the notation introduced by S. Watanabe.<sup>2)</sup> Namely, four basic matrices  $E_1$ ,  $E_2$ ,  $E_3$  and  $E_0$  are defined by

$$1/2 \cdot (E_\mu E_\nu + E_\nu E_\mu) = g_{\mu\nu},$$

and the matrix  $K$  is such one that

$$K^{-1} E_\mu K = -E_\mu^T; \quad K^T = -K.$$

Further,  $\bar{\psi}$  is defined by

$$\bar{\psi} = -\psi^\dagger E_0$$

in a hermitian system, where  $E_1$ ,  $E_2$ ,  $E_3$  and  $iE_0$  are hermitian.

axis in a certain space, which we shall call hereafter the charge space. Considering (1.1), (1.2) and (1.3), it finds out that the charge conjugation corresponds to the following transformation

$$\begin{aligned}
 \text{(Ia)} \quad & \{\bar{\psi} E_{\mu} \tau_3 \phi\} \longrightarrow -\{\bar{\psi} E_{\mu} \tau_3 \phi\} \\
 \text{(IIa)} \quad & A_{\mu}^{(3)} \longrightarrow A_{\mu}^{(3)} \\
 \text{(IIIa)} \quad & e^{(3)} \longrightarrow -e^{(3)}.
 \end{aligned} \tag{2.5}$$

Further, from the expression of (2.3), (2.5) is equivalent to the following transformation.

$$\begin{aligned}
 \text{(Ib)} \quad & \{\bar{\psi} E_{\mu} \tau_3 \phi\} \longrightarrow -\{\bar{\psi} E_{\mu} \tau_3 \phi\} \\
 \text{(IIb)} \quad & A_{\mu}^{(3)} \longrightarrow -A_{\mu}^{(3)} \\
 \text{(IIIb)} \quad & e^{(3)} \longrightarrow e^{(3)}
 \end{aligned} \tag{2.5}'$$

When we adopt the latter expression (2.5)'\*, and regard  $A_{\mu}^{(3)}$  and  $\bar{\psi} E_{\mu} \tau_3 \phi$  as the third components of the vectors in the charge space, (2.5)' shows that the charge conjugation is a reflection of the third axis in the charge space. \*

Since it seems natural from a physical point of view to consider the charge conjugation as (2.5)' rather than (1.1) - (1.3), we define it by

$$\begin{aligned}
 \text{(I)} \quad & I_c \{\bar{\psi} E_{\mu} \tau_3 \phi\} = -\{\bar{\psi} E_{\mu} \tau_3 \phi\} \\
 \text{(II)} \quad & I_c A_{\mu}^{(3)} = -A_{\mu}^{(3)},
 \end{aligned} \tag{2.6}$$

where  $I_c$  is the charge conjugation operator in the charge space. However, it is important to note that the unitary operator  $C$  in the  $P_{-1}$  space which corresponds to  $I_c$  in the charge space, i.e.,

$$I_c \{\bar{\psi} E_{\mu} \tau_3 \phi\} = \{\bar{\psi} C^{-1} E_{\mu} \tau_3 C \phi\} \tag{2.7}$$

is not uniquely determined. From (2.6) and (2.7),  $C$  must satisfy the following relations:

$$C^{-1} \tau_3 C = -\tau_3. \tag{2.8}$$

Therefore the general form of  $C$  is given by

$$C = e^{i\varphi} \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix},$$

where  $\theta$  and  $\varphi$  are the real arbitrary factors, respectively.

It should be noted here that the charge conjugation  $C$  which is equivalent to the certain unitary transformation  $R_c$  of the system is restricted only to the one whose deter-

(\*) In order that the charge conjugation is equivalent to some unitary transformation, this interpretation of the charge conjugation becomes unique, because the coupling constant remains invariant under the unitary transformation.

minant is

$$D\psi, \quad C = -1,$$

that is,

$$C = \begin{pmatrix} 0 & \rho \\ \rho^* & 0 \end{pmatrix}, \quad \rho\rho^* = 1. \quad (2.9)$$

This fact is easily explained as follows. Let us assume that there exists the following unitary transformation ;

$$\mathbf{R}_c \phi \mathbf{R}_c^{-1} = C\phi, \quad (2.10)$$

where

$$C = \begin{pmatrix} 0 & \rho' \\ \rho_1 & 0 \end{pmatrix}.$$

Then the relation between  $\rho$  and  $\rho_1$  is determined from the fact that the two components  $\phi_e$  and  $\phi_e^-$  are connected by (1.4) with a specific value  $\rho_s$ . Using (1.4) and  $\mathbf{R}_c \phi_e \mathbf{R}_c^{-1} = \rho \phi_e^-$  which follows from (2.10), we have

$$\mathbf{R}_c \phi_e^- \mathbf{R}_c^{-1} = \rho_s K \mathbf{R}_c \bar{\phi}_e \mathbf{R}_c^{-1} = \rho_s \rho^* K \bar{\phi}_e^- = \rho^* \phi_e. \quad (2.11)$$

From (2.10) and (2.11), we have

$$\rho_1 = \rho^*. \quad (2.12)$$

Thus, the above statement is verified.

For the boson field, the situation is perfectly analogous to the case of spinor field. For brevity, we treat the scalar or pseudoscalar boson, whose current four vector is given by

$$j_\mu = ic(u^\dagger \partial_\mu u - u \partial_\mu u^\dagger). \quad (2.13)$$

Here again, we can rewrite (2.13) as

$$j_\mu = ie u^\dagger \partial_\mu \tau_3 u, \quad (2.14)$$

introducing  $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ , where  $u_1$  and  $u_2$  are defined by

$$u_1 = u$$

and

$$u_2 = \eta u_1^\dagger, \quad (2.15)^{1*)}$$

where  $\eta$  is an arbitrary factor such as  $\rho$  in the case of spinors.

The equations corresponding to (2.5)-(2.12) for the spinor field hold for the boson field, only if  $\phi$ ,  $\bar{\phi}$ ,  $E_\mu$ ,  $K$  and  $\rho$  are replaced by  $u$ ,  $u^\dagger$ ,  $\partial_\mu$ , 1 and  $\eta$ , respectively.

### § 3. The superselection rule

In this section we shall show that the phase factor in the charge conjugation plays

\*) For the neutral boson,  $\eta$  is restricted to be real, that is,  $\pm 1$ . However, it is not the case for the neutral boson which is represented by the complex variable.<sup>3)</sup>



an important role in connection with the time reversal of Wigner's type. The time reversal of Wigner's type is defined by the following transformation ;

$$\Psi \rightarrow K_0 \mathbf{R}_W \Psi, \quad (3.1)$$

where  $\Psi$  is a state vector,  $K_0$  is an operator which converts the operand into its complex conjugate, and further  $\mathbf{R}_W$  is a unitary operator which has a following properties :

$$\mathbf{R}_W \phi_a(t) \mathbf{R}_W^{-1} = c E_0 K \bar{\phi}_a^T(-t) \quad (3.2)$$

$$\mathbf{R}_W \phi_a^-(t) \mathbf{R}_W^{-1} = \rho_s^2 c^* E_0 K \bar{\phi}_a^{-T}(-t) \quad (3.3)$$

$$\mathbf{R}_W u_1(t) \mathbf{R}_W^{-1} = c' u_1^{\dagger T}(-t) \quad (3.4)$$

$$\mathbf{R}_W u_2(t) \mathbf{R}_W^{-1} = \eta_s^2 c'^* u_2^{\dagger T}(-t), \quad (3.5)$$

where (3.3) and (3.5) are obtained by using (3.2), (3.4) and  $\phi_a^- = \rho_s K \bar{\phi}_a^*$ ,  $u_2 = \eta_s u_1^\dagger$ , while  $c$  and  $c'$  are arbitrary factors which satisfy the relations  $c c^* = 1$  and  $c' c'^* = 1$ .

According to (2.9), (2.10) and (3.2)-(3.5), the product of the time reversal of Wigner's type and the charge conjugation  $K_0 \mathbf{R}_W \mathbf{R}_C \equiv K_0 \mathbf{R}_P$ , the time reversal of Pauli's type has following properties, operating to the states  $\phi_a \Psi_0$ ,  $\phi_a^- \Psi_0$ , etc., where  $\Psi_0$  denotes a vacuum state.

$$\left. \begin{aligned} K_0 \mathbf{R}_P \phi_a(t) \Psi_0 &= e^{i\pi\lambda/2} \bar{\phi}_a(-t) \Psi_0^* \\ K_0 \mathbf{R}_P \phi_a^-(t) \Psi_0 &= e^{-i\pi\lambda/2} \bar{\phi}_a^-(t) \Psi_0^* \end{aligned} \right\} \quad (3.6)$$

$$\left. \begin{aligned} K_0 \mathbf{R}_P \bar{\phi}_a(t) \Psi_0^* &= -e^{-i\pi\lambda/2} \phi_a(-t) \Psi_0 \\ K_0 \mathbf{R}_P \bar{\phi}_a^-(t) \Psi_0^* &= -e^{i\pi\lambda/2} \phi_a^-(t) \Psi_0 \end{aligned} \right\}. \quad (3.7)$$

In the above expression,  $e^{i\pi\lambda/2}$  is given by

$$e^{i\pi\lambda/2} = \rho_s^* \rho_s^* c. \quad (3.8)$$

For  $u_1$  and  $u_2$ , the similar equations are fulfilled ;

$$\left. \begin{aligned} K_0 \mathbf{R}_P u_1(t) \Psi_0 &= e^{i\pi\lambda'/2} u_1^\dagger(-t) \Psi_0^* \\ K_0 \mathbf{R}_P u_2(t) \Psi_0 &= e^{-i\pi\lambda'/2} u_2^\dagger(-t) \Psi_0^* \end{aligned} \right\} \quad (3.9)$$

$$\left. \begin{aligned} K_0 \mathbf{R}_P u_1^\dagger(t) \Psi_0^* &= e^{-i\pi\lambda'/2} u_1(-t) \Psi_0 \\ K_0 \mathbf{R}_P u_2^\dagger(t) \Psi_0^* &= e^{i\pi\lambda'/2} u_2(-t) \Psi_0 \end{aligned} \right\}, \quad (3.10)$$

where

$$e^{i\pi\lambda'/2} = \eta_s^* \eta_s^* c'. \quad (3.11)$$

Operating the time reversal of Pauli's type twice in succession, we have

$$\left. \begin{aligned} (K_0 \mathbf{R}_P)^2 \phi_a \Psi_0 &= -e^{-i\pi\lambda} \phi_a \Psi_0 \\ (K_0 \mathbf{R}_P)^2 \phi_a^- \Psi_0 &= -e^{i\pi\lambda} \phi_a^- \Psi_0 \end{aligned} \right\} \quad (3.12)$$

$$\left. \begin{aligned} (K_0 \mathbf{R}_P)^2 u_1 \Psi_0 &= e^{-i\pi\lambda'} u_1 \Psi_0 \\ (K_0 \mathbf{R}_P)^2 u_2 \Psi_0 &= e^{i\pi\lambda'} u_2 \Psi_0 \end{aligned} \right\} \quad (3.13)$$

It is characteristic to the time reversal of Pauli's type that the phase factor  $e^{\pm i\pi\lambda}$  remains after this transformation operates twice in succession, while in the time reversal of Wigner's type the phase factor is cancelled out in this case.

Now we assume that the several types under the time reversal corresponding to arbitrary  $\lambda$ -values are associated with the intrinsic property of the individual elementary particles and give the qualitative difference together with the mass, spin, charge, etc. Especially let us call them the particle of the first and second kind, whose  $\lambda$ -values are 0 and  $\pm 1$ , respectively. As shown in (3.12) and (3.13), the  $\lambda$ -values of the particle and antiparticle are connected with each other by

$$\lambda_P = -\lambda_{\bar{P}}. \quad (3.14)$$

Now let us describe the states in which the sum of the existing particles of the first kind fermion and second kind boson is even and odd by  $\Psi_E$  and  $\Psi_O$ , respectively, and consider the superposed state  $\Psi_E + \Psi_O$ . If we operate the time reversal of Pauli's type twice in succession to this state, the result becomes

$$\Psi_E + \Psi_O \rightarrow \text{const.} (\Psi_E - \Psi_O), \quad (3.15)$$

according to the relation (3.12) and (3.13). While the state which is transformed by the time reversal twice in succession must be indistinguishable from the original state, it is not possible to make any statement as to the relative parity between the two states  $\Psi_E$  and  $\Psi_O$ , and the measurability of the hermitian operator which has finite matrix element between the both states would lead to a contradiction. Namely, we can say that the superselection rule operates between the both states.<sup>11)</sup> Accordingly, the phenomena where the fermions of the first kind are created or annihilated by an odd number or even number are possible only if the bosons of the second kind are created or annihilated by an odd or even number, respectively, at the same time. However, the same statement holds for the fermions of the second kind from the consideration of the spin and statistics, if all spinor particles which are considered to exist in nature are classified into either the first or the second kind.

On the other hand, the heavy particles have been discovered one after another of late years and the stability of these particles has been attached importance to. However, the current theory seems to lack the principle to determine the interaction Hamiltonian or to be phenomenological in this sense. Although these systematic transition rule between elementary particles may be solved positively in a future theory, it seems interesting to note that, if we assume that the above *nucleonic* charge  $\lambda$  of the individual elementary particles is assigned to be identical with  $\lambda$  introduced by S. Ôneda,<sup>5)</sup> then the superselection rule plays a role for the stability of the heavy particles from the fact mentioned above.

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Theoretical Analysis of Pion-nucleon Scattering<sup>\*)</sup>

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Theoretical phase shifts of pion-nucleon scattering are derived assuming the pseudo-scalar meson theory with pseudo-scalar coupling. In order to make static approximation for such a relativistic coupling, the well-known Tani-Foldy transformation is applied. Since there appears an isotopic-spin-dependent term which is negative and rather strong for the  $T=1, 2$  state, theoretical interpretation of recent experiments on the  $S$ -wave scattering seems to be hopeful. As to the  $P$ -wave scattering, the Tamm-Dancoff approximation and renormalization technique are used. The anomalous magnetic moment of the nucleon is also calculated. The cut-off method is used throughout this work and the results are very sensitive to the cut-off momentum, but the essential features of the theory would be retained qualitatively and, at lower energies, somewhat quantitatively.  $P$ -phase shifts are calculated by the Born approximation, though it is possible that the radiative effects, which always tend to make the nucleon spread out, are very important in this case.

## § 1. Introduction

The recent experimental research on the pion-nucleon scattering by Fermi *et al.*<sup>1)</sup> shows that the two  $S$ -phase shifts,  $\alpha_3$  and  $\alpha_1$ , are quite different in behavior and especially the isotopic spin  $3/2$  phase shift  $\alpha_3$  varies very rapidly with increasing energy. These facts seem to be too peculiar to be understood from the present meson theory. In the pseudo-vector coupling theory, the  $S$ -phase shifts are almost zero up to 100 Mev even if we take into account the higher order effects, because these effects would work so as to make the interaction much weaker than one expects from the Born approximation. In the pseudoscalar coupling theory, there appears not only a strong core term which contributes to the two  $S$ -phase shifts in equal magnitude, but also a second term which is a type of  $LS$  coupling in the isotopic spin space. This second term is still positive for the  $T=3/2$  state but is negative for the  $T=1/2$  state and seems at first sight to be very hopeful in order to explain the different behavior of these phase shifts.<sup>2)</sup> It turns out however that this term is too small

\*) This work had been prepared for the International Conference on Theoretical Physics held in Japan Sept. 1953.



to separate  $\alpha_0$  and  $\alpha_1$  so large as obtained from experiments, whether we apply the Dyson transformation or the Tamm-Dancoff approximation.<sup>3)</sup>

It is to be noted here that in the pseudoscalar coupling theory one must take into account higher order radiative effects much more seriously and, moreover, the usual perturbation theory might not work at all especially as to the *S*-wave interaction; pair formation, pion-pion interaction, etc. should be treated in a way quite different from the Born approximation. In this case, therefore, the renormalization problem becomes very much complicated. If one takes up the Bethe-Salpeter formalism, the mass and charge renormalization could be performed in a consistent way at least in principle,<sup>4)</sup> but still one could not expect a good answer because many important radiative effects are omitted in this formalism. Thus it seems extremely desirable to explore the interaction between the nucleon and meson field from a more general point of view.\*)

In view of these circumstances one of the authors (K. S.) and Akiba<sup>5)</sup> has studied how the interaction terms obtained by the Dyson transformation would be modified if one uses the other more appropriate transformation function. They found out that the Tani-Foldy transformation is in fact the best one among the similar kind of transformations from the variation principle and that the interaction terms are all somewhat changed compared with the Dyson transformation. In addition their result seems to show that the usual Tamm-Dancoff approximation using this Hamiltonian might be rather trustful; the series obtained by taking up higher configurations would be convergent or at least asymptotically. To speak more precisely, if we perform the mass renormalization and assume that the bare nucleon mass is very small compared with the observed value, then the core and pseudo-vector terms remain almost the same as before, whereas the *LS* coupling term becomes twice as large and the other higher order terms are considerably cut down. This result seems to be advantageous for the theoretical interpretation of the recent experimental data.

In section II we make use of the Tamm-Dancoff approximation neglecting higher configurations which contain more than two mesons and show in fact that the two *S*-phase shifts are enough separated to be in rough agreement with experiments. However, since the cut-off method has been adopted throughout this work and the result is very sensitive to the cut-off parameter, the emphasis has to be on qualitative, rather than quantitative, aspects. Moreover, the peculiar character of  $\alpha_0$ , i.e., the fact that it stays close to zero up to about 50 Mev, was not accounted for, because we did not take into account a force of longer range which may be derived from, say, the pion-pion interaction.<sup>6)</sup>

In section III the *P*-wave scattering is discussed using the Tamm-Dancoff approximation. For the state,  $T=1/2$  and  $J=1/2$ , the renormalization procedure of mass and charge is necessary, but we have here omitted the charge renormalization, because this effect

<sup>4)</sup> We are indebted to Professor Brueckner for valuable discussions on this point during his stay in Japan (Sept. 1953).

<sup>5)</sup> In the meson theory, the charge renormalization and the dissociation probability are intimately connected, namely  $(1-Z_2)$  expresses this probability, and it is not certain how much of the radiative interaction is to be renormalized.

seems to be small and in addition there is some ambiguities as to this treatment.\*<sup>(\*)</sup> The same approximation is applied to obtain the anomalous magnetic moment of the nucleon; the result, however, is not so well in agreement with the observed value with our rather high cut-off momentum and small coupling constant.

In section IV the validity of the usual Tamm-Dancoff approximation for the  $P$ -wave scattering is discussed by examining how much the three meson configuration would contribute to this scattering. The resulting coupled integral equations are too much complicated to be evaluated exactly, but the rough estimate shows that its correction amounts to 20 or at most 30 percent.<sup>(7)</sup>

It seems also worthwhile to investigate to what extent the  $D$ -phase shifts will be effective in the phase shift analysis at somewhat higher energies, because the various conclusions drawn by Fermi *et al.* are all based on the assumption that this wave contributes nothing to the scattering even at 200 Mev.<sup>(8)</sup> Thus in section V we have calculated the  $D$ -phase shifts by the Born approximation. If we take into account higher order radiative effects, the  $D$ -wave interaction might be more effective since such effects tend to make the nucleon spread out considerably, but the ordinary perturbation theoretic estimation may serve as a starting point for further investigation.

## § 2. S-phase shifts

If we perform the Tani-Foldy transformation against the pseudo-scalar meson theory with pseudo-scalar coupling and make the contraction (well-ordering) of the resulting Hamiltonian, we obtain the following interaction Hamiltonian:

$$H_{int} = H_{pv} + H_v + H_\pi + H'$$

with

$$\begin{aligned} H_{pv} &= A \cdot \frac{g}{2m} \int d\mathbf{r} \, \psi^* [\boldsymbol{\sigma} \cdot \nabla (\tau \phi) - \rho_1 (\tau \pi)] \psi, \\ H_v &= B \cdot \frac{g^2}{2m} \int d\mathbf{r} \, \psi^* \beta \psi \, \phi^2, \\ H_\pi &= C \cdot \left( \frac{g}{2m} \right)^2 \int d\mathbf{r} \, \psi^* \beta (\boldsymbol{\tau} \cdot [\boldsymbol{\phi} \times \boldsymbol{\pi}]) \psi, \end{aligned} \quad (1)$$

and  $H'$  is the remaining terms which contain more than three ordered meson variables. In this expression,  $\psi$  is the nucleon field with observed mass  $m$ ,  $\phi$  the symmetric pseudo-scalar meson field with coupling constant  $g$  and  $\pi$  the canonical conjugate of  $\phi$ .  $A$ ,  $B$  and  $C$  are all the numerical constants which are obtained by the procedure of contraction, and we can approximately put these values  $A=1$ ,  $B=1$  and  $C=2$ , independent of the cut-off parameter when the coupling is sufficiently large.

Now, in the case of low energy of the incident meson, the main contribution to the  $S$ -phase shifts comes from the terms,  $H_v$  and  $H_\pi$ , because  $H_{pv}$  contributes mainly to the  $P$ -phase shifts and, as shown in section V, we can neglect its contribution to the  $S$ -phase

shifts at lower energies. But, as well known,  $H_c$  is very large compared with  $H_\pi$  in the Born approximation and then one might suppose at first sight that this  $H_\pi$  can also be neglected compared with  $H_c$ . This is not the case, however, since the reaction of the term  $H_\pi$  will be much larger than that of  $H_c$ . This situation can be seen from the fact that  $H_\pi$  includes the variable  $\pi$ , the time derivative of  $\phi$ , in contrast with  $H_c$  which contains only  $\phi^{(*)}$ . Thus, if we neglect higher configurations, the interaction Hamiltonian can be approximated as follows:

$$H_{int} \simeq \frac{g^2}{2m} \int d^3r \, \psi^* \beta \psi \phi^2 + 2 \cdot \left( \frac{g}{2m} \right)^2 \int d^3r \, \psi^* \beta (\tau \cdot |\phi \times \pi|) \psi. \quad (1')$$

The matrix element of this expression for the scattering of pion is given by

$$\langle \mathbf{k}' | H_{scat} | \mathbf{k} \rangle = \frac{g^2}{2m} \frac{1}{(\varepsilon_k \varepsilon_{k'})^{1/2}} \left[ 1 + \frac{\varepsilon_k + \varepsilon_{k'}}{2m} (\tau \cdot \omega) \right], \quad (2)$$

neglecting the recoil of the nucleon.  $\mathbf{k}$  and  $\mathbf{k}'$  are the momentum of the meson and  $\varepsilon_k = \sqrt{k^2 + \mu^2}$ , where  $\mu$  is the mass of the meson.  $\omega$  is the charge isotopic spin operator of the meson and has the eigen values;

$$(\tau \cdot \omega) = \begin{cases} 1 & \text{for the charge triplet state } (T=3/2) \\ -2 & \text{for the charge singlet state } (T=1/2) \end{cases} \quad (3)$$

1 is the unit matrix in the isotopic space of the meson. The property of the operator  $(\tau \cdot \omega)$  appearing in the equation (3) seems to be favorable for the theoretical interpretation of the experimental results the  $S$ -phase shifts given by experiments are negative for  $T=3/2$  ( $\alpha_3 < 0$ ) and positive for  $T=1/2$  ( $\alpha_1 > 0$ ).

We now calculate the  $S$ -phase shifts by solving the Schroedinger equation for the one meson system taking the above matrix element (2) as the interaction Hamiltonian. This is the usual Tamm-Dancoff approximation taking into account only one meson configuration. We obtain the following result:

$$\tan \alpha_i = - \frac{f^2 k_0 / m}{A(k_0) + B(k_0, \lambda)} \quad (i=3, 1), \quad (4)$$

where

$$A(k_0) = \left[ M + \varepsilon_0 \log \frac{2M}{\mu} - k_0 \log \frac{k_0 + \varepsilon_0}{\mu} \right] \frac{f^2}{\pi m}$$

$$B(k_0, \lambda) = \frac{[1 + \lambda f^2 / 4\pi m^2 (M^2 - \mu^2 \log 2M/\mu)]^2}{1 + \lambda \varepsilon_0 / m - \lambda^2 f^2 / 4\pi m^3 [M^3/3 - \varepsilon_0/2 (M^2 - \mu^2 \log 2M/\mu)]},$$

and

$$f^2 = g^2 / 4\pi \quad (5)$$

<sup>\*)</sup> This may be expected from the uncertainty principle. When we take into account the reaction of  $H_c$ , the contribution of this term becomes much smaller compared with the value of the Born approximation. This fact is equivalent to the famous damping effect of  $\phi$  itself. Then, the  $\pi$ , which satisfies the commutation relation  $[\phi, \pi'] = -i\delta(x-x')$  will be enhanced by the uncertainty principle.

$$\lambda = \begin{cases} +1 & \text{for } i=3 \quad (T=3/2) \\ -2 & \text{for } i=1 \quad (T=1/2) \end{cases}$$

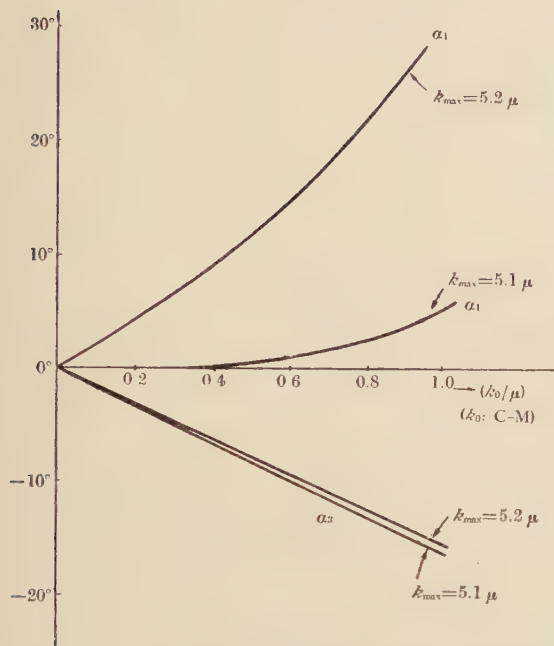


Fig. 1. Isotopic spin and energy dependence of  $S$ -phase shifts;  $\alpha_3$  and  $\alpha_1$  correspond to  $T=3/2$  and  $T=1/2$ , respectively.  $\alpha_1$  is very sensitive to the cut-off momentum. The rapid variation of  $\alpha_3$  is not obtained.  $g^2/4\pi=20$  was chosen.

which inherit the property of  $(\tau \cdot \omega)$ .  $k_0$  and  $\epsilon_0$  are the momentum and energy of the incident meson and  $M$  is the cut-off momentum of the divergent integrals which appear in the solution of the Schroedinger equation due to higher order radiative effects. The phase shifts  $\alpha_1$  given by the expression (4) are plotted as the function of the incident meson energy in Fig. 1 in which we assume  $f^2=20$  and cut-off momentum  $M=5.1 \mu$ , or  $5.2 \mu$ .

In this figure the behavior of  $\alpha_1$  is very characteristic compared with  $\alpha_3$ . At low energies, the  $S$ -phase shift is usually proportional to the momentum of the incident particle. The peculiar behavior of  $\alpha_1$  comes from the circumstances that a large cancellation occurs incidentally in the denominator of the expression (5) near the region of zero value of the numerator. Thus we are obliged to consider the adequacy of the

omission of other terms in equation (1) because of this peculiar behavior of  $\alpha_1$ . However, the contribution from higher configurations seems to be not so large because these terms are strongly damped compared with those obtained by the Born approximation. If it is true, then we shall not perhaps be able to avoid this peculiar behavior even if we consider the higher order configurations.

### § 3. P-wave scattering and the anomalous magnetic moment of the nucleon

According to the recent experimental research, the resonance phenomenon seems to occur in the  $T=3/2$  and  $J=3/2$  state at the energy of about 200 Mev, and, as shown by many authors, this point is well explained by solving the Schroedinger equation with an appropriate interaction kernel for the scattering.<sup>(9)</sup> But the self-energy should be renormalized in the  $T=1/2$  and  $J=1/2$  state even in the non-relativistic Tamm-Dancoff ap-

\*) The renormalization of the extended source theory has recently been done in a systematic way independently by G. F. Chew (private communication).



proximation.\* We have subtracted this self-energy by taking into account the fact that in the Tamm-Dancoff approximation, the reaction of the meson field in every configuration is different, since the maximum number of mesons which are present at the same time is restricted to some values; for example, if one considered only up to two meson configurations, the self-energy of the zero meson configuration is affected by the reaction up to two mesons, but that of the one-meson configuration only from the two meson state. Hence, we should add different counter terms to the Hamiltonian and determine these counter term to the Hamiltonian and determine these counter term in the following way. The usual Tamm-Dancoff equations are in this case written down as follows;

$$(E + J_0)(0|W|E) = \sum_l H_l(U|W|E) \quad (6)$$

$$(E - \varepsilon_l + J_1(l))(U|W|E) = H_l^*(0|W|E) + \sqrt{2} \sum_{\mu} H_{\mu}(U|W|E) \quad (7)$$

$$(E - \varepsilon_l - \varepsilon_{\mu})(U|W|E) = (1/\sqrt{2})H_{\mu}^*(U|W|E) + (1/\sqrt{2})H_l^*(U|W|E), \quad (8)$$

where  $H_l$  is given by

$$H_l = \frac{1}{(2\pi)^{3/2}} \left( \frac{G}{\mu} \right)^2 \frac{1}{\sqrt{2}\varepsilon_l} \tau_l(\sigma l), \quad G = \frac{\mu}{2m} g, \quad \varepsilon_l = \sqrt{\mu^2 + l^2} \quad (9)$$

in the non-relativistic approximation. And  $U$  is the wave matrix defined by Möller.<sup>10</sup> Substitution of (8) into (7) gives

$$\left( E - \varepsilon_l + J_1(l) - \sum_{\mu} \frac{|H_{\mu}|^2}{E - \varepsilon_l - \varepsilon_{\mu}} \right) (U|W|E) = H_l^*(0|W|E) + \sum_{\mu} \frac{H_{\mu} H_l^*}{E - \varepsilon_l - \varepsilon_{\mu}} (U|W|E). \quad (10)$$

Now  $J_1(l)$  is determined by the condition that at the energy  $E = \varepsilon_0$  the equation obtained by putting the right hand side equal to zero should have a solution of the form  $\hat{n}(l - l_0)$ , representing an incident plane wave. This means that equation (10) holds in the limit of infinitely large normalization volume of the meson wave, because in this case the last term of (10) gives no contribution since it is proportional to  $1/l^3$ , with  $l^3$  normalization volume. Then, we have

$$J_1(l) = \sum_{\mu} \frac{|H_{\mu}|^2}{-\varepsilon_{\mu}} \quad (11)$$

and (10) becomes

$$(E - \varepsilon_l) \left( 1 + \sum_{\mu} \frac{|H_{\mu}|^2}{(-\varepsilon_{\mu})(E - \varepsilon_l - \varepsilon_{\mu})} \right) (U|W|E) = H_l^*(0|W|E) + \sum_{\mu} \frac{H_{\mu} H_l^*}{E - \varepsilon_l - \varepsilon_{\mu}} \times (U|W|E). \quad (12)$$

Then to determine  $J_0$ , we assume that there is no incident meson, and the solution of (12) becomes

$$(U|W|E) = \sum_k (U|K(E)|k) \frac{H_k^*}{(E - \varepsilon_k) \left( 1 + \sum_{\mu} \frac{|H_{\mu}|^2}{(-\varepsilon_{\mu})(E - \varepsilon_k - \varepsilon_{\mu})} \right)} (0|W|E), \quad (13)$$

$$= G(\mathbf{l}, E) \quad (0|W|E)$$

where  $K(E)$  satisfies the integral equation ;

$$(\mathbf{l}|K(E)|\mathbf{k}) = \delta(\mathbf{l}-\mathbf{k}) + \frac{1}{(E-\varepsilon_l) \left( 1 + \sum_{\nu} \frac{|H_{\nu}|^2}{(-\varepsilon_{\nu})(E-\varepsilon_l-\varepsilon_{\nu})} \right)} \times \\ \sum_{\nu} \frac{H_{\nu} H_l^*}{E-\varepsilon_l-\varepsilon_{\nu}} (\mathbf{l}|K(E)|\mathbf{k}). \quad (14)$$

Substituting from (13) into (6), we get

$$(E + A_0 - \sum_l H_l G(\mathbf{l}, E)) (0|W|E) = 0. \quad (15)$$

The counter term in (15) is determined by the condition that the energy shift should be cancelled, namely  $E=0$  should be the solution of this equation. Then

$$A_0 = \sum_l H_l G(\mathbf{l}, 0). \quad (16)$$

Thus, obtaining the solution of stationary value problem, we can treat in the usual way the scattering problem (there is no energy shift any more). The solution of the equation (12), when there is an incident meson of momentum  $\mathbf{l}_0$ , is given by

$$(\mathbf{l}|W|\mathbf{l}_0) = \sum_k (\mathbf{l}|K(\varepsilon_{l_0})|\mathbf{k}) \left\{ \delta(\mathbf{k}-\mathbf{l}_0) + \frac{H_k^* (0|W|\mathbf{l}_0)}{(E-\varepsilon_k) \left( 1 + \sum_{\nu} \frac{|H_{\nu}|^2}{(-\varepsilon_{\nu})(E-\varepsilon_l-\varepsilon_{\nu})} \right)} \right\} \quad (17)$$

In the integral equation (14) the factor

$$1 + \sum_{\nu} \frac{|H_{\nu}|^2}{(-\varepsilon_{\nu})(E-\varepsilon_l-\varepsilon_{\nu})} \quad (18)$$

in the denominator expresses the charge renormalization term, but it was put equal to one in the following calculations since we have no definite way of treating it in the meson theory.

The equation (14) was solved by the Fredholm method, only the second order minor determinants being retained. The errors introduced are estimated to be of about 10 percent for the  $T=1/2$  and  $J=1/2$  state and of about 20 percent for the other states. The phase shifts thus obtained are ;

$$\tan \alpha_{33} = \frac{4}{3\mu^2} \left( \frac{G^2}{4\pi} \right) \frac{l_0^3}{\varepsilon_{l_0} - 4[J(\varepsilon_{l_0}) - J(\varepsilon_{l_0}/2)]},$$

$$\tan \alpha_{31} = \tan \alpha_{13} = -\frac{2}{3\mu^2} \left( \frac{G^2}{4\pi} \right) \frac{l_0^3}{\varepsilon_{l_0} + 2[J(\varepsilon_{l_0}) - J(\varepsilon_{l_0}/2)]},$$

$$\tan \alpha_{11} = \frac{1}{3\mu^2} \left( \frac{G^2}{4\pi} \right) \left[ \frac{l_0^3}{\varepsilon_{l_0} - [J(\varepsilon_{l_0}) - J(\varepsilon_{l_0}/2)]} - \frac{l_0^3}{\varepsilon_{l_0} + 9[J(\varepsilon_{l_0}/2) - J(0)]} \right] \times$$

$$\left( \varepsilon_{l_0} + \frac{[J(\varepsilon_{l_0}/2) - J(0)]}{\varepsilon_{l_0} - [J(\varepsilon_{l_0}) - J(\varepsilon_{l_0}/2)]} \right)^2, \quad (19)$$

$$J(E) = \frac{1}{3\pi\mu^2} \left( \frac{G^2}{4\pi} \right) \int_0^M \frac{l^4 dl}{\varepsilon_l(\varepsilon_l - E)}.$$

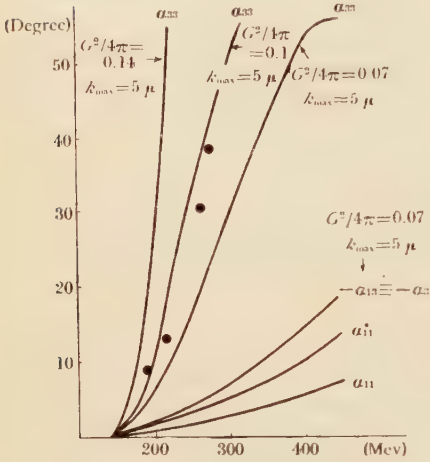


Fig. 2. Energy dependence of  $P$ -phase shifts.  $a_{11}^*$  is the value when the self-energy graph is not included. Full circles are the experimental values obtained by Fermi *et al.*

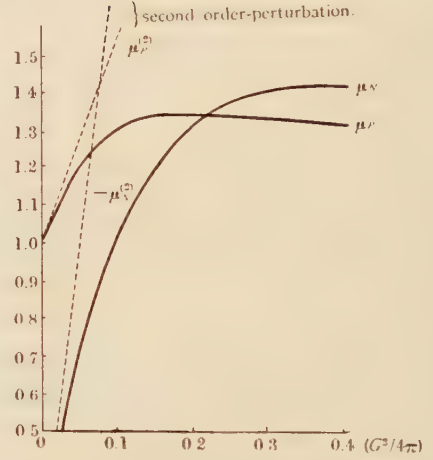


Fig. 3. Variation of the magnetic moments of nucleons with the coupling constant.  $\mu^{(2)}$ 's represent the values obtained by the second order perturbation theory. The cut-off momentum was taken to be  $5\mu$ .

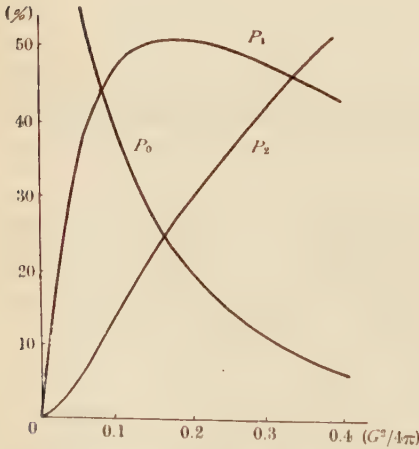


Fig. 4. Probability for each configuration.  $P_n$  indicates the probability for the  $n$ -meson state.

The results are given in Fig. 2. The last term of  $\tan \alpha_{11}$  represents the effect of mass subtraction.

It is now very easy to calculate the anomalous magnetic moment of the nucleon by using the above obtained wave matrices. In this case the accuracy of the Fredholm method is very good. The matrix  $\langle 0|H|0\rangle$  should be determined by the normalization condition  $(\Psi, \Psi) = 1$ . The interaction Hamiltonian with the electromagnetic field is given by

$$H = -e \int d\mathbf{r} \mathbf{A}_{tr} (\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2) - \frac{e}{2m} (\boldsymbol{\sigma} \cdot \mathbf{H}) \tau_P. \quad (20)$$

Actually, there exists another so-called catastrophic term

$$H' = -e (g/\mu) \int d\mathbf{r} \phi^* (\boldsymbol{\sigma} \mathbf{A}_{tr}) (\boldsymbol{\tau} \times \boldsymbol{\phi})_3 \phi$$

but, in our approximation, where the  $P$ -wave is dominant and the nucleon recoil is neglected, this term contributes nothing to the result. Magnetic moments of the nucleons are given in Fig. 3. Dissociation probabilities of the nucleon are given in Fig. 4. for the sake of convenience. The probability of one-meson state is very large, in contrast with Sachs' phenomenological theory where this state is neglected.<sup>11)</sup> This will be the main reason why we could not get so large values for anomalous magnetic moments as obtained by experiments; this situation is partly due to the rather high cut-off momentum and small coupling constant.

#### § 4. An estimate of the effects of higher-order configurations.

In most calculations using the Tamm-Dancoff method, the higher order configurations have been neglected by assuming that the coupling is rather so weak as to make these effects negligibly small, but this assumption is not plausible from the beginning. Thus Chew<sup>12)</sup> has tried to estimate the effect of three meson configuration only by taking up its contribution to the fourth order scattering potential and has proved that it amounts to less than about 20 percent compared with the two meson configuration. We have also made a similar investigation from another point of view using the non-relativistic pseudo-vector coupling.

As well known, when one expands the state function  $\Psi$  in the eigen-functions  $\Psi_n$ 's for free mesons, namely,  $\Psi = \sum_n c_n \Psi_n$ , then one obtains the simultaneous integral equations for the amplitude  $c_n$ 's. For the sake of simplicity, we neglect the four and more meson configurations, i.e., put  $c_n = 0$  for  $n \geq 4$ , and also approximate the integral equation for  $c_2$ ,  $c_3$  by the first Born approximation. This approximation seems to be too rough at first sight, but it is sufficient for an order of magnitude estimate of the higher configurations. Then, we obtain the following integral equation for the one meson amplitude  $c_1$ , or more precisely  $C_l$ , which means that there exists one meson with momentum  $l$  and energy  $\varepsilon_l$ ;

$$(E - \varepsilon_l - F_0) C_l = -i \frac{G}{(2\pi)^{3/2} \mu} (\sigma l) \tau_l C_0 + \int dV [(8F_2 + 4F_1) T_{ll'}^{3/2} L_{ll'}^{3/2} + (8F_2 - 2F_1) (T_{ll'}^{3/2} L_{ll'}^{1/2} + T_{ll'}^{1/2} L_{ll'}^{3/2}) + (26F_2 + F_1) T_{ll'}^{1/2} L_{ll'}^{1/2}] C_{ll'} \quad (21)$$

$$\text{where } F_0 = \frac{G^2}{(2\pi)^3 \mu^2} \int dV' \frac{l'^2}{(2\varepsilon_l)(E - \varepsilon_l - \varepsilon_{l'})} \times \\ \times \left[ 1 + \frac{G^2}{(2\pi)^3 \mu^2} \int dV'' \frac{l''^2}{(2\varepsilon_{l''})(E - \varepsilon_{l''} - \varepsilon_{l'})} \right], \\ F_1 = \frac{G^2}{(2\pi)^3 \mu^2} \frac{1}{(2\varepsilon_l)^{1/2} (2\varepsilon_{l'})^{1/2} (E - \varepsilon_l - \varepsilon_{l'})} \times \\ \times \left[ 1 + \frac{G^2}{(2\pi)^3 \mu^2} \int dV'' \frac{l''^2}{(2\varepsilon_{l''})(E - \varepsilon_{l''} - \varepsilon_{l'})} \right], \quad (22) \\ F_2 = \frac{G^2}{(2\pi)^3 \mu^2} \frac{1}{(2\varepsilon_l)^{1/2} (2\varepsilon_{l'})^{1/2}} \left[ \frac{G^2}{3(2\pi)^3 \mu^2} \times \right.$$



$$\int dV'' \frac{l'^2}{(2\varepsilon_{V''})(E-\varepsilon_l-\varepsilon_{V''})(E-\varepsilon_{V''}-\varepsilon_{V''})(E-\varepsilon_l-\varepsilon_{V''}-\varepsilon_{V''})} \Big],$$

and

$$\left\{ \begin{array}{l} L_{ll'}^{3/2} = (\mathbf{l} \cdot \mathbf{l}') - 1/3 (\boldsymbol{\sigma} \mathbf{l}) (\boldsymbol{\sigma} \mathbf{l}') \\ L_{ll'}^{1/2} = 1/3 (\boldsymbol{\sigma} \mathbf{l}) (\boldsymbol{\sigma} \mathbf{l}') \end{array} \right\}, \quad \left\{ \begin{array}{l} T_{ll'}^{3/2} = \delta_{ll'} - 1/3 \tau_l \tau_{l'} \\ T_{ll'}^{1/2} = 1/3 \tau_l \tau_{l'} \end{array} \right\} \quad (23)$$

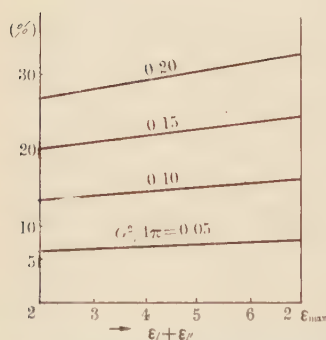
which are the projection operators;  $L^J$  is the projection operator\* for the state of angular momentum  $J$  and  $T^I$  for the state of charge  $I$ .  $E$  is the total energy of the system,  $G$  is the dimensionless coupling constant of the pseudo-vector coupling and  $\tau_l$  is the nucleon isotopic spin operator referred to the  $l$  meson.

From this expression we can estimate the order of magnitude of the effect of three-meson configuration by calculating the kernel, say,  $F_1$ , which consists of two terms; the first term in  $F_1$  comes from the two-meson configuration and the second one from that of three meson's.  $F_2$  is the proper three-meson effect. Therefore, we have calculated, as an example, for the state  $I=3/2$  and  $J=3/2$ , namely,  $8F_2+4F_1$  which is the coefficient of the projection operator  $T_{ll'}^{3/2} L_{ll'}^{3/2}$ . Then,  $F_1+2F_2$  becomes, omitting the common irrelevant factors,\*\* as

$$F_1+2F_2 \propto 1 + \frac{1}{\pi \mu^2} \frac{G^2}{4\pi} \left[ \int dV'' \frac{l'^4}{\varepsilon_{V''}(\varepsilon_{V''}+\varepsilon_{V''})(\varepsilon_l+\varepsilon_{V''}+\varepsilon_{V''})} + \frac{2}{3}(\varepsilon_l+\varepsilon_{V''}) \int dV'' \frac{l'^4}{\varepsilon_{V''}(\varepsilon_l+\varepsilon_{V''})(\varepsilon_{V''}+\varepsilon_{V''})(\varepsilon_l+\varepsilon_{V''}+\varepsilon_{V''})} \right]. \quad (24)$$

However, 
$$\int dV'' \frac{l'^4}{\varepsilon_{V''}(\varepsilon_{V''}+\varepsilon_{V''})(\varepsilon_l+\varepsilon_{V''}+\varepsilon_{V''})} < \int dV'' \frac{l'^4}{\varepsilon_{V''}^3} = I_1,$$

$$\int dV'' \frac{l'^4}{\varepsilon_{V''}(\varepsilon_l+\varepsilon_{V''})(\varepsilon_l+\varepsilon_{V''})(\varepsilon_l+\varepsilon_{V''}+\varepsilon_{V''})} < \int dV'' \frac{l'^4}{\varepsilon_l^4} = I_2.$$



Thus, we can obtain the highest estimation only by calculating the integrals  $I_1$  and  $I_2$  by elementary integration. The results are plotted in Fig. 5 for various values of coupling constant and  $\varepsilon_l + \varepsilon_{V''}$ . The cut-off momentum appearing in the calculation of  $I_1$  and  $I_2$  is assumed to be  $m/2$ , where  $m$  is the nucleon mass.

If we consider the above pseudo-vector coupling as

Fig. 5. Estimation of the scattering kernel which is derived from the three-meson configuration. The ordinate designates its ratio in percentage to the second order kernel.

\*) To speak more correctly,  $L^J|l\rangle$  is the projection operator.

\*\*)  $E$  is taken to be zero in the following evaluation, so the estimate made here may not be trustful at somewhat higher energies.

obtained from the Tani-Foldy transformation for the pseudo-scalar coupling, then the coupling constant  $G^2/4\pi$  corresponds to about 0.1 for the  $f^2=20$  in Section II. In this case we can conclude from Fig. 5. that the order of the effect of three-meson configuration, which is of course only the effect of  $P$ -wave meson will be at most about 15 percent. Although this conclusion has been obtained for the state  $J=3/2$  and  $T=3/2$  by the Born approximation, it is expected even for the other states that this effect is generally less than 25 percent. (In the case of pseudo-scalar coupling, the effects of higher configurations are of course, due to  $P$ -wave meson and  $S$ -wave meson, but we have not estimated  $S$ -wave meson effects here).

As for the  $S$ -wave interaction, there is a term, for instance, in  $H'$  in Section II

$$-D \cdot \frac{g^4}{(2m)^3} \int d\mathbf{r} \, \phi^* \beta \phi \cdot \bar{\phi}^2 \phi^2$$

which contributes to the  $S$ -wave scattering through the three and five-meson configurations, but the factor  $D$  obtained by the contraction becomes about 1/2 which is to be 1 in the Born approximation. The higher order terms seem in general to be cut down more and more by the inclusion of radiative effects. It is expected therefore that the Tamm-Dancoff approximation is also useful for the  $S$ -wave scattering. However, calculation shows that the inclusion of higher meson configurations leads to the catastrophic result, unless we choose the cut-off momentum considerably less than the nucleon mass, which seems to be very unnatural. This situation might be improved by the method of normal vibration as was discussed by Wentzel, but, in any case, a more consistent approximation should be applied before any reliable discussions could be done.

## § 5. D-phase shifts

Fermi *et al.*<sup>(13)</sup> have tried to explain their recent experimental results on the pion-nucleon scattering up to 210 Mev in terms of  $S$ - and  $P$ -phase shifts on the charge independent hypothesis. The phase shifts thus obtained exhibit rather strange features in some respects from the theoretical point of view, i.e., the peculiar behavior of  $S$ -phase shifts, strong asymmetry between  $\alpha_{11}$  and  $\alpha_{13}$ , and the non-resonance character of  $\alpha_{33}$ . Their experiments, however, have been made only at three angles ( $45^\circ$ ,  $90^\circ$  and  $135^\circ$  in the laboratory system) using the minus pion beam and so they have obtained six values at each energy just sufficient to determine the above six phase shifts. One cannot, therefore, rule out the  $D$ -wave phase shifts entirely from their results; it might be explained equally well by including the  $D$ -wave and the  $S$ - and  $P$ -phase shifts may turn out to have more reasonable values. Thus it seems to be important to study theoretically to what extent the  $D$ -wave would contribute to the scattering.

If we assume the force range of the nucleon and meson is of the order of the meson Compton wave length, it is easily seen by a general consideration that up to 100 Mev only the  $S$ - and  $P$ -waves are important, but at energies more than 100 Mev, the  $D$ -wave comes into play, its order of magnitude dependent on the special nature of force. In the usual

perturbation theoretic treatment neglecting higher order effects, this force range turns out to be of the order of the nucleon Compton wave length and the  $D$ -wave contribution is expected to be very small even at somewhat higher energies. However, the higher order radiative effects tend to make the nucleon spread out considerably and, consequently, the interaction of the meson and the clothed nucleon would in practice have a long range of as much as the meson Compton wave length, though its strength might become much smaller. Thus a more rigorous treatment of the meson cloud should be investigated before any reliable discussions could be made as to the  $D$ -wave interaction. But in this section we shall calculate the  $D$ -phase shifts only by the first Born approximation as a starting point for further investigation.

Denoting  $D$ -phase shifts as  $\beta_{2T, 2J}$ , the first indices  $T$  representing the total isotopic spin, the second  $J$  the total angular momentum, they are given by

$$\frac{1}{\pi} \tan \beta_{2T, 2J} = -\frac{k \varepsilon E}{(2\pi)^3 W} K(T, J, k), \quad (25)$$

in the center of mass system, where  $K$  is the eigen value of the reactance matrix and  $k$ ,  $\varepsilon$ ,  $E$  and  $W$  denote the incident meson momentum, its total energy, the nucleon energy and the total energy, respectively. In the first Born approximation  $K$ 's are given as follows ;

$$K(3/2, 3/2) = -f^2 \frac{\pi^2}{15} \frac{k^4}{m^3 \varepsilon^3} \left( 1 + \frac{\varepsilon}{2m} + \frac{\mu^2}{m\varepsilon} \right),$$

$$K(3/2, 5/2) = 4f^2 \frac{\pi^2}{15} \quad , \quad ,$$

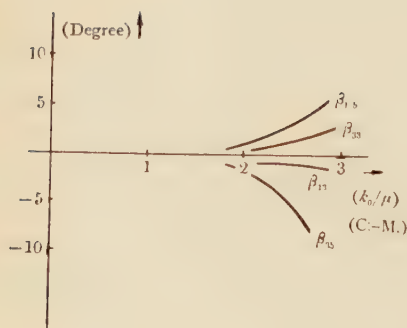


Fig. 6.  $D$ -phase shifts calculated by the first Born approximation.  $f^2=20$  was chosen. Equivalence theorem between  $PS$  and  $PV$  couplings holds.

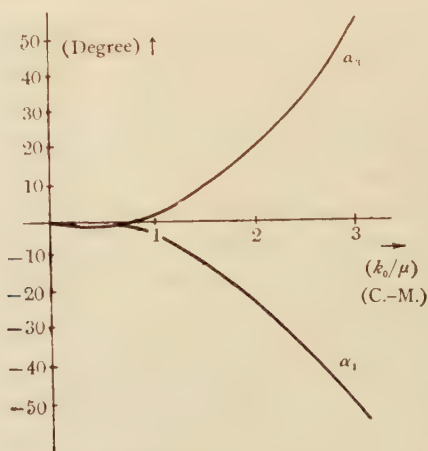


Fig. 7.  $S$ -phase shifts derived from the pseudo-vector coupling. Rapid variation with energy above 100 Mev is due to recoil effects (see text).  $f^2=20$  was chosen.

$$K(1/2, 3/2) = 1/2 f^2 - \frac{\pi^2}{15} \quad , \quad ,$$

$$K(1/2, 5/2) = -2 f^2 - \frac{\pi^2}{15} \quad , \quad . \quad (26)$$

where only the first power of  $(\varepsilon/m)$  is retained. In this approximation the equivalence theorem holds exactly, i.e., in the pseudo-vector coupling case  $f^2$  should only be replaced by  $\left(\frac{2m}{\mu}G\right)^2$ . Results are shown in Fig. 6.

The  $S$ -phase shifts derived from the equivalent pseudo-vector coupling are also given in Fig. 7. Unexpected steep rise of  $S$ -phase shifts above 100 Mev is caused by the recoil effect. For instance,  $\alpha_3$  is given by;

$$\tan \alpha_3 = -\frac{G^2}{4\pi} \frac{k\varepsilon^2}{\mu^2 m} \left[ 1 - 4\left(\frac{k}{\varepsilon}\right)^2 + \frac{2k^4}{3\varepsilon^4} \right],$$

neglecting higher powers of  $m$  in the denominator. The first term is derived from the fourth component of pseudo-vector coupling ( $\rho_1 \pi$ ) and the second term, which is dominant above 100 Mev, from the positive-positive interference of  $\rho_1 \pi$  and  $\sigma \Gamma \cdot \phi$  (recoil effect)\*. The last term is due to the fact that part of the angle-independent matrix element should be attributed to the  $D$ -wave scattering. One would therefore have to take into account the  $\rho_1 \pi$ -term at somewhat higher energies, even after the Tani-Foldy transformation is applied.

## § 5. Concluding remarks

Qualitative discussions concerning the pion-nucleon scattering are presented in this paper using the pseudo-scalar meson theory with pseudo-scalar coupling. Isotopic spin dependence of  $S$ -phase shifts and the resonance-like phenomena of  $\alpha_{33}$  are rather reasonably understood. In this connection, however, following remarks would be worth mentioning.

The first point refers to some ambiguities of determining phase shifts from the differential cross sections. Besides the existence of various possible phase shifts which can equally well explain the experimental data, there are some essential ambiguities of this determination.

For example, by performing the canonical transformation  $U = \exp \left\{ i \frac{(\sigma \cdot k)}{k} \theta \right\}$

( $\theta$  is an adjustable parameter which may depend on  $k$ ) against the meson nucleon wave function  $\psi(\mathbf{k}, \sigma_z')$ , it is easily seen that the angular distribution is invariant against the following transformations;

$$e^{J-1/2} \rightarrow e^{J-1/2} = [i e^{J-1/2} \cos \theta + e^{J+1/2} \sin \theta] e^{-i f(\theta)}$$

\*) This term is exactly the same as obtained by E. M. Henley and M. A. Ruderman (Phys. Rev. 90 (1953), 719) from the view point of Galileian invariance of pseudovector coupling.



$$e_J^{J+1/2} \rightarrow e_J^{J+1/2} = [i e_J^{J+1/2} \cos \theta + e_J^{J-1/2} \sin \theta] e^{-if(\theta)} \quad (\text{for all } J)$$

where  $J$  is the total angular momentum, upper indices are orbital angular momentum and  $e_J^l$ 's are related to the phase shift as

$$e_J^l = e^{2i\delta_{Jl}} - 1.$$

Although there exist in general no real phase shifts solutions of these equations except the special case of  $\theta = \pm \pi/2$  which has first been pointed out by Minami<sup>14)</sup>, there may be some sets of possible phase shifts which can equally well fit to the experimental data.\*

Second, if one performs the Tani-Foldy transformation against the  $\gamma_5$  coupling, the resulting Hamiltonian becomes less and less renormalizable as one takes up higher meson configurations, though the initial coupling is renormalizable and is less dependent on the cut-off parameter. We think that if one takes into account various contributions in a consistent way from the transformed Hamiltonian, then the strongly cut-off dependent terms would be cancelled out, or, in other words, the cut-off momentum could be taken rather small when the result is very sensitive to it. However, the substantial justification of the cut-off method would have to be done from different points of view. This problem will be discussed at another opportunity by one of us (N.F.)

Third, the  $\rho_1\pi$  term of pseudo-vector coupling should be taken into account in discussing  $S$ -wave scattering above 100Mev, but the consistent elimination of the odd part of this term by an appropriate canonical transformation is very difficult. We expect that there may be some powerful successive transformations for this elimination.\*\* In addition it is highly desirable to derive a reasonable pion-pion interaction which seems to be necessary to explain the energy dependence of  $\alpha_3$  at lower energies.

Fourth, the anomalous magnetic moment of the nucleon is not likely to be explained by the Tamm-Dancoff method taking into account two-meson configuration. It is one of the most important problems to see whether Sachs' phenomenological treatment could be justified or not by the current meson theory, using the Tamm-Dancoff or intermediate coupling theory. But the former method seems to be more suitable, because the mass and charge renormalization can be performed even in the non-relativistic treatment. This problem will be discussed at another opportunity, by combining the Tamm-Dancoff and the covariant theory.

In conclusion, the authors would like to express their sincere thanks to Professors S. Tomonaga and H. Fukuda for valuable discussions throughout this work. One of them (S.O.) is indebted to the Yukawa Yomiuri Fellowship for financial aid.

<sup>1)</sup> In the Hamiltonian which is obtained by the above mentioned canonical transformation  $U$ , the parity is no longer a good quantum number except Minami's special case. This is the reason why there are in general no other real phase shifts which can exactly fit to the experimental data. But, in the practical application, only the approximate phase shifts are enough to explain the experiments. It turns out that  $f(\theta)$  is to be taken equal to  $\theta$ .

\*\*) This problem is now under study by S. Tani *et al.*

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# On the Effective Hamiltonian for Pseudoscalar Meson with Pseudoscalar Coupling with Nucleon

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The method of canonical transformation is applied to the pseudo-scalar meson theory with pseudo-scalar coupling. It will be shown that the Tani<sup>1)</sup>-Foldy<sup>2)</sup> transformation results as the best one when regarded as a variational function among the similar family of transformation. The resulting Hamiltonian was ordered assuming some cut-off for the virtual meson momentum, and performing the mass renormalization. This Hamiltonian shows a good tendency to account for the S-phase shifts for the nucleon  $\pi$ -meson scattering; the  $L$ - $S$  coupling ( $\tau \cdot \varphi \times \pi$ ) in the isotopic spin space, which is positive for the  $T=3/2$  state and negative for the  $T=1/2$  state, becomes twice as large as obtained by the perturbation theory.

## § 1. Construction of effective Hamiltonian

The Hamiltonian of the system composed of pseudo-scalar mesons interacting through pseudo-scalar coupling with the nucleon system is given by;<sup>4)</sup>

$$H_T = H_0^M + H_0^N + H_1, \quad (1.1)$$

$$H_0^M = (1/2) \int (\pi^2 + \phi(\mu^2 - \Delta)\phi) d\mathbf{x}, \quad (1.2)$$

$$H_0^N = \int \psi^* (-i\rho_1 \boldsymbol{\sigma} \nabla + \rho_3 m) \psi d\mathbf{x}, \quad (1.3)$$

$$H_1 = g \int \psi^* \rho_2 (\tau \cdot \phi) \psi d\mathbf{x}. \quad (1.4)$$

Here we transform the state functional as following, with the unknown function  $f^{5)}$ ;

$$\Psi = \exp[-i \int \psi^* \rho_1 (\tau \cdot \phi) f(\sqrt{\phi^2}) \psi d\mathbf{x}] \Psi_1 = e^{-G(f)} \Psi_1, \quad (1.5)$$

We now determine the form  $f$  by requiring that, when (1.5) is regarded as a variational function with  $f$  to be determined ( $\Psi_1$  is then replaced by one of the eigen state  $(H_0^M + H_0^N) \Psi_0 = E_0 \Psi_0$ ), the energy expectation value is to be stationary.

For this purpose, we first construct the transformed Hamiltonian;

$$e^{G(f)} (H_0^M + H_0^N + H_1) e^{-G(f)}$$

<sup>4)</sup> This work was prepared for the International Conference on Theoretical Physics held in Japan (September 1953.)

$$\begin{aligned}
 &= 1/2 \int (\pi^2 + \phi(\mu^2 - A)\phi) d\mathbf{x} + \int \phi^* (-i) \rho_1 (\sigma \nabla) \phi d\mathbf{x} + \int \phi^* \rho_3 A_m(f) \phi d\mathbf{x} \\
 &+ \int \phi^* (\rho_1 \boldsymbol{\pi} + (\sigma \nabla) \phi) (\phi(\boldsymbol{\tau} \phi) A(f) + \boldsymbol{\tau} A_{pn}(f) + (\boldsymbol{\tau} \times \phi) \rho_1 A_\pi(f)) \phi d\mathbf{x} \\
 &+ \int \phi^* \rho_2 A_{ps}(f) (\boldsymbol{\tau} \phi) \phi d\mathbf{x} \\
 &+ (1/2) \int [\phi^* \rho_1 (\phi(\boldsymbol{\tau} \phi) A(f) + \boldsymbol{\tau} A_{pn}(f) + (\boldsymbol{\tau} \times \phi) \rho_1 A_\pi(f)) \phi]^2 d\mathbf{x}, \quad (1.6)^6
 \end{aligned}$$

where

$$A(f) = - \left( \frac{f'}{(\phi^2)^{1/2}} + \frac{f}{\phi^2} - \frac{\sin(2(\phi^2)^{1/2}f)}{2(\phi^2)^{3/2}} \right), \quad (1.7)$$

$$A_{pn}(f) = - \frac{\sin(2(\phi^2)^{1/2}f)}{2(\phi^2)^{1/2}}, \quad (1.8)$$

$$A_\pi(f) = \frac{1 - \cos(2(\phi^2)^{1/2}f)}{2\phi^2}, \quad (1.9)$$

$$A_m(f) = m \cos(2(\phi^2)^{1/2}f) - g'(\phi^2)^{1/2} \sin(2(\phi^2)^{1/2}f), \quad (1.10)$$

$$A_{ps}(f) = \frac{m}{(\phi^2)^{1/2}} \sin(2(\phi^2)^{1/2}f) + g' \cos(2(\phi^2)^{1/2}f). \quad (1.11)$$

Then the variation in the energy expectation value with respect to  $f$  (functional variation)

$$\delta \langle E \rangle = \delta (\Psi_0^* \cdot e^G (H_0^M + H_0^N + H_1) e^{-G} \Psi_0), \quad (1.12)$$

comes from the third term of (1.6) and from the last term, because the first and second term is independent of the  $f$  and others are odd in meson field variable.

However, the last term is, as well known, related to the derivative coupling (the 4-th term in the right hand side of (1.6)), and represents normal dependent term. So, we put this term out of consideration, because they must be considered in combination with 4-th term (pseudo-vector term).

Then, we have as a stationary expression for the energy,

$$\frac{\partial}{\partial f} (\Psi_0^* \cdot \int \phi^* \rho_3 A_m(f) \phi d\mathbf{x} \Psi_0). \quad (1.13)$$

To evaluate this expression, it is necessary to take the vacuum expectation value of  $A_m(f)$ . This can be given as follows;

$$\begin{aligned}
 \langle A_m(f) \rangle_0 &= \frac{1}{(2\pi)^3} \int \int (m \cos(2xf(x)) - g'x \sin(2xf(x))) e^{-iux} d\mathbf{x} \cdot \langle e^{iux} \rangle_0 / u \\
 &= \frac{1}{(2\pi)^3} \int \int (m \cos(2xf(x)) - g'x \sin(2xf(x))) e^{-iux} d\mathbf{x} \cdot e^{-(\langle g^2 \rangle_0 / 2) u^2} / u. \quad (1.14)
 \end{aligned}$$

So that the variation with respect to  $f(x)$  gives;

$$\frac{\partial}{\partial f} \langle A_m(f) \rangle_0 = -2xm \sin(2xf(x)) - 2g'x^2 \cos(2xf(x)) = 0, \quad (1.15)$$

namely,



$$f(x) = -\frac{1}{2x} \tan^{-1}\left(\frac{g}{m} x\right), \quad (1.16)$$

$$f(\langle \phi^2 \rangle^{1/2}) = -\frac{1}{2\langle \phi^2 \rangle^{1/2}} \tan^{-1}\left(\frac{g}{m} \langle \phi^2 \rangle^{1/2}\right), \quad (1.17)$$

and is just the transformation function due to Foldy.<sup>2)</sup>

Thus, the Tani-Foldy transformation is proved to be the best one (under the restriction mentioned above (1.13)) and in this case we have for  $A$ 's

$$A = \frac{1}{2m\phi^2} \left( \frac{1}{1 + (g/m)^2 \phi^2} - \frac{1}{\sqrt{1 + (g/m)^2 \phi^2}} \right), \quad (1.18)$$

$$A_{pv} = \frac{g}{2m} \frac{1}{\sqrt{1 + (g/m)^2 \phi^2}}, \quad (1.19)$$

$$A_\pi = \frac{1}{2\phi^2} \left( 1 - \frac{1}{\sqrt{1 + (g/m)^2 \phi^2}} \right), \quad (1.20)$$

$$A_m = m \sqrt{1 + (g/m)^2 \phi^2}, \quad (1.21)$$

$$A_{ps} = 0. \quad (1.22)$$

## § 2. Ordering of the Hamiltonian

Ordering of the  $A$ 's are performed by first making their Fourier analysis, e.g., the functional Fourier analysis with respect to meson variable  $\phi$ , and then ordering the exponential function. The following formulae are sufficient for further calculations;

$$f(\langle \phi^2 \rangle^{1/2}) = \frac{2}{(2\pi)^{1/2}} \int_0^\infty f(\langle \phi^2 \rangle_0^{1/2} y) e^{-y^2/2} \sum_{n=0}^\infty \frac{1}{(2n+1)!} y^{2n+2} \left( \frac{\bar{\phi}^2}{\langle \phi^2 \rangle_0} \right)^n e^{-(\bar{\phi}^2/2\langle \phi^2 \rangle_0)} dy, \quad (2.1)$$

$$\begin{aligned} (\phi e) f(\langle \phi^2 \rangle^{1/2}) &= (e\phi) \frac{2}{(2\pi)^{1/2}} \int_0^\infty f(\langle \phi^2 \rangle_0^{1/2} y) e^{-y^2/2} \sum_{n=0}^\infty \left( \frac{1}{2n!} - \frac{1}{(2n+1)!} \right) y^{2n+2} \times \\ &\quad \times \left( -\frac{\bar{\phi}^2}{\langle \phi^2 \rangle_0} \right)^{n-1} e^{-(\bar{\phi}^2/2\langle \phi^2 \rangle_0)} dy, \quad (2.2) \end{aligned}$$

$$\begin{aligned} (\phi e_1) (\phi e_2) f(\langle \phi^2 \rangle^{1/2}) &= \frac{2}{(2\pi)^{1/2}} \langle \phi^2 \rangle_0 \left[ (e_1 e_2) \sum_{n=0}^\infty \left( \frac{1}{2n!} - \frac{1}{(2n+1)!} \right) \times \right. \\ &\quad \times \left( -\frac{\bar{\phi}^2}{\langle \phi^2 \rangle_0} \right)^{n-1} \int_0^\infty f(\langle \phi^2 \rangle_0^{1/2} y) e^{-y^2/2} y^{2n+2} dy + \frac{(e_1 \bar{\phi})(e_2 \bar{\phi})}{\bar{\phi}^2} \sum_{n=0}^\infty \left( -\frac{3}{(2n+2)!} + \frac{1}{(2n+1)!} \right. \\ &\quad \left. \left. + \frac{3}{(2n+3)!} \right) \left( -\frac{\bar{\phi}^2}{\langle \phi^2 \rangle_0} \right)^n \int_0^\infty f(\langle \phi^2 \rangle_0^{1/2} y) e^{-y^2/2} y^{2n+4} dy \right] e^{-(\bar{\phi}^2/2\langle \phi^2 \rangle_0)}, \quad (2.3) \end{aligned}$$

(where  $\bar{\phi}$  means the ordered operators).<sup>7)</sup>

After the ordering, the effective Hamiltonian becomes;

$$\begin{aligned} \langle H_T \rangle = & H_0^M + \int \phi^* (-i \rho_1 \sigma \nabla + A \cdot m \rho_3) \phi d\mathbf{x} \\ & + \int \phi^* \rho_3 \left( \frac{B}{m} \bar{\phi}^2 + \frac{C}{m^3} (\bar{\phi}^2)^2 + \dots \right) \phi d\mathbf{x} + \int \phi^* D \frac{(\bar{\pi} \times \tau \cdot \bar{\phi})}{m^2} \phi d\mathbf{x} + \dots \\ & + \int \phi^* E \frac{1}{m} (\rho_1 \bar{\pi} + (\sigma \nabla) \bar{\phi}) \cdot \tau \phi d\mathbf{x} + \dots. \end{aligned} \quad (2.4)$$

The evaluation of the numerical coefficients are performed by putting the cutoff momentum at about nucleon mass  $m$ .

$g^2/4\pi$	$A$	$B$	$C$	$D$	$E$
$4\pi$	2.559	27.61	-91.49	7.082	2.543
$6\pi$	2.985	34.18	-119.35	8.049	2.545
$8\pi$	3.389	40.04	-143.55	8.598	2.566

Table I. Numerical coefficients appearing in the equation (2.4). The cut-off momentum is at nucleon rest mass  $m$ .

By using the abbreviation  $mA = m^*$ ,

$$\begin{aligned} \langle H_T \rangle = & H_0^M + \int \phi^* (-i \rho_a (\sigma \nabla) + m^* \rho^3) \phi d\mathbf{x} \\ & + \int \phi^* \rho_3 \left( \frac{g^2}{2m^*} B' \bar{\phi}^2 + \frac{g^4}{8m^{*3}} C' (\bar{\phi}^2)^2 + \dots \right) \phi d\mathbf{x} \\ & + \frac{g^{v,2}}{4m^{*2}} D' \int \phi^* (\bar{\pi} \times \tau \cdot \bar{\phi}) \phi d\mathbf{x} + \dots \\ & + \frac{g}{2m^*} E' \int \phi (\rho_1 \bar{\pi} + (\sigma \nabla) \bar{\phi}) \tau \phi d\mathbf{x} + \dots. \end{aligned} \quad (2.5)$$

$g^2/4\pi$	$B'$	$C'$	$D'$	$E'$
$4\pi$	0.895	-0.492	1.175	1.036
$6\pi$	0.862	-0.453	1.211	0.995
$8\pi$	0.859	-0.446	1.251	0.979
	$8/3\pi$ =0.849	$-64/15\pi^2$ =0.432	$16/3\pi$ =1.70	$8/3\pi$ =0.849

Table II. The numerical coefficients appearing in equation (2.5). The figures are the relative ratio with the coefficients which is obtained by expanding (1.18)-(1.22), and if one does not take into account the ordering, the lowest order term gives Hamiltonian (2.5) with all coefficients 1.

The last column is obtained by assuming  $\frac{g^2}{m^2} \langle \phi^2 \rangle_0 \gg 1$  (strong limit) with  $m^* = \frac{4g^2 V \langle \phi^2 \rangle_0}{(2\pi)^{1/2}}$  and in this case the result is not dependent on the cut-off.

It will be seen from Table II that the core term and pseudo-vector coupling reduces slightly

but the  $L$ - $S$  coupling term  $(\tau \cdot \phi \times \pi)$  is increased by a factor two in the strong limit. This is a very interesting result in order to explain the  $S$ -wave phase shift in the pion-nucleon scattering.

### § 3. Discussion

The Tani-Foldy transformation function was found to be the best one in the similar classes of transformations which preserves the relativistic invariance of the transformed Hamiltonian. But it is to be shown whether the remaining interaction of the Hamiltonian thus derived is small or not. As to this point we cannot say anything at present, but we may expect that the main features of the low-momentum meson reactions are well established in our effective Hamiltonian. The discussion on this point will be more carefully given in a separate paper.

Of course, if it is found that the higher powers of the transformed Hamiltonian is not so weak, we should replace the transformation functions by the more general ones; what we found is that the Foldy type of transformation is the best among the similar type of transformations.

The authors should like to express their thanks to Professor S. Tomonaga and to Professor K. Nakabayashi for kind encouragements, and to the members of Tokyo University of Education for valuable discussions on this subjects.

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- 6) The expression  $\pi : f(\phi)$  means  $\frac{1}{2} \{ \pi, f(\phi) \}_+$  and its vacuum expectation value  $\langle \frac{1}{2} \{ \pi, f(\phi) \}_+ \rangle = 0$ .
- 7)  $\langle \phi^2 \rangle_0$  means  $\langle \phi_i^2 \rangle_0$  with  $i$  any one of 1, 2, 3,  $\bar{\phi}^2 = \bar{\phi}_1^2 + \bar{\phi}_2^2 + \bar{\phi}_3^2$ .

## Letters to the Editor

### High Energy Nucleon Scattering by Nuclei

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June 23, 1954

The total cross sections and differential cross sections of elastically scattered nucleons by nuclei are analyzed by means of the so-called optical model, using the parameters recently given by Taylor<sup>1)</sup> so as to fit the total cross section data. We have used the method of calculation developed by Montroll et

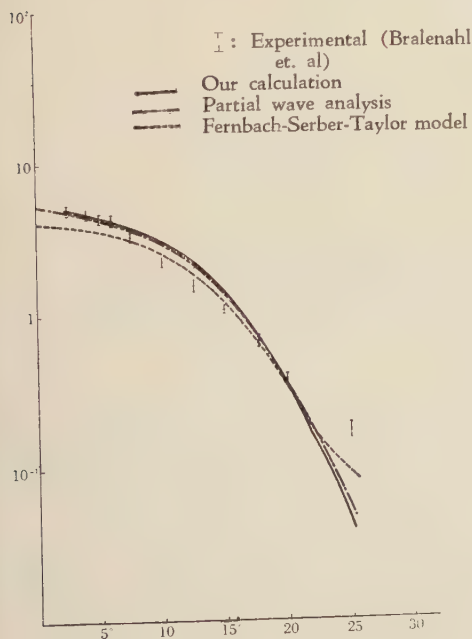


Fig. a, Angular distribution of 85 Mev neutrons elastically scattered by Al. Absorption coefficient  $K=1.97 \times 10^{12} \text{cm}^{-1}$ , increase in the magnitude of the propagation vector of a nucleon upon entering the nucleus  $k_1=2.76 \times 10^{12} \text{cm}^{-1}$ , nuclear radii  $R=4.49 \times 10^{-13} \text{cm}$ .

al.,<sup>2)</sup> which is appropriate for treating the scattering of waves by soft obstacles.

The energy range considered is from 85 Mev to 400 Mev. The shape of the potential chosen is that of square well. The calculated total cross sections are in agreement with experimental values, although slight systematic deviations are observed. The differential cross sections are calculated in the same way and compared with the 85 Mev neutron experiment of Cook et al.<sup>3)</sup> and the 340 Mev proton experiment of Richardson et al.<sup>4)</sup> Agreement between theoretical and experimental values is good, except for too deep minima on the theoretical diffraction patterns, as has already been reported by other authors.<sup>4, 5)</sup>

Some typical examples are given below.

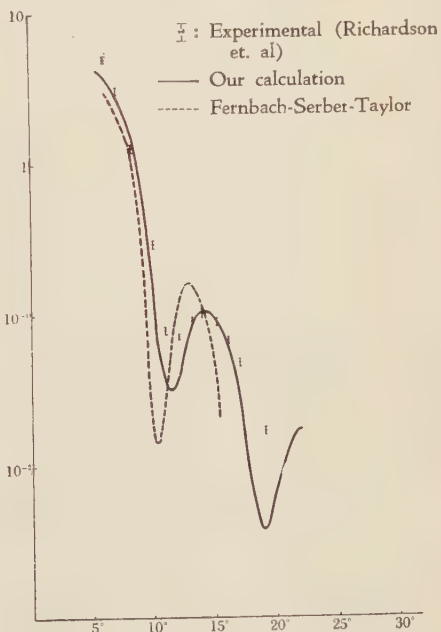


Fig. b, Angular distribution of 340 Mev protons elastically scattered by Cu.

$$K=1.76 \times 10^{12} \text{cm}^{-1}, \quad k_1=0.75 \times 10^{12} \text{cm}^{-1}, \\ R=5.80 \times 10^{-13} \text{cm}.$$

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Fig. c, Angular distribution of 340 Mev protons elastically scattered by Pb.

$$\begin{aligned} A &= 1.76 \times 10^{12} \text{cm}^{-1}, & k_1 &= 0.75 \times 10^{12} \text{cm}^{-1}, \\ R &= 8.12 \times 10^3 \text{cm}. \end{aligned}$$

Fig. (a) shows the result for 85 Mev neutron on Al. The solid line represents theoretical values and the crosses represent experimental results. The dotted line represents the result of calculation by Pasternack and Snyder<sup>(6)</sup> by means of formula given by Fermi, Serber and Taylor,<sup>(6)</sup> using the present parameters. It would also be interesting to carry out the partial wave analysis using the same parameters.

Figs. (b) and (c) show the results for 340 Mev proton on Cu and Pb, respectively. In the case of Cu agreement is good while for Pb the calculated values are systematically a little smaller than the experimental ones. It seems to be that the too deep minima of theoretical values are due to the inappropriate shape of the potential. It may perhaps be necessary to round the edge of the potential.

The calculation with Gaussian potential by Montroll, Hart and Greenberg method is now in progress. The details of the result will be reported later.

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## Electromagnetic Properties of Deuteron

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June 24, 1954

In recent years considerable progresses have been achieved in the meson theory of nuclear forces.<sup>(1)-(6)</sup> However, before comparing the theoretical results with the experiments, it is necessary to know how one expresses the magnetic and the electric quadrupole moments of deuteron in terms of its wave function. The information to this will also clarify the meaning of the wave functions used in the various theories of nuclear forces. We therefore calculate the moments of deuteron using the pseudoscalar meson theory. This problem has been investigated by Villars<sup>(7)</sup> and by Deser<sup>(8)</sup>, but their results are very different from each other. Moreover, their approximations are not sufficient to be consistent with those adopted in the recent studies of nuclear forces.

The Schroedinger equation for the system of nucleons and meson field interacting with an external electromagnetic field is given by

$$(E - H_m - K - H_{ne} - H_{nm})\Psi = 0, \quad (1)$$

where  $H_m$ ,  $K$ ,  $H_{ne}$  and  $H_{nm}$  are Hamiltonian for free meson field, kinetic energy of the nucleons, interaction between the nucleons and the external field, and interaction between the nucleons and the meson field respectively. We omit the interaction between the meson field and the external field, because it does not contribute to the moments of deuteron.<sup>9)</sup> The solution of (1) is written as

$$\Psi = [1 - a^{-1}(1-v)H']^{-1}\psi \times \omega_0$$

with

$$a = \epsilon - H_m - K - H_{ne}, \quad H' = H_{nm} - \delta E,$$

where  $\omega_0$  is the state vector representing the meson vacuum,  $v$  is projection operator to  $\omega_0$  and  $\psi$  is a state vector in the nucleon space obeying the equation

$$(\epsilon - K - H_{ne})\psi = \langle H' [1 - a^{-1}(1-v)H']^{-1} \rangle \psi, \quad (2)$$

where the symbol  $\langle A \rangle$  denotes  $(\omega_0, A\omega_0)$ . To the first order in  $\epsilon - H_{ne}$ , (2) becomes

$$[\epsilon(1+P) - K]\psi = (U + U_e)\psi \quad (3)$$

with the abbreviations

$$1+P = \langle b^*b \rangle, \quad U = \langle H'b \rangle, \quad U_e = \langle b^*H_{ne}b \rangle,$$

where

$$b = [1 + (H_m + K)^{-1}(1-v)H']^{-1}.$$

$P$  is the relative probability for the dissociation of the nucleons into the 'bare' ones and virtual mesons. It is easily shown that  $P$  is positive definite. If no external field exists, (3) reduces to

$$[\epsilon_0(1+P) - K]\psi_0 = U\psi_0. \quad (4)$$

It follows from (3) and (4) that the increment of energy due to the external field is, to the first order in  $e$ , given by

$$\epsilon_1 = (\psi_0, U_e \psi_0) / (\psi_0, (1+P)\psi_0). \quad (5)$$

The deuteron moments are easily calculated from (5).

We take

$$H_{nm} = (f/\mu) \sum_{i=1,2} \sum_{\alpha=1}^3 \tau_{\alpha}(i) \\ \times [\sigma^{(i)} \nabla \phi_{\alpha}(\mathbf{r}^{(i)}) + \rho_1^{(i)} \Pi_{\alpha}(\mathbf{r}^{(i)})].$$

We expand  $P$  and  $U_e$  in powers of  $f$ , and take only the second and the fourth order terms. In the

fourth order terms we neglect  $K$  entirely, while in the second order terms  $K$  is taken into account to the first order in  $K/H_m$ . We found that the divergences which remain after the subtraction of self energy can be eliminated by the mesic charge renormalization, but this elimination destroys the positive definiteness of  $P$ . We therefore use the cut off procedure. The cut off momentum is taken as  $k_{\max} = 3\mu$ .

The results are as follows:

The magnetic moment  $\mu_d$  and the electric quadrupole moment  $Q$  are written as

$$\begin{aligned} N\mu_d = & \int_0^{\infty} \{ (3/4)(1+B_0)v^2 \\ & + (3/2)B_T(\sqrt{2}uv - v^2) + C(u^2 - v^2 - uv|\sqrt{2}) \\ & + (D/\sqrt{2})(3uv + xuv' - xu'v) + (\mu_p + \mu_n) \\ & \times [(1+E_0)(u^2 - v^2/2) + E_T(\sqrt{2}uv + v^2)] \} dx, \\ NQ = & \mu^{-2}(1/10) \int_0^{\infty} x^2 [(1+F_0)(\sqrt{2}uv - v^2/2) \\ & + F_T(2u^2 - \sqrt{8}uv + 3v^2)] dx, \quad (6) \end{aligned}$$

$$N = \int_0^{\infty} [(1+A_0)(u^2 + v^2) + 2A_T(\sqrt{8}uv - v^2)] dx, \quad (7)$$

where

$$\begin{aligned} A_0 = & a_f[a_2 + (1 + 2a_f a_2/9)(k_0 - k_1) \\ & - (e_0 - 2e_1 + 2a_1)J_0(x) - 8(e_0 + e_1)J_T(x)] \\ & + a_f^2[6k_0^2 + 19k_0k_1 + 43k_1^2/2 \\ & - e_0(3e_1 + 2e_2 + 8e_3 + 4e_4) + 4a_1e_1/9], \\ A_T = & a_f[(1 + 2a_f a_2/9)(k_0 + 2k_1) - (e_0 + e_1)J_0(x) \\ & - (2a_1 - e_0 - 4e_1)J_T(x)] - a_f^2[7k_0k_1/2 + 7k_1^2 \\ & + e_0(2e_2 + 5e_3 + 4e_4) - 4a_1(e_1 + 3e_2 + 3e_3)/9], \\ B_0 = & A_0 - a_f\lambda(a_3 - a_1)/3 + a_f\lambda e_1/2, \\ B_T = & A_T + a_f\lambda(e_1 + 3e_2 + 3e_3)/2, \\ C = & 3a_f^2(k_0 + 2k_1)^2/2 - 3a_f(e_0 + e_1)J_T(x), \\ D = & 3a_f\lambda(e_2 + e_3)/4, \\ E_0 = & a_f[a_2 + 2\lambda(a_3 - a_1)/3 + (1 + 2a_f a_2/9)(k_0 - k_1) \\ & - (2a_1/3 + e_0 - 2e_1)J_0(x) - 2(e_0 + e_1)J_T(x)] \\ & + a_f^2[3k_0^2 + 7k_0k_1 + 19k_1^2/2 - e_0(e_1 + 2e_2 + 4e_3 + 2e_4)], \\ E_T = & a_f[(1 + 2a_f a_2/6)(k_0 + 2k_1) - (e_0 + e_1)J_0(x) \\ & - (2a_1/3 + 2e_0 - e_1)J_T(x)] + a_f^2[3k_0^2/2 + 5k_0k_1/2 \\ & - k_1^2 - e_0(e_1 + 2e_2 + e_3 - e_4)], \end{aligned}$$

$$F_0 = 2B_0 - A_0, \quad F_T = B_T - A_T$$

with the abbreviations

$$\alpha_f = f^2/4\pi, \quad \lambda = \mu/M, \quad k_0 = (2/\pi) K_0(x),$$

$$k_1 = (2/\pi x) K_1(x), \quad e_n = x^{-n} e^{-x},$$

$$a_n = (6/\pi) \mu^{-n} \int_0^{k_{\max}} k^4 (k^2 + \mu^2)^{(n-5)/2} dk.$$

The functions  $J_0$  and  $J_T$  have appeared because we replaced  $K/\mu$  appearing in  $P$  and  $U_e$  by  $J_0 + J_T S_{12}$ , assuming that the equation for  $\psi_0$  (4) can be replaced by an equivalent equation

$$(\epsilon_0 - K) \psi_0 = -\mu (J_0 + J_T S_{12}) \psi_0.$$

$J_0$  and  $J_T$  are taken from the paper of Brueckner and Watson.<sup>5)</sup> Numerical value is computed only for the quadrupole moment. In doing this, (6) and (7) are approximated by

$$Q = \int_0^\infty (Q_0 + \mu^{-2} x^2 F_T/5) u^2 dx / \int_0^\infty (1 + A_0) u^2 dx,$$

where  $Q_0$  is the quadrupole moment calculated without the radiative corrections. For the value of  $Q_0$  and the function  $u$ , we take those given in Reference 5):

$$Q_0 = 2.83 \times 10^{-28} \text{ cm}^2, \quad \text{for } \alpha_f = 0.086$$

$$u = e^{-(r-r_0)/r_D} - e^{-8(r-r_0)/r_D}.$$

We then have

$$Q = 3.19 \times 10^{-28} \text{ cm}^2,$$

$$(\psi_0, \psi_0)/(\psi_0, (1+P)\psi_0) = 0.70.$$

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## Diagonalisation of Hamiltonian

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June 25, 1954

Let us consider the next eigen-value problem.

$$H\Psi = E\Psi. \quad (1)$$

And let  $\eta$  be the projection operator for some states that are the eigen-states of the free Hamiltonian. These are taken for the two nucleon no pair and no meson states, if we consider the problem of the nuclear forces.

Put as follows,

$$\Psi_1 = \eta\Psi, \quad \Psi_2 = (1-\eta)\Psi, \quad (2)$$

then we have

$$\left. \begin{aligned} \eta H \eta \Psi_1 + \eta H (1-\eta) \Psi_2 &= E \Psi_1, \\ (1-\eta) H \eta \Psi_1 + (1-\eta) H (1-\eta) \Psi_2 &= E \Psi_2. \end{aligned} \right\} \quad (3)$$

The usual Tamm-Dancoff method is that we represent  $\Psi_2$  by  $\Psi_1$  from the second equation of (3) and obtain the equation of  $\Psi_1$  only. This formalism was given by Sawada.<sup>1)</sup> But here we consider the formalism that  $H$  is diagonalised directly. Perform the following unitary transformation.

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} (1+A^\dagger A)^{-1/2} S_1, & -A^\dagger (1+A A^\dagger)^{-1/2} S_2 \\ A(1+A^\dagger A)^{-1/2} S_1, & (1+A A^\dagger)^{-1/2} S_2 \end{pmatrix} \times \begin{pmatrix} \Psi_1' \\ \Psi_2' \end{pmatrix}. \quad (4)$$

Here  $A$  is an operator whose form is given by

$$A = (1-\eta) A' \eta \quad (5)$$

and  $S_1$  or  $S_2$  is an arbitrary unitary operator in the respective subspace. The condition that (3) is

$$(1-\eta)(H + [H, A] - AHA)\eta = 0. \quad (6)$$

Then (3) becomes to

$$\begin{aligned} (1+A^\dagger A)^{-1/2} (1+A^\dagger) H (1+A) (1+A^\dagger A)^{-1/2} \Psi_1' \\ (1+A A^\dagger)^{-1/2} (1-A^\dagger) H (1-A^\dagger) (1+A A^\dagger)^{-1/2} \Psi_2' \\ = E \Psi_1', \\ = E \Psi_2'. \end{aligned} \quad (7)$$

Here we put  $S_1$  and  $S_2$  to units for brevity. And if we put

$$J = 1 + A, \quad (8)$$

the first equation of (7) reduces to the next.

$$\langle J^+ J \rangle^{-1/2} \langle J^+ H J \rangle \langle J^+ J \rangle^{-1/2} \Psi_1' = E \Psi_1'. \quad (9)$$

Here

$$\langle J \rangle = \eta J \eta + (1 - \eta) J (1 - \eta). \quad (10)$$

$J$  defined here corresponds to  $J$  given by Sawada<sup>1)</sup> and has close relationship with the Tamm-Dancoff method.

From (2), (5) and (6), (9) is rewritten as follows.

$$\langle J^+ J \rangle^{1/2} \langle H J \rangle \langle J^+ J \rangle^{-1/2} \Psi_1' = E \Psi_1'. \quad (11)$$

Let us put

$$\Psi_1' = \langle J^+ J \rangle^{1/2} \Phi. \quad (12)$$

Then

$$\langle H J \rangle \Phi = E \Phi \quad (13)$$

(13) is equivalent to the usual Tamm-Dancoff equation that the energy  $E$  in the denominators are eliminated and  $\Phi$  corresponds to the usual Tamm-Dancoff amplitude. In that case,  $\Psi_2'$  is identically null. These will be shown in the full paper which will be published later in this journal. We can calculate the nuclear forces by (9) or (13). The latter is the method by the Tamm-Dancoff equation. Sawada noticed that the potential calculated in this way is not necessarily hermitic, and one by (11) or (9) is hermitic. The latter corresponds to the fact that we renormalise the Tamm-Dancoff amplitude so that the normalisation of the amplitude may be equal to one. Of course, the potentials calculated from (13) till now are fortunately hermitic and (9) and (13) give the same results. But when we want to calculate the higher order nuclear forces, we encounter on the non-hermitic potential. The example is given by the usual Lévy-Klein<sup>2)</sup> procedure. When they calculated the nuclear force that comes from the iteration of the second order graph, they made the approximation that the energies of two nucleon lines in the intermediate state are put to be equal, and obtained the potential which was the difference of the potentials by Taketani-Machida-Onuma and Brueckner-Watson.<sup>3)</sup> If we do not make use of the approximation, we have a non-hermitic velocity dependent force. But by (11) or (9), we have not such term.

It is convenient to distinguish the free and interaction Hamiltonian.

$$H = H_0 + H_1. \quad (14)$$

Noticing (5) and  $\eta$  and  $H_0$  commute with each other then (6) is rewritten as follows:

$$(1 - \eta) (H_1 J + [H_0, J] - J \langle H_1 J \rangle) \eta = 0$$

If we introduce the interaction representation, we get

$$i \cdot dJ/dt = (1 - \eta) (H_1 J - J \langle H_1 J \rangle) \eta. \quad (15)$$

The solution has the close connection with the adiabatic theorem if we take  $\eta$  for the projection operator for a pure state. The second term of the right hand of the above equation represents the self-energy.

At last we notice that the method of Cini-Fubini<sup>4)</sup> is to neglect the second term of the right hand of (15) and solve it under the proper initial condition. (Of course, the notation is different from ours.  $J$  here corresponds to  $\bar{J}$  there and  $\eta$  to  $1 - \eta$ .)

The details will be published later in this journal.

The author expresses his thanks to Dr. Sawada for the useful discussions and suggestions on this problem.

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# Reformulation of Brueckner-Watson Method by Means of Canonical Transformations

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The purpose of the present note is to point out that the considerations made by Okubo<sup>1)</sup> in the preceding note can be formulated by means of a canonical transformation of arctangent type and that it is in principle possible to reformulate the Brueckner-Watson<sup>2)</sup> method (B.W. method) in nuclear force or pion scattering problem by making use of suitable canonical transformations.

If we succeed in this reformulation, the question of the 'normalization of the state vector', which has been raised and studied by Fukuda, Sawada and Taketani<sup>3)</sup>, may be solved naturally. Since we resort to the method of canonical transformation, the physical interpretation of the results (e.g., that of the probability amplitude for a certain meson configuration in the cloud around a nucleon) will be straightforward, and the eigenvalue of the system under consideration will be readily evaluated.

Suppose we deal with a system which is described in terms of the total Hamiltonian,  $\mathcal{H}$ .  $\mathcal{H}$  consists of two parts; the one is concerned with non-interacting meson and nucleon,  $H_0$ , and the other is concerned with their interaction,  $H$ ;

$$\mathcal{H} = H_0 + H. \quad (1)$$

At the beginning, we must remind ourselves of the reformulation of B. W. method due to Klein<sup>4)</sup>. Following him, we classify the components of the state vector into those with even number of mesons present and those with odd number of them. Hereafter, for convenience, we introduce the classification of operators into even or odd ones. The even operator is the one by which the even-odd nature of the meson occupation number is not changed, and the odd one gives rise to the change of the even-odd nature of the meson number.  $H_0$  is even and  $H$  is

usually odd. In the reformulation of the B.W. method, we should eliminate odd operators from the transformed Hamiltonian by means of a suitable canonical transformation.

For this purpose it is recommended for us to put the transformation function in the form

$$U = e^{i \tan^{-1} G} = (1 + iG) \cdot 1/\sqrt{1 + G^2}, \quad (2)$$

and look for a suitable odd operator  $G$  (which must be Hermitian). Then we have the equation of the transformation

$$U^{-1}(H_0 + H)U = H_0 + W; \quad (3)$$

where  $W$  denotes the operator which describes the effects of self-energy, scattering, emission or absorption of even number of mesons, etc., which are caused by repeated operation of the interaction  $H$ . Since  $H_0$  and  $W$  are even, we must find odd operators on the left hand side of eq. (3) to cancel each other. In view of the last statement of eq. (2), this means

$$[H_0, iG] + H + G H G = 0. \quad (4)$$

This is the defining equation for  $G$ , from which we can compute it. If we assume, as usual, that the interaction  $H$  is proportional to the coupling constant  $g$  and composed of operators for single emission or absorption of a meson, then by means of the expansion of  $G$  into power series in  $g$ , eq. (4) may be solved uniquely. The first order term in  $g$ , which is denoted by  $G_1$ , is the solution to the equation

$$[H_0, iG_1] + H = 0, \quad (5)$$

which is easily solvable. The next term which is third order in  $g$  is then given by the solution to the equation

$$[H_0, iG_3] + G_1 H G_1 = 0. \quad (6)$$

The procedures to determine higher order terms (5th, 7th and so on) are similar.

Any transformation function of the form

$$\exp\{i[\text{an odd and Hermitian operator}]\}$$

may be available for the purpose of elimination of odd operators from the transformed Hamiltonian. But the ansatz of arctangent is most convenient, since it gives the defining equation of the transformation generator which has the simplest structure as shown in eq. (4). This point can be analysed using the method reported by the author<sup>5)</sup>.

Now in the transformed Hamiltonian  $W$  is composed of operators which are of order  $2n$  in  $g$

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After dividing the meson number by 2, the term 'odd' or 'even' will be redefined. Those parts of  $W$ , which cause emission or absorption of 2, 6, 10, ... mesons, are now called odd operators and their sum is denoted by  $W_{2\text{odd}}$ . The remaining parts are denoted by  $W_{2\text{even}}$ , and they contain some operators which cause emission or absorption of 4, 8, 12, ... mesons. Here the terms classified in this manner should include also those which are looked upon their correction terms arising from the multiple scattering effects. If we perform a second transformation by means of

$$U' = e^{i \tan^{-1} G'}, \quad (7)$$

where  $G'$  is Hermitian and odd in the redefined sense, then we can obtain the transformed Hamiltonian free from odd operators.  $G'$  turns out to be determined by the solution of the equation

$$- [H_0, iG'] = [W_{2\text{even}}, iG'] + W_{2\text{odd}} + G' W_{2\text{odd}} G', \quad (8)$$

which is derived in a similar way as eq. (4); it is composed (mainly) of terms of order  $2(2n+1)$  in  $g$  (except for correction terms caused by multiple scattering effects). The transformed Hamiltonian contains operators which cause emission or absorption of 4, 8, 12, ... mesons as well as the self-energy operators and the scattering operators.

The procedure is pushed one step further, if we divide the meson number by four and redefine its even-oddness, then, classify the terms in the twice transformed Hamiltonian into  $W_{4\text{odd}}$  and  $W_{4\text{even}}$  and perform a third transformation to eliminate odd operators; in this third step, the operators termed 'odd' are those which are responsible for the emission or absorption of  $4(2n+1)$ -mesons. The definition of the transformation function in such further steps can be done in a similar way to that in the second step which has been explained above.

The correspondence of the canonical transformation here discussed with the procedure in B.W. method refined by Sawada and Okubo<sup>1)</sup> can be found readily, if we take the operator of the form

$$\eta A(1-\eta) \text{ or } (1-\eta) A \eta \quad (\eta; \text{a projection operator})$$

in Okubo's paper as the operators which are termed 'odd' in the above discussions.

After the reformulation in terms of the 'arctangent-transformation', it can be naturally understood that one gains nothing in making use of the B.W. method to treat the neutral scalar meson field\*\*. As is well known, in this case the total Hamiltonian is made

diagonal by the canonical transformation with transformation function

$$e^{iG_1}$$

where  $G_1$  is defined by eq. (5). So, it only complicates the further treatment to perform a canonical transformation with transformation function

$$e^{i \tan^{-1} G_1}$$

at the first step of the solution.

The author is indebted much to K. Sawada and S. Okubo for interesting discussions during their stay at the Research Institute for Fundamental Physics.

- 1) S. Okubo, Prog. Theor. Phys. **12** (1954), 102 (L).
- 2) K. A. Brueckner and K. M. Watson, Phys. Rev. **90** (1953), 699.
- 3) N. Fukuda, K. Sawada and M. Taketani, Prog. Theor. Phys. (1954) (in press).
- 4) A. Klein, Phys. Rev. **91** (1953), 1285 (L).
- 5) S. Tani, Prog. Theor. Phys. **11** (1954), 190.
- 6) E. M. Henley and M. A. Ruderman, Phys. Rev. **92** (1953), 1032.

## On the Variational Solution of Bethe-Salpeter Equation in Pion-nucleon Scattering

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July 9, 1954

It is well known that the solution of B-S equation<sup>1)</sup> in pion-nucleon scattering

$$\psi = \phi + (1/F) I \psi \quad (1)$$

obeys the following variational principle<sup>2)</sup>;

$$\delta^* K' = 0, \quad (2)$$

\*\* Professor R. Marshak called our attention to this point at the Intern. Conf. Theor. Phys. held at Kyoto, Sept. 1953; cf. also Refr. 6. p. 1041.

$$\backslash K'_{ba} \equiv \frac{(\psi_b, I\phi_a)(\phi_b, I\psi_a)}{(\psi_b, I\psi_a) - R_e(\psi_b, I(1/F)\psi_a)} \quad (3)$$

where  $\psi$  describe non-interacting initial or final states of the system,  $1/F$  denotes the pion-nucleon two body propagator and  $I$  is the well known interaction kernel.

As the variational principle of this type does not require the correctly normalized trial functions in order to secure the hermiticity of the reaction operator  $\backslash K'$ , it is especially convenient for the approximate treatment of the scattering problems in B-S formalism where the physical interpretation of "wave functions" as a probability amplitude is not clear and its normalization is very troublesome.

Recently Chew<sup>3)</sup> has pointed out that a crude choice of the trial function  $\psi = \phi$  gives rather good approximation for the problems which have the almost factorable kernels such as pion-nucleon scattering. In fact, in the factorable kernel, trial function  $\psi = \phi$  gives the exact  $\backslash K'$ .

With this trial function, (3) turns out to be

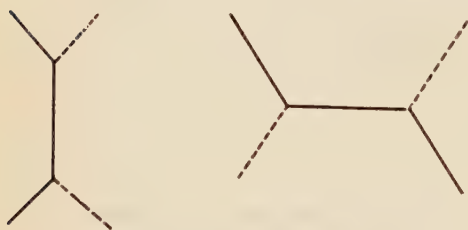


Fig. 1 Interaction kernel

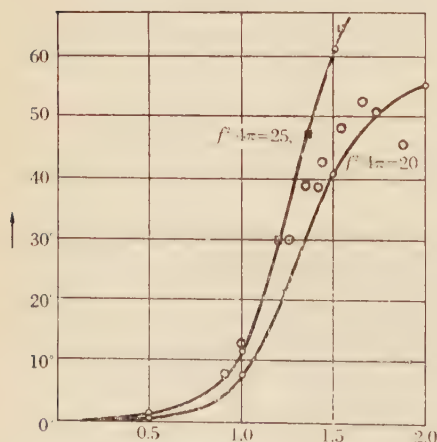


Fig. 2 ○; Fermi's results for  $a_{33}$   
■; Results of Glicksman and Marlin's Analysis

$$\backslash K'_{ba} \approx \frac{\backslash K'_{ba}^{(2)}}{1 - \backslash K'_{ba}^{(4)}/\backslash K'_{ba}^{(2)}} + \dots \quad (4)$$

where  $\backslash K^{(2)}$  and  $\backslash K^{(4)}$  are the second and fourth order reaction operators in the sense of interaction, so they may be calculated by the usual method of the covariant perturbation with the renormalization prescriptions. Therefore they are free from any ambiguity due to the non-relativistic treatment and to the cut-off.<sup>4)</sup> It is also seen that the power expansion of (4) coincides with the results of the iteration up to the 4-th order. Similar results are obtained by Cini and Fubini from the more general point of view.

We have applied this method of approximation to the case of Fig. 1, calculating  $\backslash K^{(2)}$  and  $\backslash K^{(4)}$  covariantly by the method of Feynman-Dyson and the actual calculation<sup>6)</sup> is compared with the experimental results for  $a_{33}$  phase shift in Fig. 2.

Detailed accounts of our calculation and discussion for other phase shift will be given in other part of this journal.<sup>7)</sup>

- 1) E. E. Salpeter and H. A. Bethe, Phys. Rev. **84** (1951), 1232.
- 2) B. A. Lippmann and J. Schwinger, Phys. Rev. **79** (1950), 469.  
M. L. Goldberger, Phys. Rev. **84** (1951), 929.  
G. F. Chew, Phys. Rev. **89** (1953), 591.
- 3) G. F. Chew, Phys. Rev. **93** (1954), 341.
- 4) Detailed calculation of the meson-nucleon scattering by the method of Tamm-Dancoff has been done recently by S. Chiba, N. Funkuda and M. Yamazaki, Prog. Theor. Phys. **11** (1954) in press.
- 5) M. Cini and S. Fubini, Nuovo Cimento **11** (1954), 142. After our present work was published in Japanese we have obtained their preprint sent to our institute.
- 6) The fourth order S matrix element for meson-nucleon scattering were calculated, two years ago, by the present authors and by M. Nakabayashi et al. independently. But they are not published owing to their discouraging results. In the case of Thomson limit the evaluations were done by J. Ashkin, A. Simon and R. E. Marshak, Prog. Theor. Phys. **4** (1950), 634.
- 7) Concerning the Renormalization of the B-S equation for meson-nucleon scattering, see also D. Ito and H. Tanaka, Prog. Theor. Phys. **11** (1954), 501.

## Some Remarks on the Even-odd Rule

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July 13, 1954

The large yield and striking stability of  $V$ -particles are tentatively interpreted in terms of the so-called even-odd rule<sup>1)</sup>, but since this rule is phenomenological in nature several attempts<sup>2)-4)</sup> to relate it to the charge independence hypothesis have been made.

In a previous letter<sup>4)</sup> (to be referred to as I) a possible relationship between them was proposed by assigning an integral isotopic spin to the  $A$ -particle and a half-integral one to the heavy meson. The purpose of the present note is to extend the above idea so as to include the interpretation of the so-called cascade decay.<sup>5)</sup>

Let us consider the collection of the elementary particles that obey the requirement of charge independence, then as is well known there is always a definite relationship between the third component of the isotopic spin and charge for each elementary particle, e.g.  $q = I_3 + 1/2$  for nucleon and  $q = I_3$  for pion.

In a similar way we may write

$$q = I_3 + 1/2 + 1/2 \cdot \eta_a \quad (1 \cdot a)$$

for a hyperon  $N_a$  and

$$q = I_3 + 1/2 \cdot \eta_b \quad (1 \cdot b)$$

for a heavy meson  $\Pi_b$ .

In this note we call this  $\eta$  the  $v$ -charge. The  $v$ -charge of a nucleon or a pion is zero as is clear from the definition. If we assume the charge states of  $A$  and  $K$  as  $A^+$ ,  $A^0$ ,  $A^-$  ( $I=1$ ) and/or  $A^0$  ( $I=0$ ) and  $K^+$ ,  $K^0$  ( $I=1/2$ ) following the assignments in  $I_s$ , we have  $\eta(A) = -1$  and  $\eta(K) = 1$ . In general an anti-particle has an opposite  $v$ -charge of the particle as is the case for the electric charge.

The total charge  $q$  and the third component of the total isotopic spin of a system are connected with each other by

$$q = I_3 + \frac{1}{2} \sum_a n(N_a) + \frac{1}{2} \sum_a \eta_a n(N_a) + \frac{1}{2} \sum_b \eta_b n(\Pi_b) \\ = I_3 + \frac{1}{2} b + \frac{1}{2} \eta, \quad (2)$$

where  $n$  means the no. of particles minus the no. of anti-particles and  $b$  and  $\eta$  are defined respectively by

$$b = \sum_a n(N_a), \quad \eta = \sum_a \eta_a n(N_a) + \sum_b \eta_b n(\Pi_b), \quad (3)$$

$b$  is the hyperon number and  $\eta$  is the total  $v$ -charge.

In production processes only charge independent and electro-magnetic interactions are operating for which  $q$ ,  $I_3$ , and  $b$  are conserved. Hence we have a conservation law

$$\eta = \text{const.}, \quad (4)$$

this is nothing but the conservation of  $v$ -charge. Contrary to the case of electric charge,  $v$ -charge is defined only for such particles that have strong nuclear interactions and its conservation is violated by the weak interactions responsible for decays.

The conservation of  $v$ -charge will be seen in the process<sup>6)</sup>

$$\pi^- + p \rightarrow A^0 + K^0 (\text{or } \theta^0), \quad (5)$$

$$0 \quad 0 \quad -1 \quad 1$$

whereas the following process is forbidden by this conservation law

$$N + N \rightarrow A + A. \quad (N: \text{nucleon}) \quad (6)$$

$$0 \quad 0 \quad -1 \quad -1$$

The decay of a  $A$ -particle into a proton and a negative pion

$$A^0 \rightarrow p + \pi^-, \quad (7)$$

$$-1 \quad 0 \quad 0$$

is forbidden and can take place only through a very weak interaction so that it is experimentally observable.

Let the group of particles with the  $v$ -charge  $\pm s$  be  $G_s$ , then elementary particles with strong nuclear interactions are divided into the following groups

$$G_0, G_1, G_2, \dots \quad (8)$$

Pions and nucleons belong to  $G_0$ , and  $A$  and  $K$  belong to  $G_1$ , and so on. If we denote hyperons and heavy mesons belonging to the group  $G_s$  as  $N_s$  and  $\Pi_s$ , then the even-odd rule is an inevitable consequence of the  $v$ -charge conservation law.

In Pais' original rule<sup>1)</sup>, decays such as

$$N_2 \rightarrow N_0 + \pi_0 \quad (9)$$

were not forbidden, but the  $v$ -charge conservation further imposes severe selection rules. The process (9) is forbidden due to the  $v$ -charge conservation.

Suppose that  $Y^0$  and  $Y^-$  are the hyperons belonging to  $G_2$  (i.e.  $N_2$ ), then the cascade decay

$$Y^- \rightarrow A^0 + \pi^-, \quad (A^0 \rightarrow p + \pi^-) \quad (10)$$

$$-2 \quad -1 \quad 0$$



is forbidden to occur rapidly so that we can observe the slow decay process (10).\*)

As has been shown in  $L$ , heavy mesons except for the member of  $G_0$  exhibit positive (or negative in general) excess. Hence it is tempting to identify  $K^{*+}$  as the long-lived  $K^{*7}$  and  $K^0$  as  $\theta^0$  since the long-lived  $K^*$  particles clearly show positive excess and the mass is close to that of  $\theta^0$ . Cloud chamber experiments show that positive and negative  $V^*$  particles are comparably abundant except for the long-lived  $K^*$  so that there might be heavy mesons being members of the group  $G_0$ .

Finally it will be instructive to remark that Pais'  $\omega$ -space theory<sup>8)</sup> leads to the multiplication of elementary particles within the group  $G_0$ .

- 1) A. Pais, Phys. Rev. **86** (1952), 663.
- 2) A. Pais, Physica, **19** (1953), 869; Prog. Theor. Phys. **10** (1953), 457.
- 3) M. Gell-Mann, Phys. Rev. **92** (1953), 833.
- 4) T. Nakano and K. Nishijima, Prog. Theor. Phys. **10** (1953), 581.
- 5) See, for instance, E. W. Cowan, Phys. Rev. **94** (1954), 161.
- 6) Fowler et al., Phys. Rev. **91** (1953), 1237.
- 7) Gregory et al., Nuovo Cim. **11** (1954), 292.
- 8) See ref. 2.

## Damping Effect in the Gamma-decay of a Neutral Pion

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In connection with the problem of pion-nucleon scattering, Brueckner, Gell-Mann and Goldberger<sup>1)</sup>

\* If we further assume that  $\Delta\eta=0, \pm 1$  for weak decay interactions,  $V^0$  will decay through  $V^0 \rightarrow \Lambda^0 + \pi^0$ , so that  $V^0$  will not be observable since another observable decay mode  $V^0 \rightarrow \rho + \pi^- (\Delta\eta=2)$  is highly forbidden. This assumption is an analogue of the selection rule  $\Delta I=0, \pm 1$  for electromagnetic interactions.

found in the pseudoscalar meson theory with pseudoscalar coupling that the virtual nucleon pair formation is strongly damped by the inertia of the self-field of a nucleon. Consequently, at low energies, the  $s$  state meson scattering which occurs through the virtual nucleon pair formation is weakened and the  $p$  state scattering without the pair formation is relatively enhanced since the latter is almost undamped.

This effect leads to a better qualitative understanding of the nature of pions in terms of the pseudoscalar meson theory with pseudoscalar coupling. Further it is also the case for the nuclear forces.<sup>2)</sup>

Bearing in mind of these successes we shall estimate this effect on the gamma-decay of a neutral pion. The motivation of this work is for the following reasons:

(a) In pion reactions involving real nucleons such as pion-nucleon scattering and nuclear forces, the transition amplitudes are partially damped. On the other hand, the gamma-decay of a neutral pion will be totally damped since this process cannot take place without the virtual nucleon pair formation.

The life-time of a neutral pion calculated by the lowest order perturbation theory is given by

$$\tau = 0.7 \times 10^{-15} (g^2/4\pi\hbar c)^{-1} \text{ sec.}, \quad (1)$$

where  $g^2/4\pi\hbar c$  is the coupling constant for the pion-nucleon interaction. If we adopt the value  $g^2/4\pi\hbar c = 10 \sim 20$  deduced from other reactions, the formula (1) gives a too short life-time compared with the experimental ones, i.e.  $10^{-14} \sim 10^{-15}$  sec.<sup>3)</sup> We may anticipate that the inertial effect will remedy this discrepancy.

(b) Similarly, the matrix elements for

$$\theta^0 \rightarrow \pi^+ + \pi^-, \quad \tau^+ \rightarrow \pi^+ + \pi^+ + \pi^-$$

would also be damped, if the decays would take place through the virtual nucleon pair formation.<sup>4)</sup>

The calculation was made as follows:

First we replace the nucleon propagator  $S_F$  by the modified function  $S_F'$  the concrete expression of which has already been given by several authors.<sup>5)</sup> Such a partial insertion of higher order contributions generally violates the gauge invariance.<sup>6)</sup> However, it is not the case for the present process owing to the special situation that the only possible Lorentz invariant form of the transition matrix for  $\pi^0 \rightarrow 2\gamma$  is  $\phi EH$  which is already gauge invariant, i.e. Lorentz invariant calculations always lead to gauge invariant results for this special process.

We write the modified propagator in the form

$$S_{p'}(\not{p}) = 1/(\not{p}\gamma + M + \Sigma^*)$$

$$= 1/[(\not{p}\gamma + M)(1 + b) + M(1 + a)], \quad (3)$$

where  $a$  and  $b$  are the functions of  $\not{p}^2$  and do not involve  $\gamma$  matrices. By neglecting terms of order of  $(\mu/M)^2$  compared to unity, the problem is reduced to the evaluation of an integral of the form

$$M = \int (\not{p}) f(\not{p}^2, a, b). \quad (4)$$

The problem is still considerably complicated and we further approximate this expression by expanding the integrand around certain values of  $a = \langle a \rangle$  and  $b = \langle b \rangle$ .

$$M = \int (\not{p}) [f(\not{p}^2, \langle a \rangle, \langle b \rangle) + (a - \langle a \rangle) \frac{\partial}{\partial \langle a \rangle} f(\not{p}^2, \langle a \rangle, \langle b \rangle) + (b - \langle b \rangle) \frac{\partial}{\partial \langle b \rangle} f(\not{p}^2, \langle a \rangle, \langle b \rangle) + \dots]. \quad (5)$$

The values  $\langle a \rangle$  and  $\langle b \rangle$  are so chosen as

$$\int (\not{p}) (a - \langle a \rangle) \frac{\partial}{\partial \langle a \rangle} f = \int (\not{p}) (b - \langle b \rangle) \frac{\partial}{\partial \langle b \rangle} f = 0, \quad (6)$$

and  $M$  is approximately given by

$$M \approx \int (\not{p}) f(\not{p}^2, \langle a \rangle, \langle b \rangle). \quad (7)$$

In this way the corrected life-time for  $\pi^0 \rightarrow 2\gamma$  is given by

$$\tau = 1.2 \times 10^{-15} \text{ sec.} \quad (8)^*$$

for the value of the coupling constant  $g^2/4\pi\hbar c = 10$ .

From this result one can see that the correction is in the favourable direction and we may infer that the inertial damping effect leads us to a better qualitative understanding of the nature of pions not only for processes involving real nucleons but also for processes without real nucleons.

- 1) Brueckner et al., Phys. Rev. **90** (1953), 476.
- 2) K. A. Brueckner and K. M. Watson, Phys. Rev. **92** (1953), 1023.
- 3) B. M. Anand, Proc. Roy. Soc. A. **220** (1953), 183.
- 4) See, however, T. Kinoshita, to be published.
- 5) See ref. 1). Also see, Karplus et al., Phys. Rev. **90** (1953), 1072.
- 6) Koba et al., Prog. Theor. Phys. **6** (1951), 322.

\* However, this result cannot quantitatively trusted, since the approximations employed are crude.

## A Remark on Neutron-Proton Mass Difference

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July 14, 1954

Lately Feynman and Speisman have shown that there is a reasonable possibility to explain the neutron-proton mass difference out of the difference of their electromagnetic self energies calculated by the phenomenological introduction of a Pauli term representing the anomalous magnetic moment coupling<sup>1)</sup>. The calculations based on the same idea but involving the different cutoff techniques were carried out independently by us. As the comparison between Feynman's calculation and ours seems to serve as a reference to know how sensitively the cutoff techniques affect the results and if one may give some reasonable physical meanings to the cutoff factors, our calculations are briefly summarized here.

One of us (Y.O.) used a convergence factor  $[C(k)]^2 = [K^2/(k^2 + K^2)]^2$  cutting off only the photon propagation function  $1/k^2$  at energy  $K$ , from the viewpoint of uniqueness of a convergence factor<sup>2)</sup>.

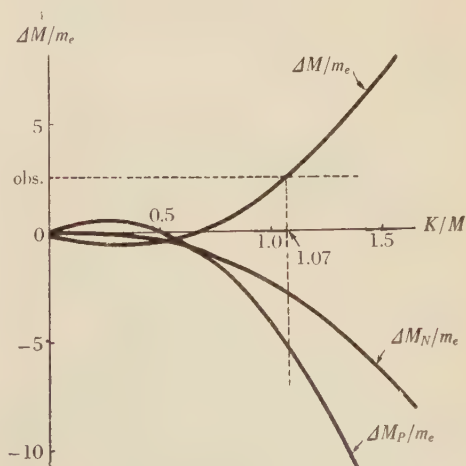


Fig. 1

The self energies are given as follows, corresponding to the interaction terms  $(\gamma_\mu \cdots \gamma_\mu)$ ,  $(\gamma_\mu \cdots \sigma_{\mu\nu})$  and  $(\sigma_{\lambda\mu} \cdots \sigma_{\mu\lambda})$ ,

$$\Delta M^{(ee)}/M = (1/8\pi)a[2\delta - \delta^2 \log \delta + 2(\delta^3 - 2\delta^2 + 4\delta)A],$$

$$\Delta M^{(e\mu)}/M = (3/8\pi)a\mu[-\delta - 2\delta^2 + (\delta^3 - 2\delta^2)\log \delta + 2(-\delta^4 + 4\delta^3 - 2\delta^2)A],$$

$$\Delta M^{(\mu\mu)}/M = (3/32\pi)a\mu^2[2\delta^2 - \delta^3 \log \delta + 2(\delta^4 - 2\delta^3 - 2\delta^2)A],$$

where  $M$  is the nucleon mass,  $a$  the fine structure constant,  $\mu$  the anomalous magnetic moment in nuclear magnetons ( $\mu_P = 1.79$ ,  $\mu_N = -1.91$ ),  $\delta = (K/M)^2$  and  $A = (1/\sqrt{4\delta - \delta^2})\cos^{-1}(\sqrt{\delta}/2)$  (for  $\delta < 4$ ),  $1/4$  (for  $\delta = 4$ ),  $(1/\sqrt{\delta^2 - 4\delta})\cosh^{-1}(\sqrt{\delta}/2)$  (for  $\delta > 4$ ). Fig. 1 shows proton self energy  $\Delta M_P = \Delta M_P^{(ee)} + \Delta M_P^{(e\mu)} + \Delta M_P^{(\mu\mu)}$ , the neutron self energy  $\Delta M_N = \Delta M_N^{(\mu\mu)}$ , and the neutron-proton mass difference  $\Delta M = M_N - M_P = \Delta M_N - \Delta M_P$  versus  $K/M = \sqrt{\delta}$ .

Another of us (H.K.) calculated the similar quantities using the straight cutoff procedure, after the observation of this possibility suggested through a rough estimation of classical self energies.<sup>3)</sup> The straight cutoff may be performed by means of a nonlocal form factor introduced by Kristensen and Møller.<sup>4)</sup> This result is<sup>5)</sup>

$$\Delta M^{(ee)}/M = (3/2\pi)a[\log(\lambda + \sqrt{1 + \lambda^2}) - a],$$

$$\Delta M^{(e\mu)}/M = -(3/4\pi)a\mu[\lambda\sqrt{1 + \lambda^2} - \log(\lambda + \sqrt{1 + \lambda^2})],$$

$$\Delta M^{(\mu\mu)}/M = -(1/32\pi)a\mu^2[21\lambda\sqrt{1 + \lambda^2} - 20 \log(\lambda + \sqrt{1 + \lambda^2}) - b],$$

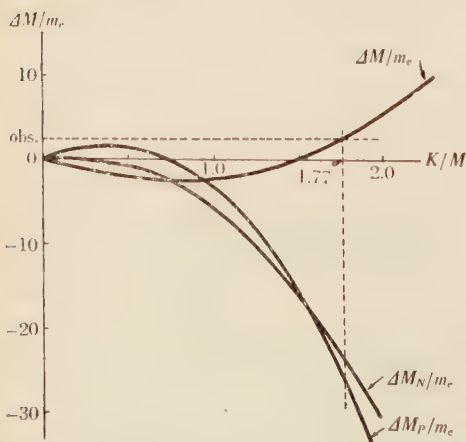


Fig. 2 shows  $\Delta M_P$ ,  $\Delta M_N$ , and  $\Delta M$  versus  $K/M = \lambda$  in this case.

where  $\lambda = K/M$ ,  $a = (1/3)\lambda(\sqrt{1 + \lambda^2} - \lambda)$  ( $a \rightarrow 1/6$ , when  $\lambda \rightarrow \infty$ ), and  $b = \sqrt{1 + \lambda^2}(\lambda + 6\lambda^3) - 6\lambda^4 - 4\lambda^3 \log[(1 + \sqrt{1 + \lambda^2})/\lambda]$  ( $b \rightarrow 5/12$ , when  $\lambda \rightarrow \infty$ ).

Concerning these results we may make some remarks in the following:

(i) We agree with Feynman and Speisman on the point that there is a possibility to explain the neutron-proton mass difference as the effect of a phenomenological Pauli moment by choosing of the cutoff momentum  $M \leq K \leq 2M$  irrespective of the details of the cutoff methods.

(ii) With regard to  $\Delta M_P$ ,  $\Delta M_N$ , which are unobservable quantities, our results have a different tendency from Feynman's one:

	$\Delta M_N/m_e$	$\Delta M_P/m_e$	$K/M$
Feynman and Speisman	1.0	-1.5	1.4
$[C(k)]^2$ -cutoff	-2.8	-5.3	1.07
straight cutoff	-23.6	-26.1	1.77
(classical bar magnet model <sup>3)</sup> )	(-10.1)	(-12.6)	( $r = 1.07 \times (\hbar/Mc)$ )

In our case,  $\Delta M^{(\mu\mu)}$  cannot be positive, whatever different cutoff value we may take. Therefore  $\Delta M_N$  is always negative, in a striking contrast to Feynman's result. In order to make clear this situation, one of us (Y.O.) has also examined  $\Delta M^{(\mu\mu)}$  using  $[C(k)]^3$ -cutoff which corresponds to the case  $A = \lambda$  in Feynman's calculation, but has not been able to obtain the positive sign. That the sign of  $\Delta M^{(\mu\mu)}$  as well as  $\Delta M^{(e\mu)}$  is negative seems to be natural from the classical self energy estimation of the bar magnet model. In case of the straight cutoff,  $\Delta M_N$  and  $\Delta M_P$  have large absolute values, which lead to the observed value of  $\Delta M$  as a small difference of large numbers. This may show that the straight cutoff is a crude method for convergence technique, though it gives the result having the similar qualitative tendency.

Some parts of Feynman and Speisman's discussion on their cutoff technique look like reasonable, but in view of many arbitrariness to determine the detail of the cutoff factor (moreover Feynman used the different cutoff factors for the photon propagation function in the case of nucleon mass difference and in the case of  $\pi$ -meson mass difference), it seems adventuresome to give the positive physical meanings special type of cutoff factors under existing conditions. We might not expect to derive something beyond the possibility stated in (i) from these phenomenological treatments.

Finally we wish to express our thanks to Dr. Y. Katayama for his kind advice.

1) R. P. Feynman and G. Speisman, Phys. Rev. **94** (1954), 500.

2) Y. Oishi, *Soryushiron-kenkyu* (in Japanese) **8** (1954), 975.

In this article, also the  $\pi$ -meson mass difference was calculated using the same cutoff factor  $[C(k)]^2$  without the approximation  $K \gg m$ . ( $m$  is the  $\pi$ -meson mass).

$\Delta m/m$

$$= (1/16\pi) a [4\delta + \delta^2 \log \delta + 2(-\delta^3 + 2\delta^2 + 8\delta) A]$$

where  $\delta = (K/m)^2$  and which gives the observed value at  $\delta = 34$  ( $K = 0.85M$ ).

3) H. Katsumori, Memoirs of Osaka Gakugei Univ. B, No. 2 (1953), 28.

4) P. Kristensen and C. Møller, Dan. Mat. Fys. Medd. **27** (1952), No. 7. In our case,  $\Pi^2$  (in the Kristensen and Møller's notation) should be modified because of the vanishing photon mass, unlike the case of neutral meson field. Using the modified  $\Pi^2$ , we may have  $\Pi^2 = \vec{k}^2$  in the rest system of the nucleon ( $\vec{k}$  means the three dimensional momentum vector of a virtual photon), and so we may assume the straight cutoff at  $|\vec{k}| = K$ .

5) H. Katsumori, *Soryushiron-kenkyu* (in Japanese) **8** (1954).

(1) normal electro-magnetic interaction, Thomson scattering,

(2) extra scattering due to mesonic effect,

(3) diffraction scattering caused by absorption (photo-meson production).

As is well-known, absorption of an incident beam is accompanied by some elastic scattering, the diffraction scattering. Thus the photo-meson production from the nucleon must cause the corresponding photon-nucleon scattering. If we assume the validity of perturbational expansion with respect to the elementary charge  $e$  and keep the lowest order terms (in our case  $e^4$ -terms), we can easily calculate processes (1) and (3) providing the cross section for photo-meson production is known. To show this, let us use the partial wave analysis. The photo-meson cross section due to the "dipole" photon absorption may be written as

$$\begin{aligned} d\sigma_{\text{prod}}/d\omega = & \frac{\lambda^2}{8} [|a|^2 + |b|^2 + \frac{1}{2}(|A|^2 + |B|^2) \\ & \times (2 + 3\sin^2\theta) - \text{Re}(A^*a + B^*b)(3\cos^2\theta - 1) \\ & + 2\text{Re}(A^*B + a^*b - A^*b - a^*B)\cos\theta], \end{aligned}$$

where \*

Re	complex conjugate,
$\theta$	real part,
$\theta$	angle between the incident photon and the produced meson,
$\lambda$	the photon wave length divided by $2\pi$ ,
$a, A, b$ and $B$	matrix elements of the $R$ -matrix, see Table I.

Thus the diffraction scattering caused by photo-meson production (again due to the "dipole" photon scattering) can readily be expressed as follows:

$$\begin{aligned} d\sigma_{\text{diff}}/d\omega = & \frac{\lambda^2}{32} [|a|^4 + |b|^4 + (|A|^4 + |B|^4) \frac{10 - 3\sin^2\theta}{4} \\ & + \frac{1}{4}(9|AB|^2 + 2|aA|^2 + 2|bB|^2)(3\cos^2\theta - 1) \\ & + \frac{1}{2}(|AB|^2 + 2|Ab|^2 + 2|aB|^2 + 4|ab|^2)\cos\theta], \end{aligned}$$

where  $\theta$  is the angle between the incident and scattered photon. A generalization to higher multipole case straightforward.

In terms of the  $R$ -matrix ( $R = S - 1 \approx 2i\delta$ , where  $S = e^{2i\delta}$  is Heisenberg's  $S$ -matrix), processes (1) and (2) correspond to the imaginary part of the  $R$ -matrix, process (3) to its real part. Therefore, the net photon-nucleon scattering will be the sum of  $d\sigma_{\text{diff}}/d\omega$  and (1) plus (2), because (1) and (2) interfere with each other but these two do not interfere with (3).

## A Note on Photon-nucleon Scattering

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Photon-nucleon scattering is a rather complicated problem which necessitates detailed examination based on specific meson theories. However, a part of the scattering cross section is so closely connected with photo-meson production that it can be calculated from the "experimental" cross section of photo-meson production. It seems worthwhile to call attention to this correlation.

We may classify the photon-nucleon scattering according to three different processes:



- 1) R. G. Sachs and L. L. Foldy, Phys. Rev. **80** (1950), 824 (there is a mistake in their calculation).

Y. Ichikawa, private communication.

H. M. Schey, unpublished.

- 2) Y. Yamaguchi, Lecture at the Summer School held at Kyoto University in July, 1952.

B. T. Feld, Phys. Rev. **89** (1952), 330 (I).

Table I

matrix element of the R-matrix for photo-meson production		total angular momentum	parity
$a$	electric dipole	1/2	odd
$A$	electric dipole	3/2	odd
$b$	magnetic dipole	1/2	even
$B$	magnetic dipole	3/2	even

# The Quantum-statistical Theory of Transport Phenomena, III

## — Coarse-grained Phase-space Distribution Functions —

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In order to introduce the coarse-grained measurement of coordinates in quantum mechanics, the whole space is divided into cubic cells, and the coordinate of the center of cell is regarded as the coarse-grained coordinate of every point inside the cell. The function which vanishes outside a certain cell and which agrees with plane wave within the same cell is called a cell function. If one imposes a certain type of boundary condition upon such cell functions, the set of these functions is complete and orthonormal, and gives a generalized type of the Wigner distribution functions, from which the coarse-grained distribution functions are derived. The change of the density matrices with time is calculated by means of the method analogous to the time-dependent perturbation theory. From this result the collision terms of the Uehling-Uhlenbeck equation are derived on the basis of the equation of motion for the generalized phase-space distribution function, under the assumption that the maximum range of the intermolecular force is sufficiently short compared with the edge length of elementary cell. Owing to the cell functions employed instead of the plane waves, the distribution functions retain dependence on the spatial coordinates irrespective of the postulate of random *a priori* phases, which plays an important role in derivation of the collision terms. In the present article, this postulate is necessary for derivation of the streaming terms, in which the differentiation operators with respect to the coarse-grained coordinates and the coarse-grained distribution functions appear. The domain of validity of the Uehling-Uhlenbeck equation is discussed, and, in conclusion, it seems doubtful that the Uehling-Uhlenbeck equation is valid even for the transport phenomena in the degenerate Bose gas, on account of the singularity of the distribution function in momentum space.

## § 1. Introduction

In the first paper of this series<sup>1)</sup>, QSTI, an attempt to deduce analytically the Uehling-Uhlenbeck equation from the first principles in quantum-statistical mechanics was made. But the distribution function (hereafter we will abbreviate this to d.f.) became independent of the spatial coordinates, if we adopt the postulate of random *a priori* phases, upon which the derivation of the collision terms was essentially based. Then, strictly speaking, this analytical method for deriving the Uehling-Uhlenbeck equation is valid only for uniform assembly. But, if the gaseous assembly is nearly uniform, a sufficiently small part of it may be considered as uniform. On the other hand, if the linear dimension of this part is sufficiently large compared with the average de Broglie wave length of particles in the assembly, the method of derivation of the Uehling-Uhlenbeck equation, developed in QSTI, is valid in this part. But, if we consider that the assembly consists of many parts of uniform density, the streaming terms vanish ex-

cept at the boundaries of two parts, where the d. f. s. are not differentiable. Thus, from such intuitive consideration, we can simply see that the Uehling-Uhlenbeck equation is valid in nearly uniform gases, but we cannot tell to what extent this equation is applicable to real systems,

The foregoing compels us to rederive the Uehling-Uhlenbeck equation by means of the method in which the spatial variation of the distribution function is properly taken into account. For this purpose we have either to abandon the postulate of random *a priori* phases, or to adopt the other orthonormal set of functions than the set of plane waves. Since we are not aware of any other suitable postulate to assign the definite values to the non-diagonal elements of the density matrix at the initial time, and since this postulate is in accordance with v. Neumann's theory of observation<sup>21</sup>, here we shall try to find out a new set of orthonormal functions which is adequate for our present purpose.

As mentioned in the first part of this section, we shall deal with the assembly of which small parts can be regarded as sufficiently uniform. This suggests us to divide the whole volume into cells in which the number density of particles is almost constant. The shape of the cells may be arbitrary, but, for simplicity's sake, we prefer the cubic cells, to which we shall refer as the elementary cells. And, in the present article, we confine ourselves to the case where the cells are sufficiently large as compared with the maximum range of intermolecular force,  $r_0$ , because the density variation becomes appreciable only in the domain sufficiently large compared with atomic scale in most cases which appear in the thermodynamical and hydrodynamical problems. And we shall call the function which is different from zero only within one of these cells by the name of cell function, and then consider a complete orthonormal set consisting of cell functions.

As stated in the second part of this series<sup>21</sup>, QSTII, if one adopts the other orthonormal sets than the set of plane waves, the modifications of the Wigner d. f. are more convenient than Wigner's original form<sup>1)</sup>. Mori<sup>22)</sup> derived the equation of motion for the generalized Wigner d. f., a generalization of the Irving-Zwanzig equation<sup>5)</sup>, of which a brief description is given in the present article (§ 2). And the generalized Wigner d. f. s. produced from a set of cell functions are related to the d. f. s. for coarse-grained coordinates and momenta (§ 3). There are, however, no unperturbed Hamiltonians whose eigenfunctions are cell functions, while the plane waves are the eigenfunctions of the total kinetic energy which is usually considered as the unperturbed Hamiltonian for a gaseous assembly of low density. But we can calculate the change of the density matrix with time from the method analogous to the time-dependent perturbation theory (§ 4), and the calculation of the generalized transition matrices for the case of the cell functions is performed (§ 5). By means of this result, the collision terms in the Uehling-Uhlenbeck equation is deduced for the case where the maximum range of intermolecular force is sufficiently short compared with the elementary cells (§ 6). In the present article, the streaming terms as well as the collision terms are derived on the basis of the postulate of random *a priori* phases (§ 7). Finally the domain of validity of Uehling-Uhlenbeck equation is discussed.

## § 2. A generalization of the Wigner d.f.

When we deal with a mixture of states of a structureless particle, it is convenient to use the density matrix  $\rho(\mathbf{x}, \mathbf{x}'; t)$ ,  $\mathbf{x}$  standing for a set of the Cartesian coordinates of the particle. And we shall consider an operator corresponding to an observable  $\mathbf{R}$ , of which eigenvalues are  $k_1, k_2, \dots$  which correspond to the eigenfunctions  $\varphi_{k1}(\mathbf{x})$ ,  $\varphi_{k2}(\mathbf{x})$ ,  $\dots$ . Now let us consider the function defined by

$$g(\mathbf{x}, k; t) = \int \varphi_k^* \left( \mathbf{x} - \frac{1}{2} \mathbf{y} \right) \varphi_k \left( \mathbf{x} + \frac{1}{2} \mathbf{y} \right) \rho \left( \mathbf{x} - \frac{1}{2} \mathbf{y}, \mathbf{x} + \frac{1}{2} \mathbf{y}; t \right) d\mathbf{y}. \quad (2.1)$$

If the set of  $\varphi_k(\mathbf{x})$ 's is complete and orthonormal, we have

$$\sum_k g(\mathbf{x}, k; t) = \rho(\mathbf{x}, \mathbf{x}; t), \quad (2.2)$$

which gives the probability density in configuration space. And the integration of  $g(\mathbf{x}, k; t)$  over  $\mathbf{x}$  gives

$$\int g(\mathbf{x}, k; t) d\mathbf{x} = \int \varphi_k^*(\mathbf{x}') \varphi_k(\mathbf{x}) \rho(\mathbf{x}', \mathbf{x}; t) d\mathbf{x} d\mathbf{x}' = \rho_{kk}(t), \quad (2.3)$$

which is the probability that the system be in the state  $\varphi_k(\mathbf{x})$ . Hence,  $g(\mathbf{x}, k; t)$  given by (2.1) may be regarded as the d.f. in  $\mathbf{x}-k$  space, although this is real but not necessarily positive.

If a plane wave function  $h^{-3/2} \exp\{-\mathbf{x} \cdot \mathbf{p}/i\hbar\}$  is chosen as  $\varphi_k(\mathbf{x})$  in (2.1), we obtain

$$f(\mathbf{x}, \mathbf{p}; t) = h^{-3} \int e^{-\mathbf{y} \cdot \mathbf{p}/i\hbar} \rho \left( \mathbf{x} - \frac{1}{2} \mathbf{y}, \mathbf{x} + \frac{1}{2} \mathbf{y}; t \right) d\mathbf{y}, \quad (2.4)$$

which is identical with the phase space d.f. first introduced by Wigner<sup>1)</sup>. Then we can say that  $g(\mathbf{x}, k; t)$  is a generalized form of Wigner's phase space d.f. This was already employed in the previous part, QSTII.

As seen from the above mentioned the Fourier transformation, which was extensively used by Moyal<sup>6)</sup> to derive the d.f.s. for non-commuting observables, has a special meaning only for the d.f.s. in the  $\mathbf{x}-\mathbf{p}$  space, because of the fact that the plane waves are the eigenfunctions of the momentum operator. It may be possible to use, for instance, the Hankel transformation instead of the Fourier transformation to derive the d.f.s. for non commuting observables.

Let us consider an assembly composed of  $N$  particles, and let us denote the set of coordinates of these  $N$  particles by  $X$ . And let  $\{\Psi_k(X)\}$  be an orthonormal set of wave functions of the assembly,  $k$  being a set of eigenvalues of a certain complete set of observables. Then, we can easily see from (2.1) that the generalized Wigner d.f. for an assembly of particles is given by

$$g^{(N)}(X, k; t) = \int \Psi_k^* \left( X - \frac{1}{2} Y \right) \Psi_k \left( X + \frac{1}{2} Y \right) \rho^{(N)} \left( X - \frac{1}{2} Y, X + \frac{1}{2} Y; t \right) dY, \quad (2.5)$$



where  $\rho^{(N)}(X, X'; t)$  is a density matrix of the assembly.

A complete orthonormal set of functions for an assembly of the  $N$  identical Bose particles may be given by the following functions:

$$\varphi_{n_1 n_2 \dots} = \det^{(+)} [\varphi_{t_1}(\mathbf{x}_1) \cdots \varphi_{t_N}(\mathbf{x}_N)], \quad (2.6)$$

where  $n_t$  is the number of particles in the state represented by  $\varphi_t(\mathbf{x})$ , and the set of  $n_t$ 's assumes sets of all non-negative values such that

$$\sum_t n_t = N, \quad (2.7)$$

and where  $\det^{(+)}$  indicates a permanent. An assembly of Fermi particles can be treated in the same way.

Let  $a_t$  and  $a_t^*$  be an annihilation and a creation operators, respectively, and the quantized wave functions introduced by Fock<sup>7)</sup> are

$$\left. \begin{aligned} \phi(\mathbf{x}) &= \sum a_t \varphi_t(\mathbf{x}) \\ \phi^*(\mathbf{x}) &= \sum a_t^* \varphi_t^*(\mathbf{x}) \end{aligned} \right\} \quad (2.8)$$

Then, as shown by Husimi and Nishiyama<sup>8)</sup>, the reduced density matrices can be written as

$$\begin{aligned} \rho^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{x}'_1, \dots, \mathbf{x}'_n; t) \\ = \text{trace}[\phi^*(\mathbf{x}'_1) \cdots \phi^*(\mathbf{x}'_n) \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n) \rho(t)], \end{aligned} \quad (2.9)$$

in which  $\rho$  is the statistical (or density) operator. The method of derivation of (2.9) is shown in QSTII. And the reduced d. f. s. are obtained from the reduced density matrices according to the relation

$$\begin{aligned} g^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n; k_1, \dots, k_n; t) \\ = \int \cdots \int \varphi_{k_1}^*(\mathbf{x}_1 - \frac{1}{2} \mathbf{y}_1) \cdots \varphi_{k_n}^*(\mathbf{x}_n - \frac{1}{2} \mathbf{y}_n) \varphi_{k_1}(\mathbf{x}_1 + \frac{1}{2} \mathbf{y}_1) \cdots \varphi_{k_n}(\mathbf{x}_n + \frac{1}{2} \mathbf{y}_n) \\ \cdot \rho^{(n)}(\mathbf{x}_1 - \frac{1}{2} \mathbf{y}_1, \dots, \mathbf{x}_n - \frac{1}{2} \mathbf{y}_n; \mathbf{x}_1 + \frac{1}{2} \mathbf{y}_1, \dots, \mathbf{x}_n + \frac{1}{2} \mathbf{y}_n; t) d\mathbf{y}_1 \cdots d\mathbf{y}_n. \end{aligned} \quad (2.10)$$

And we can prove the relation

$$\begin{aligned} g^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n; k_1, \dots, k_n; t) \\ = [N! / (N-n)!] \sum_{k_{n+1}}^1 \cdots \sum_{k_N}^1 \int \cdots \int g^{(N)}(\mathbf{x}_1, \dots, \mathbf{x}_N; k_1, \dots, k_N) d\mathbf{x}_{n+1} \cdots d\mathbf{x}_N. \end{aligned} \quad (2.11)$$

The temporal development of the quantized wave functions is determined by the quantized Schrödinger equation,

$$\begin{aligned} i\hbar \dot{\phi} &= \{ -(\hbar^2/2m) \nabla^2 + \phi(\mathbf{x}) + G(\mathbf{x}) \} \phi \\ G(\mathbf{x}) &= \int d\bar{\mathbf{x}} \phi^*(\bar{\mathbf{x}}) u(\mathbf{x}, \bar{\mathbf{x}}) \phi(\bar{\mathbf{x}}), \end{aligned} \quad (2.12)$$

where  $\phi$  and  $u$  are the potential energy of the external field and the intermolecular potential, respectively. According to (2.9), (2.10) and (2.12), we obtain the equation of motion, as shown in QSTII,

$$\begin{aligned}
 & \partial g(\mathbf{x}, \mathbf{k}; t) / \partial t \\
 & - (\hbar/mi) \int d\mathbf{y} \varphi_k^* \left( \mathbf{x} - \frac{1}{2} \mathbf{y} \right) \varphi_k \left( \mathbf{x} + \frac{1}{2} \mathbf{y} \right) \Gamma_i \Gamma_{yi} \rho \left( \mathbf{x} - \frac{1}{2} \mathbf{y}, \mathbf{x} + \frac{1}{2} \mathbf{y}; t \right) \\
 & + (\hbar/mi) \int d\mathbf{y} \varphi_k^* \left( \mathbf{x} - \frac{1}{2} \mathbf{y} \right) \varphi_k \left( \mathbf{x} + \frac{1}{2} \mathbf{y} \right) \left\{ \phi \left( \mathbf{x} - \frac{1}{2} \mathbf{y} \right) - \phi \left( \mathbf{x} + \frac{1}{2} \mathbf{y} \right) \right\} \\
 & \quad \cdot \rho \left( \mathbf{x} - \frac{1}{2} \mathbf{y}, \mathbf{x} + \frac{1}{2} \mathbf{y}; t \right) \\
 & = (1/i\hbar) \int d\mathbf{y} \int d\bar{\mathbf{y}} \left\{ u \left( \mathbf{x} - \frac{1}{2} \mathbf{y}, \bar{\mathbf{y}} \right) - u \left( \mathbf{x} + \frac{1}{2} \mathbf{y}, \bar{\mathbf{y}} \right) \right\} \varphi_k^* \left( \mathbf{x} - \frac{1}{2} \mathbf{y} \right) \varphi_k \left( \mathbf{x} + \frac{1}{2} \mathbf{y} \right) \\
 & \quad \cdot \rho^{(2)} \left( \mathbf{x} - \frac{1}{2} \mathbf{y}, \bar{\mathbf{y}}; \mathbf{x} + \frac{1}{2} \mathbf{y}, \bar{\mathbf{y}}; t \right), \quad (2.13)
 \end{aligned}$$

which is a generalization of the Irving-Zwanzig equation<sup>5)</sup>

### § 3. The cell method and the coarse-grained d.f.s.

The macroscopic measurement of the coordinate can be performed at the same time as the macroscopic measurement of its conjugate momentum, while the accuracy in a measurement of momentum is limited by the accuracy in a simultaneous measurement of position. Then, for the quantum-mechanical formulation of macroscopic measurements, we divide the whole space into cells, and regard the coordinate of the center of a cell as the macroscopic coordinate of every point inside the cell. We cannot, however, divide the momentum space in the similar cells at the same time, in contrast with the classical case.

Here, let us divide the positional space into cubic cells whose edge length is  $a$  in such a way that their centers constitute a simple cubic array. And we adopt the complete orthonormal set of the functions

$$\varphi(\mathbf{X}, \mathbf{P}; \mathbf{x}) = a^{-3/2} e^{-i\mathbf{x} \cdot \mathbf{P}/\hbar} E(\mathbf{X}, \mathbf{x}), \quad (3.1)$$

with  $\mathbf{P} = \hbar \mathbf{k}/a$  and  $\mathbf{X} = \mathbf{l}a$ , both  $\mathbf{k}$  and  $\mathbf{l}$  being the vectors with integer components. And  $E(\mathbf{X}, \mathbf{x})$  is so-called characteristic function such as

$$E(\mathbf{X}, \mathbf{x}) = \begin{cases} 1, & \text{if all of components of } \mathbf{X} - \mathbf{x} \text{ lie between } -a/2 \text{ and } a/2 \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

In other words,  $E(\mathbf{X}, \mathbf{x})$  is unity within the cubic cell of which the center is located at  $\mathbf{X}$ , and vanishes outside the cell. Then,  $\mathbf{X}$  and  $\mathbf{P}$  may be regarded as the coarse-grained (or macroscopic) coordinate and momentum, respectively. In this case, the generalized phase space d.f. defined as (2.1) becomes

$$g(\mathbf{x}, \mathbf{P}, \mathbf{X}; t) = a^{-3} \int e^{-\mathbf{y} \cdot \mathbf{P} / i\hbar} E\left(\mathbf{X}, \mathbf{x} + \frac{1}{2}\mathbf{y}\right) E\left(\mathbf{X}, \mathbf{x} - \frac{1}{2}\mathbf{y}\right) \rho\left(\mathbf{x} - \frac{1}{2}\mathbf{y}, \mathbf{x} + \frac{1}{2}\mathbf{y}; t\right). \quad (3.3)$$

And the d. f. in the coarse-grained phase space is given by

$$f(\mathbf{X}, \mathbf{P}; t) = N h^{-3} \int g(\mathbf{x}, \mathbf{P}, \mathbf{X}; t) d\mathbf{x}, \quad (3.4)$$

which is normalized to  $N$ ; and here each set of  $\mathbf{X}$  and  $\mathbf{P}$  corresponds to the phase volume  $h^3$ .

The equation of motion for  $g(\mathbf{x}, \mathbf{P}, \mathbf{X}; t)$  defined as (3.3) becomes, according to (2.13),

$$\begin{aligned} \partial g(\mathbf{x}, \mathbf{P}, \mathbf{X}; t) / \partial t - (\hbar / a^3 m i) \int d\mathbf{y} e^{-\mathbf{x} \cdot \mathbf{P} / i\hbar} \\ \cdot E\left(\mathbf{X}, \mathbf{x} + \frac{1}{2}\mathbf{y}\right) E\left(\mathbf{X}, \mathbf{x} - \frac{1}{2}\mathbf{y}\right) \nabla_x \nabla_y \rho\left(\mathbf{x} - \frac{1}{2}\mathbf{y}, \mathbf{x} + \frac{1}{2}\mathbf{y}; t\right) \\ = - (i / a^3 \hbar) \int d\mathbf{y} e^{-\mathbf{x} \cdot \mathbf{P} / i\hbar} E\left(\mathbf{X}, \mathbf{x} + \frac{1}{2}\mathbf{y}\right) E\left(\mathbf{X}, \mathbf{x} - \frac{1}{2}\mathbf{y}\right) \\ \cdot \int d\bar{\mathbf{y}} \left\{ u\left(\mathbf{x} - \frac{1}{2}\mathbf{y}, \bar{\mathbf{y}}\right) - u\left(\mathbf{x} + \frac{1}{2}\mathbf{y}, \bar{\mathbf{y}}\right) \right\} \rho^{(2)}\left(\mathbf{x} - \frac{1}{2}\mathbf{y}, \bar{\mathbf{y}}; \mathbf{x} + \frac{1}{2}\mathbf{y}, \bar{\mathbf{y}}; t\right). \end{aligned} \quad (3.5)$$

A symmetrized function is given by

$$\Phi(\kappa, \mathbf{X}) = \det^{(+)} [\varphi(\mathbf{X}_1, \mathbf{P}_1; \mathbf{x}_1) \cdots \varphi(\mathbf{X}_N, \mathbf{P}_N; \mathbf{x}_N)], \quad (3.6)$$

where  $\kappa$  stands for a set of  $u(\mathbf{X}, \mathbf{P})$ 's,  $u(\mathbf{X}, \mathbf{P})$  being the number of particles whose coarse-grained position and momentum are  $\mathbf{X}$  and  $\mathbf{P}$ , respectively.

The  $N$ -representation of the density operator is given by

$$\rho^{(N)}(\kappa, \kappa'; t) = \iint \Phi^*(\kappa, X) \rho^{(N)}(X, X'; t) \Phi(\kappa', X') dX dX'. \quad (3.7)$$

If the  $X$ -representation of the density operator is given in the form

$$\rho^{(N)}(X, X'; t) = \sum_k w_k \Psi_k^*(X') \Psi_k(X), \quad (3.8)$$

the  $N$ -representation may be written as

$$\rho^{(N)}(\kappa, \kappa'; t) = \sum_k w_k C_k^*(\kappa', t) C_k(\kappa, t), \quad (3.9)$$

where  $C_k(\kappa, t)$  is the coefficient of the expansion in the series

$$\Psi_k(X) = \sum_k C_k(k, t) \Phi(k, X). \quad (3.10)$$

# § 4. The change of the density matrix with time

In order to derive the Uehling-Uhlenbeck equation from (3.5), we shall carry out the calculation of the temporal development of the density matrix, by means of the method analogous to Dirac's time-dependent perturbation theory, which was employed in QSTI, and we shall choose the system of non-interacting particles as the unperturbed system, of which the Hamiltonian will be denoted by  $H_0$ . The functions given by (3.6) are, however, not the eigenfunctions of  $H_0$ , contrary to the case of plane waves used in QSTI.

The unperturbed Hamiltonian  $H_0$  is given by

$$H_0 = -(\hbar^2/2m) \sum_{i=1}^2 \nabla_i^2. \quad (4.1)$$

And we introduce the function defined by

$$\Phi_\tau(\kappa, X) = e^{H_0\tau/\hbar} \Phi(\kappa, X) \quad (4.2)$$

where  $\Phi_\tau(\kappa, X)$ 's form a complete set of orthonormal functions.

Let us consider other orthonormal set of functions  $\Psi_j(X, t)$ . Then, the functions  $\Psi(X_j, t + \tau)$  form an orthonormal set, and are given by

$$\Psi_j(X, t + \tau) = e^{H\tau/\hbar} \Psi_j(X, t), \quad (4.3)$$

where  $H$  is the true Hamiltonian including the perturbation,

$$H = H_0 + U(X), \quad (4.4)$$

$U(X)$  being the potential energy of the system. We expand  $\Psi_j(X, t + \tau)$  in the following series :

$$\Psi_j(X, t + \tau) = \sum_{\kappa} C_j(\kappa, \tau) \Phi_\tau(\kappa, X). \quad (4.5)$$

Differentiating both sides of (4.5) with respect to  $\tau$ , we obtain

$$\frac{\partial \Psi_j(X, t + \tau)}{\partial \tau} = \sum_{\kappa} \frac{dC_j(\kappa, \tau)}{d\tau} \Phi_\tau(\kappa, X) + \sum_{\kappa} C_j(\kappa, \tau) \frac{\partial \Phi_\tau(\kappa, X)}{\partial \tau}. \quad (4.6)$$

Inserting (4.2) and (4.3) in (4.6), and using (4.4) and (4.5) we have

$$i\hbar \sum_{\kappa} \frac{dC_j(\kappa, \tau)}{d\tau} \Phi_\tau(\kappa, X) = \sum_{\kappa} C_j(\kappa, \tau) U(X) \Phi_\tau(\kappa, X). \quad (4.7)$$

Since the functions  $\Phi_\tau(\kappa, X)$  are orthogonal, we immediately obtain

$$i\hbar \frac{dC_j(\kappa, \tau)}{d\tau} = \sum_{\kappa' \neq \kappa} \langle \kappa | U | \kappa' \rangle_\tau C_j(\kappa', \tau), \quad (4.8)$$

where

$$\langle \kappa | U | \kappa' \rangle_\tau = \int \Phi_\tau^*(\kappa', X) U(X) \Phi_\tau(\kappa, X) dX, \quad (4.9)$$

As in Dirac's theory of time-dependent perturbation, regarding  $C_j(\kappa', \tau)$ 's in the right hand side of (4.8) as constants and integrating both sides of (4.8) with respect to  $\tau$ ,



we obtain

$$C_j(\kappa, \tau) = C_j(\kappa, 0) - i\hbar \sum_{\kappa \neq \kappa'} C_j(\kappa', 0) \int_0^\tau \langle \kappa' | U | \kappa \rangle_\tau d\tau. \quad (4.10)$$

Then we observe that the probability that the system which is found in the state  $\phi_\tau(\kappa, X)$  at the time  $t$  will be found in the state  $\phi_\tau(\kappa', X)$  at the time  $t+\tau$  is given by  $\hbar^{-2} |\int_0^\tau \langle \kappa' | U | \kappa \rangle_\tau d\tau|^2$ , which is the transition probability discussed by Feynman in more general form<sup>9)</sup>.

Now let  $\varphi_\tau(X, P; x)$  be a function which agrees with  $\varphi(X, P; x)$  defined by (3.1) when  $\tau=0$  and which obeys the equation

$$-(\hbar^2/2m)P^2\varphi_\tau(X, P; x) = i\hbar\partial\varphi_\tau(X, P; x)/\partial\tau. \quad (4.11)$$

Then from (3.6) and (4.1),  $\phi_\tau(\kappa, X)$  given by (4.2) may be expressed in the form

$$\phi_\tau(\kappa, X) = \det^{(+)}[\varphi_\tau(X_1, P_1; x_1) \cdots \varphi_\tau(X_N, P_N; x_N)]. \quad (4.12)$$

As an  $N$ -representation of the density matrix  $\rho^{(N)}(X, X'; t+\tau)$ , we may take, according to (3.9) and (4.5),

$$\rho^{(N)}(\kappa, \kappa'; t+\tau) = \sum_k \omega_k C_k^*(\kappa', \tau) C_k(\kappa, \tau). \quad (4.13)$$

If the postulate of the random *a priori* phases is realized at the time  $t$ , we obtain, from (4.10) and (4.13),

$$\begin{aligned} \rho^{(N)}(\kappa, \kappa'; t+\tau) &= \rho^{(N)}(\kappa, \kappa; t) \delta_{\kappa\kappa'} \\ &- i\hbar \left\{ \rho^{(N)}(\kappa', \kappa'; t) - \rho^{(N)}(\kappa, \kappa; t) \right\} \int \langle \kappa | U | \kappa' \rangle_\tau d\tau. \end{aligned} \quad (4.14)$$

Adopting  $\varphi_\tau(X, P; x)$  as  $\varphi_t(x)$  in (2.6), we have, from (2.9),

$$\begin{aligned} \rho^{(2)}(x_1, x_2; x_1', x_2'; t+\tau) &= S\varphi_\tau^*(X_1', P_1'; x_1')\varphi_\tau^*(X_2', P_2'; x_2') \\ &\cdot \varphi_\tau(X_1, P_1; x_1)\varphi_\tau(X_2, P_2; x_2) \text{trace} [\alpha^*(X_1', P_1')\alpha^*(X_2', P_2') \\ &\cdot \alpha(X_1, P_1)\alpha(X_2, P_2)]\rho^{(N)}(t+\tau), \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} \rho(x, x'; t+\tau) &= S\varphi_\tau^*(X', P'; x')\varphi_\tau(X, P; x) \\ &\cdot \text{trace} [\alpha^*(X', P')\alpha(X, P)]\rho^{(N)}(t+\tau), \end{aligned} \quad (4.16)$$

for the doublet and singlet density matrices, respectively, where  $S$  indicates the summation over such variables that  $X, X', P$  and  $P'$ , and where  $\alpha(X, P)$  and  $\alpha^*(X, P)$  are an annihilation and a creation operators, respectively.

## § 5. Calculation of the elements of the generalized transition matrices

For the purpose of finding out the temporal behavior of the density matrix, we have at first to calculate the matrix element  $\langle \kappa | U | \kappa' \rangle_\tau$ , as seen from (4.14), (4.15) and (4.16). Here we shall assume that the potential energy may be written as the sum

of terms,  $u(\mathbf{x}_i, \mathbf{x}_j)$ , which represents the interaction between the  $i$ 'th and  $j$ 'th particles :

$$U(X) = \sum_{i>j} u(\mathbf{x}_i, \mathbf{x}_j). \quad (5.1)$$

Making use of (4.2), we obtain, from the definition (4.9),

$$\begin{aligned} \langle \kappa | U | \kappa' \rangle_\tau = & \frac{1}{2} S [n(\mathbf{X}, \mathbf{P}) \{n(\mathbf{Y}, \mathbf{Q}) - \delta(\mathbf{X}, \mathbf{Y}) \delta(\mathbf{P}, \mathbf{Q})\} \\ & \cdot \{n(\mathbf{X}', \mathbf{P}') + 1 - \delta(\mathbf{X}, \mathbf{X}') \delta(\mathbf{P}, \mathbf{P}') - \delta(\mathbf{X}', \mathbf{Y}) \delta(\mathbf{P}', \mathbf{Q})\} \\ & \cdot \{n(\mathbf{Y}', \mathbf{Q}') + 1 - \delta(\mathbf{X}, \mathbf{Y}') \delta(\mathbf{P}, \mathbf{Q}') - \delta(\mathbf{Y}, \mathbf{Y}') \delta(\mathbf{Q}, \mathbf{Q}') \\ & + \delta(\mathbf{X}', \mathbf{Y}') \delta(\mathbf{P}', \mathbf{Q}')\}]^{1/2} \langle \mathbf{X}\mathbf{P}, \mathbf{Y}\mathbf{Q} | u | \mathbf{X}'\mathbf{P}', \mathbf{Y}'\mathbf{Q}' \rangle_\tau \delta \{n(\mathbf{X}, \mathbf{P}) \\ & + \delta(\mathbf{X}, \mathbf{X}') \delta(\mathbf{P}, \mathbf{P}') - 1, n'(\mathbf{X}, \mathbf{P})\} \delta \{n(\mathbf{Y}, \mathbf{Q}) + \delta(\mathbf{Y}, \mathbf{Y}') \delta(\mathbf{Q}, \mathbf{Q}') \\ & - 1, n'(\mathbf{Y}, \mathbf{Q})\} \delta \{n(\mathbf{X}', \mathbf{P}') + \delta(\mathbf{X}, \mathbf{X}') \delta(\mathbf{P}, \mathbf{P}') + 1, n'(\mathbf{X}', \mathbf{P}')\} \\ & \cdot \delta \{n(\mathbf{Y}', \mathbf{Q}') - \delta(\mathbf{Y}, \mathbf{Y}') \delta(\mathbf{Q}, \mathbf{Q}') + 1, n'(\mathbf{Y}', \mathbf{Q}')\}, \end{aligned} \quad (5.2)$$

where

$$\begin{aligned} \langle \mathbf{X}\mathbf{P}, \mathbf{Y}\mathbf{Q} | u | \mathbf{X}'\mathbf{P}', \mathbf{Y}'\mathbf{Q}' \rangle_\tau = & \iint \varphi_\tau^*(\mathbf{X}, \mathbf{P}; \mathbf{x}_1) \varphi_\tau(\mathbf{Y}, \mathbf{Q}; \mathbf{x}_2) \\ & \cdot u(\mathbf{x}_1, \mathbf{x}_2) \varphi_\tau(\mathbf{X}', \mathbf{P}'; \mathbf{x}_1) \varphi_\tau(\mathbf{Y}', \mathbf{Q}'; \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2. \end{aligned} \quad (5.3)$$

$\delta(\mathbf{X}, \mathbf{X}')$  and  $\delta(n, n')$  are Kronecker's  $\delta$  in somewhat extended sense.  $\varphi_\tau(\mathbf{X}, \mathbf{P}; \mathbf{x})$  defined as (4.11) can be expressed in the form

$$\varphi_\tau(\mathbf{X}, \mathbf{P}; \mathbf{x}) = h^{-3/2} \int_{-\infty}^{\infty} \sigma(\mathbf{X}, \mathbf{P}; \mathbf{p}) \exp \left\{ -\frac{\mathbf{x} \cdot \mathbf{p}}{i\hbar} + \frac{\mathbf{p}^2 \tau}{2im\hbar} \right\} d\mathbf{p}, \quad (5.4)$$

where

$$\sigma(\mathbf{X}, \mathbf{P}; \mathbf{p}) = h^{-3/2} \int_{-\infty}^{\infty} \varphi(\mathbf{X}, \mathbf{P}; \mathbf{x}) e^{i\mathbf{x} \cdot \mathbf{p}/\hbar} d\mathbf{x}. \quad (5.5)$$

Substitution of (3.1) into (5.5) leads to

$$\sigma(\mathbf{X}, \mathbf{P}; \mathbf{p}) = a^{-3/2} h^{3/2} \prod_{i=1}^3 \frac{\sin \alpha(P_i - p_i)/i\hbar}{(P_i - p_i)} e^{i\mathbf{x} \cdot \mathbf{p}/i\hbar}, \quad (5.5)$$

where suffices 1, 2, and 3 stand for  $x$ -,  $y$ - and  $z$ - components, respectively.  $|\sigma(\mathbf{X}, \mathbf{P}; \mathbf{p})|$  has a maximum at  $\mathbf{p} = \mathbf{P}$ . This function may be treated as a function having a single maximum, since the magnitude of the other maxima are sufficiently small compared with the one at  $\mathbf{P} = \mathbf{p}$ . And this function becomes sufficiently small as  $|\mathbf{P} - \mathbf{p}|$  increases. Then, since (5.4) is of the form of wave packet, it can be written as

$$\begin{aligned} \varphi_\tau(\mathbf{X}, \mathbf{P}; \mathbf{x}) = & h^{-3/2} \exp \{ -\mathbf{x} \cdot \mathbf{P}/i\hbar + \omega^2 \tau / 2im\hbar \} \\ & \cdot \int_{-\infty}^{\infty} \sigma(\mathbf{X}, \mathbf{P}; \mathbf{P} + \omega) \exp \left\{ -\frac{\boldsymbol{\eta} \cdot \omega}{i\hbar} + \frac{\omega^2 \tau}{2im\hbar} \right\} d\omega \end{aligned} \quad (5.7)$$

according to the manipulation as shown in Kemble's text-book<sup>10)</sup>. Here  $\eta_p$  denotes  $\mathbf{x} - (\mathbf{P}/m)\tau$ , and we shall restrict the discussion to the values of  $\tau$  which are not too large. As seen from (5.6),  $\sigma$  decreases rapidly as soon as any one of  $\omega_i$ 's becomes the order of magnitude of  $\hbar/a$ . Hence, if  $P_i \gg \omega_i$ , the factor  $\omega^2\tau/2im\hbar$  is negligible, and consequently if  $P_i \gg \hbar/a$  ( $i=1, 2, 3$ ), (5.7) becomes

$$\varphi_\tau(\mathbf{X}, \mathbf{P}; \mathbf{x}) = \hbar^{-3/2} \exp \left\{ -\frac{\mathbf{x} \cdot \mathbf{P}}{i\hbar} + \frac{\mathbf{P}^2\tau}{2im\hbar} \right\} \int_{-\infty}^{\infty} \sigma(\mathbf{X}, \mathbf{P}; \mathbf{P} + \boldsymbol{\omega}) e^{-i\eta_p \cdot \boldsymbol{\omega}/i\hbar} d\boldsymbol{\omega}, \quad (5.8)$$

which implies that the wave packet propagates with the constant group velocity  $\mathbf{P}/m$ .

If  $a$  is sufficiently larger than the maximum range of the intermolecular force  $r_0$ , and if the time interval  $\tau$  is sufficiently shorter than  $ma/P_i$  ( $i=1, 2, 3$ ), the time interval necessary for a particle with the momentum  $\mathbf{P}$  to pass through a cell, then (5.3) becomes

$$\left. \begin{aligned} \langle \mathbf{XP}, \mathbf{YQ} | u | \mathbf{X}'\mathbf{P}', \mathbf{Y}'\mathbf{Q}' \rangle_\tau &= a^{-3} \delta(\mathbf{X}, \mathbf{X}') \delta(\mathbf{Y}, \mathbf{Y}') \delta(\mathbf{X}, \mathbf{Y}') \delta(\mathbf{P} - \mathbf{P}', \mathbf{Q} - \mathbf{Q}') \\ &\cdot \exp \left[ \{ \epsilon(\mathbf{P}') + \epsilon(\mathbf{Q}') - \epsilon(\mathbf{P}) - \epsilon(\mathbf{Q}) \} / i\hbar \right] I(|\mathbf{P} - \mathbf{P}'|), \\ &\epsilon(\mathbf{P}) = \mathbf{P}^2/2m, \\ I(p) &= 4\pi \int_0^\infty [\sin(pr/\hbar) / (pr/\hbar)] u(r) r^2 dr. \end{aligned} \right\} \quad (5.9)$$

## § 6. Calculation of the collision terms

Integrating (3.5) over the coordinate  $\mathbf{x}$ , and using (3.1), (3.2) and (3.4), we obtain

$$\begin{aligned} \hbar^3 \partial f(\mathbf{X}, \mathbf{P}; t) / \partial t &= (\hbar/2mi) \iint d\mathbf{x} d\mathbf{x}' \varphi(\mathbf{X}, \mathbf{P}; \mathbf{x}) \varphi^*(\mathbf{X}, \mathbf{P}; \mathbf{x}') \\ &\cdot \{ \mathbf{p}_{x'}^2 - \mathbf{p}_x^2 \} \rho(\mathbf{x}, \mathbf{x}'; t) d\mathbf{x} d\mathbf{x}' \\ &= - (i/\hbar) \iint d\mathbf{x} d\mathbf{x}' \varphi(\mathbf{X}, \mathbf{P}; \mathbf{x}') \varphi^*(\mathbf{X}, \mathbf{P}; \mathbf{x}) \\ &\cdot d\mathbf{y} \{ u(\mathbf{x}, \mathbf{y}) - u(\mathbf{x}', \mathbf{y}) \} \rho^{(2)}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}; t). \end{aligned} \quad (6.1)$$

The time-average of the both sides of (6.1) over the interval  $\tau$  becomes

$$\begin{aligned} (\hbar^3/\tau) \{ f(\mathbf{X}, \mathbf{P}; t+\tau) - f(\mathbf{X}, \mathbf{P}; t) \} &= (\hbar/2mi) \iint d\mathbf{x} d\mathbf{x}' \\ &\cdot \varphi(\mathbf{X}, \mathbf{P}; \mathbf{x}') \varphi^*(\mathbf{X}, \mathbf{P}; \mathbf{x}) \{ \mathbf{p}_{x'}^2 - \mathbf{p}_x^2 \} \bar{\rho}(\mathbf{x}, \mathbf{x}'; t) \\ &= - (i/\hbar) \iint d\mathbf{x} d\mathbf{x}' \varphi(\mathbf{X}, \mathbf{P}; \mathbf{x}') \varphi^*(\mathbf{X}, \mathbf{P}; \mathbf{x}) \\ &\cdot \int d\mathbf{y} \{ u(\mathbf{x}, \mathbf{y}) - u(\mathbf{x}', \mathbf{y}) \} \bar{\rho}^{(2)}(\mathbf{x}', \mathbf{y}; \mathbf{x}', \mathbf{y}; t), \end{aligned} \quad (6.2)$$

where the bar indicates the time-average between  $t$  and  $t + \tau$ . Integrating (4.15) with respect to  $\tau$  from 0 to  $\tau$ , and using (4.14), (5.2), (5.3) and (5.9), we obtain, according to the properties of the creation and annihilation operators,

$$\bar{\rho}^{(2)}(x_1, x_2; x'_1, x'_2; t) = S\bar{\rho}^{(2)}(X'_1 P'_1, X'_2 P'_2; X_1 P_1, X_2 P_2; t) \cdot \varphi^*_{\tau}(X'_1, P'_1; x'_1) \varphi^*_{\tau}(X'_2, P'_2; x'_2) \varphi(X_1, P_1; x_1) \varphi(X_2, P_2; x_2), \quad (6.3)$$

where

$$\begin{aligned} \bar{\rho}^{(2)}(X'_1 P'_1, X'_2 P'_2; X_1 P_1, X_2 P_2; t) &= (i\hbar/a^3\tau) \sum_{\kappa} \{ \omega[\epsilon(P'_1) + \epsilon(P'_2) - \epsilon(P_1) - \epsilon(P_2)] \} \\ &\cdot \{ \rho^{(\Lambda)}(\kappa, \kappa; t) - \rho^{(\Lambda)}(\dots, n(X'_1, P'_1) + 1, n(X'_2, P'_2) + 1, \dots; \\ &\dots n(X_1, P_1) - 1, n(X_2, P_2) - 1, \dots; t) \delta(X_1, X'_1) \delta(X_2, X'_2) \delta(X_1, X_2) \\ &\cdot \delta(P_1 - P'_1, P_2 - P'_2) \} \{ I(|P_2 - P'_2|) + I(|P_1 - P'_1|) \}, \end{aligned} \quad (6.4)$$

and

$$\omega(x) = \frac{1 - e^{-x\tau/i\hbar}}{x^2} - \frac{\tau}{i\hbar x}. \quad (6.5)$$

The quantities of the order of magnitude of  $\tau$  are neglected, and the method of derivation of the above equation is exactly same that used in derivation of (4.11) of QSTI.

Let us consider that the interval  $\tau$  is so short that we may neglect the quantities which are at least proportional to  $\tau$ . Then we can replace  $\varphi_{\tau}(X, P; x)$  in  $\bar{\rho}_2$  by  $\varphi(X, P; x)$ , and obtain

$$\begin{aligned} & - (i/\hbar) \iint dxdx' \varphi(X, P; x') \varphi^*(X, P; x) \\ & \cdot \int d\mathbf{y} \{ u(x, \mathbf{y}) - u(x', \mathbf{y}) \} \bar{\rho}^{(2)}(x, \mathbf{y}; x', \mathbf{y}; t) \\ & = (i/\hbar) S\bar{\rho}^{(2)}(X'_1 P'_1, X'_2 P'_2; X_1 P_1, X_2 P_2; t) \delta(X'_1, X) \delta(X'_2, X_2) \\ & \cdot \delta(X_1, X) \delta(P, P_1) \iint dxd\mathbf{y} \varphi^*(X, P'_1; x) \varphi^*(X_2, P'_2; \mathbf{y}) \\ & \cdot \varphi(X, P_1; x) \varphi(X_2, P_2; \mathbf{y}) u(x, \mathbf{y}) + \text{c.c.}, \end{aligned} \quad (6.6)$$

If  $r_0 \ll a$ , and if  $u(x, \mathbf{y})$  depends only on the difference  $x - \mathbf{y}$ , we have

$$\begin{aligned} & \iint dxd\mathbf{y} \varphi^*(X, P'_1; x) \varphi^*(X_2, P'_2; \mathbf{y}) \varphi(X_1, P_1; x) \varphi(X_2, P_2; \mathbf{y}) u(x, \mathbf{y}) \\ & = a^3 \delta(P_1 - P'_1, P_2 - P'_2) \delta(X, X_2) I(|P_2 - P'_2|). \end{aligned} \quad (6.7)$$

Substitution of (6.4) and (6.7) into (6.6) leads to

$$\begin{aligned} & - (i/\hbar) \iint dxdx' \varphi(X, P; x') \varphi^*(X, P; x) \int d\mathbf{y} \{ u(x, \mathbf{y}) - u(x', \mathbf{y}) \} \\ & \cdot \bar{\rho}^{(2)}(x, \mathbf{y}; x', \mathbf{y}; t) \end{aligned}$$



$$\begin{aligned}
&= a^{-6} \sum_{\mathbf{K}} \sum_{\mathbf{P}, \mathbf{P}'} n(\mathbf{X}, \mathbf{P}_1) n(\mathbf{X}, \mathbf{P}_2) \{n(\mathbf{X}, \mathbf{P}) + 1\} \{n(\mathbf{X}, \mathbf{P}') + 1\} \\
&\quad - n(\mathbf{X}, \mathbf{P}) n(\mathbf{X}, \mathbf{P}') \{n(\mathbf{X}, \mathbf{P}_1) + 1\} \{n(\mathbf{X}, \mathbf{P}_2) + 1\} \{I(|\mathbf{P}_2 - \mathbf{P}'|) \\
&\quad + I(|\mathbf{P}_1 - \mathbf{P}'|)\}^2 [\mathcal{Q}\{\epsilon(\mathbf{P}) + \epsilon(\mathbf{P}') - \epsilon(\mathbf{P}_1) - \epsilon(\mathbf{P}_2)\} / \tau] \rho(\kappa, \kappa; t) \quad (6.8)
\end{aligned}$$

where  $\mathbf{P}_2$  is given by  $\mathbf{P}_2 = \mathbf{P} + \mathbf{P}' - \mathbf{P}_1$ , and where

$$\mathcal{Q}(x) = \omega(x) + \omega^*(x) = (\tau/\hbar) \sin^2(x\tau/2\hbar) / (x\tau/2\hbar)^2, \quad (6.9)$$

which is twice the real part of (6.5).

Assuming  $a$  to be sufficiently large, we shall replace the summation with respect to  $\mathbf{P}_1$  and  $\mathbf{P}'$  by the integration over  $\mathbf{P}_1$  and  $\mathbf{P}'$ . It is necessary to multiply the factor  $a^3/\hbar^3$  because there are  $a^3/\hbar^3 d\mathbf{P}$  states in the range  $d\mathbf{P}$ . After such replacement, (6.8) becomes

$$\begin{aligned}
&- (i/\hbar) \iint d\mathbf{x} d\mathbf{x}' \varphi(\mathbf{X}, \mathbf{P}; \mathbf{x}') \varphi^*(\mathbf{X}, \mathbf{P}; \mathbf{x}) \int d\mathbf{y} \{u(\mathbf{x}, \mathbf{y}) - u(\mathbf{x}', \mathbf{y})\} \\
&\quad \cdot \bar{p}^{(2)}(\mathbf{x}, \mathbf{y}; \mathbf{x}', \mathbf{y}; t) \\
&= \hbar^{-6} \sum_{\mathbf{x}} \int d\mathbf{P}' \int d\omega [n(\mathbf{X}, \mathbf{P}_1) n(\mathbf{X}, \mathbf{P}_2) \{n(\mathbf{X}, \mathbf{P}) + 1\} \{n(\mathbf{X}, \mathbf{P}') + 1\} \\
&\quad [\mathbf{P} + \mathbf{P}' = \mathbf{P}_1 + \mathbf{P}_2, \mathbf{P}' - \mathbf{P} = \frac{1}{2}\mathbf{P}_1 - \mathbf{P}' - \mathbf{P}] \\
&\quad - n(\mathbf{X}, \mathbf{P}) n(\mathbf{X}, \mathbf{P}') \{n(\mathbf{X}, \mathbf{P}_1) + 1\} \times \\
&\quad \{n(\mathbf{X}, \mathbf{P}_2) + 1\}] \alpha(|\mathbf{P} - \mathbf{P}'|, \theta) \rho(\kappa, \kappa; t), \quad (6.10)
\end{aligned}$$

where

$$\alpha(p, \theta) = (m\pi/4\hbar) \left\{ I\left(p \sin \frac{1}{2}\theta\right) + I\left(p \cos \frac{1}{2}\theta\right) \right\}^2, \quad (6.11)$$

$\theta$  being the angle of deflection and  $d\omega$  the solid angle. If this deflection is given,  $\mathbf{P}$  and  $\mathbf{P}'$  are determined from  $\mathbf{P}_1$  and  $\mathbf{P}_2$ . It is assumed in derivation of (6.1) that the maximum of  $\mathcal{Q}(x)$  at  $x=0$  is so sharp that  $\sum_{\mathbf{x}} [n(\mathbf{X}, \mathbf{P}_1) \cdot n(\mathbf{X}, \mathbf{P}_2) \{n(\mathbf{X}, \mathbf{P}) + 1\} \{n(\mathbf{X}, \mathbf{P}') + 1\} - n(\mathbf{X}, \mathbf{P}) n(\mathbf{X}, \mathbf{P}') \{n(\mathbf{X}, \mathbf{P}_1) + 1\} \{n(\mathbf{X}, \mathbf{P}_2) + 1\}] \rho(\kappa, \kappa; t)$  may be regarded as a constant. This assumption cannot be justified for the collision between a molecule and a zero molecule in the condensed Bose-Einstein gas.

## § 7. Derivation of the Uehling-Uhlenbeck equation

As seen from the mentioned in the preceding section, we may immediately obtain the collision terms of the Uehling-Uhlenbeck equation from the right-hand side of (6.2). And let us deal with the second term of the left-hand side of the same equation. From (4.11) we obtain the relation

$$\begin{aligned}
&\varphi_{\tau}(\mathbf{X}, \mathbf{P}; \mathbf{x}) \Gamma_{\tau}^2 \varphi_{\tau}(\mathbf{X}', \mathbf{P}'; \mathbf{x}') - \varphi_{\tau}^*(\mathbf{X}', \mathbf{P}'; \mathbf{x}') \Gamma_{\tau}^2 \varphi(\mathbf{X}, \mathbf{P}; \mathbf{x}) \\
&= \frac{2mi}{\hbar} \frac{\partial}{\partial \tau} \{\varphi_{\tau}^*(\mathbf{X}', \mathbf{P}'; \mathbf{x}') \varphi_{\tau}(\mathbf{X}, \mathbf{P}; \mathbf{x})\}. \quad (7.1)
\end{aligned}$$

According to (4.14) and (4.16), we obtain

$$(\nabla_{x'}^2 - \nabla_x^2) \bar{\rho}(x, x'; t) = (2m\hbar^3 / \hbar\tau) S f((X, P; t) \cdot [\varphi_\tau^*(X, P; x') \varphi_\tau(X, P; x) - \varphi^*(X, P; x') \varphi(X, P; x)], \quad (7.2)$$

the terms of the order of magnitude of  $\tau$  being neglected. And here we used the following relation :

$$\hbar^3 f(X, P; t) = \sum_{\kappa} n(X, P) \rho^{(\kappa)}(\kappa, \kappa; t), \quad (7.3)$$

which can be obtained from (3.3), (3.4) and (4.16). If we use (7.2), the second term of the right-hand side of (6.2) becomes

$$\begin{aligned} & -(\hbar/2m) \int \int d\mathbf{x} d\mathbf{x}' \varphi(X, P; x') \varphi^*(X, P; x) \{ \nabla_{x'}^2 - \nabla_x^2 \} \bar{\rho}(x, x'; t) \quad (7.4) \\ & = -(\hbar^3/\tau) S f(X', P'; t) \{ | \int \varphi^*(X, P; x) \varphi_\tau(X', P'; x) d\mathbf{x} |^2 \\ & \quad - \delta(X, X') \delta(P, P') \}. \end{aligned} \quad (7.4)$$

Since this term vanishes if one replaces  $\varphi_\tau(X, P; x)$  by  $\varphi(X, P; x)$ , we use (5.8), in contrast with the case for the collision terms. After somewhat tedious but elementary calculations, we obtain

$$| \int \varphi^*(X, P; x) \varphi_\tau(X', P'; x) d\mathbf{x} |^2 = \prod_{i=1}^3 | J_i(P_i, P'_i; X'_i - X_i; \tau) |^2, \quad (7.5)$$

where

$$\begin{aligned} & J_i(P_i, P'_i; X'_i - X_i; \tau) \\ & = \begin{cases} 1 - \left| \frac{X'_i - X_i}{a} + \frac{P_i \tau}{ma} \right|, & \text{if } P_i = P'_i, \left| (X'_i - X_i) + \frac{P'_i \tau}{m} \right| \leq a \\ \frac{\hbar}{\pi a} \frac{\sin[\pi P'_i (P'_i - P_i) \tau / m \hbar]}{P'_i - P_i}, & \text{if } P_i \neq P'_i, \left| (X'_i - X_i) + \frac{P'_i \tau}{m} \right| \leq a \\ 0, & \text{if } \left| (X'_i - X_i) + \frac{P'_i \tau}{m} \right| \geq a. \end{cases} \end{aligned}$$

For sufficiently short interval  $\tau$  to satisfy the condition  $\tau \ll ma/P_i$ ,  $J_i(P_i, P'_i; X'_i - X_i)$  vanishes unless  $X'_i - X_i$  is 0 or  $\pm a$ . And, neglecting the terms of the order of magnitude of  $\tau^3$  we have

$$\begin{aligned} & S f(X', P'; t) \{ | \int \varphi^*(X, P; x) \varphi_\tau(X', P'; x) d\mathbf{x} |^2 - \delta(X, X') \delta(P, P') \} \\ & = - \sum_{i=1}^3 \left\{ f(X, P; t) \left( 2 \frac{|P_i| \tau}{m\hbar} + \frac{P_i^2 \tau^2}{m^2 a^2} \right) - S f_{P'} \left( X - \frac{a P_i e_i}{|P_i|}, P + e_i (P'_i - P_i), t \right) \right. \\ & \quad \left. \cdot \frac{\sin^2 \{ \pi P'_i (P'_i - P_i) \tau / m \hbar \}}{(P'_i - P_i)^2} \right\} \end{aligned}$$

$\mathbf{e}_i$  being the unit vector of the  $i$ -direction. Since  $\sin^2\{\pi P'_i(P'_i - P_i)\tau/m\hbar\}/(P'_i - P_i)^2$  has a sharp maximum at  $P_i = P'_i$ , one may replace  $P'_i$  and  $P'_i(P'_i - P_i)$  in the other factors by  $P_i$  and  $P_i(P'_i - P_i)$ , respectively. Then replacing the summation over  $P'_i$  by the integration, and using the relation  $\int_{-\infty}^{\infty} \{\sin^2 qx/x^2\} dx = |\pi|$ , we have

$$\begin{aligned} & \sum_{\mathbf{X}', \mathbf{P}'} f(\mathbf{X}', \mathbf{P}'; t) \left\{ \left| \int \varphi^*(\mathbf{X}, \mathbf{P}; \mathbf{x}) \varphi_{\tau}(\mathbf{X}', \mathbf{P}'; \mathbf{x}) d\mathbf{x} \right|^2 - \partial(\mathbf{X}, \mathbf{X}') \partial(\mathbf{P}, \mathbf{P}') \right\} \\ &= \sum_{i=1}^3 \frac{|P_i| \tau}{ma} \left\{ f\left(\mathbf{X} - \frac{aP_i \mathbf{e}_i}{|P_i|}, \mathbf{P}; t\right) - f(\mathbf{X}, \mathbf{P}; t) \right\}, \end{aligned} \quad (7.6)$$

only terms up to the order of magnitude of  $\tau$  being retained.

Differentiation operator with respect to the macroscopic coordinate  $\mathbf{X}$  may be defined by

$$\frac{\partial f(\mathbf{X}, \mathbf{P}; t)}{\partial X_i} = \pm \frac{1}{a} \{ f(\mathbf{X}, \mathbf{P}; t) - (\mathbf{X} \pm a\mathbf{e}_i, \mathbf{P}; t) \} \quad (7.7)$$

However, one cannot define this differentiation operator, when two forms of the above expression give the different results. Since  $P_i/|P_i|$  assumes the values  $\pm 1$ , we can rewrite (7.6) in the form

$$\begin{aligned} & \sum_{\mathbf{X}', \mathbf{P}'} f(\mathbf{X}', \mathbf{P}'; t) \left\{ \left| \int \varphi^*(\mathbf{X}, \mathbf{P}; \mathbf{x}) \varphi_{\tau}(\mathbf{X}', \mathbf{P}'; \mathbf{x}) d\mathbf{x} \right|^2 - \partial(\mathbf{X}, \mathbf{X}') \partial(\mathbf{P}, \mathbf{P}') \right\} \\ &= -\tau \sum_i \frac{P_i}{m} \frac{\partial}{\partial X_i} f(\mathbf{X}, \mathbf{P}; t). \end{aligned} \quad (7.8)$$

On the other hand, the differentiation of the d. f. with respect to the macroscopic time may be defined as

$$\frac{1}{\tau} \{ f(\mathbf{X}, \mathbf{P}; t+\tau) - f(\mathbf{X}, \mathbf{P}; t) \} = \frac{\partial \bar{f}(\mathbf{X}, \mathbf{P}; t)}{\partial t}. \quad (7.9)$$

If one utilizes (6.10), (7.4), (7.8) and (7.9), (6.2) can be cast into the following form:

$$\begin{aligned} & \frac{\partial \bar{f}(\mathbf{X}, \mathbf{P}; t)}{\partial t} + \frac{\mathbf{P}}{m} \cdot \nabla_{\mathbf{X}} f(\mathbf{X}, \mathbf{P}; t) = \hbar^{-1} \sum_{\mathbf{x}} \int d\mathbf{P}' \int d\omega [n(\mathbf{X}, \mathbf{P}_1) n(\mathbf{X}, \mathbf{P}_2) \\ & \quad [ \mathbf{P} + \mathbf{P}' = \mathbf{P}_1 + \mathbf{P}_2, |\mathbf{P}' - \mathbf{P}| = |2\mathbf{P}_1 - \mathbf{P}' - \mathbf{P}| ] \\ & \quad \cdot \{ n(\mathbf{X}, \mathbf{P}) + 1 \} \{ n(\mathbf{X}, \mathbf{P}') + 1 \} - n(\mathbf{X}, \mathbf{P}) n(\mathbf{X}, \mathbf{P}')] \\ & \quad \cdot \{ n(\mathbf{X}, \mathbf{P}_1) + 1 \} \{ n(\mathbf{X}, \mathbf{P}_2) + 1 \} \alpha(|\mathbf{P} - \mathbf{P}'|, \theta) \rho^{(N)}(\kappa, \kappa; t). \end{aligned} \quad (7.10)$$

As in Kirkwood's theory<sup>(11)</sup> on the transport properties of the classical assemblies, if the assumption of molecular chaos is fulfilled for the time average of the d. f. s., we have, according to (7.3)\*,

\* cf. Eq. (5.10) of QSTI.

$$\begin{aligned} & \hbar^6 \bar{f}(\mathbf{X}, \mathbf{P}_1; t) \bar{f}(\mathbf{X}, \mathbf{P}_2; t) \{ \hbar^3 \bar{f}(\mathbf{X}, \mathbf{P}; t) + 1 \} \{ \hbar^3 \bar{f}(\mathbf{X}, \mathbf{P}'; t) + 1 \} \\ & = \sum_{\kappa} n(\mathbf{X}, \mathbf{P}_1) n(\mathbf{X}, \mathbf{P}_2) \{ n(\mathbf{X}, \mathbf{P}) + 1 \} \{ n(\mathbf{X}, \mathbf{P}') + 1 \} \rho^{(\Lambda)}(\kappa, \kappa; t). \end{aligned} \quad (7 \cdot 11)$$

After applying the second time-averaging procedure, as in Kirkwood's theory, to (7·10), we obtain the Uehling-Uhlenbeck equation:

$$\begin{aligned} & \frac{\partial f(\mathbf{X}, \mathbf{P}; t)}{\partial t} + \frac{\mathbf{P}}{m} \nabla_{\mathbf{X}} f(\mathbf{X}, \mathbf{P}; t) = \hbar^{-3} \int d\mathbf{P}' \int d\omega [ \bar{f}(\mathbf{X}, \mathbf{P}_1; t) \bar{f}(\mathbf{X}, \mathbf{P}_2; t) \\ & \quad [ \mathbf{P} + \mathbf{P}' = \mathbf{P}_1 + \mathbf{P}_2, \mathbf{P}^2 + \mathbf{P}'^2 = \mathbf{P}_1^2 + \mathbf{P}_2^2 ] \\ & \quad \cdot \{ \hbar^3 \bar{f}(\mathbf{X}, \mathbf{P}; t) + 1 \} \{ \hbar^3 \bar{f}(\mathbf{X}, \mathbf{P}'; t) + 1 \} - \bar{f}(\mathbf{X}, \mathbf{P}; t) \bar{f}(\mathbf{X}, \mathbf{P}'; t) \\ & \quad \cdot \{ \hbar^3 \bar{f}(\mathbf{X}, \mathbf{P}_1; t) + 1 \} \{ \hbar^3 \bar{f}(\mathbf{X}, \mathbf{P}_2; t) + 1 \} ] \alpha(|\mathbf{P} - \mathbf{P}'|, \theta). \end{aligned} \quad (7 \cdot 12)$$

## § 8. Conclusion

It is seen from the discussions in the present article that Uehling-Uhlenbeck equation can be derived analytically from the principles of quantum statistical mechanics under certain restrictive conditions. Among these conditions the one that the maximum range of intermolecular force is sufficiently short compared with the edge length of the cell,  $a$ , is certainly satisfied in the ordinary transport phenomena in fluids. And the condition that most particles have such momenta that  $P_i \gg \hbar/a$  implies that the average de Broglie wave length is sufficiently short compared with  $a$ , i. e.  $kT \gg \hbar^2/a^2m$ . As is known, the diffraction effects become important when the average de Broglie wave length exceeds the maximum range of intermolecular force,  $r_0$ , and the symmetry effects become important when the average de Broglie wave length exceeds the average distance between the molecules in the gas. Hence, if we assume that  $a$  is the order of magnitude of  $100\text{\AA}$ , the second condition may be fulfilled even at considerably low temperature, until the Bose-Einstein condensation begins to take place.

In a condensed Bose gas, the present method of derivation of the Uehling-Uhlenbeck equation becomes, invalid, not only because of extreme growth of the average de Broglie wave length, which possibly becomes as large as the linear dimension of the vessel, but because of the following important fact. In derivation of (7·10),  $\sum_{\kappa} [n(\mathbf{X}, \mathbf{P}_1) n(\mathbf{X}, \mathbf{P}_2) \{ n(\mathbf{X}, \mathbf{P}) + 1 \} \{ n(\mathbf{X}, \mathbf{P}') + 1 \} - n(\mathbf{X}, \mathbf{P}) n(\mathbf{X}, \mathbf{P}') \{ n(\mathbf{X}, \mathbf{P}_1) + 1 \} \{ n(\mathbf{X}, \mathbf{P}_2) + 1 \} \cdot \rho^{(\Lambda)}(\kappa, \kappa; t)]$  was approximately considered as a constant, but in a condensed Bose gas, this factor is of the order of magnitude of the total number of particles if at least any one of  $\mathbf{P}, \mathbf{P}', \mathbf{P}_1$ , and  $\mathbf{P}_2$  is zero, whereas this factor falls to the normal value as far as  $\mathbf{P}, \mathbf{P}', \mathbf{P}_1$  and  $\mathbf{P}_2$  are all different from zero. Therefore, Ishihara's treatment<sup>12)</sup>, in which the collision with the zero atoms are introduced in the Uehling-Uhlenbeck equation, does not seem to be justified, and the manipulation to introduce the singularity of  $\delta$ -function type in d. f. s., as done by Nakajima<sup>13)</sup>, does not seem to be self-evident.

As far as we wish to derive the super-fluidity of liquid helium II, based upon the quantum-statistical mechanics, we are obliged to deal with the collisions with the zero



atoms. It does not, however, seem to be all concerning the problem of superfluidity, since abnormal growth of the de Broglie wave length may play some role in this phenomena.

The similar situation may be true of the Fermi particles at immediate neighborhood of the Fermi surface. But the number of the particles which do not satisfy the requirement for derivation of the Uehling-Uhlenbeck equation is far smaller than the number of the zero atoms in the condensed Bose gas.

It seems noteworthy that the postulate of random *a priori* phases plays a important role not only in derivation of the collision terms but in derivation of the streaming terms. This postulate is, however, unattractive, but is indispensable to the present stage of quantum-statistical theory of transport and irreversible processes.

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## On Equivalent Observers, II

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This is a continuation of the previous paper of the same title.<sup>1)</sup> From the view-point that our theory provides a new method of introducing the Lorentz transformation, some new kinds of E. O. are propounded. It is shown that one of these kinds characterizes the Lorentz transformation and another the Galilei transformation as their admissible relative transformations respectively. Some considerations on these results are made. Lastly this theory is applied to the E. O. of one dimensional space frame.

### § 1. Introduction

Recently the present writer and Ueno have proposed a theory concerning equivalent observers<sup>1)</sup>. The present paper is a continuation of it. Since detailed discussions were made in [ 1 ] concerning the importance of the concept of the equivalency of the observers, the analysis of the characteristic properties of E. O. (abbreviation of equivalent observers), the relation between E. O. and physical laws, etc., we shall not touch on these problems here. However, it is only with the connection between our research and the Lorentz transformation that we shall here concern ourselves.

Various investigations have been made by many authors concerning the introduction of the Lorentz transformation. For example, we have the researches of Einstein,<sup>2)</sup> Pauli<sup>3)</sup>, Ignatowski<sup>3)</sup>, Iwatsuki-Mimura<sup>4)</sup>, etc., and, recently, of Stiegler,<sup>5)</sup> Ueno,<sup>6)</sup> etc. In the investigations hitherto made the linearity of transformation plays an important rôle in introducing the Lorentz transformation. In [ 1 ], however, the transformation was introduced in quite a different way. First we assumed the form of the group of the frame transformations  $\mathcal{G}_0$  suitably. Then, we have, on the basis of the theory of the continuous group of transformations, introduced the Lorentz transformation from the condition that the *relative transformations*, that is the transformations connecting E. O. to each other,\* form a group  $\mathcal{G}$  together with  $\mathcal{G}_0$ . By similar methods Galilei transformation, the transformation connecting the nebulae obtained in wave geometry, etc. were also obtained in [ 1 ]. In this sense, we may say, our theory provides a new method of introducing the Lorentz transformation.

However, in [ 1 ], we could not introduce the E. O. characterizing the Lorentz trans-

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\* In other words a relative transformation is nothing but the transformation stated in (b) of [1],

formation as the unique relative transformation. We shall explain this more in detail: In [1] we have defined two kinds of E.O., i.e., the E.O. of the first kind and the E.O. of the second kind. The former admits as its relative transformations the Galilei transformation, the Lorentz transformation and the transformation between two observers whose relative velocity is proportional to their relative distance, and the latter also admits various transformations as well as the Lorentz transformation. It is difficult, in fact, to exclude the Galilei transformation in introducing the Lorentz transformation since the former is obtained as a limiting case of the latter. But as for the other transformations the above result is unsatisfactory from our standpoint in that our theory may be considered as providing a new method of introducing the Lorentz transformation. In this paper, taking such a circumstance into consideration, we shall introduce another four kinds of E.O. First we shall show that in order to introduce the Lorentz transformation it is most natural to deal with the E.O. of the third kind defined in § 2, and shall make some considerations about this result. Then we shall further introduce the remaining three kinds, of which the E.O. of the fourth kind characterizes the Galilei transformation as the unique relative transformation. Lastly we shall apply our theory to the case of one dimensional space frame and shall show that the dimension of the space frame has serious effects on the property of the E.O.

## § 2. E.O. of the third kind

First we shall write down the postulates for our E.O. given in [1]:

(P. I) Any E.O.  $M$  is furnished with a three dimensional space frame  $S$  with origin  $M$  and a one dimensional time frame  $T$ , and can give one and only one set of space coordinates  $(x, y, z)$  and time coordinate  $(t)$  to any point event  $E$  to within frame transformations.

(P. II) Any observer  $M$  can observe all other E.O. which are moving relative to  $M$ .

(P. III) The group of frame transformations  $(S)_0$  forms a continuous group of transformations together with relative transformations.

(P. IV) If  $M$  and  $M'$  are any two E.O. they make radial motions relative to each other and, furthermore, if  $M$  observes any point event  $E$  on the straight line  $MM'$ , then  $M'$  also observes the event  $E$  on the straight line  $M'M$  independent of each time coordinate  $t$  and  $t'$ .

Of these postulates, (P. I), (P. II) and (P. IV) are the same as those given in [1], but, for convenience' sake we have slightly modified the expression of (P. III) using the term 'relative transformations'.

The E.O. of the first and the second kind treated in [1] are the E.O. whose groups of frame transformations are given by

$$(R_1, R_2, R_3; T_1, T_2, T_3; U) \text{ and } (R_1, R_2, R_3) \quad (2.1)$$

respectively, where  $R_a, T_a$ , ( $a=1, 2, 3$ ), and  $U$ , the same notations as in [1], give the rotations of space-frame, the translations of space frame, and the translation of time frame

respectively. Now we shall define the E. O. of the third kind by the one whose  $\mathfrak{G}_0$  is given by

$$\mathfrak{G}_0: R_1, R_2, R_3; T_1, T_2, T_3; U; D \quad (2.2)$$

where

$$D = x\partial_x + y\partial_y + z\partial_z + t\partial_t \quad (2.3)$$

gives the dilatation of the space-time frame.

Hence the only difference between the E. O. of the third kind and that of the first kind lies in the fact that former group of the frame transformations contains  $D$  while the latter group does not contain  $D$ . For (2.2) it holds that

$$\begin{aligned} (R_a, R_b) &= \epsilon^{abc} R_c, \quad (T_a, T_b) = (R_a, U) = (T_a, U) = (R_a, D) = 0, \\ (R_a, T_b) &= \epsilon^{ab} T_c, \quad (T_a, D) = T_a, \quad (U, D) = U, \quad (a, b = 1, 2, 3). \end{aligned} \quad (2.4)$$

As is easily seen, the special Lorentz transformation forms a group together with the above  $\mathfrak{G}_0$  and the same holds for the Galilei transformation  $G$ . Now we shall consider the converse problem: Is there any relative transformation connecting the E. O. of the third kind besides these two transformations? The answer is given by the following:

*The most general relative transformation between two E. O. of the third kind is given by the following two types to within the frame transformation.*

[a]  $L$ , i. e.  $P_1 = t\partial_x + mx\partial_t$ , cycl.,\* ( $m$  being a const.). In this case it holds that

$$\begin{cases} (P_a, P_b) = m\epsilon^{abc} R_c, & (R_a, P_b) = \epsilon^{abc} P_c, & (U, P_a) = T_a, \\ (T_a, P_b) = m\delta_{ab} U, & (D, P_a) = 0. \end{cases} \quad (2.5)$$

[b]  $G$ , i. e.  $P_1 = t\partial_x$ , cycl., and it holds that

$$(P_a, P_b) = (T_a, P_b) = (D, P_a) = 0, \quad (R_a, P_b) = \epsilon^{abc} P_c, \quad (U, P_a) = T_a, \quad (2.6)$$

which is obtained by putting  $m=0$  in (2.5).

Here we have used the symbol  $L$  especially to denote the special Lorentz transformation when the physical meaning of its parameter is not under consideration. Accordingly,  $L$  and the special Lorentz transformation are quite the same mathematically. To prove the above we have only to establish the condition which is necessary and sufficient in order that the eleven transformations  $R_a, T_a, U, D$  and  $P_a$  may form a group, and then to solve it, as was done in [1]. The proof being somewhat long we shall only give its summary in the Appendix (1).

From this result we know that the admissible relative transformations of this kind of E. O. are given by  $L$  and  $G$ . On the other hand, we have seen in [1] that a relative transformation  $K$  of the form

\* The notation 'cycl.' means the cyclic changes of  $x, y, z$  and the corresponding changes of 1, 2, 3.



$$K: e^{mt}\partial_x, \text{ cycl., } (m \text{ being a const. } \neq 0), \quad (2.7)$$

can exist besides  $G$  and  $L$  for the E. O. of the first kind and that the relative velocity of the observers connected by this  $K$  is proportional to their relative distance. Now from the above result we know that this  $K$  cannot survive as an admissible relative transformation if we add the dilatation  $D$  to the  $\mathfrak{G}_0$  of the E. O. of the first kind. Later in § 4, we shall determine the form of the group of frame transformations by which we can remove  $L$  further and obtain  $G$  as the unique relative transformation.

### § 3 Discussion of the result obtained in § 2

In this section we shall make some considerations about the result obtained in the preceding section. If we should succeed in introducing  $L$  by some method, then, at the same time,  $G$  would also be introduced automatically as a limiting case of  $L$  unless we put some additional condition on the constant contained in  $L$ . This fact, together with the result obtained in the previous section, will suggest that it is most natural to adopt the E. O. of the third kind to introduce  $L$ . In the following we shall consider the physical meanings of this result.

One of the guiding principles of special relativity is the invariance of the physical laws under the Lorentz group. The full Lorentz group is given by

$$\mathfrak{G}_L: R_a, T_a, U, L_a, (\alpha=1, 2, 3), \quad (3.1)$$

where  $L_a$  ( $\alpha=1, 2, 3$ ), denote the special Lorentz transformations in the directions of  $x, y$  and  $z$  axes respectively, and hence, mathematically, they are quite equivalent to  $L$ 's in the respective directions. The invariance for  $D$ , the dilatation of the space-time frame, is not requested in general. For example, the equation for the meson field

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - 1/c^2 \cdot \partial^2/\partial t^2 - \kappa)\psi = 0 \quad (3.2)$$

where  $\kappa$  is a constant, is not invariant under  $D$  though it is invariant under all the transformations of  $\mathfrak{G}_L$ .<sup>7)</sup> However, the physical meaning of  $D$  is as follows: The unit of the coordinates of the space frame is multiplied by some constant and at the same time the unit of the coordinate of the time frame is also multiplied by the same constant. Hence if we take into consideration the physical dimension of the constant  $\kappa$  in (3.2) and adopt as the new  $\kappa$  in the new transformed frame the constant obtained from  $\kappa$  by multiplying suitable factor, in other words, if we introduce the transformation formula of  $\kappa$  under  $D$  suitably, we can say that (3.2) is invariant under  $D$  also. It seems to the writer that, in some sense, such a point of view is more natural than the usual one in which  $\kappa$  is treated as an invariant number. Thus by making a similar consideration with respect to another physical laws we may say with good reason that the fundamental group of the special relativity is not  $\mathfrak{G}_L$  but  $\{\mathfrak{G}_L + D\}$ . (Here it is evident that  $\{\mathfrak{G}_L + D\}$  forms a group.) If we take such a standpoint the result obtained in the preceding section

is not unnatural.\*

On the other hand, however, there exists a point of view from which one considers that the observers connected by the 'motions', in terms of geometry, of the four dimensional curved space-time are equivalent to each other.<sup>9)</sup> This way of thinking is that which underlies the relativistic cosmology<sup>10)</sup>, and Möller called such a motion the generalized Lorentz transformation.<sup>11)</sup> Mathematically speaking, a motion is a transformation which keeps the line element of the space-time form-invariant, and all motions together form what is called the group of motions. Further it is well known that the group of motions of the Minkowski space-time is given by the  $\mathfrak{G}_L$  in the coordinate system in which the line element takes the form

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2. \quad (3.3)$$

Evidently,  $D$  does not belong to this group, but belongs to the group of conformal transformations, namely the transformations which keep the equation  $ds^2=0$  form-invariant. Accordingly, the point of view which asserts the equivalency of the observers connected by motions and that which considers  $\{\mathfrak{G}_L + D\}$  as the fundamental group of special relativity are not the same, and there is much room left for investigating the physical meanings of their mutual relation. Here, however, we shall not touch this problem further.

Lastly we shall point out an interesting property of the group  $\mathfrak{G}$  which we have not touched in [1]. We shall explain this with respect to the E. O. of the third kind: Among sub-groups of  $\mathfrak{G}_0$ , both  $\{T_a, U, D\}$  and  $\{T_a, U\}$  are invariant sub-groups of  $\mathfrak{G}$  but this is not the case for  $\{R_a\}$ . Similar circumstances also hold for every kind of E. O. It seems to the writer that such a mathematical property is closely connected with the physical meanings of the frame transformations. At the present stage, however, we can not arrive at any definite conclusion.

#### § 4. Another three kinds of E. O.

In this section, first we shall give E. O. whose only relative transformation is given by  $G$ , and further shall introduce another two kinds of E. O. relating to it.

Now we shall call the E. O. whose  $\mathfrak{G}_0$  is given by

$$\mathfrak{G}_0: R_a, T_a, U, \bar{D} \quad (4.1)$$

where  $\bar{D}=t\partial_t$  is the dilatation of the time frame, the E. O. of the fourth kind. Then we can prove that:

*The most general relative transformation between two E. O. of the fourth kind is given by  $G$  (i. e.  $P_1=t\partial_x$ , cycl.) and it holds that*

$$(R_a, P_b) = \epsilon^{abc} P_c, (T_a, P_b) = (P_a, P_b) = 0, (U, P_a) = T_a, (\bar{D}, P_a) = P_a. \quad (4.2)$$

\* Such a consideration is closely connected with Hoffmann's research concerning the guiding principle of his similarity geometry.<sup>8)</sup> But his opinion is not the same as the one stated here.

From this result we know that the  $\mathfrak{G}_0$  given by (4.1) is the group of frame transformations which characterizes  $G$  as the unique relative transformation. The proof of this result is quite similar to that used in § 2, so we shall omit it here.

Next we shall call the E. O. whose group of frame transformations is given by (4.1) in which  $\bar{\bar{D}}$  is replaced by the dilatation of space frame  $\bar{D} = x\partial_x + y\partial_y + z\partial_z$  (or by the two dilatations  $D$  and  $\bar{D}$ ), the E. O. of the fifth (or the sixth) kind. For  $D$  and  $\bar{\bar{D}}$  we have

$$\begin{aligned}(R_a, \bar{D}) &= (U, \bar{D}) = 0, \quad (T_a, \bar{D}) = T_a; \\ (R_a, \bar{\bar{D}}) &= (T_a, \bar{\bar{D}}) = (\bar{D}, \bar{\bar{D}}) = 0, \quad (U, \bar{\bar{D}}) = U.\end{aligned}\quad (4.3)$$

By calculating  $(I_a, D)$ , we can easily see that the  $\mathfrak{G}_0$  of the E. O. of the fifth kind i. e.  $\{R_a, T_a, U, \bar{D}\}$  does not form a group together with  $I_a$ . From this fact one might feel inclined to conclude that this  $\mathfrak{G}_0$  also characterizes  $G$  as relative transformation. In point of fact, however, this is not the case and we can show that:

*The most general relative transformation between two E. O. of the fifth kind is given by the two types  $G$  and  $K$ , and it holds that*

$$\text{for } G: \quad (R_a, P_b) = \epsilon^{abc} P_c, \quad (T_a, P_b) = (P_a, P_b) = 0, \quad (U, P_a) = T_a, \quad (\bar{D}, P_a) = -P_a, \quad (4.4)$$

$$\text{and for } K: \quad (R_a, P_b) = \epsilon^{abc} P_c, \quad (T_a, P_b) = (P_a, P_b) = 0, \quad (U, P_a) = mP_a, \quad (\bar{D}, P_a) = -P_a. \quad (4.5)$$

*Next the most general relative transformation for the E. O. of the sixth kind is given by  $G$  and*

$$P_1 = (1/2) (x^2 - y^2 - z^2) \partial_x + xy \partial_y + xz \partial_z, \quad \text{cycl.} \quad (4.6)$$

*For  $G$  we have (4.4) and*

$$(\bar{\bar{D}}, P_a) = P_a, \quad (4.7)$$

*and for (4.6) we have*

$$\begin{aligned}(R_a, P_b) &= \epsilon^{abc} P_c, \quad (T_a, P_b) = \delta_{ab} \bar{D} + \epsilon^{ab} R_c, \\ (\bar{D}, P_a) &= P_a, \quad (U, P_a) = (\bar{\bar{D}}, P_a) = (P_a, P_b) = 0.\end{aligned}\quad (4.8)$$

Hence  $\mathfrak{G}_0 = \{R_a, T_a, U, D\}$  admits not only  $G$  but also  $K$  as relative transformations. Similar circumstances hold also for the E. O. of the sixth kind.

As is seen in the above it is difficult to presume the relative transformations on the basis of the group of frame transformations. This is due to the reason that the condition ' $P_a$ , ( $a=1, 2, 3$ ), make a group together with  $\mathfrak{G}_0$ ' is not so simple as it looks. In fact, an increase of a frame transformation means an increase of the equations to be satisfied by  $P_a$  together with an increase of undetermined constants in these equations at the same time, and consequently it is not so restrictive as it looks. Lastly we shall point out

that we can also discuss about the invariant subgroups of  $\mathfrak{G}$  concerning the three kinds of E. O. treated in this section as was done about the E. O. of the third kind in § 3.

### § 5. Review of the six kinds of E. O.

By the investigations of the preceding sections some group theoretical properties of  $G, L$ , etc. were made clear. Here, in order to clarify the mutual relations of the six kinds of E. O. hitherto considered, we shall give the following table showing their groups of frame transformations and their admissible relative transformations :

Number of kind	Group of frame transformations	Admissible relative transformations
1	$R_a, T_a, U$	$G, L, K$
3	$R_a, T_a, U, D$	$G, L$
4	$R_a, T_a, U, \bar{D}$	$G$
5	$R_a, T_a, U, \bar{D}$	$G, K$
6	$R_a, T_a, U, \bar{D}, \bar{\bar{D}}$	$G$ , transf. given by (4.6)
2	$R_a$	Many transformations containing $L, G, K, (4.6)$ etc.

As is seen from the above examples, if we specify  $\mathfrak{G}_0$  the corresponding relative transformations are determined. Hence by designating diverse kinds of  $\mathfrak{G}_0$  we can obtain various kinds of E. O. Though some general discussions were made in [ 1 ] concerning the physical meanings of this fact, we can not say that we have made clear the meanings of the concrete results such as given in the above table. At the present stage, we can only give the following correspondence between the frame transformations and the physical properties of the frames.

$R_a$ ...isotropy of space frame,  $T_a$ ...homogeneity of space frame,

$U$ ...homogeneity of time frame,

$D$ ...invariancy for the change of unit of space-time frame,

$\bar{D}$ ... „ „ of space frame,

$\bar{\bar{D}}$ ... „ „ of time frame,

### § 6. E. O. whose space frame is one dimensional

All the space frames of the E. O. so far treated are three dimensional. In this section, by giving an example, we shall show that we can also introduce similarly the 'linear equivalent observers', i. e. the E. O. whose space frame is one dimensional, and further shall point out that the dimension of the space frame has an important bearing upon the admissible relative transformations. In order to distinguish the dimensions of the space frames we shall denote the linear E. O. by  $(E. O.)_1$  while the E. O. so far considered, by  $(E. O.)_3$ .



Now, correspondingly to the four postulates for  $(E.O.)_3$ , we shall put the following for  $(E.O.)_1$ :

(P. I)<sub>1</sub> The same as (P. I)<sub>3</sub> with a proviso that the space frame is one dimensional and that the space coordinate is given by  $x$  only.

(P. II<sub>a</sub>)<sub>1</sub> Any observer  $M$  can observe all other observers, and

(P. II<sub>b</sub>)<sub>1</sub> they are moving relative to  $M$ .

(P. III)<sub>1</sub> The same as (P. III)<sub>3</sub>.

These postulates are the same as the first three in § 2 if we ignore the difference between dimensions of space frames, and as a matter of course the one corresponding to (P. IV)<sub>3</sub> is unnecessary. (P. II)<sub>1</sub> is divided into two parts for the following reason: As was seen in [1], the condition that the observers are not at rest relative to each other plays an important rôle in determining the relative transformations of  $(E.O.)_3$ . On the other hand, in the case of  $(E.O.)_1$ , we can easily determine the relative transformations without using this condition (P. II<sub>b</sub>)<sub>1</sub> and further the result obtained is of much interest as will be seen in the following. This is the reason why we divided (P. II)<sub>1</sub>.

As was the case of  $(E.O.)_3$ , we can obtain various kinds of  $(E.O.)_1$  by designating diverse kinds of  $\mathfrak{G}_0$ . Hence the first question to be answered is what kind of  $\mathfrak{G}_0$  we have to take. As the first trial, however, we shall tentatively take the  $\mathfrak{G}_0$  corresponding to that of the  $(E.O.)_3$  of the third kind, that is

$$\mathfrak{G}_0: T_1 = \partial_x, U = \partial_t, D_1 = x\partial_x + t\partial_t. \quad (6.1)$$

In other words, we shall require the homogeneity of the space frame, that of the time frame, and the invariance for the change of the unit of the space-time frame. For (6.1) we have

$$(T_1, U) = 0, (T_1, D_1) = T_1, (U, D_1) = U. \quad (6.2)$$

Then we can easily show that\*

When  $\mathfrak{G}_0$  is given by (6.1), the most general group  $\mathfrak{G}$  for the  $(E.O.)_1$  satisfying (P. I)<sub>1</sub>, (P. II<sub>a</sub>)<sub>1</sub> and (P. III)<sub>1</sub> is given by

$$P = (p_1x + p_2t + p_3)\partial_x + (q_1x + q_2t + q_3)\partial_t \quad (6.3)$$

where the  $p$ 's and  $q$ 's are arbitrary constants. Hence  $\mathfrak{G}$  coincides with the affine group in  $(x, t)$ -space.  $\mathfrak{G}_0$  is an invariant sub-group of  $\mathfrak{G}$ .

If the  $(E.O.)_1$  satisfies the remaining postulates (P. II<sub>b</sub>)<sub>1</sub> further, then by removing suitable frame transformation we have

$$P = t\partial_x + (px + qt)\partial_t \quad (6.4)$$

where  $p$  and  $q$  are any constants. The procedure by which (6.4) is obtained from (6.3) is as follows: First we make a displacement of the space frame of  $M$  so as its spatial origin may coincide with that of  $M'$  at  $t=0$  ( $M'$  being the observer connected by  $P$

\* The outline of the proof is given in the Appendix (2).

with the original observer  $M$ ), next a dilatation of the space-time frame of  $M$  so as the scales of both space frames may coincide at  $t=0$ , and lastly a translation of the time frame of  $M'$  so as the  $t'$  coordinate corresponding to the event  $t=x=0$  may become 0. Then the transformation between both observers is given by (6.4).

In spite of the fact that the (E.O.)<sub>1</sub> above treated is the one corresponding to the (E.O.)<sub>3</sub> of the third kind, the admissible relative transformation (6.4) contains a term corresponding to  $\bar{D}$  as well as  $G$  and  $L$ , contrary to our expectation. Furthermore this transformation has an interesting property which we shall show in the following:

We postulate the invariance of the light velocity as a physical law to be satisfied by the (E.O.)<sub>1</sub>, and formulate this in the following form in order to obtain its mathematical expression:

$[A^+]$  ( $[A^-]$ ): An event  $E^+(E^-)$  of constant velocity  $c(-c)$  relative to  $M$  has also the same velocity  $c(-c)$  relative to  $M'$ , where  $c$  is a positive constant.

By expressing this mathematically we have

$$[A^+]: \text{ If } dx/dt=c \text{ then } dx'/dt'=c,$$

$$[A^-]: \text{ ,, ,, } =-c \text{ ,, ,, } =-c.$$

If we apply these conditions to (6.4) we have

$$c(pc+q)=1, \text{ and } c(pc-q)=1 \quad (6.5)$$

respectively, and  $P$  becomes

$$P=t\partial_x + (x/c^2)\partial_t. \quad (6.6)$$

Thus the special Lorentz transformation is obtained.

Though this result is quite natural it is to be noticed that we can never obtain the Lorentz transformation (6.6) by using  $[A^+]$  alone or  $[A^-]$  alone. For instance, if we apply  $[A^+]$  to (6.4), we have

$$P=t\partial_x + (1/c^2 - q/c)x + qt'\partial_t, \quad (6.7)$$

and this transformation never satisfies  $[A^-]$  so far as  $q \neq 0$  holds. (6.7) is a linear combination of (6.6) and  $Q=(t-x/c)\partial_t$ . The finite form of  $Q$  is given by

$$x'=x, \quad t'=x/c + (t-x/c)e^\tau, \quad (6.8)$$

where  $\tau$  is the parameter of the transformation, and the velocities  $+c$  and  $-c$  for  $M$  correspond to  $+c$  and  $-c(2e^\tau-1)$  for  $M'$  respectively.\* The reason why such a circumstance is not the case for (E.O.)<sub>3</sub> is that the group of rotations  $\{R_a\}$  is contained in every  $\mathbb{G}_0$  considered by us.

By the above discussion it is clear that there is a great difference between circumstances for (E.O.)<sub>3</sub> and (E.O.)<sub>1</sub>. As a further example we shall consider (E.O.)<sub>1</sub> corresponding to the (E.O.)<sub>3</sub> of the fourth kind which characterizes  $G$ . For this purpose

\* It is to be noted that under the condition that  $ds^2 = -dx^2 + c^2 dt^2 = 0$  is kept invariant, this transformation  $Q$  can not survive.



$$D\gamma_1 - \gamma_1 = a_8\gamma_1 + b_8\beta_2 + c_8\alpha_3 - e_8y + f_8x + n_8 + u_8z, \quad (\text{A} \cdot 5)$$

$$D\delta_1 - \delta_1 = a_8\delta_1 + b_8\delta_2 + c_8\delta_3 + p_8 + u_8t,$$

where  $\alpha_8, \dots, u_8$  are again undetermined constants.

As in [ 1 ], using

$$\beta(x, 0, 0, t) = \gamma(x, 0, 0, t) = 0, \quad \alpha(0, 0, 0, t) \neq \text{const.}, \quad (\text{A} \cdot 6)$$

we have from  $E_5$ ,  $E_6$  and  $E_7$

$$\alpha_1 = x^2\rho + \tau - u_6x, \quad \beta_1 = xy\rho + z\theta - u_6y, \quad \gamma_1 = xz\rho - y\theta - u_6z, \quad \delta_1 = x\sigma - p_6 - u_6t, \quad (\text{A} \cdot 7)$$

where  $\rho, \tau, \theta$  and  $\sigma$  are any functions of  $x$  and  $t$ , and  $u_6$  and  $p_6$  are any constants. Then denoting  $p_1 - p_6U + u_6D$  by  $P_1$  anew, we have

$$P_1 = (x^2\rho + \tau)\partial_x + (xy\rho + z\theta)\partial_y + (xz\rho - y\theta)\partial_z + x\sigma\partial_t. \quad (\text{A} \cdot 8)$$

For this  $P_1$  it holds that  $(R_a, P_b) = \epsilon^{abc}P_c$  and  $E_5$ ,  $E_6$ ,  $E_7$  are all satisfied.

Substituting (A·8) into  $E_2$ , we have

$$\begin{aligned} \rho &= u_2, \quad \sigma = p_2 + u_2t, \quad u_2 + \tau'/r = 0, \quad a_2u_2 = b_2u_2 = c_2u_2 = 0, \\ a_2\theta + e_2 &= b_2\theta + f_2 = c_2\theta + g_2 = 0, \quad a_2\tau + l_2 = b_2\tau + m_2 = c_2\tau + n_2 = 0, \\ a_2\sigma &= b_2\sigma = c_2\sigma = 0, \quad \theta = \theta(t), \end{aligned} \quad (\text{A} \cdot 9)$$

where  $a_2, b_2, \dots, u_2$  are arbitrary constants. From (A·9) and the remaining equations it follows that :

- (i) When  $\sigma \neq 0$ . Denoting  $P_1 - v_1R_1 - v_2T_1$ , where  $v_1$  and  $v_2$  are constants suitably chosen, by  $P_1$  anew, we have [ a ] of § 2.
- (ii) When  $\sigma = 0$ , i. e. when  $p_2 = u_2 = 0$ . By denoting  $P_1 - v_3R_1$ , where  $v_3$  is a constant suitably chosen, by  $P_1$  anew, we have [ b ] of § 2.

## (2) Proof of the deduction of (6·3)

Again we shall give the outline. For the  $P$  given by (6·3) it holds that

$$(P, T_1) = -(\dot{p}_1T_1 + q_1U), \quad (P, U) = -(\dot{p}_2T_1 + q_2U), \quad (P, D_1) = \dot{p}_3T_1 + q_3U. \quad (\text{A} \cdot 10)$$

Therefore (6·3) satisfies the required condition. To prove the necessity of (6·3) we have only to solve the equations obtained from the condition that  $(P, T_1)$ ,  $(P, U)$ ,  $(P, D_1)$  be linear combinations with constant coefficients of  $T_1$ ,  $U$ ,  $D_1$  and  $P$ . If we put

$$P = \xi^i\partial_i = \xi^1\partial_x + \xi^2\partial_t, \quad (\text{A} \cdot 11)$$

we have

$$\begin{aligned} -(P, T_1) &= (\partial_x \xi^1)\partial_x + (\partial_t \xi^2)\partial_t, \quad -(P, U) = (\partial_t \xi^1)\partial_x + (\partial_t \xi^2)\partial_t, \\ -(P, D_1) &= (x\partial_x \xi^1 + t\partial_t \xi^1 - \xi^1)\partial_x + (x\partial_x \xi^2 + t\partial_t \xi^2 - \xi^2)\partial_t, \end{aligned} \quad (\text{A} \cdot 12)$$

by using which (6·3) is easily obtained.



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## Reduced Widths on Collective Model

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The reduced width is calculated on the basis of the collective model. The two extreme approximations, weak and strong couplings are adopted. The weak coupling approximation predicts its magnitude similar to but a little smaller than that predicted by the individual particle model. On the other hand the strong coupling approximation predicts a very different value from that predicted by the single particle model. We have treated only the case in which the compound nucleus has one or two particles outside the core. It is shown that the experimental data of the reduced widths of light nuclei lie between those derived on the two extreme approximations.

### § 1. Introduction

The width of a nuclear energy level of compound nuclei is considered to be decomposed into two factors, one being the penetration probability which depends only on the wave function in the outer region of a nucleus and the other the reduced width. In the case where the nuclear reaction does not pass through the compound state, as in  $d/p$  stripping reactions, the cross section can also be expressed in terms of reduced widths. Although there is a distinction according to whether the level is virtual or bound, reduced width is in either case determined by a kind of overlapping of the wave functions of the compound or final state and the particle state which is smoothly connected to the state at infinite separation of colliding systems.

Weisskopf<sup>(1)</sup> and Wigner<sup>(2)</sup> have treated the reduced width statistically and obtained the well-known sum rule and the relation between the reduced widths and the level distances. Their methods are useful to analyze experimental data independent of specific nuclear models but are not suitable to study individual levels in detail. For this purpose the investigation based on some models is desirable.

Lane<sup>(3)</sup> has calculated the reduced width for light nuclei on the basis of the individual particle model. He has shown that the reduced width occasionally possesses a very different value from that predicted by the single particle model when the orbit of an incident nucleon is the same as that of the unfilled shell of a target nucleus and particularly when the extreme  $LS$  or  $jj$ -coupling is not a good approximation. On the contrary, when the orbit of an incident nucleon is different from that of the unfilled shell of a target nucleus the individual particle model predicts the same reduced width as that predicted by the single particle model apart from a factor due to  $\tau$ -spins. In this case a fair agreement between theoretical and experimental reduced widths are found in light nuclei.

In the individual particle model nucleus are considered to move in a common potential well which is formed by other nucleons and the interaction between nucleons outside the

core is considered to be a small perturbation, while the deformation of the core is usually neglected. On the contrary this deformation is taken into account in the collective model rather than the interaction between nucleons outside the core, and the success in the interpretation of low energy nuclear phenomena in terms of the collective model suggests the importance of the effect of the core deformation. Therefore it is likely that the reduced width can be explained more nicely by the collective model than the individual particle model in the case where the orbit of an incident nucleon is different from that of the nucleon outside core in the target nucleus. But the collective model as well as the individual particle model have been believed to be applicable only to low energy levels. Lane has, however, applied the individual particle model to relatively high energy levels with success that this gives unexpectedly good results. Encouraged by this, we are led to presume that the collective model may also be applicable to the relatively high energy levels. Moreover, there are low lying levels of light nuclei which are considered to be the collective mode of excitation, such as the rotational level. For these levels the collective model is more suitable than the individual particle model. Since these reduced widths are supposed to be available in future due to the development of theories as well as the accumulation of experimental data we think it worth while to study the reduced width on the collective model.

In calculating the reduced width on the basis of the collective model, the two extreme approximations, weak and strong couplings, are adopted following the work by Bohr and Mottelson<sup>1)</sup>. The weak coupling approximation gives us the reduced width similar to but smaller than that on the individual particle model, because of the deformation of a core. The strong coupling approximation predicts the magnitude of the reduced width considerably different from that of the single particle model even if the target nucleus is of a closed shell without nucleon outside it. When there are many nucleons outside the closed shell, these nucleons should be treated as the nucleons outside the core in the collective model. But we treat only the case where a few nucleons exist outside the core, because it is supposed that the present collective model is not yet so refined as to be eligible to a detailed calculation in other complex cases. However the effect of the mixing of levels due to the deformation of a core is taken into account in the weak coupling approximation.

Comparison with the experimental data is restricted to the nuclear reaction at light nuclei in this paper. The deuteron stripping data are not referred because the theory of this reaction is not yet so refined as to allow us a quantitative discussion of reduced widths.

## § 2. Definition of the reduced width

When an incident nucleon with spin  $\sigma$  and orbital angular momentum  $l$  collides with a target nucleus with spin  $I$ , making the transition to a compound or a final nucleus state with spin  $I'$  and specified by other quantum number  $\lambda$ , the reduced width for channel spin  $s$  is given by<sup>2)</sup>

$$\gamma_{\lambda sl} = (\hbar^2/2M)^{1/2} \int_S X_\lambda^* \phi_{sl} dS \quad (2.1)$$

where  $X_\lambda$  is the wave function of the compound state normalized in the internal region.  $S$  means the boundary which divides the whole space into an internal and external regions, and  $\int_S dS$  denotes the integration over the boundary  $S$ .  $M$  is the reduced mass of nucleon and  $\hbar$  Planck's constant divided by  $2\pi$ . The spin-angle wave function  $\phi_{sl}$  is defined in terms of Dirac's bracket notation as

$$\phi_{sl} = \sum_{m_s m_l} (slm_s m_l | I' M') | sm_s \rangle | lm_l \rangle. \quad (2.2)$$

In the above equation  $M'$ ,  $m_s$ , and  $m_l$  are the  $z$ -components of angular momenta  $I'$ ,  $s$ , and  $l$  respectively,  $(slm_s m_l | I' M')$  is the Clebsch-Gordan coefficient.  $|sm_s\rangle$  and  $|lm_l\rangle$  represent the wave functions for channel spin and orbital motion respectively.  $|sm_s\rangle$  is further expanded as

$$|sm_s\rangle = \sum_{M m_\sigma} (I \sigma M m_\sigma | sm_s) | IM \rangle | \sigma m_\sigma \rangle. \quad (2.3)$$

where  $M$  and  $m_\sigma$  are the  $z$ -components of angular momenta  $I$  and  $\sigma$  respectively.

As the collective model is usually described in reference to the  $jj$ -coupling scheme, it is convenient to express  $\phi_{sl}$  in terms of the basic wave function  $|jm_j\rangle$ , that describes a state with total angular momentum  $j$  and its  $z$ -component  $m_j$  of the incident nucleon. This can be done by use of Racah coefficient  $W^{(b)}$  as

$$\phi_{sl} = \sum_j \sqrt{(2s+1)(2j+1)} W(I \sigma I' l; sj) \sum_{M m_j} (I j M m_j | I' M') | IM \rangle | jm_j \rangle. \quad (2.4)$$

Since the last summation in (2.4) corresponds to the wave function of a particle of total angular momentum  $j$  around the target nucleus, we denote it by

$$\phi_j = \sum_{M m_j} (I j M m_j | I' M') | IM \rangle | jm_j \rangle. \quad (2.5)$$

Then the reduced width is expressed in a way suitable to the  $jj$ -coupling scheme as

$$\gamma_{\lambda sl} = \left( \frac{\hbar^2}{2M} \right)^{1/2} \sum_j \sqrt{(2s+1)(2j+1)} W(I \sigma I' l; sj) \int_S X_\lambda^* \phi_j dS. \quad (2.6)$$

Now we can obtain the quadratic channel sum of reduced widths by making use of the orthogonality relation of  $W'S$  as follows,

$$\sum_s \gamma_{\lambda sl}^2 = \frac{\hbar^2}{2M} \sum_j \left| \int_S X_\lambda^* \phi_j dS \right|^2. \quad (2.7)$$

In order that this quantity is directly compared with experiments, the further reduction is needed, as in a current manner. The wave function of the compound state  $X_\lambda$  may be decomposed into the single particle states which are connected at the surface  $S$  to the outer wave function of the nucleon with radial part  $1/r \cdot u_{sl}(r)$ . Hence we can factorize the compound wave function  $X_\lambda$  by the boundary value of the single particle wave function in which we are interested;

$$X_\lambda = u_{sl}(\alpha) \Psi_\lambda \quad (2.8)$$

where  $\alpha$  is the radius of the sphere surrounded by  $S$  and may be set equal to the nuclear



radius.  $\Psi_\lambda$  has the unit amplitude at the surface and is defined in the internal region. Then (2.7) is reduced to

$$\sum_s \Psi_{\lambda st}^2 = \sum_j \frac{\hbar^2}{2M} |\mathcal{U}_{st}(a)|^2 \theta_{\lambda j}^2 \quad (2.9)$$

where

$$\theta_{\lambda j} = \int_S \Psi_\lambda^* \phi_j dS \quad (2.10)$$

is dimensionless and provides the probability amplitude to what extent the single particle mode  $\phi_j$  appears in the internal state  $\Psi_\lambda$ . This can in principle be obtained, provided  $\mathcal{U}_{st}(r)$  is known. The latter is investigated by Lane<sup>30</sup>, to which we shall refer in the following analysis.

### § 3. Weak coupling approximation.

First we calculate the reduced width in the weak coupling approximation, in which the wave function is given by Bohr and Mottelson<sup>41</sup>. Our calculation is restricted to the case where the number of nucleons  $n$  outside the core in the compound nucleus is one and two.

(A)  $n=1$ .

The wave function in the compound state\* is given according to B-M (II, 11)<sup>41</sup> by

$$\Psi = N[|j'; 00; I'M'\rangle + \sum_{j''} |j''; 12; I'M'\rangle \langle j''; 12; I'| \rangle]. \quad (3.1)$$

The first term in the right hand side represents the state in which a nucleon has total angular momentum  $j'$  and a core is not excited, so that  $j''=I'$ . The second term represents the state in which a nucleon with total angular momentum  $j''$  couples with a core and one phonon of angular momentum 2 is excited so that the resultant spin is  $I'$ . All these functions are normalized to 1. Normalization constant  $N$  is given by

$$N^{-2} = 1 + \sum_{j''} |\langle j''; 12; I'| \rangle|^2 \quad (3.2)$$

The expansion coefficient  $\langle j''; 12; I'| \rangle$  is given by B-M (Ap. II. 3) as

$$\langle j''; 12; I'| \rangle = k \sqrt{\frac{\hbar\omega}{2C}} \frac{\langle j'|k|j'' \rangle}{\hbar\omega + \mathcal{A}_{j'j''}}, \quad (3.3)$$

where  $C$  is the deformability of the core,  $\hbar\omega$  the energy of a phonon,  $k$  the coupling constant, and  $\mathcal{A}_{j'j''}$  the separation between two particle levels of angular momenta  $j'$  and  $j''$ .  $\langle j'|k|j'' \rangle$  is the angular part of the matrix element of the interaction between the core and the nucleon, as given by B-M (Ap. II. 2). If we neglect the off-diagonal elements of  $\langle j'|k|j'' \rangle$ , the normalization constant is simplified as

\*) The suffix  $\lambda$  which represents quantum numbers of a compound state is dropped hereafter for simplicity.

$$N^{-2} = 1 + \frac{k^2}{2C\hbar\omega} - \frac{5}{64\pi} \frac{(2j'-1)(2j'+3)}{j'(j'+1)} \quad (3.2')$$

The spin-angle wave function  $\phi_j$  is expressed for the ground and the excited states of a target nucleus respectively as

$$\begin{aligned} \phi_j &= |j; 00; I' M'\rangle \\ \phi_j &= |j; 12; I' M'\rangle \end{aligned} \quad (3.4)$$

The non-dimensional reduced width is easily obtained with the aid of the orthonormality of wave functions as

$$\theta_j = N \delta_{jj'} \quad (3.5)$$

for the ground state of the target nucleus, and

$$\theta_j = N \langle j; 12; I' | \rangle \quad (3.5')$$

for the state of  $I=2$  of the target nucleus. The sum of  $\theta_j^2$  over various target nucleus states and the angular momenta of the incident nucleon results in

$$\sum \theta_j^2 = 1 \quad (3.6)$$

as it should be. This is nothing but the sum rule.

(B)  $n=2$

The wave functions are obtained in a similar way to the case  $n=1$ . Corresponding to (3.1), (3.2) and (3.4) we have

$$\Psi = N' [ |j_1' j_2'; 00; I' M'\rangle + \sum_{j_1'' j_2''} |j_1'' j_2''(s); 12; I' M'\rangle \langle j_1'' j_2''(s); 12; I' | \rangle ] \quad (3.7)$$

$$N'^{-2} = 1 + \sum_{j_1'' j_2''} | \langle j_1'' j_2''(s); 12; I' | \rangle |^2 \quad (3.8)$$

and

$$\begin{aligned} \phi_j &= \sum_{m_1 m} N [ |j; 00; j_1 m_1\rangle + \sum_{j_1''} |j_1''; 12; j_1 m_1\rangle \langle j_1''; 12; j_1 | \rangle ] \\ &\quad \times |j m\rangle \langle j_1 j m_1 m | I' M' \rangle \end{aligned} \quad (3.9)$$

$$N^{-2} = 1 + \sum_{j_1''} | \langle j_1''; 12; j_1 | \rangle |^2 = 1 + k^2 \frac{\hbar\omega}{2C} \sum_{j_1''} \left| \frac{\langle j_1 | h | j_1'' \rangle}{\hbar\omega + \mathcal{A}_{j_1 j_1''}} \right|^2 \quad (3.10)$$

In equation (3.7) the second terms represent the state with excited core and  $s$  is the resultant of angular momenta of particles.

The interaction Hamiltonian between particles and a core,  $H_{int}$ , is the sum of  $H^{(1)}$  and  $H^{(2)}$ , the former representing the interaction between particle 1 and the core and the latter the corresponding one for particle 2. The matrix elements of  $H^{(1)}$  and  $H^{(2)}$  can be obtained by expanding the wave functions appearing in equation (3.7) as follows:

$$\begin{aligned} |j_1'' j_2''(s); 12; I' M'\rangle &= \sum_{s_1} [(2s+1)(2s_1+1)]^{1/2} W(s 2 j_2'' s_1; I' j_1'') |j_1'' 2(s_1) j_2'' I' M'\rangle \\ &= \sum_{s_2} [(2s+1)(2s_2+1)]^{1/2} W(j_1'' j_2'' I' 2; s s_2) |j_1'' j_2'' 2(s_2) I' M'\rangle. \end{aligned} \quad (3.11)$$

Then the matrix element is obtained as

$$\begin{aligned} & \langle j_1'' j_2''(s); 12; I' M' | H_{int} | j_1' j_2'; 00; I' M' \rangle \\ &= \sum_{s_1} [(2s+1)(2s_1+1)]^{1/2} W(s j_2'' s_1; I' j_1'') \delta_{s_1 j_1'} \delta_{j_2'' j_2'} \langle j_1''; 12; s_1 | H^{(1)} | j_1'; 00; j_1' \rangle \\ &+ \sum_{s_2} [(2s+1)(2s_2+1)]^{1/2} W(j_1'' j_2'' I' 2; s s_2) \delta_{j_1'' j_1'} \delta_{s_2 j_2'} \langle j_2''; 12; s_2 | H^{(2)} | j_2'; 00; j_2' \rangle \end{aligned}$$

On account of (3.3) the expansion coefficient is expressed as

$$\begin{aligned} \langle j_1'' j_2''(s); 12; I' | \rangle &= k \sqrt{\frac{\hbar \omega}{2C}} \frac{[\delta_{j_2'' j_2'} \frac{\langle j_1' | k | j_1'' \rangle}{\hbar \omega + \mathcal{A}_{j_1' j_1''}}] (-)^{I' + j_1'' - s - j_1'}}{\hbar \omega + \mathcal{A}_{j_1' j_1''}} \\ &\times [(2s+1)(2j_2'+1)]^{1/2} W(j_2' j_1'' I' 2; s j_1') \\ &+ \delta_{j_1'' j_1'} \frac{\langle j_2' | k | j_2'' \rangle}{\hbar \omega + \mathcal{A}_{j_2' j_2''}} [(2s+1)(2j_2'+1)]^{1/2} W(j_1' j_2'' I' 2; s j_2') \end{aligned} \quad (3.12)$$

Substituting the above equation into equation (3.8), the normalization constant is reduced to

$$\begin{aligned} N'^{-2} &= 1 + k^2 \frac{\hbar \omega}{2C} \sum_s \left[ \sum_{j_1'} \left| \frac{\langle j_1' | k | j_1'' \rangle}{\hbar \omega + \mathcal{A}_{j_1' j_1''}} \right|^2 (2s+1)(2j_1'+1) W^2(j_2' j_1'' I' 2; s j_1') \right. \\ &\quad + \sum_{j_2'} \left| \frac{\langle j_2' | k | j_2'' \rangle}{\hbar \omega + \mathcal{A}_{j_2' j_2''}} \right|^2 (2s+1)(2j_2'+1) W^2(j_1' j_2'' I' 2; s j_2') \\ &\quad + 2 \frac{\langle j_1' | k | j_1' \rangle}{\hbar \omega} \frac{\langle j_2' | k | j_2' \rangle}{\hbar \omega} (-)^{I' - s} (2s+1) [(2j_1'+1)(2j_2'+1)]^{1/2} \\ &\quad \times W(j_2' j_1' I' 2; s j_1') W(j_1' j_2' I' 2; s j_2') \left. \right] \quad (3.13) \\ &= 1 + k^2 \frac{\hbar \omega}{2C} \left[ \sum_{j_1'} \left| \frac{\langle j_1' | k | j_1'' \rangle}{\hbar \omega + \mathcal{A}_{j_1' j_1''}} \right|^2 + \sum_{j_2'} \left| \frac{\langle j_2' | k | j_2'' \rangle}{\hbar \omega + \mathcal{A}_{j_2' j_2''}} \right|^2 + 2 \frac{\langle j_1' | k | j_1' \rangle}{\hbar \omega} \frac{\langle j_2' | k | j_2' \rangle}{\hbar \omega} \right. \\ &\quad \times [(2j_1'+1)(2j_2'+1)]^{1/2} W(j_2' I' 2 j_1'; j_2' j_1') \left. \right] \end{aligned}$$

By using the equations (3.11) the wave function (3.7) is expanded as,

$$\begin{aligned} \Psi &= N' [ | j_1' j_2'; 00; I' M' \rangle + \sum_{j_1''} \sum_{j_2''} [(2s+1)(2s_1+1)]^{1/2} W(s j_2'' s_1; I' j_1'') \\ &\quad \times | j_1'' 2(s_1) j_2'' I' M' \rangle \langle j_1'' j_2''(s); 12; I' | \rangle ] \end{aligned}$$

which together with the equation (3.9) give the reduced width as

$$\begin{aligned} \theta_j &= N N' [\delta_{j_1 j_1'} \delta_{j_2 j_2'} + \frac{k^2 \hbar \omega}{2C} \sum_{j_1''} \sum_{j_2''} \delta_{j_2'' j_2'} (2s+1) [(2j_1+1)(2j_1'+1)]^{1/2} (-)^{2j_1'' - j_1 - j_1'} \\ &\quad \times W(j_1'' I' 2; s j_1) W(j_1'' I' 2; s j_1') \frac{\langle j_1' | k | j_1'' \rangle}{\hbar \omega + \mathcal{A}_{j_1 j_1''}} \frac{\langle j_1' | k | j_1'' \rangle}{\hbar \omega + \mathcal{A}_{j_1 j_1'}} \\ &\quad + k^2 \frac{\hbar \omega}{2C} \sum_s (2s+1) [(2j_1+1)(2j_2'+1)]^{1/2} (-)^{I' + j_1'' - s - j_1} \end{aligned}$$

$$\begin{aligned}
& \times W(j_1 I' 2; s j_2') W(j_1 j I' 2; s j_2') \frac{\langle j_1 | l | j_1' \rangle}{\hbar \omega + \mathcal{A}_{j_1 j_1'}} \frac{\langle j_2' | l | j \rangle}{\hbar \omega + \mathcal{A}_{j_2' j}} \Big] \\
& = N N' [\delta_{j_1 j_1'} \delta_{j_2 j_2'} + k^2 \frac{\hbar \omega}{2C} \{ \delta_{j_2' j} \delta_{j_1 j_1'} \sum_{j_1''} \left| \frac{\langle j_1 | \hbar | j_1'' \rangle}{\hbar \omega + \mathcal{A}_{j_1 j_1''}} \right|^2 + (-)^{-2j_1 - j - j_2'} \\
& \quad \times [(2j_1 + 1)(2j_2' + 1)]^{1/2} W(j_2 I' j'; j_1 j_2') \frac{\langle j_1 | l | j_1' \rangle}{\hbar \omega + \mathcal{A}_{j_1 j_1'}} \frac{\langle j_2' | l | j \rangle}{\hbar \omega + \mathcal{A}_{j_2' j}} \} ]
\end{aligned} \quad (3.14)$$

When the  $\tau$ -spin is taken into account, the wave function in equation (3.7) must be symmetrized or antisymmetrized with respect to the exchange of particles and the reduced width  $\theta_j$  obtained above must be multiplied by factor  $\sqrt{2}$ , and further by the factor  $1/\sqrt{2}$  when the  $z$ -component of the  $\tau$ -spin of the compound state is 0. In particular case  $j_1'' = j_2'' = j_1' = j_2' = j$  the sum over  $s$  in equations (3.13) and (3.14) should be restricted to odd or even number, in order that the wave function of the compound state is symmetric or antisymmetric. The equations (3.13) and (3.14) reduce to

$$\theta_j = \sqrt{2} N N' [1 + k^2 \frac{\hbar \omega}{2C} \left| \frac{\langle j | l | j \rangle}{\hbar \omega} \right|^2 \{1 - (2j + 1) W(jjj; I' 2)\}] \quad (3.15)$$

$$N'^{-2} = 1 + 2k^2 \frac{\hbar \omega}{2C} \left| \frac{\langle j | l | j \rangle}{\hbar \omega} \right|^2 \{1 - (2j + 1) W(jjj; I' 2)\} \quad (3.16)$$

We can prove the following sum rule

$$\sum \theta_j^2 = 1$$

where the sum must include the excited states of the target nucleus as in (A).

#### § 4. Strong coupling approximation

The wave function of the system in the strong coupling approximation is, according to B-M (II. 15)

$$\Psi = \varphi'(\beta, \gamma) \sqrt{\frac{2I' + 1}{8\pi^2}} \frac{1}{\sqrt{2}} \{ \chi_{\omega'} D_{M' K'}'' + \bar{\chi}_{-\omega'} D_{M' -K'}'' \} \quad (4.1)$$

where  $\varphi'(\beta, \gamma)$  represents vibrations,  $D_{M' K'}''$  is the eigen-function for the symmetric top and describes the nuclear rotation state which is normalized so as to give the unitary transformation from the coordinate system fixed in space to that attached to the nucleus under consideration. Thus<sup>(6)</sup>

$$\int |D_{M' K'}''|^2 dR = \frac{8\pi^2}{2I' + 1}. \quad (4.2)$$

$M$  and  $K'$  are the components of spin  $I$  with respect to the  $z$ -axis of the fixed and the nuclear coordinate systems respectively.  $\chi_{\omega'}$  describes the nucleons outside the core in the nuclear coordinate system with  $\omega'$  being the sum of the  $z$ -components of nucleon angular momenta in this system. This is expressed in terms of individual particle states:



$$\chi_{\Omega} = \frac{1}{\sqrt{n!}} \sum \pm \chi_{\Omega_1}^{j_1}(1) \chi_{\Omega_2}^{j_2}(2) \cdots \chi_{\Omega_n}^{j_n}(n) \quad (4.3)$$

where  $\Omega = \sum_{i=1}^n \Omega_i$ . The summation in equation (4.3) must be carried out so that  $\chi_{\Omega}$ 's have an appropriate symmetry character for particle exchange and the Pauli principle be satisfied. The normalization for  $\chi_{\Omega}$  is appropriately chosen so as to satisfy equation (2.8).  $\bar{\chi}_{-\Omega}$  is introduced for the whole system to have symmetry character for the exchange of the coordinate axis and given by

$$\bar{\chi}_{-\Omega} = \frac{1}{\sqrt{n!}} (-)^{I'-\Sigma j} \sum \pm \chi_{-\Omega_1}^{j_1}(1) \chi_{-\Omega_2}^{j_2}(2) \cdots \chi_{-\Omega_n}^{j_n}(n).$$

The symmetry character for the exchange of axes further implies that

$$K' - \Omega' = 0, \pm 2, \pm 4, \cdots \quad (4.5)$$

Now we can construct a wave function  $\psi_j$  as

$$\psi_j = \sum_{sM} \varphi(\beta, \gamma) \sqrt{\frac{2I+1}{8\pi^2}} \frac{1}{\sqrt{2}} \{ \chi_{\Omega} D_{MK}^I + \bar{\chi}_{-\Omega} D_{M-K}^I \} \chi_{M'-M}^j (IjMM' - M | I' M') \quad (4.6)$$

where  $\chi_{M'-M}^j$  describes the incident nucleon in the fixed coordinate system. If the wave function in the nuclear coordinate system is denoted by  $\chi_s^j$ , that in the space coordinate system  $\chi_{M'-M}^j$  is expressed as

$$\chi_{M'-M}^j = \sum_s D_{M'-Ms}^j \chi_s^j \quad (4.7)$$

Now we are ready to derive the reduced width as

$$\begin{aligned} \theta_j = & \frac{\sqrt{(2I'+1)(2I+1)}}{16\pi^2} \iint \varphi'^* \varphi \sum_{Ms} \{ \chi_{\Omega}^* D_{M'M'}^{*I'} + \bar{\chi}_{-\Omega}^* D_{M'-K'}^{*I'} \} \{ \chi_{\Omega} D_{MK}^I + \bar{\chi}_{-\Omega} D_{M-K}^I \} \\ & \times D_{M'-Ms}^j \chi_s^j (IjMM' - M | I' M') \end{aligned} \quad (4.8)$$

This can be reduced by using the formula for the representation of the rotational group<sup>(6)</sup>

$$\int D_{MK}^I D_{M'-Ms}^j D_{M'K'}^{*I'} dR = \frac{8\pi^2}{2I'+1} \delta_{K'-K'} \delta_{K'-K'} (IjMM' - M | I' M') (IjK'K' - K | I' K') \quad (4.9)$$

to

$$\begin{aligned} \theta_j = & \frac{1}{2} \sqrt{\frac{2I+1}{2I'+1}} \int \varphi'^* \varphi \{ \chi_{\Omega}^* \chi_{\Omega} \chi_{K'-K}^j + (-)^{I'+j-I} \chi_{-\Omega}^* \chi_{-\Omega} \chi_{-K'-K}^j \} (IjK'K' - K | I' K') \\ & + \{ \chi_{\Omega}^* \chi_{\Omega} \chi_{-K'-K}^j + (-)^{I'+j-I} \chi_{-\Omega}^* \chi_{-\Omega} \chi_{K'-K}^j \} (IjK', -K' - K' | I' - K'). \end{aligned} \quad (4.10)$$

There holds

$$\Omega' - K' = \Omega - K = 0, \pm 2, \pm 4, \cdots \quad (4.11)$$

due to the conservation of the  $z$ -component of angular momentum in the nuclear system and (4.5).

In the special cases  $\Omega' = K' = 0$  or  $\Omega = K = 0$ , expression (4.10) turns to be

$$\theta_j = \frac{1}{\sqrt{2}} \sqrt{\frac{2I+1}{2I'+1}} \int \varphi'^* \varphi \left[ \left\{ \chi_{\Omega'}^* \chi_{\Omega} \chi_{-K}^j + (-)^{I+I'-I'} \int \bar{\chi}_{\Omega'}^* \chi_{-\Omega} \chi_K^j \right\} (IjK - K | I'0) \right]$$

for  $\Omega' = K' = 0$  (4.11')

$$\theta_j = \frac{1}{\sqrt{2}} \sqrt{\frac{2I+1}{2I'+1}} \int \varphi'^* \varphi \left[ \left\{ \chi_{\Omega'}^* \chi_{\Omega} \chi_{K'}^j + (-)^{I+I'-I'} \int \bar{\chi}_{\Omega'}^* \chi_{\Omega} \chi_{-K}^j \right\} (Ij0K' | I'K') \right]$$

for  $\Omega = K = 0$ .

In the case of  $j=3/2$ , a special treatment is necessary because  $\Omega$  is not a good quantum number. For simplicity, however, we neglect here the effect of mixing of  $\Omega$ .

We shall show a number of simple examples for the reduced width in what follows.

$$(A) \quad n=1, \quad I' = \Omega' = K' = j, \quad \Omega = K = 0$$

From equation (4.11') we get

$$\begin{aligned} \theta_j &= \frac{1}{\sqrt{2}} \sqrt{\frac{2I+1}{2j+1}} \int \varphi'^* \varphi \left[ \int \chi_j^* \chi_j^j + (-)^I \int \bar{\chi}_{-j}^* \chi_j^j \right] (Ij0j | jj) \\ &= \sqrt{2} \int \varphi'^* \varphi (jj - jj | I0) \end{aligned}$$

for  $I = \text{even}$  (4.12)

To obtain the sum rule we take the quadratic sum of reduced widths over  $I$ ,

$$\sum_j \theta_j^2 \leq 2 \sum_{I=\text{even}} (jj - jj | I0)^2, \quad (4.13)$$

where the summation of  $(jj - jj | I0)^2$  over  $I = \text{even}$  is easily evaluated as

$$\sum_{I=\text{even}} (jj - jj | I0)^2 = 1/2$$

on account of the following formulae

$$\begin{aligned} \sum_I (jj - jj | I0)^2 &= 1 \\ \sum_I (jj - jj | I0) (jjj - j | I0) &= 0. \end{aligned}$$

Substitution of this into (4.13) yields

$$\sum_I \theta_j^2 \leq 1. \quad (4.14)$$

$$(B) \quad n=2, \quad \Omega' = K' = 0 \quad \Omega = K$$

The wave functions for nucleons are

$$\begin{aligned} \chi_{\Omega T'} &= \frac{1}{\sqrt{2}} \{ \chi_{\Omega}(1) \chi_{-\Omega}(2) + (-)^{T'} \chi_{-\Omega}(1) \chi_{\Omega}(2) \} \chi_{M T'}^T \\ \chi_{\Omega T} &= \chi_{\Omega}(1) \chi_{M T}^T \end{aligned} \quad (4.15)$$

where  $T'$  and  $T$  are  $\tau$ -spins of the compound and target nuclei respectively and  $\chi_{M T'}^T$  and

$\chi_{M_T}^T$  are corresponding  $\tau$ -spin wave functions. Putting these wave functions into equation (4.11') we get

$$\theta_j = \sqrt{2} \sqrt{\frac{2I+1}{2I'+1}} \int \varphi'^* \varphi (IjK-K'|I'0) (TtM_T m_t | T' M_{T'}) \text{ for } T'+I'=\text{odd} \quad (4.16)$$

In the last Clebsch-Gordan coefficient  $t$  and  $m_t$  are  $\tau$ -spin of the nucleon and its  $z$ -component respectively. This results in the sum rule

$$\sum_I \theta_j^2 \leq 2 (TtM_T m_t | T' M_{T'})^2 \quad (4.17)$$

$$(B') \quad n=2, \quad \Omega'=K' \neq 0, \quad \Omega=K$$

With the particle wave functions

$$\chi_{\omega'} = \frac{1}{\sqrt{2}} \{ \chi_{\omega_1}^{j_1}(1) \chi_{\omega_2}^{j_2}(2) + (-)^{T'} \chi_{\omega_2}^{j_2}(1) \chi_{\omega_1}^{j_1}(2) \} \chi_{M_{T'}}^{T'} \quad (4.18)$$

$$\bar{\chi}_{-\omega'}^* = (-)^{-I'+2j} \frac{1}{\sqrt{2}} \{ \chi_{-\omega_1}^{*j_1}(1) \chi_{-\omega_2}^{*j_2}(2) + (-)^{T'} \chi_{-\omega_2}^{*j_2}(1) \chi_{-\omega_1}^{*j_1}(2) \} \chi_{M_{T'}}^{*T'} \quad (4.18')$$

$$\chi_{\Omega} = \chi_{\omega'}^{j'}(1) \chi_{M_T}^T \quad (4.19)$$

$$\bar{\chi}_{-\Omega} = (-)^{I-j} \chi_{-\omega'}^{j_1}(1) \chi_{M_T}^T \quad (4.19')$$

we can easily obtain the reduced width as

$$\theta_j = \sqrt{\frac{2I+1}{2I'+1}} \int \varphi'^* \varphi (IjKK'-K|I'K') (TtM_T m_t | T' M_{T'}) \quad (4.20)$$

and the sum rule

$$\sum_I \theta_j^2 \leq (TtM_T m_t | T' M_{T'})^2.$$

In the case  $n \geq 3$  the particle wave function must be antisymmetric for all exchanges of particles, as in the case of the individual particle model, and the treatment becomes complex. We here only explain the treatment briefly taking the case  $n=3$  as an example.

$$(C) \quad n=3$$

The wave function of particles  $\chi_{\omega'}$  is expanded in terms of the coefficient of fractional parentage (c. f. p.) into various single particle states

$$\chi_{\omega'} = \chi(\Omega_1' \Omega_2' \Omega_3' T') = \sum_{T_{12}', \Omega_{12}', \Omega_3'} \chi(\Omega_1'' \Omega_2'' (T_{12}') \Omega_3'' T') (\Omega_1'' \Omega_2'' (T_{12}') \Omega_3'' T' | \Omega_1' \Omega_2' \Omega_3' T')$$

where  $T_{12}'$  is the resultant  $\tau$ -spin of particle 1 and 2,  $\Omega_1'' \Omega_2'' \Omega_3''$  are obtained by making permutations to  $\Omega_1' \Omega_2' \Omega_3'$ . The c. f. p.  $(\Omega_1'' \Omega_2'' (T_{12}') \Omega_3'' T' | \Omega_1' \Omega_2' \Omega_3' T')$  can be calculated in a similar way to the case of atomic spectra following the method of Racah<sup>5)</sup>. The elements which do not vanish are:

$$(\Omega_1' \Omega_2'(1) \Omega_3' \frac{3}{2} | \Omega_1' \Omega_2' \Omega_3' \frac{3}{2}) = \frac{1}{\sqrt{3}} \quad (\Omega_1' \Omega_2'(1) \Omega_3' \frac{1}{2} | \Omega_1' \Omega_2' \Omega_3' \frac{1}{2}) = \frac{1}{2}$$

$$\begin{aligned}
(\mathcal{Q}_1 \mathcal{Q}_3(1) \mathcal{Q}_2 \frac{3}{2} | \} \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_3 \frac{3}{2}) &= -\frac{1}{\sqrt{3}} & (\mathcal{Q}_2 \mathcal{Q}_3(1) \mathcal{Q}_1 \frac{1}{2} | \} \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_3 \frac{1}{2}) &= -\frac{1}{2} \\
(\mathcal{Q}_2 \mathcal{Q}_3(1) \mathcal{Q}_1 \frac{3}{2} | \} \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_3 \frac{3}{2}) &= \frac{1}{\sqrt{3}} & (\mathcal{Q}_1 \mathcal{Q}_2(0) \mathcal{Q}_3 \frac{1}{2} | \} \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_3 \frac{1}{2}) &= \frac{1}{2\sqrt{3}} \\
(\mathcal{Q}_1 \mathcal{Q}_3(0) \mathcal{Q}_2 \frac{1}{2} | \} \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_3 \frac{1}{2}) &= -\frac{1}{\sqrt{3}} \\
(\mathcal{Q}_2 \mathcal{Q}_3(0) \mathcal{Q}_1 \frac{1}{2} | \} \mathcal{Q}_1 \mathcal{Q}_2 \mathcal{Q}_3 \frac{1}{2}) &= \frac{1}{2\sqrt{3}}
\end{aligned}$$

If the overlapping of particle wave functions are evaluated by using these c. f. p, the reduced widths can easily be obtained. But here we have explained only the method of the treatment and do not go into detail.

### § 5. Discussion

We will here compare the results obtained in the preceding sections with those predicted by the individual particle model and also with experimental data. First of all, it is easily shown that the individual particle model predicts reduced widths equal to those predicted by the single particle model apart from the factor due to  $\tau$ -spins in the cases  $n=1$  and 2. The nuclear state in the  $j\bar{j}$ -coupling scheme is determined for  $n=1$  and 2 only if the resultant angular momentum is determined. Consequently the compound state is described by one single particle state. The result is the same in the  $LS$  and intermediate couplings. On the other hand in the strong coupling approximation the nuclear state can not be determined only by the total angular momentum but also by the other quantum numbers. Therefore a compound state is described as a superposition of various single particle states. The predicted reduced widths are smaller than those predicted by the single particle model. The weak coupling approximation has a nature intermediate between the independent particle model and the strong coupling approximation. These situations are shown in the following.

The numerical constants  $C$ ,  $\hbar\omega$ ,  $k$  and others which appear in the collective model are evaluated by Ford<sup>7)</sup> and by Bohr and Mottelson<sup>4)</sup> on the basis of the classical hydrodynamical model. These constants are not always correct, as the quadrupole moments and the first excitation levels of even-even nuclei derived on these constants do not always agree with experimental data. But we shall refer to them, since the order of magnitude of these constants seems to be correct<sup>4,7)</sup>.

(A)  $n=1$

The individual particle model predicts

$$\theta_j^2 = 1 \quad (5.1)$$

without concerning the core deformation. In the weak coupling approximation of the collective model we take this effect into account and the reduced width is obtained from



equation (3.5) as

$$\theta_j^2 = \left[ 1 + \frac{k^2}{2C\hbar\omega} \frac{5}{64\pi} \frac{(2j-1)(2j+3)}{j(j+1)} \right]^{-1} \quad (5.2)$$

provided the contribution from the mixed orbits is negligible. The constant  $k^2/C\hbar\omega$  which appears in equation (5.2) varies approximately from 8 to 20 with the variation of the atomic numbers  $A$ . These numerical values are tablated in Table 1. It is clear that  $\theta_{1/2}^2=1$ , because the single particle with  $j=1/2$  does not couple with the deformed core of angular momentum 2. In the cases  $j \geq 3/2$ , this coupling makes the reduced widths small depending slightly on  $j$  and  $A$ .

On the contrary, the strong coupling approximation predicts the reduced width which depends strongly on  $j$ ,

$$\theta_j^2 = \frac{2}{2j+1} \left| \int \varphi'^* \varphi \right|^2 \quad (5.3)$$

In Table 2 the numerical values for these reduced widths are listed. Numerical calculations based on the constants used by Ford show that the overlapping integral  $|\int \varphi'^* \varphi|^2$  is approximately unity over all  $j$  and  $A$ .

Table 1.  $\theta_j^2$  in the weak coupling approximation for  $n=1$ .

$A \backslash j$	1/2	3/2	5/2	7/2	9/2
10	1	0.75	0.72	0.72	0.71
30	1	0.75	0.73	0.72	0.71
50	1	0.76	0.73	0.72	0.72
100	1	0.70	0.67	0.66	0.66
200	1	0.55	0.52	0.51	0.51

Table 2.  $\theta_j^2$  in the strong coupling approximation for  $n=1$ .

$j$	1/2	3/2	5/2	7/2	9/2
$\theta_j^2 / \left  \int \varphi'^* \varphi \right ^2$	1	0.50	0.33	0.25	0.20

### (B) Two identical particles. ( $n=2$ )

As in the case of  $n=1$  the individual particle model predicts reduced widths independent of the spins  $I'$  of compound states. The explicit expressions for the reduced widths neglecting the mixture of levels are obtained as:

$$\theta_j^2 = 2.$$

In the weak coupling approximation, the reduced width is given by

$$\theta_j^2 = 2 \left[ 1 + \frac{k^2}{2C\hbar\omega} \left( \frac{5}{64\pi} \right) \frac{(2j-1)(2j+3)}{j(j+1)} \{ 1 - (2j+1) W(jjjj; I'2) \} \right]^2$$

$$\times \left[ 1 + \frac{k^2}{2C\hbar\omega} \left( \frac{5}{64\pi} \right) \frac{(2j-1)(2j+3)}{j(j+1)} \right]^{-1} \left[ 1 + \frac{k^2}{C\hbar\omega} \left( \frac{5}{64\pi} \right) \frac{(2j-1)(2j+3)}{j(j+1)} \right]^{-1} \\ \times \{ 1 - (2j+1) W(\ddot{jj}j; I'2) \}^{-1} \quad (5.4)$$

The numerical values of the reduced widths in the weak coupling approximation are shown in Table 3. It is seen that  $\theta_j^2$ s, for  $j \geq 3/2$  are rather insensitive to the values of  $j$  and  $A$ .

The results in the strong coupling approximation depend strongly on the spin of compound states  $I'$ . If we assume  $\Omega' = K' = 0$  and  $I = K = \Omega$ , then we get from equation (4.19)

$$\theta_j^2 = \frac{4[(2j)!]^2(2j+1)}{(I'+2j+1)!(2j-I')} |\{\varphi'^* \varphi\}|^2 \quad (5.5)$$

The numerical values for  $\theta_j^2$  in this approximation are listed in Tables 4 and 5.

Table 3.  $\theta_j^2$  for two identical nucleons in weak coupling approximation

$A=10$

$j' \backslash I'$	1/2	3/2	5/2	7/2	9/2
0	2.0	1.8	1.8	1.8	1.8
2		1.5	1.6	1.7	1.7
4			1.5	1.5	1.5
6				1.5	1.5
8					1.5

$A=50$

$j \backslash I'$	1/2	3/2	4/2	7/2	9/2
0	2.0	1.8	1.8	1.8	1.8
2		1.5	1.6	1.6	1.7
4			1.5	1.5	1.6
6				1.5	1.5
8					1.6

$A=100$

$j \backslash I'$	1/2	3/2	5/2	7/2	9/2
0	2.0	1.8	1.8	1.8	1.8
2		1.4	1.5	1.6	1.7
4			1.4	1.4	1.5
6				1.4	1.4
8					1.4

$A=200$

$j \backslash I'$	1/2	3/2	5/2	7/2	9/2
0	2.0	1.8	1.8	1.8	1.8
2		1.1	1.4	1.5	1.6
4			1.2	1.3	1.3
6				1.3	1.2
8					1.2

Table 4.  $\theta_j^2/|\int \varphi'^* \varphi|^2$  for two identical nucleons in strong coupling approximation

$j \backslash I'$	1/2	3/2	5/2	7/2	9/2
0	2	2	2	2	2
2		2/5	5/7	14/15	12/11
4			1/63	14/99	36/143
6				2/429	3/143
8					1/2431

(B') Two nucleons. ( $n=2$ )

We assume that  $\Omega'=K'\neq 0$  and  $\Omega=K'=I$ . This is the case which appears often in odd-odd nuclei. In reference to the explicit form of the Clebsch-Gordan coefficient in the equation (4.20) we obtain

$$\theta_j^2 = |\int \varphi'^* \varphi|^2 \frac{(2I+1)(2I)!(2I')!}{(I+j+I'+1)!(I-j+I')!}, \quad (5.6)$$

Table 5.  $|\int \varphi'^* \varphi|^2$  for two identical nucleons  $j=5/2$  and  $A=100$ .

$I'$	0	2	4
$\int \zeta^{-1/2} \zeta^{-2}$	0.90	0.75	0.32
$\theta^{25/2}$	1.80	0.54	0.0051

When the  $\tau$ -spin state is taken into account, it must be divided by 2 for odd-odd nuclei.

We see that the predicted reduced widths in the strong and weak coupling approximations are considerably different. In particular in the case of the rotational states of even-even nuclei the strong coupling approximation predicts reduced widths which decrease with the increase of spins, whereas the weak coupling approximation predicts the same reduced widths for all values of spins.

These reduced widths can be obtained from the  $d/p$  stripping reaction data, although there exist inevitable ambiguities in the present theories that neglect the various effects which may be of importance. We presume, however, that the order of magnitude may make a sense and can be compared with experiments. We hope that many experimental data will be available.

The reduced widths for light nuclei have been listed by Lane<sup>3</sup> and compared with the theoretical values deduced from the individual particle model. We have calculated the reduced widths for  $s$  and  $d$  wave nucleons on the weak and strong coupling approximations assuming that  $n=1$  or 2 according to the compound nucleus is odd or even nucleus. These results together with experimental data accumulated by Lane and the theoretical values of the individual particle model are shown in Tables 6 and 7. In column 5 the experimental values are shown and in column 6, 7, and 8 the theoretical value in the individual particle model and the weak and strong approximation of the collective model. For the  $s$ -wave nucleon the collective model predicts the reduced widths almost identical with those of the individual particle model in agreement with the experimental ones. For the  $d$ -wave nucleon the individual particle model predicts a value slightly larger than the experimental one in many cases. Lane ascribed this trend to the sensitivity of the penetration factor to the choice of nuclear radius. But the experimental data have intermediate values between the ones predicted by the weak and strong coupling approximations. If the nuclear radius chosen by Lane is appropriate it would be concluded that the deformation of the core makes the reduced width small. The experiments for medium and heavy nuclei would be useful for the test of the collective model.

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Table 6. Reduced widths of  $s$ -wave nucleon for light nuclei

Nucleus	Excitation of State (MeV)	Spin	Reaction	$\theta_{exp}^2$	$\theta_{ind}^2$	$\theta_{weak}^2$	$\theta_{strong}^2$
Be <sup>7</sup>	~6.4	1/2, 3/2 <sup>+</sup>	Li <sup>6</sup> + $p$	~0.9	1	1	1
B <sup>10</sup>	6.89	1, 2 <sup>-</sup>	Be <sup>9</sup> + $p$	~0.4	0.5	0.5	0.5, 0.4
C <sup>12</sup>	16.57	2 <sup>-</sup>	B <sup>11</sup> + $p$	$\geq 0.4$	0.5	0.5	0.4
C <sup>12</sup>	17.22	1 <sup>-</sup>	B <sup>11</sup> + $p$	$\geq 0.6$	0.5	0.5	0.5
N <sup>13</sup>	2.37	1/2 <sup>+</sup>	C <sup>12</sup> + $p$	1	1	1	1
N <sup>14</sup>	8.06	1 <sup>-</sup>	C <sup>13</sup> + $p$	0.5	0.5	0.5	0.4
N <sup>14</sup>	8.70	0 <sup>-</sup>	C <sup>13</sup> + $p$	$\geq 0.5$	0.5	0.5	0.5
O <sup>15</sup>	~8.0	1/2, 3/2 <sup>+</sup>	N <sup>14</sup> + $p$	$\geq 0.5$	1	1	1
O <sup>15</sup>	~9.8	1/2, 3/2 <sup>+</sup>	N <sup>14</sup> + $p$	$\geq 0.5$	1	1	1
O <sup>16</sup>	13.09	1 <sup>-</sup>	N <sup>15</sup> + $p$	0.4	0.5	0.5	0.4
F <sup>17</sup>	0.54	1/2 <sup>+</sup>	O <sup>16</sup> + $p$	~1	1	1	1

Table 7. Reduced widths of  $d$ -wave nucleon for light nuclei

Nucleus	Excitation of State (MeV)	Spin	Reaction	$\sum \theta_{exp}^2$	$\sum \theta_{ind}^2$	$\sum \theta_{weak}^2$	$\sum \theta_{strong}^2$
Be <sup>10</sup>	7.37	3 <sup>-</sup>	Be <sup>9</sup> + $n$	1	1	0.8	0.6~0.2
B <sup>12</sup>	4.53	3 <sup>-</sup>	B <sup>11</sup> + $n$	0.7	1	0.8	0.6~0.2
N <sup>13</sup>	3.56	5/2 <sup>+</sup>	C <sup>12</sup> + $p$	0.5	1	0.7	0.3
C <sup>13</sup>	7.56	3/2 <sup>+</sup>	C <sup>12</sup> + $n$	0.9	1	0.7	0.5
N <sup>14</sup>	8.90	3 <sup>-</sup>	C <sup>13</sup> + $p$	0.4	0.5	0.4	0.2
N <sup>14</sup>	9.49	2, 3 <sup>-</sup>	C <sup>13</sup> + $p$	0.3	0.5	0.4	0.2~0.1, 0.2
O <sup>16</sup>	12.51	2 <sup>-</sup>	N <sup>15</sup> + $p$	0.4	0.5	0.4	0.2~0.1
O <sup>16</sup>	12.95	2 <sup>-</sup>	N <sup>15</sup> + $p$	0.5	0.5	0.4	0.2~0.1
O <sup>17</sup>	5.08	3/2 <sup>+</sup>	O <sup>16</sup> + $n$	0.5	1	0.7	0.5

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## On the Construction of Potential in Field Theory<sup>†</sup>

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The general formal construction of potential in the field theory is given. It is constructed so as to conserve the normalization of the wave function, irrespective of the switch-off or -on of the potential. The potential is analyzed into its normal part and probability operator. The probability operator is shown to be related to the probability of the system staying in the same state as it does when the interaction is switched off.

From the probability character of the probability operator, we can get a measure of the applicable region of the power series expansion of potential in the coupling constant. The intense meson cloud around the nucleon gives rise to strong singularities in the normal part of the potential, but, on the other hand, the probability of the nucleon to be bare becomes much smaller, and thus it turns out that the actual potential has a lesser singularity. Both potentials derived previously by Taketani-Machida-Ōnuma<sup>1)</sup> (T.M.O.) and its corrected form by Brueckner-Watson<sup>2)</sup> (B.W.) are shown to be two extreme cases in handling this probability operator, but the latter one is more appropriate qualitatively in many cases. The reasonable way of constructing the potential is given in the following paragraphs and the justification of this argument will be illustrated in a separate paper for the pseudo-scalar meson theory<sup>3)</sup>

### § 1. General remark

There are various ways to get the field theoretical potential, most of which give the same results in the so-called static approximation. The method of using the canonical transformation was first applied to this problem by Möller and Rosenfeld in a general form and developed by many authors<sup>1</sup>. This leads necessarily to the Taketani-Machida-Ōnuma<sup>1)</sup> potential, as shown by Nishijima<sup>4)</sup>. Several authors have also attempted to derive the potential from the scattering matrix; usually, however, this method has some very ambiguous points. Especially, the derivation of the potential needs the off-energy-shell matrix elements of the scattering matrix; But usually only the diagonal-energy-shell matrix elements are used expecting that the off- and diagonal-elements are not so different in the adiabatic limit<sup>†</sup>. The third one is due to Tamm and Dancoff<sup>6)</sup> and formulated in a com-

† The idea of this article had been reported by one of the authors (K.S.) at the International Conference on Theoretical Physics held in Japan (1953, Sept.). The contents were read at the meeting of the Physical Society of Japan held in April in Tokyo (1954).

†† On account of this ambiguity, it is very dangerous to estimate recoil effects or non-adiabatic corrections from this sort of calculation. See also, I. Sato, Prog. Theor. Phys. **10** (1953), 323.

pact form by Brueckner and Watson<sup>21)</sup>; this is called the non-adiabatic treatment. This method gives some correction to the above mentioned T. M. O.-potential but this correction turned out to be only effective in the triplet even states in the case of non-relativistic ps-pv coupling theory<sup>22)</sup>. In this treatment, however, the state functionals, on which the potential acts, are neither normalized to unity nor are orthogonal. Therefore the scattering cross-section derived by this treatment cannot give a direct idea about the potential which is defined in the usual scattering problem; the switching off of the non-adiabatic potential disturbs the normalization condition of the state functional\*. Such a cumbersome situation does not occur in the method of canonical transformation.

Nevertheless, the Tamm-Dancoff method has its greater advantage over that of canonical transformation, since it enables one to construct the potential not in the power series expansion of the coupling constant.

We, here, want to analyze the problem of constructing the potential along the same line suggested by the Tamm-Dancoff method, but take into account of the correct normalization and orthogonality of the wave functions on which the potential acts. For this purpose, it will be sufficient to express the Tamm-Dancoff wave functions (on which the potential acts, i. e. the amplitude of the reduced Tamm-Dancoff wave functions) by some appropriate orthonormal wave functions. Then, it is found out that the potential in the usual sense which is derived in this way is composed of two parts. The one is the normal part and the other is the probability operator. The probability operator comes from the expression giving the Tamm-Dancoff wave functions in terms of the ortho-normal wave functions. It will be shown that the normal part corresponds to the two-nucleon analogy of the contribution from Feynman graphs which gives the expectation value of the self-energy and are not reducible to single-nucleon graphs, whereas the probability operator corresponds to the two-nucleon analogy of the self-energy contribution with mass subtracted<sup>23)</sup>. The probability operator is intimately related to the probability of two nucleons not accompanied with mesons.

In the T.M.O. or other perturbation theoretic treatment, this probability operator is also expanded along with the normal part. Thus, for example, the 4-th order potential contains a product of the 2nd order potential and 2nd order probability operator. If one takes into account the power series expansion of the probability operator up to infinite order, then its expectation value, i. e. the probability of the nucleons to be bare, should be less than one and positive. In the ordinary treatment, however, this series is necessarily retained only to some order, and hence, for example, the potential, the 2nd plus 4th, is meaningful only at most in the region where the expectation values of this expanded probability operator is less than one and larger than zero. In order that there may exist such an applicable region at all, it is absolutely necessary to include in the probability operator the probability of two nucleons to be bare even in an infinite separation as well as the probability depending on the two-nucleon separation. The former quantity is divergent in any present theory without applying some cut-off method. Thus,

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\* Hence, the potential obtained in this way is not Hermitian in general.

we are forced in this case to adopt the cut-off hypothesis about the higher momentum mesons to push our analysis further. We might expect that the potential of T.M.O. is constructed assuming the probability operator could be expanded in the power series and approximated by finite terms, which is of course the one extreme case of treatment.

On the other hand, if one assumes the probability operator to be not so different from one, which amounts to the same assumption as in T.M.O., but here one does not use its expansion but replace this operator always by one, one gets the same potential as obtained by Brueckner and Watson<sup>2)</sup> (B.W.). The fact that the term discarded in Brueckner and Watson's treatment is related to the change of the probability of two nucleons to be bare can be shown in the following argument; the potential constructed by reducing Tamm-Dancoff equations to the two-nucleon sub-space contains no intermediate states where only two nucleons are present; this corresponds to the contribution of the proper single nucleon self-energy (Fig. 1a). Especially the 2nd order graph (Fig. 1b)

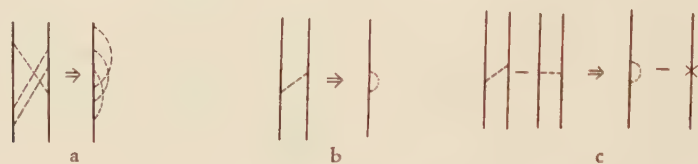


Fig. 1.

corresponds to the 2nd order self-energy. But since this contribution contains the total energy of the system which represents the fact that two nucleons are not free, there remain some observable effects even after the static potential is subtracted which is independent of the total energy of the system; it corresponds to the mass subtraction in the single nucleon case (Fig. 1c). The latter procedure gives the quantity which is proportional to the change of the probability of the nucleon to be bare as has already been shown by Feynman<sup>7)</sup>. Thus the neglect of the difference in Fig. 1c. corresponds to the neglect of the change of the probability of nucleons to be bare. We can see in this way that the modified T.M.O. potential obtained by Brueckner and Watson<sup>2)</sup> is also another extreme case of approximation.

In the single nucleon problem, the contribution of graphs such as given in Fig. 1c. is always bigger than other radiative corrections, e.g. vertex correction (Fig. 2). In the case of non-relativistic pseudoscalar meson theory with pseudovector coupling, (a) is 9 times larger than (b), and in the two nucleon problem the difference in Fig. 1c. is



Fig. 2.

large enough to give the difference between the T.M.O. potential<sup>1)</sup> and its modified form (B.W.). Thus, we must handle the probability operator more carefully, first because it is related to the probability of nucleons to be bare, second because it may give the largest

contribution when expanded.\* As to the singularities of the potential, we see immediately that if the normal part exhibits a very singular character, then the probability of nucleons to be bare becomes also very small near the singular region of the normal part since the meson cloud is rather intense in this region, and so the potential may not be so singular. In fact, in the case of non-relativistic neutral scalar meson theory, the normal part contains singularities such as  $1/r$ ,  $1/r^2$ , ..., but the probability of the two nucleons to be bare also becomes smaller and smaller when the two nucleons approach together and there remains only  $1/r$  singularity in the potential. (In the case of neutral scalar theory, the probability operator simply expresses the probability). The usual perturbation theory expands the probability operator into power series and the potential thus obtained becomes apparently much singular than the real one, except in the above case where each power series cancels exactly. The B.W. method\*\* does not take into account the decrease of the probability and so this treatment will also be sometimes misleading<sup>5)</sup>.

One could imagine the true potential to lie between the T.M.O. potential and its modified form (B.W.) from the above arguments, because the correct potential in the ordinary sense will be obtained by multiplying the probability operator upon the modified T.M.O. potential, whereas the T.M.O. potential is correct when this probability is really small and can be expanded into power series.

## § 2. Construction of potential

The Schrodinger equation of the system composed of two nucleons and mesons is given by

$$(E - H_0) \Psi = H \Psi \quad (2.1)$$

where  $E$  is the total energy of the system,  $H_0$  the free Hamiltonian for the meson and nucleon field, and  $H$  the interaction Hamiltonian between them. In order to solve Eq. (2.1) along the line suggested by Tamm and Dancoff<sup>6)</sup>, we take a somewhat generalized condition for  $\Psi$  in the limit  $H \rightarrow 0$ ; when  $H \rightarrow 0$ ,  $\Psi$  is in a subspace  $\bar{n}$  which is complete in some sense. For the purpose of constructing the nuclear potential,  $\bar{n}$  is taken to be the subspace with no meson which however contains a complete set of two nucleon wave functions. Introducing the projection operator  $V_{\bar{n}}$  which has the property;

$$\begin{cases} V_{\bar{n}} \Phi = \Phi & \text{for } \Phi \text{ in } \bar{n}; \\ V_{\bar{n}} \Phi = 0 & \text{for } \Phi \text{ orthogonal to } \bar{n}, \end{cases} \quad (2.2)$$

eq. (2.1) can be separated into two parts;

\* This is one of the reasons why the perturbation theory may not be applicable to the probability operator (coupling is rather strong in this part), though it is generally verified that this theory can safely be applied to the normal part at least in the outer region.

\*\* We use here the "B.W." method not in its original form but in the sense of omitting the probability change, because in this way we can get the same potential in the 4-th order adiabatic limit,



$$(E-H_0) V_n^- \Psi = V_n^- H V_n^- \Psi + V_n^- H (1 - V_n^-) \Psi, \quad (2.3)$$

$$(E-H_0) (1 - V_n^-) \Psi = (1 - V_n^-) H (1 - V_n^-) \Psi + (1 - V_n^-) H V_n^- \Psi. \quad (2.4)$$

Then, by our condition for  $\Psi$ , we can solve eq. (2.4) for  $(1 - V_n^-) \Psi$  regarding the term  $(1 - V_n^-) H V_n^- \Psi$  as a source; the formal solution can be written as follows:

$$(1 - V_n^-) \Psi = \frac{1}{1 - \frac{1}{E-H_0} (1 - V_n^-) H} \frac{1}{E-H_0} (1 - V_n^-) H V_n^- \Psi. \quad (2.5)$$

Putting this expression into eq. (2.3), we have

$$(E-H_0) V_n^- \Psi = \left( V_n^- H \frac{1}{1 - \frac{1}{E-H_0} (1 - V_n^-) H} V_n^- \right) V_n^- \Psi, \quad (2.6)$$

with

$$\Psi = \frac{1}{1 - \frac{1}{E-H_0} (1 - V_n^-) H} V_n^- \Psi. \quad (2.7)$$

The quantity in the bracket in eq. (2.6) is nothing but the potential in the nonadiabatic treatment given by T.D. or B.W.<sup>(2)</sup>. It should be noted however that the wave functions  $V_n^- \Psi$ 's are neither normalized to unity nor orthogonal, because we have from eq. (2.7):

$$\begin{aligned} \langle \Psi_i | \Psi_j \rangle &= \langle V_n^- \Psi_i | V_n^- \frac{1}{1 - H(1 - V_n^-) \frac{1}{E_i - H_0}} \\ &\times \frac{1}{1 - \frac{1}{E_j - H_0} (1 - V_n^-) H} V_n^- | V_n^- \Psi_j \rangle = \delta_{ij}. \end{aligned} \quad (2.8)$$

Thus the switching off of the potential of (2.6) changes the normalization of the wave function  $V_n^- \Psi$ , and, as a consequence this potential is in general non-Hermitian and has not the usual property of the potential.\*

In order to obtain the relation between  $V_n^- \Psi$ 's and the wave functions in the ordinary sense, we first rewrite the relation (2.7) in a form independent of " $E$ ". This can be done easily by the iterated use of equation (2.1) and the result is as follows:

$$\Psi = J V_n^- \Psi \quad (\equiv J(E) V_n^- \Psi), \quad (2.9)$$

\* The situation is just similar to the case where one reduces the 4-component Dirac wave equation to the two-component one. If one does not take into account the normalization of the two component wave function correctly, one meets the well-known imaginary potential.

$$J = \left[ 1 + \frac{1}{1 - \frac{1 - V_n^-}{H_0^2 - H_0} H} \frac{1 - V_n^-}{H_0^2 - H_0} H \right]^{-1} \cdot \left[ \frac{1}{1 - \frac{1 - V_n^-}{H_0^2 - H_0} H} \frac{1 - V_n^-}{H_0^2 - H_0} H \right];$$

$$\left( J(E) = \frac{1}{1 - \frac{1 - V_n^-}{E - H_0} H} \right), \quad (2.10)$$

$$FV_n^-G = \frac{1}{1 - \frac{1 - V_n^-}{H_0^2 - H_0} H} V_n^-GV_2 - \frac{1}{1 - \frac{1 - V_n^-}{H_0^2 - H_0} H} \frac{1 - V_n^-}{H_0^2 - H_0} FV_n^-GV_2H, \quad (2.11)$$

where  $G$  is an arbitrary operator. Supersufixed  $H_0^i (=H_0)$  operates only on  $V_{\bar{n}_i}^-$  and  $V_{\bar{n}_2}^-$  with  $\bar{n}$  omitted is an identity operator. Then, we have from eq. (2.8)

$$\langle \Psi_i | \Psi_j \rangle = \langle V_n^- \Psi_i | V_n^- (J^+ J) V_n^- | V_n^- \Psi_j \rangle = \delta_{ij}. \quad (2.12)$$

Remembering that  $\bar{n}$  spans a complete sub-space, we can define the wave functions in the ordinary sense as follows;

$$| \chi^i \rangle = V_n^- (J^+ J)^{1/2} V_n^- | \Psi_j \rangle; \quad \langle \chi_i | \chi_j \rangle = \delta_{ij}. \quad (2.13)$$

The equation for our wave function  $\chi$  is from eq. (2.6) :

$$E\chi = V_n^- (J^+ J)^{1/2} V_n^- (H_0 + HJ(E)) V_n^- (J^+ J)^{-1/2} V_n^- \chi. \quad (2.14)$$

Then, expressing  $J(E)$  by  $J$  (since in the right hand side  $HJ(E) V_n^- (J^+ J)^{-1/2} V_n^- \chi = HJ(E) V_n^- \Psi = HJV_n^- \Psi$ )\*

$$E\chi = V_n^- (J^+ J)^{-1/2} V_n^- (J^+ (H_0 + H) J) V_n^- (J^+ J)^{-1/2} V_n^- \chi. \quad (2.15)$$

Hence

$$(E - H_0^N) \chi = v \chi,$$

$$v = V_n^- (J^+ J)^{-1/2} V_n^- (J^+ (H_0 + H) J) V_n^- (J^+ J)^{-1/2} V_n^- - V_n^- H_0 V_n^-, \quad (2.16)$$

where  $H_0^N = V_n^- H_0 V_n^-$  is, say, the free two-nucleon Hamiltonian. The potential obtained in (2.16) corresponds to the potential in the ordinary sense. We can find the meaning of the operator  $J$  from the expression (2.7) and

$$\Psi = JV_n^- (J^+ J)^{-1/2} V_n^- \chi (\equiv J(E) V_n^- (J^+ J)^{-1/2} V_n^- \chi). \quad (2.17)$$

This shows that the operator  $J(E)$  involves virtual transitions only to states not in  $\bar{n}$ . Thus, it gives contributions from proper graphs in the  $S$ -matrix terminology. The potential  $V_n^- (J^+ (H_0 + H) J) V_n^-$  may be called the normal part in this sense. On the other hand, the expression

\* It is not necessary to write  $J(E)$  in terms of  $J$ ; the following considerations can be applied in the form (2.14) as well as (2.15). For clearness of discussion, we took the expression (2.15).

$$\sum_{\bar{n}} |\langle \phi_{\bar{n}} | \Psi \rangle|^2 = \langle V_{\bar{n}} \Psi | V_{\bar{n}} \Psi \rangle = \langle \chi | V_{\bar{n}} (J^+ J)^{-1} V_{\bar{n}} | \chi \rangle, \\ (\phi_{\bar{n}}; \text{an arbitrary complete set in subspace } \bar{n}), \quad (2.18)^*$$

represents the probability of the system to be found in the state  $\bar{n}$  when the interaction is acting. Thus, one may call the operator  $V_{\bar{n}} (J^+ J)^{-1} V_{\bar{n}}$  the probability operator. Its expectation value gives just this probability.

The above procedure has the advantage to make it possible to separate proper contributions and the probability operator. The resulting potential is just the same as that obtained by the canonical transformation (when expanded into power series).

### § 3. The analysis of potential in static limit

We now discuss the potential construction in the static limit. Denoting no meson sub-space by 0, we take  $V_{\bar{n}}$  as  $V_0$ . The above obtained Schroedinger equation (2.16) becomes in this case:

$$(E - H_0^N) \chi = \tau \chi, \\ v = V_0 (J^+ J)^{-1/2} V_0 (J(H_0 + H)J) V_0 (J^+ J)^{-1/2} V_0 - H_0^N, \quad (3.1)$$

with the state functional (from eq. (2.17))

$$\Psi = J V_0 (J^+ J)^{-1/2} V_0 \chi. \quad (3.2)$$

To obtain the power series expression for  $\tau$ , we expand  $J$  (from eq. (2.10)) into the following form (assuming  $H$  is linear in meson variables):

$$J = \left[ 1 + \frac{1 - V_0}{H_0^1 - H_0} H + \frac{1 - V_0}{H_0^1 - H_0} H \frac{1 - V_0}{H_0^1 - H_0} H \right. \\ + \frac{1 - V_0}{H_0^1 - H_0} H \frac{1 - V_0}{H_0^1 - H_0} H \frac{1 - V_0}{H_0^1 - H_0} H \\ \left. + \frac{1 - V_0}{(H_0^3 - H_0)(H_0^2 - H_0)(H_0^1 - H_0)} H V_{03} H V_{02} H + \dots \right] V_{01}. \quad (3.3)$$

Then, the static potential can be obtained by ignoring  $H_0^N$  from the expression of  $v$ . In this approximation  $H_0^1$  and  $H_0^3$  appearing in eq. (3.3) are also neglected (because the state 0 is that of no meson, and  $H_0^1$  and  $H_0^3$  become simply  $H_0^N$ ). Thus, we have after simple computations:

$$\{V_0 (J^+ (H_0 + H)J) V_0\}_{(S, L)} \\ = \left( H \frac{1 - V_0}{-H_0} H \right)_{00} + \left( H \frac{1 - V_0}{-H_0} H \frac{1 - V_0}{-H_0} H \frac{1 - V_0}{-H_0} H \right)_{00} + \dots, \quad (3.4)$$

\*  $\sum_{\bar{n}} \langle \phi_{\bar{n}} | \Psi \rangle^2 = \sum_{\bar{n}} \langle \phi_{\bar{n}} | V_{\bar{n}} | \Psi \rangle^2 = \langle V_{\bar{n}} \Psi | V_{\bar{n}} \Psi \rangle$ , because of the presence of  $V_{\bar{n}}$ , and  $V_{\bar{n}} (J^+ J)^{-1/2} V_{\bar{n}} (J^+ J)^{-1} V_{\bar{n}} = V_{\bar{n}} (J^+ J)^{-1} V_{\bar{n}}$  by definition.

and

$$[V_0(J^+J)^{-1/2}I_0^+](S.L.) = \frac{1}{\left(1 + \left(H \frac{1-V_0}{(-H_0)^2} H\right)_{00} + \dots\right)^{1/2}}, \quad (3.5)$$

where  $(\ )_{00}$  means the matrix element in the no-meson state and can be understood as operators containing two-nucleon variables. Thus we have

$$[\tau^+](S.L.) = \frac{1}{\left(1 + \left(H \frac{1-V_0}{(-H_0)^2} H\right)_{00} + \dots\right)^{1/2}} \left[ \left( H \frac{1-V_0}{-H_0} H \right)_{00} + \left( H \frac{1-V_0}{-H_0} H \frac{1-V_0}{-H_0} H \frac{1-V_0}{-H_0} H \right)_{00} + \dots \right] \times \frac{1}{\left(1 + \left(H \frac{1-V_0}{(-H_0)^2} H\right)_{00} + \dots\right)^{1/2}}, \quad (3.6)$$

where (S.L.) means static limit. We are now in a position to discuss the situation for the T. M. O.<sup>(1)</sup> and its modified (B. W.) potentials.

First of all, we note that  $[V_0(J^+J)^{-1}V_0]$  is the probability operator in the static limit and so

$$\langle \chi_{(S.L.)} | [V_0(J^+J)^{-1}V_0]_{(S.L.)} | \chi_{(S.L.)} \rangle = 1 - P_{D(S.L.)}, \quad (3.7)$$

is the probability of nucleons to be bare in this limit. Therefore the condition,

$$0 < 1 - P_{D(S.L.)} < 1, \quad (0 < P_{D(S.L.)} < 1), \quad (3.8)$$

should hold. If we expand this quantity appearing in eq. (3.6) into power series in the coupling constant, we obtain finally,

$$[v]_{(S.L.)}^{T.M.O.} = \left( H \frac{1-V_0}{-H_0} H \right)_{00} + \left( H \frac{1-V_0}{-H_0} H \frac{1-V_0}{-H_0} H \frac{1-V_0}{-H_0} H \right)_{00} - \frac{1}{2} \left( H \frac{1-V_0}{(-H_0)^2} H \right)_{00} \left( H \frac{1-V_0}{-H_0} H \right)_{00} - \left( H \frac{1-V_0}{-H_0} H \right)_{00} \frac{1}{2} \left( H \frac{1-V_0}{(-H_0)^2} H \right)_{00} + \dots, \quad (3.9)$$

The expression (3.9) is just the T. M. O. potential up to 4-th order. But, in order that this potential may be workable, the expansion of (3.7) should not contradict with the condition (3.8):

$$0 < \langle \chi_{(S.L.)} | 1 - \left( H \frac{1-V_0}{(-H_0)^2} H \right)_{00} | \chi_{(S.L.)} \rangle < 1, \quad (3.10)$$



because in eq. (3.9) only the second order change in the probability operator is included, i. e. we have approximated it by its 2nd order. The condition (3.10) gives a measure of an applicable region of the power series of the potential expanded up to 4-th order. Of course, if one takes into account its expansion up to infinite order in (3.5), the condition is always satisfied, but if one uses the series broken at somewhere, then the similar condition to (3.10) should be imposed for the applicable region of this broken potential. Obviously, in (3.10) the dissociation probability which does not depend on the two-nucleon separation must be included; otherwise (3.10) gives rise to an apparent contradiction in some state where  $\left(H \frac{1-V_0}{(-H_0)^2} H\right)_{00} < 0$ ; and here, we are forced to take some sort of cut-off hypothesis. Even if we assume a somewhat lower cut-off (e. g.  $\sim 3\mu$ ), the pseudoscalar meson theory with pseudovector coupling gives the potential which is not compatible with the condition (3.10) inside the range of  $r \lesssim \frac{1}{2\mu}$ .<sup>31)</sup>

On the other hand, it will be obvious that if we put

$$[V_0(J^+J) - V_0]^{(S,L)} \rightarrow 1, \quad (3.11)$$

then the potential becomes

$$[v]_{(S,L)}^{B.W.} = \left(H \frac{1-V_0}{-H_0} H\right)_{00} + \left(H \frac{1-V_0}{-H_0} H \frac{1-V_0}{-H_0} H \frac{1-V_0}{-H_0} H\right)_{00} + \dots, \quad (3.12)$$

and just coincide with the modified T. M. O. potential (B. W.). The difference between (3.12) and (3.9) corresponds to Klein's velocity dependent correction, but not exactly so, because Klein treated the problem with T. D. wave function, instead of our normalized wave function, and the potential obtained was

$$[v]_{(S,L)}^{B.W.} - \left(H \frac{1-V_0}{(-H_0)^2} H\right)_{00} \cdot \left(H \frac{1-V_0}{-H_0} H\right)_{00}, \quad (3.13)$$

and is obviously non-Hermitian in general, whereas our last two terms in (3.8) are Hermitian. The reason why the condition (3.11) gave the same potential as the modified T. M. O. potential (B. W.) was already given in § 1. Writing the omitted term explicitly in their treatment (4-th order):

$$\left[ \lim_{E \rightarrow 0} \frac{1}{E} \left\{ \left(H \frac{1-V_0}{E-H_0} H\right)_{00}^{(S,L)} - \left(H \frac{1-V_0}{-H_0} H\right)_{00}^{(S,L)} \right\} \right] E(V_0 \Psi), \quad (3.14)$$

it is easily seen that this corresponds to the self-energy subtraction, and so the quantity in the bracket is related to the decrease of the probability of two nucleons not accompanied with mesons.<sup>7)</sup> The assumption (3.11) may be expected from the physical argument that the dissociation probability is not so large<sup>31)</sup>. It may be said from this consideration that the modified T. M. O. potential would have some correct features in a limited sense. But the complete neglect of space dependence of  $P_{D(S,L)}$  is not relevant, since obviously  $1 - P_{D(S,L)}$  becomes considerably smaller when the meson cloud around the nucleons becomes intense.

Thus, it turns out that the above two treatments are two extreme cases in handling the probability operator. That the latter standpoint is not always consistent was illustrated by Ruderman and Henley<sup>5)</sup> by an example of neutral scalar meson theory; with the normal part of (3.6) only, the potential contains singularities such as  $1/r$ ,  $1/r^2$ ,  $1/r^3$ , ... and  $g^2$ ,  $g^4$ , ... dependence, quite different from the exact solution which is of  $1/r$  and  $g^2$  dependence. Moreover, the power series expansion of the probability operator may be also misleading in general as to the singularity, because from the expression (3.6) the singularity appearing in the denominator is not always cancelled in the power series expansion; the rest gives too singular potential; the neutral scalar meson theory is only an exceptional case.

We are thus led to the conclusion that the probability operator (appearing in the expression (3.6)) should be treated separately. This will suggest to keep the probability operator as in the expression (3.6). In the case of the neutral scalar meson theory, (3.6) can be rewritten as

$$[v]_{(S.L.)}^{N.S.} = \frac{1}{\left(1 + \left(H \frac{1-V_0}{(-H_0)} H\right)_{00} + \dots\right)} \left( \left(H \frac{1-V_0}{-H_0} H\right)_{00} + \left(H \frac{1-V_0}{-H_0} H \frac{1-V_0}{-H_0} H \frac{1-V_0}{-H_0} H\right)_{00} + \dots \right), \quad (3.15)$$

and if one takes into account the probability of two nucleons to be bare (first factor in the right hand side of (3.15) in this case the probability operator just expresses the probability) up to 2nd order in the denominator and the normal part up to 4-th, or, in general, the probability up to  $2n$ -th order and the normal part up to  $2(n+1)$ -th order, one obtains always a correct answer. The condition (3.8) is always satisfied. This procedure will give the potential which is just between the two extreme cases, i.e. T.M.O.'s and of B.W.'s, and of lesser singularity. The actual problem will be handled in a separate paper with this formalism; the one illustration will be found in the case of charged scalar non-relativistic meson theory attacked from slightly different treatment<sup>3)</sup>.

It is to be noted here that the expansion of the probability operator (3.6) gives (3.9) and the difference between (3.9) (T.M.O.) and (3.12) (B.W.), i.e. the last two terms in (3.9), is large enough to change the attractive potential derived by B.W. in the triplet-even state to the repulsive T.M.O. potential in the pseudoscalar, pseudo-vector coupling meson theory. The treatment suggested above however gives the potential with the same sign as the corrected T.M.O. (B.W.), because  $P_{D(S.L.)}$  is always between 0 and 1; in this sense, the Modified T.M.O. potential (B.W.) is qualitatively correct, and the sign change above mentioned is due to the contradiction with the condition (3.10). Thus, we might expect the correct potential in this case has a lesser singularity (owing to the multiplication of probability operator) with the same sign as the modified T.M.O. (B.W.) potential. This expectation will also be justified by the fact that the correction to the change in the probability due to mesons present is larger than the other corrections such as vertex

correction, when the freedom of coupling of the nucleon and meson is large. (see argument concerning Fig. 2 in I).

#### § 4. Concluding remarks

We have analyzed the potential constructed from the field theory which has the usual meaning of potential into its normal part and the probability operator. Our arguments lead to the conclusion that the probability operator is very largely affected by the interaction even more than the normal part. The potentials given by T.M.O. and the one corrected by B.W. are considered as two extreme cases in handling this probability operator. We met in the course of our discussion the divergence difficulties of the approximate expression for the probability of nucleons to be bare and are obliged to use some cut-off hypothesis. Without this hypothesis, the potentials derived would keep us from making any reliable analysis and interpretation. We expect this point may give rise to a new difficulty even in the present renormalizable field theory.

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# The Effects of Heavy Particles on $\pi$ -meson-proton Scattering

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Recent experimental results on total  $\pi^-p$  scattering cross section show that the broad second maximum exists at about 900 MeV. This high energy region lies above the threshold of the production of heavy particles, so that the radiative corrections due to heavy unstable particles to pion-nucleon scattering would be fairly large, as suggested by Yang, since the intermediate state exists in which the energy-momentum conservation holds. Hence we calculate here the radiative corrections due to heavy particles to  $\pi^-p$  scattering.

Assuming  $A$ -particle is a spin  $1/2$  particle with isotopic spin 0 or 1 and  $\theta$ -particle is a spin 0 or 1 particle with isotopic spin  $1/2$ , and introducing interactions  $(\pi NN)$ ,  $(\theta AN)$  and  $(\pi AA)$  which are considered as strong interactions by the even-odd rules, we carry out the fourth order perturbational calculations.

The results are as follows:

- (1)  $SE$  type  $S(S)$  corrections are larger at lower energies and they do not seem to be able to make the cross section larger in the high energy region.
- (2)  $SE$  type  $V(V)$  corrections have more desirable tendencies than in the  $S(S)$  case, yet they make only the cross section flat at the high energy end.
- (3)  $V$  type  $S(S)$  corrections have possibilities to raise the cross section to be compared with the experimental values.
- (4) Considering that the radiative corrections due to heavy particles to  $\pi^-p$  scattering are rather large, we might conclude that it gives us a support to the validity of the charge independence for heavy unstable particles.

## § 1. Introduction

The fact that the frequency of occurrence of heavy particles is about three per-cent of that of  $\pi$ -mesons<sup>1)</sup> seems to indicate that the strength of coupling of heavy particles is of the same order of magnitude as that of  $\pi$ -mesons. Hence we could not neglect the effects of heavy particles on pion reactions.

On the other hand the recent experimental data on the total cross sections of negative pion-proton scattering measured by attenuation using very high energy pion beams from the Brookhaven cosmotron show that, in  $\pi^-p$  scattering, the broad second maximum exists at about 900 MeV of pion kinetic energy in laboratory system.<sup>2)\*</sup>

This high energy region lies above the threshold of the production of heavy particles. Therefore, the radiative corrections due to heavy particles to pion-nucleon scattering would be fairly large, as suggested by Yang,<sup>3)</sup> since the intermediate state exists where energy-momentum

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\*) On the contrary, in the  $\pi^-n$  scattering, they report that the broad minimum exists in the same energy region.



conservation holds. Hence we calculate here the radiative corrections due to heavy particles to  $\pi^-p$  scattering.

At present we do not know much about the types and strengths of couplings of heavy particles as well as their spins, and so, concerning their isotopic spins, we adopt here the Nakano-Nishijima model,<sup>1)</sup> according to which we assume that the  $A$ -particle has isotopic spin 0 (neutral theory) or 1 (symmetrical theory) and the  $\theta$ -particle has isotopic spin 1/2. Then we take interactions  $(\pi NN)^{**}$ ,  $(\theta AN)$  and  $(\pi AA)$ , which are considered as strong interactions by the even-odd rules.<sup>4)5)</sup>

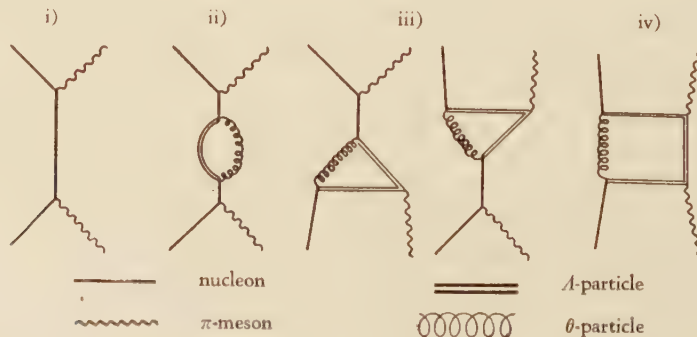
The experimental data<sup>1)</sup> suggest an angular correlation between  $A$ -decay planes and production planes, which may mean that  $A$ -particles have large spin, but as the statistics are still poor and we cannot treat with large spin, we assume that the  $A$ -particle is a Fermion with spin 1/2 which obeys the usual Dirac equation and the  $\theta$ -particle is a Boson with spin 0 or 1. Further, in order that the decay process  $\theta \rightarrow \pi^+ + \pi^-$  may be allowed,  $\theta$  must have either even spin and even parity or odd spin and odd parity, so that we choose here scalar and vector theories for  $\theta$ -particle.

The coupling constant of the interaction  $(\pi NN)$  will be designated as  $g$  and those of  $(\theta AN)$  and  $(\pi AA)$ , which are considered of the same order of magnitude, as  $G$ .

In this paper we carry out the fourth order perturbational calculations to estimate the effects to  $\pi^-p$  scattering due to heavy particles, introducing the above three interactions phenomenologically. Of course, as is well known, this approximation gives extremely large  $S$  wave contributions at low energies and cannot be used at all below about 200 MeV.

## § 2. Notations

First, we summarize the notations and nomenclatures which will later be used frequently. There are three types of the possible lowest-order Feynman-Dyson diagrams which contain heavy particles in the intermediate states, namely, (1) Self-Energy Type, (2) Vertex Type and (3) Convergent Type, as shown in Fig. 1, together with the uncorrected lowest-order diagram.



\*\*) Here  $N$  stands for nucleon, not neutron which is designated by  $n$  above.

Next the various energy-momentum four-vectors and masses are denoted as follows :

particle	energy-momentum four-vector (in C. M. system)	mass
incident $\pi^-$	$q_0(E(q_0), \vec{q}_0)$	$\mu$
final $\pi$	$q(E(q), \vec{q})$	$\mu$
initial $p$	$p_0(E(p_0), \vec{p}_0)$	$m$
final nucleon	$p(E(p), \vec{p})$	$m$
virtual $A$		$M$
virtual $\theta$	$k$	$\kappa$

The values of masses we adopt are

$$\begin{aligned}
 m &= 1836 m_e, \\
 \mu &= 272 m_e, & (m_e: \text{electron mass}) \\
 M &= 2200 m_e, \\
 \kappa &= 1000 m_e,
 \end{aligned}$$

Other notations are quite similar to Feynman's.<sup>(6)</sup> For example,  $\not{p} = \not{p}_\mu \gamma_\mu$ , and  $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$ ,  $\delta_{11} = \delta_{22} = \delta_{33} = -1$ ,  $\delta_{44} = 1$  and  $\gamma_5^2 = -1$ .

Wave functions of nucleon,  $\pi$ -meson,  $A$ - and  $\theta$ -particle are denoted by  $\psi$ ,  $\phi$ ,  $\Psi$  and  $\Phi$ , respectively.

Here we must classify the various processes. The results are :

Theory Type	neutral theory			symmetrical theory		
	SE	V	C	SE	V	C
elastic	0	X	X	0	0	0
charge exchange	0	0	X	0	0	0

(0 : possible, X : impossible)

In order to calculate the total cross section, we must, of course, consider all possible processes for each theory, this is, however, very hard to be done and, what is worse,  $C$  type can not be calculated for the symmetrical theory in a closed form (Ashkin, Simon and Marshak<sup>(7)</sup> calculated  $C$  type in the case of corrections due to  $\pi$ -mesons in pion-scattering in the Thomson limit. But we could not find any suitable approximation in our case where the wide energy region should be taken into account.). Hence we pick up some of the above processes as typical examples and combine each of them with the uncorrected lowest order process to find the effect of heavy particles in each case.

### § 3. Neutral SE type

Two interactions,

$(\pi N N)$ : symmetrical  $Ps(Ps)$

$(\theta \Lambda N)$ : neutral  $S(S)$  or neutral  $V(V)$

are taken, that is, the interaction Hamiltonians are

$$H_{\pi NN} = g \bar{\Psi} \boldsymbol{\tau} \gamma_5 \boldsymbol{\phi} \cdot \boldsymbol{\phi}, \quad (3.1)$$

$$H_{\theta \Lambda N} = G \bar{\Psi} \boldsymbol{\phi} \boldsymbol{\phi} + \text{Herm. conj.} \quad \text{for } S(S) \quad (3.2)$$

$$= G \bar{\Psi} \gamma_\mu \boldsymbol{\phi} \boldsymbol{\phi}_\mu + \text{Herm. conj.} \quad \text{for } V(V) \quad (3.2')$$

where bold letters are the vectors in isotopic space and dot means scalar product of these vectors.

a) Elastic scattering  $(\pi^- + p \rightarrow \pi^- + p)$

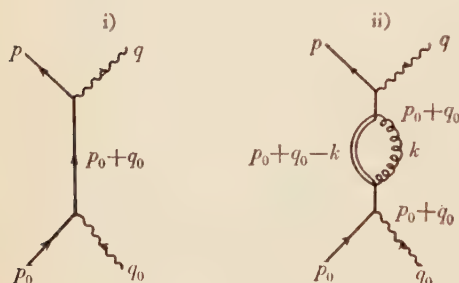


Fig. 2.

In this case there are two kinds of diagrams as shown in Fig. 2 and the matrix elements can be put into the following form:

$$M_{\alpha} = -g^2 \bar{u}(p) \tau_i \gamma_5 S_F'(p_0 + q_0) \tau_j \gamma_5 u(p_0). \quad (3.3)$$

Renormalizing mass and charge by the criterion that  $S_F'$  becomes  $S_F$  for free particle case, we know according to Brueckner, Gell-Mann and Goldberger<sup>5)</sup>,  $S_F'$  can be written as

$$S_F'(p) = \frac{i}{p-m} \left\{ 1 + \frac{G^2}{4\pi} \int_0^1 dx \left[ \frac{(1-x)p + M}{p-m} \log \frac{\phi(p^2)}{\phi(m^2)} + \frac{2mx(1-x)[m(1-x) + M]}{\phi(m^2)} \right] \right\} \quad (3.4)$$

for  $S(S)$  and

$$S_F'(p) = \frac{i}{p-m} \left\{ 1 + \frac{G^2}{4\pi} \int_0^1 dx \left[ \frac{2(1-x)p - 4M}{p-m} \log \frac{\phi(p^2)}{\phi(m^2)} + \frac{4mx(1-x)[m(1-x) - 2M]}{\phi(m^2)} \right] \right. \\ \left. + \frac{G^2}{4\pi} \frac{1}{\kappa^2} \int_0^1 dx \left[ \frac{-p(1-x) + 2M}{p-m} \left\{ \phi(p^2) \log \frac{\phi(p^2)}{\phi(m^2)} + \frac{1}{2} (p^2 - m^2) - m^2 x(1-x)^2 + 2mMx(1-x) \right\} \right] \right\} \quad (3.4')$$

for  $V(V)$ , where

$$\phi(p^2) = -p^2 x(1-x) + M^2 x + \kappa^2(1-x).$$

The above parametric integrations can be carried out easily in the C. M. system, as

Karplus, Kivelson and Martin<sup>9)</sup> have done. As the poles occur in the integration interval  $[0, 1]$  of the parameter  $x$  above the threshold of the production of heavy particles (about 800 MeV), we must choose the suitable contours to represent the outgoing waves in the residue calculations<sup>9)</sup>. After the parametric integrations are carried out,  $S_F'$  becomes as follows

$$S_F'(\mathbf{p}_0 + \mathbf{q}_0) = \frac{im^3}{[\mu^2 + 2(p_0 q_0)]^2} \left\{ A + B \frac{\mathbf{p}_0 + \mathbf{q}_0}{m} \right\}, \quad (3.5)$$

where

$$A = G + iH, \quad B = D + iF, \quad (3.6)$$

$G$  and  $D$  can be represented by the combinations of various real parametric integrals, and  $H$  and  $F$  come from residues and vanish for energies less than the threshold of the production of heavy particles.

Using the above  $S_F'$ , we get, after the conventional spur calculations, the total elastic scattering cross section  $\sigma_{elas}$ ;

$$\begin{aligned} \sigma_{elas} = & \frac{8\pi g^4}{m^2} \left[ \frac{m}{E(p_0) + E(q_0)} \right]^2 \left[ \frac{m^2}{\mu^2 + 2E(p_0)E(q_0)} \right]^4 \times \\ & \times \left[ \left\{ |A|^2 + |B|^2 \frac{[E(p_0) + E(q_0)]^2}{m^2} \right\} \left( 1 + \frac{E^2(p_0)}{m^2} \right) - \right. \\ & \left. - \frac{2E(p_0)[E(p_0) + E(q_0)]}{m^2} \Re(A^* B) \right]. \end{aligned} \quad (3.7)$$

b) Charge exchange scattering ( $\pi^- + p \rightarrow \pi^0 + n$ )

In this case there are two more graphs besides those in Fig. 2 (Fig. 3). Fig. 3 (ii) does not contribute appreciably, and so we neglect (ii) here.

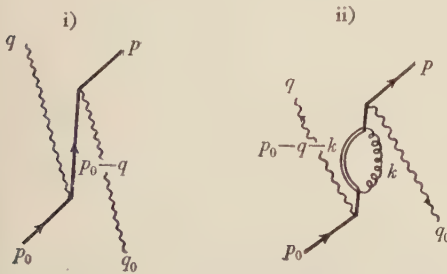


Fig. 3.

The matrix element is given by

$$\begin{aligned} M_{ch\ ex} = & -g^2 \bar{u}(\mathbf{p}) \tau_4 \gamma_5 \left[ \frac{i}{\mathbf{p}_0 - \mathbf{q} - m} + \right. \\ & \left. S_F'(\mathbf{p}_0 + \mathbf{q}_0) \right] \tau_4 \gamma_5 u(\mathbf{p}_0). \end{aligned} \quad (3.8)$$

The total charge exchange scattering cross section is

$$\sigma_{ch\ ex} = \frac{1}{2} \sigma_{elas} + \sigma_c + \sigma_{int} \quad (3.9)$$

where  $\sigma_c$  corresponds to (i) in Fig. 3 and  $\sigma_{int}$  is the interference term between  $\sigma_{elas}$  and  $\sigma_c$ . Their explicit forms are as follows:

$$\sigma_c = \frac{2\pi g^4}{m^2} \left[ \frac{m}{E(p_0) + E(q_0)} \right]^2 \left[ \frac{2\vec{q}_0^2 + 2E(p_0)E(q_0) + \mu^2}{4\vec{q}_0^2} \times \right.$$



$$\times \log \frac{2E(p_0)E(q_0) + 2\vec{q}_0^2 - \mu^2}{2E(p_0)E(q_0) - \vec{q}_0^2 - \mu^2} + \quad (3 \cdot 10)$$

$$+ \frac{\mu^4}{[2E(p_0)E(q_0) + 2\vec{q}_0^2 - \mu^2][2E(p_0)E(q_0) - 2\vec{q}_0^2 - \mu^2]},$$

$$\sigma_{int} = -\frac{4\pi g^4}{m^2} \left[ \frac{m}{E(p_0) + E(q_0)} \right]^2 \left[ \frac{m^2}{\mu^2 + 2E(p_0)E(q_0)} \right]^2 \left[ \left\{ \left[ \frac{E(p_0) + E(q_0)}{m} \right]^2 D - G \right\} + \right.$$

$$+ \left. \left[ \left\{ \left[ \frac{E(p_0) + E(q_0)}{m} \right]^2 D - G \right\} (2\vec{q}_0^2 + \mu^2) + 4[E(p_0) + E(q_0)][E(q_0)D - \right. \right.$$

$$- 2E(p_0)E(q_0) \left. \left\{ \left[ \frac{E(p_0) + E(q_0)}{m} \right]^2 D + G \right\} \right] \cdot [4\vec{q}_0^2]^{-1} \times \quad (3 \cdot 11)$$

$$\times \log \frac{2E(p_0)E(q_0) + 2\vec{q}_0^2 - \mu^2}{2E(p_0)E(q_0) - 2\vec{q}_0^2 - \mu^2} \Big].$$

Here  $D$  and  $G$  are those given by (3.6). All quantities are those in C. M. system.

Finally, the total cross section (elastic plus charge exchange scattering)  $\sigma_{tot}$  is given by

$$\sigma_{tot} = \sigma_{elas} + \sigma_{ch\ ex}. \quad (3 \cdot 12)$$

## § 4. Symmetrical vertex type

Here we take the following three interaction Hamiltonians:

$$H_{\pi NN} = g \bar{\psi} \tau \gamma_5 \psi \cdot \phi, \quad (4 \cdot 1)$$

$$H_{\theta \Lambda N} = G \Psi \cdot \psi \tau \Psi + Herm. \ conj., \quad (4 \cdot 2)$$

$$H_{\pi \Lambda \Lambda} = (-i) G (\Psi \times \gamma \Psi) \cdot \phi \quad (4 \cdot 3)$$

where bold letters and dot have the same meanings as in § 3 and the cross means the vector product.

We consider elastic scattering only. Therefore two more diagrams given in Fig. 4 appear besides those in Fig. 2 (i).

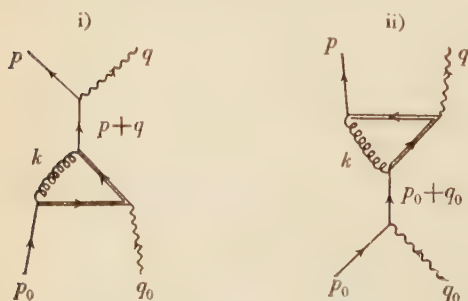


Fig. 4.

Corresponding to (i) and (ii) we get two same matrix elements  $M(I)$  and  $M(II)$ , respectively, by consideration of isotopic space.

$$M(I) = (-i) g G^3 \frac{1}{\mu^2 + 2(p_0 q_0)} \bar{u}(p)$$

$$\left[ q \frac{1}{\pi} - \frac{1}{i} V(I) \right] u(p_0),$$

$$M(II) = (-i)g^3 \frac{1}{\mu^2 + 2(\mathbf{p}_0 q_0)} \bar{u}(\mathbf{p}) \left[ \frac{1}{\pi} \frac{1}{i} V(II) \mathbf{q}_0 \right] u(\mathbf{p}_0) \quad (4.4)$$

where

$$V(I) = \int \frac{(\mathbf{p}_0 + \mathbf{q}_0 - \mathbf{k} - M)(M + m - \mathbf{k})}{[(\mathbf{p}_0 + \mathbf{q}_0 - \mathbf{k})^2 - M^2][(\mathbf{p}_0 - \mathbf{k})^2 - M^2][k^2 - \kappa^2]} d^4 k,$$

$$V(II) = \int \frac{(M + m - \mathbf{k})(\mathbf{p}_0 + \mathbf{q}_0 - \mathbf{k} - M)}{[(\mathbf{p} - \mathbf{k})^2 - M^2][(\mathbf{p}_0 + \mathbf{q}_0 - \mathbf{k})^2 - M^2][k^2 - \kappa^2]} d^4 k. \quad (4.5)$$

The evaluation of these expressions and the renormalization of charge are easily done using the usual methods. Then the finite vertex contribution from  $V(I)$ , denoted as  $\bar{V}(I)$ , is

$$(1/i) \bar{V}(I) = J_1 + J_2 + J_3 + J_4,$$

$$J_1 = \frac{1}{2} \int_0^1 dt (1-t) \log \frac{m^2(1-t)^2 - (M^2 + m^2 - \kappa^2)(1-t) + M^2}{m^2},$$

$$J_2 = -\frac{1}{2} \int_0^1 dt (1-t) \log \frac{m^2(1-t)^2 + [M^2 + m^2 - \kappa^2 + 2(\mathbf{p}_0 q_0)t](1-t) + M^2 - \mu^2(1-t)t}{m^2},$$

$$J_3 = -\frac{1}{4} \int_0^1 dt \int_0^{1-t} dx \frac{3m^2 x^2 - 2(M^2 + m^2 - \kappa^2)x + M^2}{m^2 x^2 - (M^2 + m^2 - \kappa^2)x + M^2},$$

$$J_4 = \frac{1}{4} \int_0^1 dt \int_0^{1-t} dx \times$$

$$\times \frac{3m^2 x^2 - \{2(M^2 + m^2 - \kappa^2) + 2(\mathbf{p}_0 q_0)t + m\mathbf{q}_0\}x + M^2 + \mu^2(1-t)t - M\mathbf{q}_0}{m^2 x^2 - \{M^2 + m^2 - \kappa^2 + 2(\mathbf{p}_0 q_0)t\}x + M^2 - \mu^2(1-t)t}.$$

$\bar{V}(II)$  can be obtained similarly, by changing  $\mathbf{q}_0$  into  $\mathbf{q}$  in  $\bar{V}(I)$ .

Hence the vertex correction can be put into the following form

$$(1/i) \bar{V}(I) = A' + B' \mathbf{q}_0/m,$$

$$(1/i) \bar{V}(II) = A' + B' \mathbf{q}/m. \quad (4.7)$$

In (4.7)  $A'$  and  $B'$  do not contain any  $\gamma$ -matrices and can be represented by the combinations of various parametric integrals. Most of them can be calculated analytically but some of them must be treated numerically. Below the threshold for heavy particles  $A'$  and  $B'$  are real and above the threshold they become complex by the appearance of poles just as in § 3.

The lowest order (Fig. 2(i)) matrix element is

$$(-i)g^2 \frac{2}{\mu^2 + 2(\mathbf{p}_0 q_0)} \bar{u}(\mathbf{p}) \mathbf{q}_0 u(\mathbf{p}_0), \quad (4.8)$$

so that the matrix element including  $V$  type corrections is given by

$$M_{el} = (-i)g^2 \frac{2}{\mu^2 + 2(p_0 q_0)} \bar{u}(p) [A'' q_0 + B''] u(p_0), \quad (4.9)$$

where

$$A'' = g^2 + \frac{4}{\pi} G^3 (A' - 2B'),$$

$$B'' = \frac{4}{\pi} G^3 B' \frac{1}{m} \{\mu^2 + 2(p_0 q_0)\}. \quad (4.10)$$

Hence the total elastic scattering cross section  $\sigma_{el}$  becomes (expressed in terms of quantities in the C. M. system)

$$\sigma_{el} = g^2 \frac{8\pi}{m^2} \left[ \Re(A''^2) + \Im\left(\frac{B''}{m}\right)^2 + \Im \cdot 2A'' \frac{B''}{m} \right], \quad (4.11)$$

where  $\Re$  and  $\Im$  and  $\Im$  are given by

$$\begin{aligned} \Re &= \left[ \frac{m}{E(p_0) + E(q_0)} \right]^2 \left[ \frac{m^2}{\mu^2 + 2(p_0 q_0)} \right]^2 \times \\ &\quad \times \frac{\{2(p_0 q_0)^2 + [\mu^2 - E(q)E(q_0)][\mu^2 + 2(p_0 q_0)]\}}{m^4}, \\ \Im &= \left[ \frac{m}{E(p_0) + E(q_0)} \right]^2 \left[ \frac{m^2}{\mu^2 + 2(p_0 q_0)} \right]^2 \frac{2m^2 - \mu^2 + E(q)E(q_0)}{m^2}, \quad (4.12) \\ \Im &= \left[ \frac{m}{E(p_0) + E(q_0)} \right]^2 \left[ \frac{m^2}{\mu^2 + 2(p_0 q_0)} \right] \left[ 1 - \frac{E(q)E(q_0)}{\mu^2 + 2(p_0 q_0)} \right]. \end{aligned}$$

In the above expressions  $A''$  and  $B''$  are very complicated expressions and can not be written in compact forms, because they contain numerical integrals. Of course, above the threshold,  $A'$  and  $B'$  (therefore  $A''$  and  $B''$ ) are complex and the squares or products should be changed as in  $SE$  type.

## § 5. Results and discussions

Fig. 5 shows the experimental data on  $\pi^-p$  scattering up to 1.5 BeV (see Reference 1).

Fig. 6 shows the calculated results for neutral  $SE$  type. The curves named "uncorrected" mean the curves which are not corrected by heavy particles ( $g^2=6$ ), and  $S(S)$  and  $V(V)$  mean the calculated results with the neutral  $S(S)$  ( $g^2=6$ ,  $G^2=2$ ) and  $V(V)$  ( $g^2=6$ ,  $G^2=0.5$ ) types corrections, respectively.

The present choices of  $G$  values amount to 4.1% for  $S(S)$  and 13.7% for  $V(V)$  as the ratios of the  $A$ -particle production cross section to  $\pi^-p$  scattering cross section, both in the lowest order. With these choices the corrections at 200 and 900 MeV for  $S(S)$  and  $V(V)$  are shown numerically in Table I.

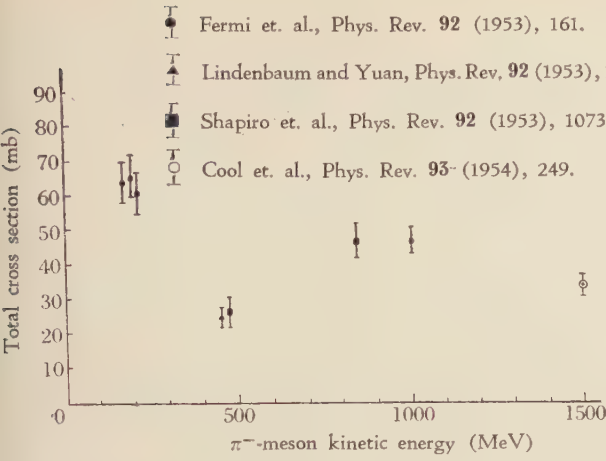


Fig. 5. Experimental data

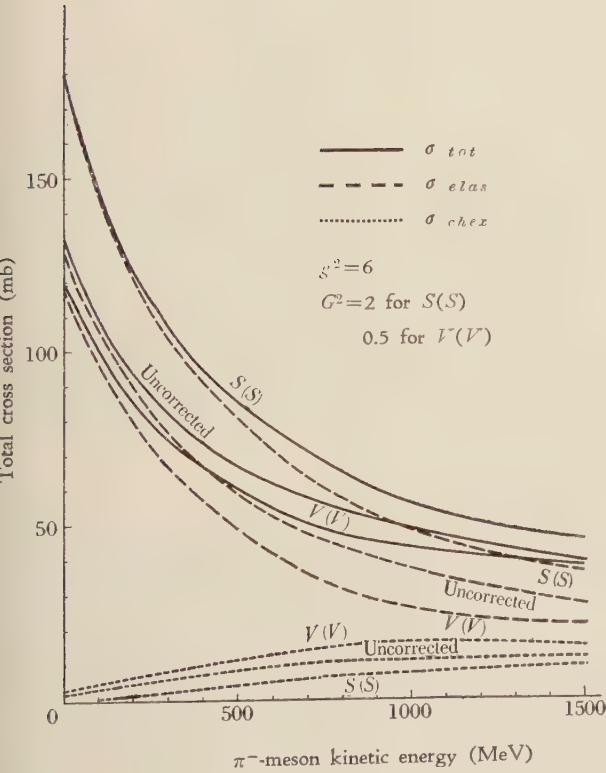


Fig. 6. SE type (neutral)

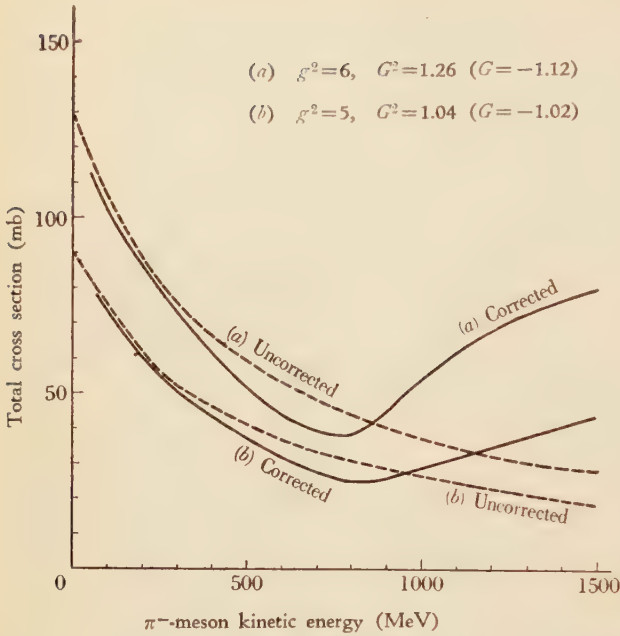
Table I. Corrections for SE type

	200MeV	900 MeV
$S(S)$	32.0%	16.6%
$V(V)$	9.2%	12.8%

Fig. 7 shows the results for symmetrical  $V$  type elastic scattering and in this case we choose two pairs of coupling constants, namely,  $g^2=6$  &  $G^2=1.26$  ( $G=-1.12$ ) and  $g^2=5$  &  $G^2=1.04$  ( $G=-1.02$ ). These choices are consistent with the SE case. However, in the  $V$  case,  $\sigma_{el}$  depends fairly sensitively on the choices of  $g$  and  $G$  values. Here we take negative signs for  $G$ , making use of the arbitrariness of the relative sign for  $H_{\pi NN}$  and  $H_{\pi\Lambda\Lambda}$ , and, if we take positive signs for  $G$ , the results would tend to be more similar to the SE case.

From the above results one can see that the radiative corrections in the  $S(S)$  case in SE type, are larger at lower energies and they do not seem to be able to make the cross section larger in the high energy region, as exhibited by experiments. On the other hand, in the  $V(V)$  case, the corrections have more desirable tendencies than in the  $S(S)$  case, yet they make only the cross section flat at the high energy end. On the contrary,  $V$  type has possibilities to raise the cross section to be compared with the



Fig. 7.  $V$  type (symmetrical)

experimental values. In this case the poles in the parametric integration have greater effects than in  $SE$  type. Finally, as for  $\sigma_{ch\ ex}$  of  $V$  type and  $\sigma_{el}$  and  $\sigma_{ch\ ex}$  of  $C$  type, we did not here carry out calculations, because they can not be done without fairly incorrect approximations.

Here a question arises whether a peak or valley exists in the case of  $\pi^+-p$  scattering as in  $\pi^-p$  scattering, if we assume  $\Lambda$ -particles have isotopic spin 1. But we can not answer this until detailed calculations are performed.

Considering the results the radiative corrections by heavy

particles to  $\pi^-p$  scattering are rather large, we are naturally led to the conclusion: If the charge independence, which holds for  $\pi$ -mesons, were not to hold for heavy particles, this breakdown would disturb the charge independence for  $\pi$ -mesons. But, as we know the charge independence holds in the low energy phenomena, we might safely conclude that the above results give us a support to the validity of the charge independence for heavy unstable particles.

The authors express their sincere thanks to Prof. M. Kobayasi and Dr. K. Nishijima for their continual encouragement and helpful discussions. Thanks are also due to Prof. S. Takagi and Dr. Y. Katayama for their kind criticism.

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## An Attempt to the Unified Description of Elementary Particles\*

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An attempt is made to give a unified and a more positive definition of elementary particles by introducing a substance of higher level than them. This substance is named "Urmaterie", and is assumed to be described by a non-local field proposed by Yukawa. Various states of the internal motion of Urmaterie are classified by eigenvalues of a complete set of mutually commutative operators with respect to internal coordinates, two of which can be taken as the spin and the mass operators. Each eigenstate thus classified is assumed to correspond to the elementary particle of the present theory.

In § 1, the present situation of the theory of elementary particles is analysed, and the phenomenological feature of the present theory and the necessity of introducing a substance of higher level to overcome it is stressed. In § 2, the spin and the mass operators are introduced. It is done according to the principle that the structure of elementary particles should determine uniquely the transformation properties of the wave functions under Lorentz-transformations. The equations of motion of Urmaterie are then derived from *korrespondenzmässig* considerations and from the principle of reciprocity. In § 3, the eigenvalue and the eigenfunction of the mass operator is given in the simplest case when the Urmaterie field is assumed as scalar. In § 4, the relation of this Urmaterie field to the local field is discussed, and it is shown that, in the case of the scalar as before, the equation of motion of Urmaterie reduces to that of Fierz for eigenstates of the spin and the mass operators. In § 5, a qualitative discussion on the interaction of Urmaterie is given. In § 6, an extension to the case of spinor is discussed. It is shown that the existence of a new structure constant other than the spin and the rest mass inevitably follows. It is shown that this is interpretable as expressing the essential difference of the heavy and light particles, and in its connection an attempt is made to deduce the conservation law of heavy particles. In § 7, the deductive perspective and the further outlook of the theory is briefly given.

### § 1. Introduction

The present theory of elementary particles, which has been formulated in a perfectly Lorentz covariant form, has succeeded to express beautifully one aspect of elementary particles, and to obtain numerous brilliant results. What is underlying the ground of the present theory is of course the concept of elementary particles, which has been defined through the course of its development as that characterized by structure constants such as the spin or the rest mass, and satisfying conservation laws such as of the energy or of the momentum by their mutual transformations. Needless to say that these definitions have been an excellent abstraction of one aspect of elementary particles, and have played an essential role in the development of the theory.

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\* This is a detailed account of our preliminary reports<sup>1)</sup>.

The recent progress of the theory of elementary particles, however, seems to have exposed its essential limitation that such definitions are no longer sufficient for the better understanding of elementary particles. For example, the present theory can not give answers to questions such as, "What kind of structure of elementary particles do the present structure constants express?" or, "What kind of intrinsic correlation of elementary particles does the present interaction scheme express?". In fact, in the present theory, the structure constants or the interaction Lagrangians are introduced phenomenologically merely as parameters or additive terms into the theory, and the assignment of the spin or the rest mass values to elementary particles or the introduction of the mutual interaction between them are done entirely *ad hoc*. In this sense, the present theory may be said to remain at the *phenomenological stage*\* in the course of the development of the theory of elementary particles. A remarkable regularity between the rest masses of elementary particles as first pointed out by Nambu<sup>1)</sup>, the universal Fermi interaction, or the divergence difficulties inherent in the quantum theory of the wave field from the day of its birth——a series of these facts suggest clearly such limitation of the present theory and also the necessity of *more positively* defining the concept of elementary particles.

Under this circumstance, it would be of great interest, as promoting a step in overcoming the limitation of the present theory, to introduce an internal structure to elementary particles, and thus try to grasp structure constants of elementary particles and the intrinsic correlation between them through its mediation. Of course there is no definite guiding principle in doing it. It would be reasonable, however, to suppose that the future theory will satisfy on one hand the requirement of relativity and on the other hand will reduce to the present theory in the approximation in which we disregard such internal freedom.

Along this line attempts have been done by many authors. We may quote among them the names of Heisenberg<sup>5)</sup>, Bopp<sup>6)</sup>, Wessel<sup>7)</sup>, and Hönl and Papapetrou<sup>7)</sup>.\*\* Heisenberg's attempt is a very ambitious one, but it seems to us that his theory contains too speculative elements in its foundation and is difficult to try any further development. Bopp, Wessel, or Hönl and Papapetrou's attempts, although interesting in as much as they attack directly the study of the internal structure, also seems to lack firm principles in constructing their basic equations.

In previous paper on Yukawa's theory of non-local field<sup>10)</sup>, one of the authors suggested to regard elementary particles as corresponding to various states of the internal motion of a kind of "*Urmaterie*", a substance of higher level than elementary particles,

\* This is after the nomenclature of Taketani<sup>2)</sup>, who pointed out that the development of the physical theory is made spirally through three stages of *phenomenological*, *substantialistic* and *essentialistic*, each of which corresponds to those of *an sich, für sich*, and *an und für sich* of Hegel. In the case of Newtonian mechanics, for example, they are Ticho Brahe's, Kepler's, and Newton's stage respectively. See also S. Sakata<sup>3)</sup>, "The Theory of the Interaction of Elementary Particles".

\*\* Recently, Pais<sup>9)</sup> proposed to introduce a internal structure in connection with  $\tau$ -spin, and pointed out that *selection* rules concerning with this new freedoms play important roles in explaining the longevity of  $V$  particles.

and pointed out that the non-local field would be nothing but the one that would describe this Urmaterie.<sup>12)\*</sup> As emphasized by Yukawa<sup>11)</sup>, the non-local field is an elegant tool of introducing the internal freedom in a Lorentz covariant form, and the use of the non-local field is expected to give a new scope in investigating the internal structure of elementary particles. It is the purpose of this paper to discuss the development of this idea.

The outline of our idea is as follows; We introduce a new substance which is described by a non-local field  $U(X_\mu, r_\mu)$ . Of course  $X_\mu$  can be identified to positional coordinates of elementary particles of the present theory, but  $r_\mu$  can not. This is assumed to describe the internal motion of Urmaterie. Various states of this internal motion will be classified by assigning eigenvalues of a complete set of mutually commutative operators with respect to internal coordinates, and two of them may be taken as the spin and the mass operators respectively.\*\* Each eigenstate thus classified may be assumed to correspond to the elementary particle of the present theory.

The actual development is done through following steps;

- i) To define the spin and the mass operators as constants of internal motion.
- ii) To solve eigenvalue equations for the spin and the mass operators.
- iii) To make clear the mathematical relation of the non-local Urmaterie field to the local field.
- iv) To check the qualitative features of the interaction of Urmaterie.

(i) is done in § 2. What we introduce there as a principle of defining them is that the structure of elementary particles should determine uniquely the transformation properties of the wave functions under Lorentz-transformations. Using it, almost unique definitions of the spin and the mass operators are given. The former agrees with that of Fierz<sup>13)</sup>, and the latter is shown to reduce to an expectation value of the energy of the internal motion if a suitable condition is imposed. Equations of motion of Urmaterie are then derived from korrespondenzmässig considerations and from the principle of reciprocity. (ii) is done in § 3. (iii) is done in § 4, and it is shown that in the case when  $U(X_\mu, r_\mu)$  is assumed as scalar, the equation of motion of Urmaterie reduces to Fierz's equation. (iv) is done in § 5 using the S-matrix formalism given by Yukawa<sup>14)</sup>.

Finally, the concept of particle family of Fermi particles<sup>15)</sup> is studied in § 6 to show an advantage of our theory. It is shown that in our theory the concept of particle family follows very naturally. This is because when we extend our theory to include the spinor non-local field, which is necessary to get particles of half-integer spin, the spin and the mass operators no longer compose a complete set. Thus the introduction of a new structure constant becomes inevitable, and we can show that this new structure constant can be interpreted as expressing the essential difference of heavy and light particles.

\* Recently, Yukawa<sup>12)</sup> published a similar attempt.

\*\* It can be shown that the spin and the mass operators compose a complete set if  $U(X_\mu, r_\mu)$  is taken as scalar, but this is not the case when it is spinor. (See § 2 and § 5)



## § 2, The introduction of the spin and the mass operators

As stated in the introduction, the most essential part of our discussion is to try a unified and a more positive definition of elementary particles by introducing Urmaterie. Thus, the first task is to introduce constants of internal motion which are responsible to structure constants such as the spin or the rest mass.

As for the spin, we have a fine analysis due to Fierz<sup>13)</sup> that the angular momentum of the internal motion of the non-local field just corresponds to the spin. Therefore, we may expect that the spin operator of our theory may be obtained by generalizing his angular momentum operator into an invariant form in such a way that it reduces to the ordinary one in the rest system of the external motion.

Contrary to the case of the spin, it is very difficult to define the mass operator. The clue to it, however, seems to be found in the concept of the wave function and of the space-time underlying the present theory.

The present theory of elementary particles is constructed by at the beginning ascribing a wave function of the specified transformation property to the elementary particle of the specified structure. This implies that it forms an integral part of the present theory to assume that the transformation property of the wave function is one of the most direct expression of the structure of the elementary particle and therefore is determined uniquely by the structure of the elementary particle.

Partly, this assumption is realized in the present theory as a relation between the spin and the transformation property of the wave function. Asserting this to the full, it seems to us very natural to assume that the rest mass, which is a very fundamental structure constant abreast with the spin, is also closely related to the transformation property of the wave function.

It seems instructive to study here the irreducible representation of the Lorentz group  $D_{jj'}$ . The wave function of the elementary particle is naturally assumed to be transformed according to it.  $D_{jj'}$  being not irreducible under spatial rotations, however, the spin of the elementary particle whose wave function is transformed according to  $D_{jj'}$  is not unique, and is given by

$$s = j + j' - \lambda, \\ \lambda = 0, 1, 2, \dots, 2j' \quad (j \geq j'). \quad (2.1)$$

This means that the transformation property of the wave function is not determined uniquely by assigning the value of spin only. The gap can be filled to some extent, since of  $j$  and  $j'$  what is physically significant is its combination  $(j + j')$  only. Even

\* In Dirac's theory of the generalized wave equation<sup>16)</sup>, special one of (2.1) corresponding to  $\lambda=0$  is chosen out by imposing a subsidiary condition. The correspondence between the spin and the transformation property of the wave function is made unique by this. Although his theory is the materialization of the general view point of the present theory, it seems to us that this restriction is not along the right course. An origin of the fact that the grasp of the rest mass in the present theory is very phenomenological seems to lie here.



taking into account this situation, however, (2.1) is not sufficient to establish a unique correspondence.

Our assumption suggests here the existence of a new relation between  $(j+j')$ ,  $\lambda$ ,  $s$ , and the rest mass  $m$ ;

$$f(j+j', \lambda, s, m) = 0. \quad (2.2)$$

If this is the case, the transformation property of the wave function is determined uniquely by solving (2.1) and (2.2) when  $s$  and  $m$  is given.

Thus, it would be reasonable to make it the guiding principle of introducing the mass operator to try to find such constant of internal motion which, when mediated to external coordinates, its eigenvalue determines uniquely with that of the spin the transformation property of the wave function.

In the following, we restrict our attention to the simplest case, and assume that Urmaterie is described by a scalar non-local field  $U(X_\mu, r_\mu)$ . Noticing the Fierz's remark that the internal angular momentum of the non-local field must corresponds to the spin, we introduce an infinitesimal rotational operator in four dimensional space;

$$R_{\mu\nu} = (1/i) (r_\mu \partial / \partial r_\nu - r_\nu \partial / \partial r_\mu). \quad (2.3)^*$$

Of course the square of  $R_{\mu\nu}$  contains spin part, since it includes spatial rotation as its special case. In general, however, it is not merely the spin and is given by a sum of the spin part and a quantity independent of it corresponding to rotations including time axis.

In order to separate these two components in a Lorentz-invariant way, we introduce  $v_{\lambda\mu\nu}$  and  $\Gamma_\nu$  defined by

$$v_{\lambda\mu\nu} = P_\mu R_{\nu\lambda} + P_\nu R_{\lambda\mu} + P_\lambda R_{\mu\nu} \quad (2.4)$$

$$\Gamma_\nu = P_\lambda R_{\lambda\nu}, \quad (2.5)$$

where  $P_\mu$  is the energy momentum four vector of the external motion

$$P_\mu = (1/i) \cdot \partial / \partial X_\mu. \quad (2.6)$$

Defining  $S^2$  and  $M^2$  by

$$S^2 = (v_{\lambda\mu\nu} v_{\lambda\mu\nu}) / 6 P_\mu^2 \quad (2.7)$$

$$M^2 = -\Gamma_\nu \Gamma_\nu / P_\mu^2, \quad (2.8)$$

we can show by a direct calculation that

$$(1/2) R_{\mu\nu} R_{\mu\nu} = S^2 - M^2. \quad (2.9)$$

$P_\mu$  is commutative with  $S^2$  and  $M^2$ . Therefore, expanding  $U(X_\mu, r_\mu)$  into Fourier series

\*  $x_4$  denotes *ict*.  $x_0 = x_4/i$  is also used when to make hermitic character explicit is needed.

$$U(X_\mu, r_\mu) = \int u(k_\mu, r_\mu) e^{ikX} (dk), \quad (2.10)$$

we consider in the following only one component with the wave number  $k_\mu$ . We further assume that  $k_\mu^2 < 0$ .\* Then we see, in the rest system of the external motion,

$$\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2 \quad (2.11)$$

$$\mathbf{M}^2 = \mu_1^2 + \mu_2^2 + \mu_3^2, \quad (2.12)$$

where  $\mathbf{S}$  and  $\mu$  are two space vectors forming a skew symmetric tensor

$$R_{\mu\nu} = \begin{pmatrix} 0, & S_3, & -S_2, & i\mu_1 \\ -S_3, & 0, & S_1, & i\mu_2 \\ S_2, & -S_1, & 0, & i\mu_3 \\ -i\mu_1, & -i\mu_2, & -i\mu_3, & 0 \end{pmatrix} \quad (2.13)$$

and each component of  $\mathbf{S}$  and  $\mu$  satisfies the commutation relations

$$\left. \begin{aligned} [S_i, S_j] &= iS_k \\ [\mu_i, \mu_j] &= -iS_k \\ [\mu_i, S_j] &= i\mu_k \quad (i, j, k; \text{cyclic}) \end{aligned} \right\} \quad (2.14)$$

From (2.14), and from the fact that  $\mathbf{S}^2$  and  $\mathbf{M}^2$  behave as scalar under Lorentz-transformations, we can see at once that\*\*

$$[\mathbf{S}^2, \mathbf{M}^2] = 0. \quad (2.15)$$

Thus, (2.9) can be regarded as a Lorentz-invariant separation of  $R_{\mu\nu}R_{\mu\nu}$  into two independent components.

It is clear from (2.11) that  $\mathbf{S}^2$  is just the required generalization of Fierz's spin operator. Thus, in our theory, to the spin is given an intuitive image that it corresponds to the rotation of a rigid sphere. This is in marked contrast to that of the present theory, where it is given only as the number of independent components of the wave function in the rest system of the center of mass.

It would be natural, from the symmetry in (2.9), to conjecture that  $\mathbf{M}^2$  is also responsible to the structure of elementary particles just as  $\mathbf{S}^2$  was. This conjecture seems not misdirected, since we can show in fact that the transformation property of the wave function is determined uniquely by assigning the eigenvalue of  $\mathbf{S}^2$  and  $\mathbf{M}^2$ . The proof is given in § 4.

It seems useful to investigate the relation of the eigenvalue of  $\mathbf{M}^2$  to dynamical variables of the internal motion to show clearer that  $\mathbf{M}^2$  should be taken as the mass operator. Of course, at the present, we know nothing of the law governing the internal motion. But the

\* This assumption is self-consistent in the sense that the eigenvalue of the mass is given in fact in a positive definite form (see § 5). An essential revision seems to be necessary to include consistently the case of vanishing rest mass.

\*\* The proof is given in Appendix IV. (b).

fact that the angular momentum of the internal motion corresponds just to the spin seems to suggest that, at least partly, we can expect that the law of the external motion may be applicable to that of the internal motion. We adopt this as a *korrespondenzmässig* suggestion, and specifically assume that the law of quantum mechanics can be applied to the internal motion in the sense that  $u(k_\mu, r_\mu)$  describes the state of the internal motion, and that  $\hbar/i \cdot \partial/\partial r_i$  ( $i=1, 2, 3$ ) and  $-\hbar c/i \cdot \partial/\partial r_0$  correspond to the momentum and the energy operator of the internal motion respectively.

Thus, defining the normalization of  $u(k_\mu, r_\mu)$  by

$$\int u^*(k_\mu, r_\mu) \delta(n_\mu r_\mu) u(k_\mu, r_\mu) (dr_\mu) = 1, \quad (2.16)$$

the expectation value of various dynamical variables is assumed to be given by

$$\begin{aligned} & \left\langle F\left(r_i, \frac{\hbar}{i} \frac{\partial}{\partial r_i}, -\frac{\hbar c}{i} \frac{\partial}{\partial r_0}, \dots\right) \right\rangle \\ &= \int u^*(k_\mu, r_\mu) \delta(n_\mu r_\mu) F\left(r_i, \frac{\hbar}{i} \frac{\partial}{\partial r_i}, -\frac{\hbar c}{i} \frac{\partial}{\partial r_0}, \dots\right) \\ & \quad \times u(k_\mu, r_\mu) (dr_\mu), \end{aligned} \quad (2.17)$$

where  $n_\mu$  is an arbitrary time-like vector.

Under these assumptions, we try to study further the meaning of  $\mathbf{M}^2$ . In the rest system of the external motion, where  $k_\mu$  takes the form  $(0, 0, 0, i\kappa)$ ,  $\mathbf{M}^2$  is given by

$$\mathbf{M}^2 u(k_\mu, r_\mu) = m^2 u(k_\mu, r_\mu) \quad (2.18)$$

$$-\mathbf{M}^2 = (r_i \partial / \partial r_0 + r_0 \partial / \partial r_i)^2, \quad (2.19)$$

from which we have, after some calculations,

$$m^2 = - \int u^*(k_\mu, r_\mu) \delta(r_0) \left( r_0 \frac{\partial}{\partial r_i} + r_i \frac{\partial}{\partial r_0} \right)^2 u(k_\mu, r_\mu) (dr_\mu), \quad (2.20)$$

where we have put  $n_\mu$  as  $(0, 0, 0, i)$ . Expanding the right hand side of (2.20),

$$\begin{aligned} (r_i \partial / \partial r_0 + r_0 \partial / \partial r_i)^2 &= 2r_0 r \partial^2 / \partial r_0 \partial r + 3r_0 \partial / \partial r_0 + r_0^2 \Delta + r \partial / \partial r + r^2 \partial^2 / \partial r_0^2 \\ & \quad (r = \sqrt{r_1^2 + r_2^2 + r_3^2}). \end{aligned} \quad (2.21)$$

Of terms of (2.21), first three vanish owing to  $\delta(r_0)$  appearing in (2.20), and the fourth also vanishes if we assume Yukawa's second equation for  $U(X_\mu, r_\mu)$  ;\*

$$(r_\mu r_\mu - \lambda^2) U(X_\mu, r_\mu) = 0. \quad (2.22)$$

Thus, under this assumption, we get

$$m^2 = (\lambda / \hbar c)^2 \langle E_{in}^2 \rangle, \quad (2.23)$$

where  $E_{in}^2$  means the square of the energy of the internal motion ;

\* As for details see Appendix IV. (a).

$$L_{in}^2 = (\hbar c/i)^2 \partial^2 / \partial r_0^2.$$

An analogous discussion can also be repeated for an operator defined by

$$\mathbf{M} = \sqrt{\mathbf{M}^2}, \quad (2.24)$$

and the result is

$$m = \lambda / \hbar c \cdot \langle \sqrt{E_{in}^2} \rangle. \quad (2.25)$$

The rest mass of elementary particles may be defined as such a constant which

- (i) firstly is a scalar specifying the structure of elementary particles,
- (ii) secondly expresses the pool of the energy of the system in the rest system of center of mass, which means that it agrees there numerically with the fourth component of the energy and momentum four vector,

and

- (iii) thirdly determines the transformation property of the wave function abreast with the spin.

Collecting above results, it seems perfectly reasonable to assume that  $\mathbf{M}$  is the mass operator, and that  $m$  represents the rest mass of elementary particle measured in unit  $(\hbar/c\lambda)$ . The invariance of  $m$  follows at once from that of  $\mathbf{M}$ .

Moreover, we can show that  $r_\mu r_\mu$ ,  $\mathbf{S}^2$  and  $\mathbf{M}^2$  compose a complete set with respect to internal coordinates.\* This means that, so long as we assume Yukawa's non-local field and his second equation, there is no alternative for the mass operator other than  $\mathbf{M}$ . At least we can say that the mass operator must be a function of it.

Thus, in our theory,  $\mathbf{S}^2$  and  $\mathbf{M}^2$  give the spin and the mass spectrum of elementary particles.

Introducing polar coordinates,

$$\left. \begin{aligned} r_1 &= \lambda \cosh \tilde{\xi} \sin \theta \cos \varphi \\ r_2 &= \lambda \cosh \tilde{\xi} \sin \theta \sin \varphi \\ r_3 &= \lambda \cosh \tilde{\xi} \cos \theta \\ r_0 &= \lambda \sinh \tilde{\xi} \end{aligned} \right\}, \quad (2.26)$$

$\mathbf{S}^2$  is written in the form

$$\mathbf{S}^2 = - \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) \quad (2.27)$$

in the rest system of the external motion. Thus, eigenfunction of  $\mathbf{S}^2$  is the usual spherical harmonics, and its eigenvalue is given by

$$s(s+1), \quad s=0, 1, 2, \dots \quad (2.28)$$

\* The commutability of  $r_\mu r_\mu$  with  $\mathbf{S}^2$  and  $\mathbf{M}^2$  is shown in Appendix IV. (c). Completeness of these set follows from the fact that their simultaneous eigenfunction is not degenerated except for  $n$ . (See (3.10)).  $n$  is not scalar, and therefore can not be adopted as a structure constant. We can readily see that it corresponds to the freedom of polarization.

The eigenvalue equation of  $\mathbf{M}^2$  leads to a hypergeometric equation, and it will be shown in the next section that discrete eigenvalues are given by solving it under a suitable boundary condition.

The scalar non-local field  $U(X_\mu, r_\mu)$  provides us, therefore, with a unified description of Bose particles, and when expanded into simultaneous eigenfunctions of  $\mathbf{S}^2$  and  $\mathbf{M}^2$ , each component describes elementary particles of definite spin and rest mass.

In view of the fact that the rest mass should be related to the external motion as being equal to the magnitude of the energy momentum four vector, it is legitimate to assume

$$(\square - (m/\lambda)^2)U(X_\mu, r_\mu; m, s) = 0, \quad (2.29)$$

which must further be generalized to

$$(\square - \mathbf{M}^2/\lambda^2)U(X_\mu, r_\mu) = 0 \quad (2.30)$$

for the general non-local field which is not the eigenfunction of  $\mathbf{S}^2$  and  $\mathbf{M}^2$ .

(2.29) or (2.30) play an important role of connecting the internal motion to the external, and is nothing but the Yukawa's first equation. This stands to (2.22) just reciprocally adjoint. Although in Yukawa's theory (2.22) was introduced from a rather formal requirement of Born's<sup>(7)</sup> reciprocity, it appears in our theory as an indispensable equation to secure that the eigenvalue of  $\mathbf{M}$  is equal to an expectation value of the energy of the internal motion.

Merely to introduce the internal freedom, it would be sufficient to increase the number of variables, and to introduce the non-local field is not necessarily needed. The fact that the reciprocity appears in such a form, however, seems to suggest the necessity of introducing it when we go to describe the internal structure of elementary particles.

### § 3. The solution of eigenvalue problem

In this section we try to solve the eigenvalue equation for the mass spectrum

$$\mathbf{M}^2 u(k_\mu, r_\mu) = m^2 u(k_\mu, r_\mu). \quad (3.1)$$

$\mathbf{M}^2$  being scalar under Lorentz-transformations, it is convenient to solve it in the rest system of the external motion, where  $\mathbf{M}^2$  can be expressed as

$$-\mathbf{M}^2 = \sum_{i=1,2,3} (r_i \partial / \partial r_0 + r_0 \partial / \partial r_i)^2.$$

In terms of polar coordinates introduced in (2.29), this is simplified to

$$-\mathbf{M}^2 = (1+x^2)d^2/dx^2 + 3xd/dx + x^2/(1+x^2)\mathcal{Q}, \quad (3.2)$$

where  $x = \sinh \xi$  and  $-\mathcal{Q}$  is given by (2.27). Substituting (3.1) into (3.2), and replacing  $-\mathcal{Q}$  by  $s(s+1)$ , which means that we restrict our attention to the mass spectrum of elementary particles of spin  $s$ , the equation to be solved becomes

$$\left\{ \frac{d^2}{dx^2} + \frac{3x}{1+x^2} \frac{d}{dx} - \frac{x^2}{(1+x^2)^2} s(s+1) + \frac{m^2}{1+x^2} \right\} \varphi(x) = 0 \quad (3.3)$$

$$(-\infty < x < \infty).$$



This is a hypergeometric differential equation, and performing a transformation

$$\begin{aligned} y &= 1 + x^2 \\ \varphi &= y^{-(1+\varepsilon)/2} u(y), \end{aligned} \quad (3.4)$$

is readily written in the standard form ;

$$\left\{ \frac{d^2}{dy^2} + \frac{\gamma - (\alpha + \beta + 1)y}{y(1-y)} \frac{d}{dy} - \frac{\alpha\beta}{y(1-y)} \right\} u(y) = 0, \quad (3.5)$$

where

$$\left. \begin{aligned} \alpha &= 1/2 \cdot \{ -s + \sqrt{s(s+1) + 1 - m^2} \} \\ \beta &= 1/2 \cdot \{ -s - \sqrt{s(s+1) + 1 - m^2} \} \end{aligned} \right\} \quad (3.6)$$

and

$$\gamma = 1/2 - s.$$

To get explicit solutions, it is necessary to impose boundary conditions. What boundary conditions should be imposed is a problem to be decided in a relation with the law of internal motion, and at the present we can say nothing about it. Here we assume tentatively that they should be square integrable. Although this seems to be a natural consequence of extending quantum mechanics into the internal world, its precise meaning is not clear, and must seriously be re-examined at the next stage.

Under this boundary condition, the eigenfunction is given by (As for the details of the calculation, see Appendix 1),

$$\varphi_{s,l} = (1+x^2)^{-\frac{1}{2}(s-l+1)} \left( \frac{x^2}{1+x^2} \right)^{\varepsilon} G_{\frac{l}{2}-\varepsilon} \left( s + \frac{1}{2}, \frac{1}{2} + 2\varepsilon, \frac{x^2}{1+x^2} \right), \quad (3.7)$$

with the corresponding eigenvalue

$$\begin{aligned} m_{s,l}^2 &= s + 1 + 2sl - l^2, \\ s &\geq 1, \quad l = 0, 1, 2, \dots, s-1, \end{aligned} \quad (3.8)^*$$

where  $G_{\frac{l}{2}-\varepsilon}$  is Jacobi's polynome of  $(l/2 - \varepsilon)$ th order and  $\varepsilon$  stands for

$$\varepsilon = \frac{1 + (-1)^{l+1}}{4} = \begin{cases} 0 & \text{for even } l \\ 1/2 & \text{for odd } l. \end{cases}$$

It should be noted that the difficulty of the infinite degeneracy as pointed out by Schrödinger<sup>(8)</sup> and by Yukawa<sup>(12)</sup> does not occur in our theory. This is because in our theory the eigenfunctions of  $\mathbf{M}^2$  is classified once more according to the eigenvalue of  $\mathbf{S}^2$ . Group theoretically, these eigenfunctions transform according to an *unitary* representation

<sup>†</sup> Discrete eigenvalues appear below  $s(s+1)$ . Above  $s(s+1)$  appear continuous ones instead. We neglected the latter since it is of less interest. As for details see Appendix I.

of the Lorentz-group. As shown by Bargman<sup>(20)</sup>, the unitary representation of this group is specified by two parameters  $\xi$  and  $\eta$ , which are eigenvalues of

$$\begin{aligned}\xi &\equiv R_{01}R_{23} + R_{02}R_{31} + R_{03}R_{12} \\ \eta &\equiv R_{12}^2 + R_{23}^2 + R_{31}^2 - R_{01}^2 - R_{02}^2 - R_{03}^2\end{aligned}\quad (3.9)$$

respectively. Negative sign before  $\sum R_{0i}^2$  is the characteristic of the pseudo-Euclid space, and is closely related to the above difficulty. That is, this makes it possible for infinitely large number of combinations of  $\sum R_{ij}^2$  and  $\sum R_{0i}^2$  to give the same  $\eta$ . Such degeneracy, however, can be removed at once if it is possible to assign eigenvalue of  $\sum R_{ij}^2$  and  $\sum R_{0i}^2$  separately. In our case, this is made possible since the separation of (2.9) is Lorentz invariant, and each of  $\mathbf{S}^2$  and  $\mathbf{M}^2$  is scalar. Note that this is a consequence of introducing the external momentum which suffer the same Lorentz-transformation with  $r_\mu$  in the definition of  $\mathbf{S}^2$  and  $\mathbf{M}^2$ .

(3.8) represent the rest mass of Bose particles measured in unit  $(\hbar/c\lambda)$ . In particular, Nambu's mass unit<sup>(1)</sup>  $137 \times (\text{electron mass})$  is obtained by taking  $\lambda$  to be of the order of the classical electron radius. Scalar Urmaterie discussed here, however, is an academic model adopted only for its symplicity as the starting point of the theory, and the detailed comparison of this result with experiment does not seem so meaningful.

The complete eigenfunction is given by a product of (3.7) and the corresponding spherical harmonics, and that in arbitrary system can be obtained at once by transforming it by a suitable Lorentz-transformation. Postponing again detailed calculations to Appendix III, the result is given by

$$\begin{aligned}\Phi_{s,m,n}(k_\mu, r_\mu) &= \frac{c(s, n)}{\{\lambda^2 - (k_\mu r_\mu)^2 / k_\mu k_\mu\}^{s/2}} g_{l,\epsilon} \left( -\frac{k_\mu r_\mu}{-(\lambda^2 k_\mu k_\mu)^{1/2}} \right) \times \\ &\times \int_{-\pi}^{\pi} du (a_{3\mu} r_\mu + i a_{1\mu} r_\mu \cos u + i a_{2\mu} r_\mu \sin u)^s e^{inu}, \\ |n| &\leq s\end{aligned}\quad (3.10)$$

where

$$\begin{aligned}c(s, n) &= \frac{(s + |n|)!}{2\pi i^{|n|} s!} \\ g_{l,\epsilon}(x) &= (1 + x^2)^{-\frac{s-l+1}{2}} \left( \frac{x^2}{1+x^2} \right)^\epsilon G_{\frac{l}{2}-\epsilon}^\epsilon \left( s + \frac{1}{2}, \frac{1}{2} + 2\epsilon, \frac{x^2}{1+x^2} \right),\end{aligned}$$

and  $a_{\mu\nu}$  are coefficients of Lorentz-transformation which transforms  $k_\mu$  into rest. The explicit form is given by

$$a_{\mu\nu} = \begin{pmatrix} -1 + (k_1/K)^2, & k_1 k_2 / K^2, & k_1 k_3 / K^2, & -i k_1 / \kappa \\ k_1 k_2 / K^2, & -1 + (k_2/K)^2, & k_2 k_3 / K^2, & -i k_2 / \kappa \\ k_1 k_3 / K^2, & k_2 k_3 / K^2, & -1 + (k_3/K)^2, & -i k_3 / \kappa \\ i k_1 / \kappa, & i k_2 / \kappa, & i k_3 / \kappa, & -i k_4 / \kappa \end{pmatrix}$$

with

$$K = \sqrt{\kappa(k_0 - \kappa)}.$$

$(2s+1)$  eigenfunctions belonging to the same eigenvalues of  $\mathbf{S}^2$  and  $\mathbf{M}^2$  span an irreducible subspace. The transformation property of  $\Phi_{s,m,n}(k_\mu, r_\mu)$  is determined essentially by that of spherical harmonics, namely

$$\Phi'_{s,m,n'}(k'_\mu, r'_\mu) = \sum_{n=0}^{2s+1} \alpha_{n'n} \Phi_{s,m,n}(k_\mu, r_\mu),$$

where  $\alpha_{n'n}$  are the transformation matrix of  $P_s^n(\cos \theta)$  by spatial rotations. The introduction of  $n$  is merely to distinguish  $(2s+1)$  independent components of this subspace. As will be easily anticipated, this expresses the freedom of polarization.

As is seen from (3.8), the eigenvalue of the mass exists only when  $s \geq 1$ . That is, particles of spin 0 can not be given in our theory so long as we assume that  $\varphi$  is square integrable and  $U(X_\mu, r_\mu)$  is scalar. To obtain particles of spin 0, we must either modify the boundary condition or introduce vector or higher tensor Urmaterie. At the present, we can not say which is the preferable one. As for the former, the provisional nature of our boundary condition should be stressed. When higher tensor Urmaterie is introduced, on the other hand, the spin of elementary particles becomes a combination of the "intrinsic" and the "orbital" angular momentum of the internal motion of Urmaterie. To take Urmaterie as scalar means to neglect intrinsic part entirely, and it is also very likely that such a simplification is not permissible.

#### § 4. The relation between Urmaterie field and local field

The discussions of §2 enables us to define the irreducible Urmaterie field  $U(X_\mu, r_\mu, s, m)$  as a simultaneous eigenstate belonging to the specified eigenvalue of  $\mathbf{S}^2$  and  $\mathbf{M}^2$ .

It was our fundamental assumption that  $U(X_\mu, r_\mu; s, m)$  describes an elementary particle of spin  $s$  and mass  $m$ . In the local theory, however, such particle was described by generalized Dirac's equation or Fierz's equation. Thus, it would be important to investigate the relation between the equations of motion of the irreducible Urmaterie field and the corresponding local equations in clarifying the constitution of the Urmaterie field, or in clarifying the physical meaning of internal coordinates.

$U(X_\mu, r_\mu; s, m)$  is obtained by expanding at first the internal wave function into eigenfunctions of  $\mathbf{S}^2$  and  $\mathbf{M}^2$

$$u(k_\mu, r_\mu) = \sum_{s,m,n \leq s} A(k_\mu; s, m, n) \Phi_{s,m,n}(k_\mu, r_\mu), \quad (4.1)$$

and then picking up from them a component belonging to a set of specified value of  $s$  and  $m$ ;

$$U(X_\mu, r_\mu; s, m) = \int A(k_\mu; s, m, n) \Phi_{s,m,n}(k_\mu, r_\mu) e^{ikX} (dk). \quad (4.2)$$

To make the correspondence to the local theory clear, it is necessary to use a rearranged form of (4.2). Using the explicit form given by (3.10),

$$\begin{aligned}
 u(k_\mu, r_\mu) &= \sum_{s, m, n | \epsilon \leq s} A(k_\mu; s, m, n) \\
 &\times \frac{c(s, n)}{\{r_\mu r_\mu - (k_\mu r_\mu)^2 / k_\mu k_\mu\}^{s/2}} g_{1, \epsilon} \left( -\frac{k_\mu r_\mu}{(-k_\mu^2 k_\mu k_\mu)^{1/2}} \right) \\
 &\times \int_{-\pi}^{\pi} du (a_{0\mu} r_\mu + i a_{1\mu} r_\mu \cos u + i a_{2\mu} r_\mu \sin u)^s e^{i n u}.
 \end{aligned} \quad (4.3)$$

Picking up coefficients of  $r_\lambda r_\mu r_\nu \dots$  from this expression under the specified values of  $s$  and  $m$ , we see that they transform under Lorentz-transformations as a symmetrical tensor of rank  $s$  contragradient to  $\underbrace{r_\lambda r_\mu r_\nu \dots}_s$ , since  $u(k_\mu, r_\mu)$  is scalar. Denoting it as  $\underbrace{A_{\lambda, \mu, \nu, \dots}}_s(k_\mu; s, m)$  to make this character explicit, it is given by

$$\begin{aligned}
 &\underbrace{A_{\lambda, \mu, \nu, \dots}}_s(k_\mu; s, m) \\
 &= \sum_{|n| \leq s} \int_{-\pi}^{\pi} du c(s, n) A(k_\mu; s, m, n) e^{i n u} (a_{01} + i a_{11} \cos u + i a_{21} \sin u)^\alpha
 \end{aligned} \quad (4.4)$$

$$\times (a_{02} + i a_{12} \cos u + i a_{22} \sin u)^\beta (a_{03} + i a_{13} \cos u + i a_{23} \sin u)^\gamma (a_{04} + i a_{14} \cos u + i a_{24} \sin u)^\delta,$$

where  $\alpha, \beta, \gamma$  and  $\delta$  denote the numbers of 1, 2, 3 and 4 respectively appearing in suffixes of  $A_{\lambda, \mu, \nu, \dots}(k_\mu; s, m)$ .

It is clear that  $A_{\lambda, \mu, \nu, \dots}(k_\mu; s, m)$  must satisfy  $\{(s+1)(s+2)(s+3)/6 - (2s+1)\}$  subsidiary conditions. This is because  $(s+1)(s+2)(s+3)/6$  components appear under  $A_{\lambda, \mu, \nu, \dots}(k_\mu; s, m)$ , whereas the number of independent component of  $A_{\lambda, \mu, \nu, \dots}(k_\mu; s, m)$  is only  $(2s+1)$ . The explicit form of these subsidiary conditions are given after some calculations, and the results are\*

$$\left. \begin{aligned} A_{\lambda\lambda\nu\dots}(k_\mu; s, m) &= 0 \\ k_\lambda A_{\lambda\mu\nu\dots}(k_\mu; s, m) &= 0 \end{aligned} \right\}. \quad (4.5)**$$

It is clear on the other hand that (2.29) is equivalent to the following equations for  $A_{\lambda, \mu, \nu, \dots}(k_\mu; s, m)$ ;

$$(k_\mu^2 - (m/\hbar)^2) A_{\lambda, \mu, \nu, \dots}(k_\mu; s, m) = 0. \quad (4.6)$$

Eqs. (4.5) and (4.6) are nothing but the equations of motion and subsidiary conditions for elementary particle of spin and mass  $m(\hbar/c\lambda)$  as given by Fierz. Therefore  $A_{\lambda, \mu, \nu, \dots}(k_\mu; s, m)$  can be regarded as a local field for particle of spin  $s$  and mass  $m(\hbar/c\lambda)$ .

\* For details of the calculation, see Appendix III.

\*\* As will be seen through the calculations of Appendix III, (4.5) is a direct consequence of the fact that the eigenfunction of  $S^2$  is essentially spherical harmonics, and therefore is closely related to the rotation of a rigid sphere. In this point, too, the intuitive image given in the non-local field theory is very satisfactory.



The above result shows that the Urmaterie field is equal to the superposition of local fields with various spins and rest masses, the values of which are mediated by the internal motion described by eigenfunctions accompanying to each of them. Such constitution of the Urmaterie field suggests a way for its quantization, since the quantization is a procedure to reproduce particle aspect from that of the wave, and particles that appear in our observation seem to have definite spin and rest mass at least in the case of no interaction. Thus, the quantization of the Urmaterie field may be achieved by quantizing those parts of it which correspond to local fields. The eigenfunctions of the internal motion will, on the other hand, be responsible to the law of interaction between such local fields. A speculative discussion of it is given in § 6.

### § 5. Survey over the interaction of Urmaterie

Thus far we have dealt exclusively with the case of no interaction, and focussed our attention on the understanding of individual elementary particle.

We shall try to see in this section what a new scope is expected when we introduce the interaction of Urmaterie. In particular we want to see to what extent our theory is expected to succeed in elucidating the structure of the interaction of elementary particles, or in dissolving divergence difficulties. Unfortunately, satisfactory theory of the interaction of non-local fields has not been given, and therefore discussions in this section are restricted to very provisional ones. What is attempted here is to see the qualitative feature of these problems that can be seen without entering upon details of the formalism of treating the interaction of non-local fields.

The structure of the interaction of elementary particles is a problem recently proposed by Sakata<sup>21)</sup>, who emphasized the importance of elucidating the qualitative difference as well as the intimate relationships between various kinds of interactions in nature. As the first step to attack this within the framework of the local theory, he suggested "*the principle of renormalisability*", and he and his collaborators classified the interactions into the first kind for which the renormalization procedure can be performed in a closed form, and the second for which this is not the case. Their work played an important role in clarifying the limit of the applicability of the present theory.

It is clear, however, that this work must go beyond such phenomenological stage, and in particular recent experiments seem to suggest strongly the necessity of it. A remarkable example is the establishment of the universal Fermi interaction. Although recent data show that the coupling constants of various direct Fermi interactions are nearly equal, in the current theory the introduction of the mutual interaction is done entirely ad hoc, and this remarkable fact can not help being regarded as accidental.

It will be shown that, by introducing the interaction in the form of the interaction of Urmaterie, a way of introducing various interactions in a unified way and of deducing such relationship is suggested in our theory.

As the formalism of treating the interactions of non-local fields we assume Yukawa's S-matrix<sup>22)</sup>, and investigate the interaction of Urmaterie by assuming appropriate interaction

Lagrangian density and therefrom constructing  $S$ -matrix according to him.

Yukawa's  $S$ -matrix is given by

$$S=1+i\langle L\rangle+i^2\langle LD_+L\rangle+\cdots, \quad (5.1)$$

the notations being the same with those used by him. As the interaction Lagrangian density we assume\*\*

$$L=g\bar{\psi}O_{\mu\nu\dots}\psi\cdot\bar{\psi}O_{\mu\nu\dots}\psi, \quad (5.2)$$

where  $\psi$  is a spinor non-local field describing spinor Urmaterie, and  $O_{\mu\nu\dots}$  some Dirac matrices whose explicit form is, for example, one of Bethe's five covariants of  $\beta$ -interaction. As discussed in § 4,  $\psi$  is a superposition of local fields with various spins and rest masses

$$(x_\mu|\psi|x'_\mu)=\sum_{s,m}\psi_{s,m}(X_\mu)\varphi_{s,m}(r_\mu), \quad (5.3)$$

where  $\varphi_{s,m}(r_\mu)$  is the eigenfunction of the spin and the mass operators which compose the form factor of this interaction as will be seen below. Substituting (5.3) into (5.2), (5.2) is written in the form

$$L=g\sum_{\substack{s,m;s',m'\\s'',m'';s''',m'''}}\bar{\psi}_{s,m}(X_\mu)O_{\mu\nu\dots}\psi_{s',m'}(X'_\mu) \\ \times\bar{\psi}_{s'',m''}(X''_\mu)O_{\mu\nu\dots}\psi_{s''',m'''}(X'''_\mu)\times(\text{form factors}). \quad (5.4)$$

(5.4) represents various interactions of Fermi particles according to which eigenvalues of the spin and the rest mass Urmaterie takes. It gives, for example, the interaction of the nucleon and the electron, of the nucleon and the  $\mu$  meson, or of the  $\mu$  meson and the electron, ... according to the eigenvalue of  $s$  and  $m$ . Here an important fact is that they all appear with the same coupling constant  $g$ . This explains at once the equality of coupling constants of various Fermi interactions, since form factors are expected to reduce approximately to  $\delta$ -functions in low energy region. (By low energy region we mean energy region below about  $(\hbar c/\lambda)$  1-Bev corresponding to  $\lambda$  taken of the order of the compton wave length of the nucleon.)

Thus, the universal Fermi interaction follows as an immediate consequence of introducing the interaction in the form of the interaction of Urmaterie. We hope that our theory thus might serve as a first step to the substantialistic study of the structure of the interaction of elementary particles.

Next we go over to divergence difficulties. This has been one of the most serious difficulties of the quantum theory of the wave field, and many attempts have been done for it.

\* Of course the interaction Lagrangian density must be so chosen that it satisfies required conservation laws. What kind of conservation laws should be satisfied can not be decided at the beginning. As the least requirement, however, it would be necessary that it should satisfy the conservation law of the energy and momentum, of the electric charge, and of the heavy particles. These requirements are satisfied by imposing invariance under translations and gauge transformations performed in a relation with electric and with mesic charge. As for mesic charge see also the discussion of § 6.

\*\* Strictly speaking, (5.2) should not be called interaction Lagrangian density since all  $\psi$  appearing there refer to the same Urmaterie. It should be regarded as a sort of self-stress,

Although remarkable progress has been made by the idea of renormalization, it seems a general feeling of recent years that the satisfactory solution of this problem would not be obtained within the framework of the local theory. The theory of non-local interaction was then proposed, and in particular Kristensen and Möller<sup>21)</sup> have shown that there is a hope of eliminating all divergences if a suitable form is taken for the form factor. The theory of the non-local interaction is thus promising in as much as this point is concerned, but it is clear that this theory has a serious limitation in that it does not give the principle of determining the form of form factors. If we could find such principle, therefore, it would mean a great advance.

It was shown by many authors<sup>10), 21)</sup> that the interaction of non-local fields leads to non-local interactions with definite form of form factors. Of course these conclusions can not be definite at the present when reliable theory of treating the interaction of non-local field is entirely lacking. But the analysis of § 4 suggests that this is a very general feature of the interaction of non-local fields, and a way of overcoming the above limitation of the theory of the non-local interactions may be found from an approach of this line.

It must of course be stressed that discussions in this section should not be regarded more than an optimistic conjecture. But two problems quoted here are the most direct evidences of the limitation of the present theory, and it seems to us not meaningless that at least a clue of overcoming them is suggested by introducing the interaction in the form of the interaction of Urmaterie.

## § 6. The deduction of particle family

The conservation of heavy particles recently emphasized by Oneda<sup>15)</sup> is certainly one of the most important problems to be solved in the theory of the interaction of elementary particles.

For its theoretical explanations have been proposed by many authors under the guiding principle of restricting the type of the interactions by assuming the invariance of the theory against possible transformations<sup>24), 25), 26)</sup> such as the charge conjugation or the time reversal. Of course such attempts of seeking for selection rules within the framework of the current theory are orthodox ones.

But it seems to us that this is a problem to be understood in a relation with a more intrinsic structure of elementary particles. The fact that such disconnected families exist seems to suggest that all Fermi-particles are classified into two families by some unknown structure constants, and the selection rules concerning with this new degrees of freedom play important roles by their mutual transformations. From this view point, the conservation of heavy particles seems to be an evidence for the incompleteness of the definition of elementary particles in the present theory. We shall show in this section that the existence of such new structure constant follows very naturally if we introduce Urmaterie which is described by a spinor non-local field.

Starting with a non-local spinor field  $\psi_\nu(X_\nu, r_\mu)$  ( $\nu=1, 2, 3, 4$ ) which describes the spinor Urmaterie, we assume that the spin and the mass operators are defined in an

analogous way as that of the scalar.

The transformation law of  $\psi_p(X_\mu, r_\mu)$  is of course given by

$$\begin{aligned}\psi'_p(X'_\mu, r'_\mu) &= \gamma_{\rho\sigma} \psi'_\sigma(X_\mu, r_\mu) \\ X'_\mu &= \alpha_{\mu\nu} X_\nu, \quad r'_\mu = \alpha_{\mu\nu} r_\nu,\end{aligned}\tag{6.1}$$

and explicit form of  $\gamma$  is given by

$$\left. \begin{aligned}\gamma &= \cos \theta/2 - \alpha_i \alpha_j \sin \theta/2 \\ \bar{\gamma} &= \cos \theta/2 + \alpha_i \alpha_j \sin \theta/2\end{aligned} \right\}\tag{6.2}$$

for the special rotation in  $(x_i x_j)$  plane  $(i, j=1, 2, 3)$

$$\left. \begin{aligned}x'_i &= x_i \cos \theta + x_j \sin \theta \\ x'_j &= -x_i \sin \theta + x_j \cos \theta\end{aligned} \right\},\tag{6.3}$$

and

$$\gamma = \bar{\gamma} = \cos \theta/2 - \alpha_i \sinh \theta/2\tag{6.4}$$

for the translation

$$\left. \begin{aligned}x'_0 &= x_0 \cosh \theta - x_i \sinh \theta \\ x'_i &= -x_0 \sinh \theta + x_i \cosh \theta \\ \theta &= \log \sqrt{\frac{c+v}{c-v}}\end{aligned} \right\}.\tag{6.5}$$

Thus, we have

$$R_{ij} = (L + 1/2 \cdot \sigma)_k \quad (i, j, k \text{ cyclic}),\tag{6.6}^{*,**}$$

and

$$R_{i0} = (\mu + 1/2 i \cdot \rho_i \sigma)_i,\tag{6.7}$$

where  $L$  and  $\mu$  are given by (2.13), and  $\rho$  and  $\sigma$  are usual Dirac matrices. Definitions of  $S^2$  and  $M^2$  are made in an analogous way as that of the scalar, and separating  $R_{\mu\nu} R_{\mu\nu}$  into  $S^2$  and  $M^2$

$$1/2 \cdot R_{\mu\nu} R_{\mu\nu} = S^2 - M^2,\tag{6.8}$$

we can show at once the commutability of  $S^2$  and  $M^2$ .\*\*\* $S^2$  and  $M^2$  are to be interpreted as the spin and the mass operators for this Urmaterie. In the rest system of the center of mass,  $S^2$  reduces to

\* Here we introduced  $L$  to distinguish the "orbital part" of the spin. This is what was written as  $S$  in § 2.

\*\* The definition of  $R_{\mu\nu}$  is according to Yennie;<sup>(27)</sup>

$$(i/2) \epsilon_{\mu\nu} R_{\mu\nu} \psi_p(X_\mu, r_\mu) = \psi_{p'}(X_\mu, r_\mu) - \psi_p(X_\mu, r_\mu).$$

\*\*\* The proof is given in Appendix IV. (b).



$$(\mathbf{L} + 1/2 \cdot \boldsymbol{\sigma})^2. \quad (6.9)$$

Thus, the spin is given in this case as the sum of the "orbital" and the "intrinsic" angular momentum of the internal motion of spinor Urmaterie.

As for the eigenvalue of the mass operator, we assume as before that it gives the rest mass of elementary particles measured in unit  $(\hbar/c\lambda)$ , where of course an equation analogous to (2.22) is assumed;

$$(r_\mu r_\mu - \lambda^2) \psi(X_\mu, r_\mu) = 0.$$

Writing  $\mathbf{M}^2$  in terms of polar coordinate introduced in (2.26),

$$\mathbf{M}^2 = -(1+x^2)d^2/dx^2 - 2xd/dx - x^2\mathcal{Q} + 3/4 + x\rho_1(\boldsymbol{\sigma}\mathbf{p}), \quad (6.10)$$

where  $p_i$  are "momentum of the internal motion" conjugate to  $r_i$ . With these spin and mass operators, and proceeding in an exactly analogous way as that of the scalar case, a unified description of particles of spin half integer is given.\*

If  $\mathbf{S}^2$  and  $\mathbf{M}^2$  compose a complete set of mutually commuting operators with respect to internal coordinates, the specification of elementary particles with the spin and the rest mass is complete, and no other structure constant appears. If, however, another scalar operator that commutes with them exists, it means that the elementary particles possess a structure constant other than the spin and the rest mass.

In the case of the spinor, such an operator is provided by contracting  $R_{\mu\nu}$  with its dual tensor  $\tilde{R}_{\mu\nu}$ . Explicitly written,

$$R_{\mu\nu}\tilde{R}_{\mu\nu} = (\mathbf{L} + 1/2 \cdot \boldsymbol{\sigma}) \cdot (\boldsymbol{\mu} + 1/2 \cdot (2i) \cdot \rho_1 \boldsymbol{\sigma}), \quad (6.11)$$

which is simplified to

$$R_{\mu\nu}\tilde{R}_{\mu\nu} = -i\rho_1/2 \cdot (i\rho_1(\boldsymbol{\mu}\boldsymbol{\sigma}) + (\mathbf{L}\boldsymbol{\sigma}) + 3/2), \quad (6.12)$$

since we can show by direct calculations that  $(\mathbf{L}\boldsymbol{\mu})$  vanishes. It can be further shown that (see Appendix IV(d))

$$\left. \begin{aligned} [\mathbf{S}^2, R_{\mu\nu}\tilde{R}_{\mu\nu}] &= 0 \\ [\mathbf{M}^2, R_{\mu\nu}\tilde{R}_{\mu\nu}] &= 0 \end{aligned} \right\}. \quad (6.13)$$

Thus, the states of the internal motion which are specified by the eigenvalues of  $\mathbf{S}^2$  and  $\mathbf{M}^2$  is always degenerated with respect  $R_{\mu\nu}\tilde{R}_{\mu\nu}$ . Some parts of  $R_{\mu\nu}\tilde{R}_{\mu\nu}$  are not independent of  $\mathbf{S}^2$  and  $\mathbf{M}^2$ , and omitting such irrelevant term, we are left with

$$\theta = \rho_1. \quad (6.14)$$

\* It should be noted that  $R_{\mu\nu}$  introduced in (6.7) is not hermitic, therefore the mass operator is also not hermitic in spinor case. This is because the representation of the Lorentz group by spinor is not unitary. A way of avoiding this difficulty is to use unitary trick, which consists in replacing  $\psi_0$  by  $i\psi_0$ . In this case, however, the square integrability of eigenfunctions is violated. These difficulties are overcome only by introducing quantity like expensor recently introduced by Dirac<sup>(19)</sup>. Its actual study will be reported elsewhere. What is intended in this section is to give the simplest model needed to understand better the concept of particle family.

Although  $\rho_1$  is pseudoscalar, it is easy to construct a scalar operator that inherits the essential feature of it, and we disregard this odd character in the following to simplify the discussion.\*

$\theta$  has eigenvalues  $\pm 1$ . As will be seen from (6.10),  $\theta$  is closely related to the eigenvalue of  $m$ , and in classifying all spinor particles into two families according to this eigenvalue, the eigenvalue  $\pm 1$  determine the minimum value of the rest mass appearing in these two families. Although eq. (6.10) has not yet been solved exactly, a preliminary estimation treating  $x\rho_1(\sigma\mathbf{p})$  as a small perturbation yields

$$m_{m\hat{m}}^2 \simeq \begin{cases} 5 & \text{for eigenvalue of } \theta=1 \\ 0 & \text{,,} \quad \quad \quad =-1 \end{cases} \quad (6.15)$$

As was discussed before,  $m$  represents the rest mass of elementary particles measured in unit  $(\hbar/c\lambda)$ . Therefore, by taking  $\lambda$  to be of the order of the compton wave length of the nucleon, this mass separation becomes comparable to that of the nucleon and lepton families.

Thus,  $\theta$  is interpreted as expressing the intrinsic difference of these two families, and in our theory the classification of Fermi particles into heavy and light follows as an inevitable consequence of introducing spinor Urmaterie. This means that in our theory the concept of particle family must be considered as very fundamental. We think that this result may be considered as a remarkable advantage of our theory.

In our theory, the difference of heavy and light particles is reduced to that of  $\theta$ . Therefore, the conservation of  $\theta$  leads at once to the conservation of both families. To find a reason of assuming the conservation of  $\theta$ , we introduce  $\theta$  defined by

$$\theta = 1/2 \cdot (1 + \Theta). \quad (6.16)$$

$\theta$  takes eigenvalues 1 and 0 for the nucleon and lepton family respectively, and just corresponds to  $\lambda$  introduced by Oneda<sup>15)</sup>. Therefore, it would be natural to interpret it as mesic charge.\*\*\*,\*\*\*\*

$\theta$  thus being interpreted as mesic charge, it is natural to require its conservation. Formally, this is satisfied by assuming the invariance of the theory under the "gauge transformation" performed in a relation with it;

$$\left. \begin{aligned} \psi &\rightarrow \psi e^{i\theta\alpha} \\ \psi^* &\rightarrow \psi^* e^{-i\theta\alpha} \end{aligned} \right\}, \quad (6.17)$$

\* A way is to double the components of  $\psi_\rho(X_\mu, r_\mu)$  into eight, and to introduce an independent set of Pauli matrices  $\omega$  other than  $\rho$  and  $\sigma$ . It seems interesting to identify  $\omega$  to  $\tau$  spin, and to try to elucidate a deeper relation between the electric and mesic charge, or between the conservation of the electric and mesic charge.

\*\* (6.15) was obtained using unitary trick. In this case, unperturbed eigenfunctions and boundary conditions differ from those of § 3, and as remarked before, square integrability is violated. But here we do not touch these difficulties.

\*\*\* Thus, in our theory, to  $\theta$  is given two fold meanings; it classifies on the one hand the heavy and light particles, and on the other hand mediates the interaction of  $\pi$ -meson with Fermi particles. We are tempted to consider this fact as an answer to the question why only heavy particles interact strongly with  $\pi$ -meson.

\*\*\*\* The concept of mesic charge and its conservation was first introduced by Okayama<sup>26)</sup> and by Wigner<sup>27)</sup>.

where  $\alpha$  is an arbitrary constant. From this requirement follows the conservation of mesic current, which is nothing but the conservation of heavy particles.

### § 7. Summary of results and concluding remarks

It would be convenient to give first a brief summary in clarifying the logical construction of the theory we have developed thus far.

What is the most fundamental in our theory is a substance we named "*Urmaterie*", which we assumed is described by a non-local field satisfying the equations of motion

$$(\square - \mathbf{M}^2/\lambda^2)U(X_\mu, r_\mu) = 0, \quad (7.1)$$

and

$$(r_\mu r_\mu - \lambda^2)U(X_\mu, r_\mu) = 0. \quad (7.2)$$

Various states of the internal motion of *Urmaterie* are classified by the eigenvalues of a complete set of mutually commuting scalar operators with respect to internal coordinates, two of which can be taken as the spin and the mass operators, and each eigenstate thus classified is assumed to correspond to the elementary particle of definite structure. What determine the structure of *Urmaterie* is the transformation property of the non-local field used in describing it, and the parameter  $\lambda$  appearing in (7.2). In this way a unified description of elementary particles is given, and the structure of *Urmaterie* determines the structure of individual elementary particle and the mutual correlation between them. That is, it determines the spectrum of the spin, of the rest mass, or of other structure constants, the structure of interactions, or the form of the form factors when interaction is introduced. For example, by taking  $U(X_\mu, r_\mu)$  as scalar, a unified description of Bose particles is given each of which is characterized by two structure constants, the spin and the rest mass, and which are subjected to the spin and the mass spectrum given by (2.28) and (3.8), and by taking it as spinor, that of Fermi-particles which are classified into two families according to mesic charge 1 and 0.

Thus, in our theory, elementary particles are regarded as *phenomenal forms* of *Urmaterie*. That the definition of elementary particles is made more positive by introducing *Urmaterie* is clear. For example, in our theory, the answer can be given at once to the question such as, "Is the specification of elementary particles by the spin and the rest mass complete?". This is never a self-evident thing, and in fact the discovery of particle families show its incompleteness at least in the case of Fermi particles. It may be a remarkable advantage of our theory that it can not only give answer to these questions, but also succeed in deducing just the required new structure constant.

In addition, the possibility of deducing the spectrum of various structure constants, the structure of the interactions, or of the form of form factors must also be stressed. Note that, as discussed in § 2, these are the most direct evidences of the limitation of the present theory.

Of course our theory is only a starting point for the unified theory and is far from satisfactory goal. The most serious limitation of our theory is the lack of law governing the internal motion. Keenly we feel that the attention of the future theory must be

focussed on elucidating it. The use of Yukawa's non-local field must also be criticized. Although his non-local field is a very elegant, probably the most elegant way of introducing internal coordinates known up to the present, there is no inevitable reason to use it at least at the present stage, and we can not rule out the possibility that the use of Yukawa's non-local field impose a severe restriction to the whole of our theory.

In conclusion, it would be important to make clear the position of our theory in the course of the development of the theory of elementary particles.

The most essential point of our discussion was that the moment to the further development would only be found in *more positively* defining elementary particles. We regarded the present theory as "*phenomenological*", and it was as the first step to overcome this limitation that we introduced Urmaterie. In this sense, our theory may be regarded as "*substantialistic*" to the present theory, and at the same time as "*phenomenological*" to the unified theory we are aiming at the next stage\*.

In fact, such limitation of our theory appears already in following forms. The first, our theory gives no answer to questions such as "What kind of Urmaterie should we take?", or "What kind of interaction should we introduce between them?". The second, the introduction of spinor Urmaterie as we did in § 6 means to introduce elements not completely analyzable in our stage.

At the present, we can say nothing to these questions. But it is our strong feeling that a series of facts we quoted in this paper as the evidences of the limitation of the present theory are problems not to be attacked separately, but should be regarded all as suggesting the necessity of introducing a substance of higher level. In this connection, we hope that our theory might play some role as the first step toward an approach of such line.

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### Appendix I

A detailed discussion of solving (3.5) is given here. In order to get a solution defined in the interval  $0 < z < 1$ , we put

$$\left. \begin{aligned} (y-1)/y &= z \\ u &= (1-z)^{\alpha} v \end{aligned} \right\}.$$

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\* In this paper we disregarded the degree of freedom concerning with charge states entirely. Of course this degree of freedom must be taken in the work of next approximation, and in particular the amalgamation of our theory with that of Pais<sup>9)</sup> seems very promising.



Then (3.5) turns into

$$z(1-z)v'' + \{(\alpha + \beta + 1 - \gamma) - (\alpha + \alpha + 1 - \gamma + 1)z\}v' - \alpha(\alpha + 1 - \gamma)v = 0,$$

where  $y; 1 \rightarrow \infty$  corresponds to  $z; 0 \rightarrow 1$ .

Expanding  $v$  into power series

$$v = z^\rho \sum_k a_k z^k,$$

we get two independent solutions in terms of hypergeometric series

$$v_0 = F(\alpha, \alpha + 1 - \gamma, \alpha + \beta + 1 - \gamma, z) \quad \text{for } \rho = 0$$

$$v_1 = z^{\gamma - \alpha - \beta} F(\gamma - \beta, 1 - \beta, \gamma + 1 - \alpha - \beta, z) \quad \text{for } \rho = \gamma - \alpha - \beta = 1/2,$$

both of which are convergent in the interval  $0 < z < 1$ .

Thus, the general solution of (3.3) is given by

$$\varphi = A\varphi_0 + B\varphi_1,$$

where

$$\left. \begin{aligned} \varphi_0 &= (1-z)^{\frac{1}{2}(s+1)+\alpha} F(\alpha, \alpha + 1 - \gamma, \alpha + \beta + 1 - \gamma, z) \\ \varphi_1 &= (1-z)^{\frac{1}{2}(s+1)+\alpha} z^{\gamma - \alpha - \beta} F(\gamma - \beta, 1 - \beta, \gamma + 1 - \alpha - \beta, z) \end{aligned} \right\}, \quad (\text{AI} \cdot 1)$$

and  $A$  and  $B$  arbitrary constants. These solutions must satisfy the condition of square integrability

$$\int_{-\infty}^{\infty} |\varphi|^2 \sqrt{1+x^2} dx = \text{finite},$$

where  $\sqrt{1+x^2}$  is the Jacobian that appeared in transforming variable from  $r_\mu$  into  $\theta$ ,  $\varphi$  and  $x$ .

In order that this condition be satisfied, the following is necessary and sufficient.

- i)  $\varphi = \text{finite}$  at  $z=0$ ,  
and  
ii)  $\varphi = 0$  at  $z=1$ .

For i) it is easily seen that the condition is automatically satisfied since

$$\gamma - \alpha - \beta = 1/2 > 0,$$

but for ii) a detailed discussion is needed. It is because although  $(1-z)^{\frac{s+1}{2}+\alpha}$  tends to zero since

$$\frac{1}{2}(s+1) + \alpha = \frac{1}{2} + \frac{\sqrt{s(s+1)+1-m^2}}{2} > 0,$$

another factors  $F(\alpha, \alpha + 1 - \gamma, \alpha + \beta + 1 - \gamma, z)$  and  $F(\gamma - \beta, 1 - \beta, \gamma + 1 - \alpha - \beta, z)$  diverge.

The following identity is useful to investigate the behavior of  $\varphi_0$  and  $\varphi_1$  near  $z=1$ ;

$$F(\alpha', \beta', \gamma', z) \equiv (1-z)^{\gamma' - \alpha' - \beta'} F(-\alpha' + \gamma', \gamma' - \beta', \gamma', z).$$

Using this, (AI.1) is expressed as

$$\left. \begin{aligned} \varphi_0 &= (1-z)^{\frac{1}{2}(s+1)+\beta} F(\beta+1-\gamma, \beta, \alpha+\beta+1-\gamma, z) \\ \varphi_1 &= (1-z)^{\frac{1}{2}(s+1)+\beta} z^{\gamma-\alpha-\beta} F(1-\alpha, \gamma-\alpha, \gamma+1-\alpha-\beta, z) \end{aligned} \right\}. \quad (\text{AI.2})$$

Both  $F$  appearing in the right hand side of (AI.2) being finite at  $z=1$  on account of

$$(\alpha+\beta+1-\gamma) - (\beta+\beta+1-\gamma) = \alpha-\beta = \sqrt{s(s+1)+1-m^2} > 0$$

and

$$(\gamma+1-\alpha-\beta) - (\gamma-\alpha+1-\alpha) = \alpha-\beta > 0,$$

the behavior of  $\varphi_0$  and  $\varphi_1$  near  $z=1$  is determined completely by whether the power index of  $(1-z)$  is positive or is negative.

Thus, for  $m^2 \geq s(s+1)$ , namely for

$$Re \left[ \frac{s+1}{2} + \beta \right] = Re \left[ \frac{1}{2} (1 - \sqrt{s(s+1)+1-m^2}) \right] \geq 0$$

both  $\varphi_0$  and  $\varphi_1$  converge. This yields continuous spectrum.

For  $m^2 < s(s+1)$ , on the other hand,  $(1-z)^{\frac{1}{2}(s+1)+\beta}$  diverges at  $z=1$ . In order to get allowable solutions in this case, therefore, we must choose  $F$  in (AI.2) so that it tends to zero at  $z=1$ . This means that we must choose  $F$  in (AI.1) so that it breaks at finite terms. Then we meet following two case;

i) For  $\varphi_0$ ,

$\alpha$  must be negative integer;

$$\alpha = \frac{1}{2} \{ -s + \sqrt{s(s+1)+1-m^2} \} = -n,$$

$$n=0, 1, 2, \dots < \frac{s}{2}.$$

Then the eigenvalue of the mass is given by

$$m_{s,n}^2 = s+1+4ns-4n^2,$$

with the corresponding eigenfunction

$$\varphi_{s,n}^0 = (1+x^2)^{-\frac{s-2n+1}{2}} F\left(-n, s-n+\frac{1}{2}, \frac{1}{2}, \frac{x^2}{1+x^2}\right).$$

ii) For  $\varphi_1$

$\gamma-\beta$  must be negative integer.

In this case the eigenvalue of the mass is given by

$$m_{s,n}^2 = 3s+4ns-4n-4n^2$$

$$n=0, 1, 2, \dots < \frac{s-1}{2},$$

with the corresponding eigenfunction

$$\varphi'_{s,n} = (1+x^2)^{-\frac{s-2n+1}{2}} x F\left(-n, s-n+\frac{1}{2}, \frac{3}{2}, \frac{x^2}{1+x^2}\right).$$

Other cases, for example,  $1-\beta=-n$ , or  $\alpha+1-\gamma=-n$  is not permitted, since it contradicts to  $m^2 < s(s+1)$ . Moreover it is impossible that both  $\varphi_0$  and  $\varphi_1$  become eigenfunctions belonging to the same value of mass. This is easily seen from the fact that for  $\alpha=-n$ , neither  $\gamma-\beta$  nor  $1-\beta$  can be negative integer.

Above two cases are expressed in a compact form, if we introduce another parameter which is given by  $2n$  for case i) and  $(2n+1)$  for case ii); Then

$$m_{s,l}^2 = s+1+2sl-l^2, \quad l=0, 1, 2, \dots < s, \quad (\text{A}\cdot\text{I}3)$$

and the eigenfunction is

$$\begin{aligned} \varphi_{s,l} &= (1+x^2)^{-\frac{s-l+1}{2}} \left(\frac{x^2}{1+x^2}\right)^\varepsilon F\left(-\frac{l}{2}+\varepsilon, s+\frac{1-l}{2}+\varepsilon, \frac{1}{2}+2\varepsilon, \frac{x^2}{1+x^2}\right) \\ &\equiv (1+x^2)^{-\frac{s-l+1}{2}} \left(\frac{x^2}{1+x^2}\right)^\varepsilon G_{\frac{l}{2}-\varepsilon}^{\frac{l}{2}-\varepsilon}\left(s+\frac{1}{2}, \frac{1}{2}+2\varepsilon, \frac{x^2}{1+x^2}\right). \end{aligned} \quad (\text{AI}\cdot 4)$$

where  $G_{\frac{l}{2}-\varepsilon}^{\frac{l}{2}-\varepsilon}$  is the Jacobi's polynomial of  $(l/2-\varepsilon)$ th order and

$$\varepsilon = \frac{1+(-1)^{l+1}}{4} = \begin{cases} 0 & \text{for even} \\ 1/2 & \text{for odd.} \end{cases}$$

Although it seems apparently possible that the solution of the form  $\varphi = (A\varphi_0 + B\varphi_1)$  can tend to zero at  $x = \pm\infty$  even in the case when both  $\varphi_0$  and  $\varphi_1$  diverges (since their order of divergence is the same), we can see at once that this does not occur. It is because  $\varphi_0$  is an even and  $\varphi_1$  is an odd function of  $x$  as will be easily seen from (AI.1). From it we can conclude at once that constants  $A$  and  $B$ , non of which is zero, satisfying

$$A\varphi_0(\infty) + B\varphi_1(\infty) = 0$$

$$A\varphi_0(-\infty) + B\varphi_1(-\infty) = A\varphi_0(\infty) - B\varphi_1(\infty) = 0$$

do not exist.

Next we examine the orthogonality of eigenfunctions given by (AI.4). The first equation (3.5) can be written in a self-adjoint form

$$\{(1+x^2)^{3/2}\varphi'\}' - \frac{x^2}{\sqrt{1+x^2}} s(s+1)\varphi + m^2\sqrt{1+x^2}\varphi = 0$$

Then we get for the same value of  $s$

$$(m_{s,k}^2 - m_{s,l}^2) \int_{-\infty}^{\infty} \varphi_{s,k} \varphi_{s,l} \sqrt{1+x^2} dx = [(1+x^2)^{3/2} \{\varphi_{s,k} \varphi'_{s,l} - \varphi'_{s,k} \varphi_{s,l}\}]_{-\infty}^{\infty}.$$

Substituting explicit form of  $\varphi$ , we find at once

$$\begin{aligned} \varphi'_{s,l} = & 2x \left\{ \left( 1 - \frac{s-l+1}{2} \right) (1+x^2)^{-1} + \varepsilon \left( \frac{x^2}{1+x^2} \right)^{-1} \frac{1}{1+x^2} \right\} \varphi_{s,l} \\ & + (1+x^2)^{-\frac{s-l+1}{2}} \left( \frac{x^2}{1+x^2} \right)^{\varepsilon} F \left( -\frac{l}{2} + \varepsilon + 1, \quad s + \frac{3}{2} - \frac{l}{2} + \varepsilon, \quad \frac{3}{2} + 2\varepsilon, \quad \frac{x^2}{1+x^2} \right) \\ & \times \frac{2x}{1+x^2}, \end{aligned}$$

and noticing that in the third term  $F$  does not contribute to divergence since it is a polynomial in  $x^2/(1+x^2)$  and is finite everywhere, we see that the term that diverges most weakly in limits  $x \rightarrow \pm \infty$  is the first, which is of the order of  $1/x$ . Hence, putting,

$$\varphi'_{s,l} = O(1/x) \varphi_{s,l},$$

we obtain

$$(1+x^2)^{3/2} \{ \varphi_{s,k} \varphi'_{s,l} - \varphi'_{s,k} \varphi_{s,l} \} \sim (1+x^2)^{-s-1+\frac{l+k}{2}} \times O(1/x) \times (1+x^2)^{3/2}.$$

The first term is  $O\{(1/x)^{2(s+1)-(l+k)}\}$ , the second  $O(1/x)$ , and the third is  $O\{(1/x)^{-3}\}$ , from which we get as the orthogonality condition for the same  $s$  and different  $l, k$ ,

$$2(s+1) - (l+k) + 1 - 3 > 0,$$

or

$$2s > l+k,$$

This condition is satisfied by  $s > l$ , and  $s > k$ , which was just the condition of square integrability.

## Appendix II. Group theoretical investigation of the eigenvalue equation\*

The eigenvalue problem for the mass spectrum can be solved group-theoretically. The result agrees with that of § 3 as it should be, but it is useful in clarifying the mathematical feature of the problem.

The internal eigenfunction solved under the boundary condition of square integrability transform under Lorentz transformations according to a unitary representation of the Lorentz group. The unitary representation of the Lorentz group was studied in detail by Wigner<sup>(31)</sup> and by Bargmann,<sup>(20)</sup> and in particular Bargmann gave an explicit solution of it. According to him, there are two such irreducible representations labeled by two parameters  $\xi$  and  $\eta$ , which are eigenvalues of  $\hat{\xi}$  and  $\hat{\eta}$  of (3.9) respectively.

- i)  $\xi=0$  and  $\eta$  is any positive number. In this case  $s$  takes all values of 0, 1, 2, ..., where  $s$  is the eigenvalue of  $R_{ij}R_{ij}; \frac{1}{2}R_{ij}R_{ij}=s(s+1)$ .

\* The content of this appendix is due to Mr. Murai<sup>(30)</sup>.



ii)  $\xi$  is any real number, and  $\eta = 1 - k^2 + (\xi/k)^2$ .

In this case  $s = k, k+1, k+2, \dots$  with

$$k = 1/2, 1, 3/2, \dots$$

The representation space is a Hilbert space defined on the surface of a unit sphere except for  $0 < \eta < 1$  of case i).

Our discrete case corresponds to  $\xi = 0$ , and  $s = \text{integer of ii})^*$ . Thus, required eigenvalue is given by

$$\mathbf{M}^2 = s(s+1) + 1 - k^2$$

$$k = 1, 2, 3, \dots$$

$$s = k+1, k+2, k+3, \dots$$

or, if  $s$  is specified first,

$$\mathbf{M}^2 = s(s+1) + 1 - (s-l)^2$$

$$s \geq 1, l = 0, 2, \dots, s-1,$$

which agrees with (3.8) completely.

### Appendix III

In this appendix the calculations outlined in § 3 will be explained in details.

We first construct eigenfunctions of the mass and the spin operators in arbitrary reference system, and then derive Fierz's subsidiary conditions for each decomposed parts of the Urmaterie field.

In what follows quantities refering to the rest system of the external motion will be distinguished by primes. Thus, the eigenfunction belonging to the eigenvalue  $s$  and  $m$  of the spin and mass operators satisfies

$$\text{and} \quad \left. \begin{aligned} \mathbf{M}^2 \phi'_{s,m,n} &= m^2 \phi'_{s,m,n} \\ \mathbf{S}^2 \phi_{s,m,n} &= s(s+1) \phi'_{s,m,n} \end{aligned} \right\}, \quad (\text{AIII} \cdot 1)$$

where

$$\phi'_{s,m,n} = \varphi_{s,m}(x') P_s^n(\cos \theta') e^{in\varphi'}; \quad x' = \sinh \xi'.$$

The explicit form of  $\varphi_{s,m}$  is given in Appendix I. (See eq. (AI.4).)  $\mathbf{S}^2$  and  $\mathbf{M}^2$  being invariant under Lorentz-transformations, required eigenfunctions can be obtained by generalizing  $\phi'_{s,m,n}$  into an invariant form. This can be done at once with the help of the formula<sup>19)</sup>

<sup>19)</sup> That  $\tilde{A}'_{\mu\nu} \tilde{A}'_{\mu\nu}$  vanishes can be seen at once, for example, by taking  $r'_\mu = (r_1, 0, 0, 0)$ . The appearance of such restriction is a little queer. We think that this related to the use of Yukawa's second equation, which stipulates  $r'_\mu$  to be space-like. A fuller study of it will appear elsewhere.

$$P_s^n(\cos \theta) e^{in\varphi} = \frac{c(s, n)}{r^s} \int_{-\pi}^{\pi} du (r_3 + ir_1 \cos u + ir_2 \sin u) e^{inu}. \quad (\text{AIII} \cdot 2)$$

The result is

$$\begin{aligned} \Phi_{s,m,n} &= \frac{c(s, n)}{\{r_u r_u - (k_\mu r_\mu)^2 / k_\mu k_\mu\}^{s/2} g_{\mu, s}} \left( -\frac{k_\mu r_\mu}{(-k_\mu^2 k_\mu)^{1/2}} \right) \times \\ &\times \int_{-\pi}^{\pi} du (a_{3\mu} r_\mu + ia_{1\mu} r_\mu \cos u + ia_{2\mu} r_\mu \sin u)^s e^{inu}, \end{aligned} \quad (\text{AIII} \cdot 3)$$

where  $a_{\mu\nu}$  are coefficients of Lorentz-transformation given in (3.11);

$$a_{\mu\lambda} k_\lambda = \begin{cases} 0 & \text{for } \mu = 1, 2, 3, \\ im(\hbar/c\lambda) & \text{for } \mu = 4, \end{cases} \quad (\text{AIII} \cdot 4)$$

and  $s$  and  $m$  are assumed as scalar.

It is clear that  $\Phi_{s,m,n}(k_\mu, r_\mu)$  reduces to  $\Phi'_{s,m,n}(k'_\mu, r'_\mu)$  when written in terms of  $r'_\mu$ , and therefore composes the complete set of eigenfunctions in arbitrary reference systems;

$$S^2 \Phi_{s,m,n}(k_\mu, r_\mu) = s(s+1) \Phi_{s,m,n}(k_\mu, r_\mu)$$

$$M^2 \Phi_{s,m,n}(k_\mu, r_\mu) = m^2 \Phi_{s,m,n}(k_\mu, r_\mu).$$

The orthogonality of  $\Phi_{s,m,n}(k_\mu, r_\mu)$  follows at once from that of  $\Phi'_{s,m,n}(k'_\mu, r'_\mu)$ .

As pointed out in § 4,  $A_{\lambda\mu\nu\dots}(k_\mu; s, m)$  must satisfy  $\{(s+1)(s+2)(s+3)/6 - (2s+1)\}$  subsidiary conditions. We try next to deduce explicit form of these subsidiary conditions. For this purpose we construct

$$\begin{aligned} k_\lambda A_{\lambda\mu\nu\dots}(k_\mu; s, m) &= \sum_{|n| \leq s} c(s, n) A(k_\mu; s, m, n) \times \\ &\times \int_{-\pi}^{\pi} du e^{inu} [a_{3\mu} k_\mu + ia_{1\mu} k_\mu \cos u + ia_{2\mu} k_\mu \sin u] \\ &\times (a_{31} + ia_{11} \cos u + ia_{21} \sin u)^{\alpha-1} (a_{32} + ia_{12} \cos u + ia_{22} \sin u)^\beta \\ &\times (a_{33} + ia_{13} \cos u + ia_{23} \sin u)^\gamma (a_{34} + ia_{14} \cos u + ia_{24} \sin u)^\delta, \\ \alpha &\geq 1, \end{aligned} \quad (\text{AIII} \cdot 5)$$

and

$$\begin{aligned} A_{\lambda\mu\nu\dots}(k_\mu; s, m) &= \sum_{|n| \leq s} c(s, n) A(k_\mu; s, m, n) \int_{-\pi}^{\pi} du e^{inu} \times \\ &\times [\{a_{3\mu} a_{3\mu} - a_{1\mu} a_{1\mu} \cos^2 u - a_{2\mu} a_{2\mu} \sin^2 u\} \\ &+ 2\{ia_{3\mu} a_{1\mu} \cos u + ia_{3\mu} a_{2\mu} \sin u - a_{1\mu} a_{2\mu} \sin u \cos u\}] \times \end{aligned} \quad (\text{AIII} \cdot 6)$$

$$\begin{aligned} & \times (a_{31} + ia_{11} \cos u + ia_{21} \sin u)^{\alpha-2} (a_{32} + ia_{12} \cos u + ia_{22} \sin u)^{\beta} \\ & \times (a_{33} + ia_{13} \cos u + ia_{23} \sin u)^{\gamma} (a_{34} + ia_{14} \cos u + ia_{24} \sin u)^{\delta}, \\ & \alpha \geq 2. \end{aligned}$$

Terms in [ ] in the right hand side of (AIII·5) vanish on account of (AIII·4), and those in [ ] in the right hand side of (AIII·6) also vanish which can be seen at once from

$$a_{u\lambda} a_{v\lambda} = \delta_{uv}.$$

Thus, we get

$$k_{\lambda} A_{\lambda\mu\nu\dots}(k_{\mu}; s, m) = 0, \quad (\text{AIII} \cdot 7)$$

and

$$A_{\lambda\lambda\nu\dots}(k_{\mu}; s, m) = 0. \quad (\text{AIII} \cdot 8)$$

(AIII·7) and (AIII·8) are nothing but the subsidiary conditions for particle of spin  $s$  and mass  $m(\hbar/c\lambda)$  as given by Fierz.

The number of independent subsidiary conditions resulting from (AIII·7) and (AIII·8) is  $s(s+1)(s+2)/6$  and  $s(s-1)/2$  respectively. This assures that other conditions can never appear.

## Appendix IV

In this appendix details of calculations not touched in the paper is given.

(a) The proof of (2·23)

(22·2) restricts the internal world to the surface of a hyperboloid, and  $u(r_{\mu})$  to be of the form

$$u(r_{\mu}) = \delta(r_{\mu}r_{\mu} - \lambda^2) u'(r_{\mu}),$$

where it should be understood that  $r_{\mu}$  in  $u'(r_{\mu})$  is restricted by  $r_{\mu}^2 = \lambda^2$ . This suggests a modification of quantum mechanics we assumed to the internal motion. Our new assumption is that the essential part of the internal wave function is  $u'(r_{\mu})$ , and the expectation values of various dynamical variables are given by

$$\begin{aligned} \left\langle F\left(r_i, \frac{1}{i} \frac{\partial}{\partial r_i}, \dots\right) \right\rangle &= \int \delta^*(r_{\mu}r_{\mu} - \lambda^2) \delta(r_{\mu}r_{\mu} - \lambda^2) \delta(r_{\mu}n_{\mu}) \\ &\times u^{*'}(r_{\mu}) F\left(r_i, \frac{1}{i} \frac{\partial}{\partial r_i}, \dots\right) u'(r_{\mu}) (dr_{\mu}). \end{aligned}$$

This gives for  $m^2$

$$\begin{aligned} m^2 &= - \int \delta^*(r_{\mu}r_{\mu} - \lambda^2) \delta(r_{\mu}r_{\mu} - \lambda^2) \delta(r_0) u^{*'}(r_{\mu}) \\ &\times \left( r_i \frac{\partial}{\partial r_0} + r_0 \frac{\partial}{\partial r_i} \right)^2 u'(r_{\mu}) (dr_{\mu}), \end{aligned} \quad (\text{AIV} \cdot 1)$$

where we have put  $n_\mu$  as  $(0, 0, 0, i)$ . Of terms of (2.21) what remained were  $r\partial/\partial r$  and  $r^2\partial^2/\partial r_0^2$ .

The vanishing or  $r\partial/\partial r$  can be shown at once if we rewrite it in terms of polar coordinates introduced in (2.26)

$$\frac{\partial}{\partial r} = \cosh \xi \frac{\partial}{\partial A} - \frac{\sinh \xi}{A} \frac{\partial}{\partial \xi}, \quad (\text{AIV} \cdot 2)$$

where  $A$  should be regarded as variable. An important fact is that  $A$  does not appear in  $u'(r_\mu)$  owing to (2.22). The first term of (AIV.2) therefore vanishes, and the second also vanishes by virtue of  $\partial(r_0)$  appearing in (AIV.1).

Thus, we are left with

$$m^2 = - \int \partial^*(r_\mu r_\mu - \lambda^2) \partial(r_\mu r_\mu - \lambda^2) \partial(r_0) u^{*'}(r_\mu) \\ \times r_0^2 \frac{\partial}{\partial r_0^2} u'(r_\mu) (dr_\mu).$$

Here  $r^2$  can be replaced by  $\lambda^2$ , and under the assumption that  $-\partial^2/\partial r_0^2$  corresponds to the operator of the square of the energy of the internal motion, we get finally

$$m^2 = (\lambda/\hbar c)^2 \langle E_{\text{int}}^2 \rangle.$$

(b) Proof of  $[\mathbf{M}^2, \mathbf{S}^2] = 0$ .

It is sufficient to prove it in center of mass system since  $\mathbf{M}^2$  and  $\mathbf{S}^2$  are scalar.

1) The case of scalar.

$$\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2,$$

and

$$\mathbf{M}^2 = \mu_1^2 + \mu_2^2 + \mu_3^2.$$

Using the commutability relations in (2.14), we get

$$[S_i, \sum_j \mu_j^2] = \sum_j \{ [S_i, \mu_i] \mu_j + \mu_j [S_i, \mu_j] \} \\ = i \sum_j \{ \mu_k \mu_j - \mu_j \mu_k + \mu_j \mu_k - \mu_k \mu_j \} = 0.$$

Hence

$$[\sum_j S_i^2, \sum_j \mu_j^2] = 0.$$

2) The case of spinor.

$$\mathbf{S}^2 = \mathbf{L}^2 + (\boldsymbol{\sigma} \mathbf{L}) + 3/4,$$

and

$$\mathbf{M}^2 = \mu^2 - i \rho_1 (\mu \boldsymbol{\sigma}) - 3/4.$$

Then

$$[\mathbf{S}^2, \mathbf{M}^2] = \sum_i \{ -i \rho_i [L, (\mu \boldsymbol{\sigma})] L_i - i \rho_i L_i [L, (\mu \boldsymbol{\sigma})] \}$$



$$-\sum_i \{ [\mu_i, (\mathbf{L}\sigma)] \mu_i + \mu_i [\mu_i, (\mathbf{L}\sigma)] \} \\ + \rho_1(\sigma, \mathbf{L} \times \mu - \mu \times \mathbf{L}),$$

terms in the second bracket vanish by virtue of (2.14), and the first is rewritten as

$$\rho_1(\sigma, \mu \times \mathbf{L} - \mathbf{L} \times \mu),$$

and compensates with the third.

(c) The proof of  $[\mathbf{S}^2, r_\mu r_\mu] = 0$  and  $[\mathbf{M}^2, r_\mu r_\mu] = 0$ .

$$[R_{\nu\lambda}, r_\mu r_\mu] = 1/i \cdot \{ r_\nu [\partial/\partial r_\lambda, r_\mu r_\mu] - r_\lambda [\partial/\partial r_\nu, r_\mu r_\mu] \} \\ = 1/i \cdot \{ r_\nu r_\mu \partial_{\lambda\mu} - r_\lambda r_\mu \partial_{\nu\mu} \} = 0,$$

thus  $r_\mu r_\mu$  and  $\mathbf{S}^2$ ,  $\mathbf{M}^2$  are commutable.

(d) The proof of  $[\mathbf{S}^2, R_{\mu\nu} \tilde{R}_{\mu\nu}] = 0$ .

Again, we can evaluate it in the center of mass system utilizing the scalar character of various operators. Thus,

$$[\mathbf{S}^2, R_{\mu\nu} \tilde{R}_{\mu\nu}] = 1/2 \cdot [\mathbf{L}^2 + (\mathbf{L}\sigma) + 3/4, (\sigma\mu) - i\rho_1(\sigma\mathbf{L}) - 3/2 \cdot i\rho_1] \\ = 1/2 \cdot \{ [\mathbf{L}^2, (\mu\sigma)] + [(\mathbf{L}\sigma), (\mu\sigma)] \} \\ = 1/2 \cdot \{ i(\sigma, \mu \times \mathbf{L} - \mathbf{L} \times \mu) + i(\sigma, \mathbf{L} \times \mu - \mu \times \mathbf{L}) \} \\ = 0.$$

$$[\mathbf{M}^2, R_{\mu\nu} \tilde{R}_{\mu\nu}] = 1/2 [\mu^2 - i\rho_1(\mu\sigma) - 3/4, (\mu\sigma) - i\rho_1(\mathbf{L}\sigma) - 3/2 \cdot i\rho_1] \\ = 1/2 \cdot \{ [\mu^2, (\mu\sigma)] - [(\mu\sigma), (\mathbf{L}\sigma)] \} \\ = 1/2 \cdot \{ i(\sigma, \mu \times \mathbf{L} - \mathbf{L} \times \mu) - i(\sigma, \mu \times \mathbf{L} - \mathbf{L} \times \mu) \} \\ = 0.$$

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# Wave Equations with New Degree of Freedom

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General wave equations of Pais type, which are invariant under the group of the product space of Minkowski space and the  $\omega$ -space, are introduced and properties of such general fields are investigated, as a model of fields with a new degree of freedom. Possibilities of space time approach to the new degree of freedom are also discussed.

## § 1. Introduction

According to the recently discovered events where new unstable particles play important roles, it seems to be necessary to introduce a new freedom which offers a new indication specifying elementary particles. Indeed recently interesting trials of new type spins have been made by several authors<sup>1)</sup> among them Pais' ingenuous proposal forms a systematized theory. This may be called as one complete model for such freedom. This theory is based upon the direct product space of the Minkowski space and the Pais'  $\omega$ -space. In this paper we will introduce general wave equations which are invariant under the group of the above stated direct product space, or the group which is the direct product ( $L \times \mathcal{Q}$ ) of the Lorentz group  $L$  and the  $\omega$ -rotation group  $\mathcal{Q}$ . These general equations describe elementary particles with arbitrary ordinary spin values  $S$  and arbitrary  $\omega$ -spin values  $T$  and may be called as general Pais' type wave equations. About these generalized fields we will investigate what happens for them comparing with the ordinary fields, for example concerning to the conservation laws, the relation of spin and statistics. These<sup>2)</sup> results will offer some tools for analysis of unfound particles to be found in the future (Part I).

However we cannot deny that it is a defect that we cannot say what the  $\omega$ -space means. So we investigated about several possibilities for introduction of the new degree of freedom by the refinement of the space time treatments (Part II). In the future it will be decided what sort of formulation should remain as the complete one. Nevertheless Pais' original method will also remain as an original model, in our opinion. This is the reason why we worked on the general Pais' type wave equations in details.

### *Part I. General Pais' Type Wave Equations*

## § 2. General wave equations for arbitrary values of ordinary spin and $\omega$ -spin

Pais' wave equations for particles involves two sorts of spins, namely the ordinary spins,

say  $S$ , and the  $\omega$ -spins, say  $T$ , as the nature of the wave functions with respect to the Lorentz group in the Minkowski space and the rotation group in three dimensional Euclidean  $\omega$ -space respectively. In his theory baryons (members of the nucleon family) are described as particles with  $S=1/2$ ,  $T=1/2$  and the wave equation for the baryon family is given as

$$\{\gamma^\lambda \partial_\lambda + M(\mu_0, \tau, \mathbf{K})\} \psi = 0, \quad (2.1)$$

where  $M$  is an operator depending on the rest mass  $\mu_0$ , the isotopic spin matrix  $\tau$  and the  $\omega$ -rotation operator  $\mathbf{K}$ , and its simplest form is given as

$$M = \mu_0 + \tau \mathbf{K} / A \quad (A: \text{a constant}). \quad (2.2)$$

The mesons are described by him as particles with  $S=0$ ,  $T=1$  and the wave equation for the meson family is given as

$$\{\square - M^2(\mu_0, \tau', \mathbf{K})\} \psi = 0, \quad (2.3)$$

where the operator  $M^2$  depends on  $\mu_0$ ,  $(\tau' \mathbf{K})$ ,  $(\tau' \mathbf{K})^2$ ,  $\mathbf{K}^2$  and so on and its form is given, for example, like

$$M^2 = \mu_0^2 \{1 + \alpha_1 (\tau' \mathbf{K}) + \alpha_2 (\tau' \mathbf{K})^2 + \alpha_3 (\mathbf{K})^2\}. \quad (2.4)$$

The matrices  $\tau$ ,  $\tau'$  are the ones corresponding to the irreducible representations  $D_{1/2}$ ,  $D_1$  of the  $\omega$ -rotation group respectively and their commutation relations are

$$\tau_l \tau_m + \tau_m \tau_l = 2\delta_{ml}, \quad (2.5)$$

$$\tau'_l \tau'_m + \tau'_m \tau'_l = \delta_{lm} \tau'_n + \delta_{nm} \tau'_l, \quad (2.6)$$

$$(l, m, n = 1, 2, 3).$$

Our first purpose is to generalize these equations so as to involve arbitrary  $S$  and  $T$  spins. There are two ways of generalization of ordinary spins, namely the one due to Dirac<sup>(2)</sup> and Fierz<sup>(3)</sup> and the one due to Bhabha<sup>(4)</sup> and Chandra<sup>(5)</sup>. These two methods are equivalent for lower spin values  $S=0, 1/2, 1$ , but they differ from each other for higher spin values  $S \geq 3/2$ . The former method gives a natural meaning for the general spin but it involves the so-called subsidiary conditions, which forces complicated forms of interaction with the electromagnetic field. The latter method gives up the subsidiary conditions and so the interaction has a simple form, but the meaning of the spin becomes somewhat obscure, or it is the highest spins in the meaning of the former method. In our opinion we should follow the former method in the present stage of development. In this part we apply the direct product space and in the last section we will give results of application of the latter method.

In order to generalize the Pais' equations in the Dirac-Fierz's method, we write them down in spinor forms. The equation (2.1) can be expressed in the form

$$\partial_{rs}^* \psi_{s,p}^* = m_p^0 \varphi_{s,\sigma}, \quad (2.7)$$

$$\partial_{rs}^* \varphi_{s,\sigma} = m_p^0 \psi_{s,p}^*,$$

where the spinor indices of Latin letter  $r, s, \dots$  are concerned with the Lorentz group and those of Greek letter  $\rho, \sigma, \dots$  with the  $\omega$ -rotation group. The wave function is a spinor of the first rank with respect to the Lorentz group and it is also the first rank spinor with respect to the  $\omega$ -rotation group. The operator  $\partial_r^*$  is the ordinary spinor differential operator, and the operator  $m_\rho^\sigma$  is the spinor operator corresponding to the operator  $M$  and for the case of (2.2) it takes the form

$$m_\rho^\sigma = i\mu_0 \delta_\rho^\sigma + \frac{1}{A} k_\rho^\sigma, \quad (2.8)$$

where  $k_\rho^\sigma$  is the symmetric spinor operator corresponding to the antisymmetric tensor operator  $K$  of the rotation in the  $\omega$ -space:

$$\begin{aligned} k_{11} &= (i/2)(K_1 - iK_2), \\ k_{22} &= -(i/2)(K_1 + iK_2), \\ k_{12} &= k_{21} = -iK_3, \\ (K_1 &= K^{23}, K_2 = K^{31}, K_3 = K^{12}), \\ (K^{\lambda\mu} &= \omega^\lambda \partial^\mu - \omega^\mu \partial^\lambda). \end{aligned} \quad (2.9)$$

The equation (2.3) can be written in the form

$$\begin{aligned} \partial_{rs} \psi_{t, \rho\nu}^r &= m_\rho^\sigma \varphi_{st, \sigma\nu}, \\ \partial^{rs} \varphi_{st, \sigma\nu} &= m_\sigma^\rho \psi_{t, \rho\nu}^r, \end{aligned} \quad (2.10)$$

where  $\psi_{t, \rho\nu}^r$  is a vector and  $\varphi_{st, \sigma\nu}$  is an antisymmetric spinor or a scalar with respect to the Lorentz group and they are both symmetric spinors of the second rank with respect to the  $\omega$ -rotation group. The operator  $m_\rho^\sigma$  corresponding to  $M$  depends on  $\tau' K$ ,  $(\tau' K)^2$ ,  $K^2$  and these operations can be substituted for in the following way

$$\begin{aligned} \tau' K &\rightarrow k_\rho^\sigma, \\ (\tau' K)^2 &\rightarrow k_\rho^\nu k_\nu^\sigma, \\ K^2 &\rightarrow \delta_\rho^\sigma k_{\lambda\nu} k^{\lambda\nu}. \end{aligned} \quad (2.11)$$

Now we propose the following equations as the general field equations for arbitrary spins  $S, T$ :

$$\begin{aligned} \partial_{rs} \psi_{a, \dots, \rho\nu}^{t, \dots} &= m_\rho^\sigma \psi_{a, \dots, \sigma\nu}^{t, \dots}, \\ \partial^{rs} \varphi_{a, \dots, \sigma\nu}^{t, \dots} &= m_\sigma^\rho \psi_{a, \dots, \rho\nu}^{t, \dots}. \end{aligned} \quad (2.12)$$

The wave functions have  $2S$  Latin letters and  $2T$  Greek letters and symmetric with respect to the three kinds of indices pairs, namely among the undotted Latin letters, among the dotted Latin letters and among the Greek letters. These wave functions exhaust all finite dimensional irreducible representations of the above stated direct product group  $L \times Q$  if we include scalar fields as antisymmetric spinors. These equations are invariant under the



group  $L \times Q$ , for they are written in spinor forms with respect to both groups  $L$  and  $Q$ . It can be shown easily in the method of Fierz that the wave functions in the rest system transforms according to the irreducible representation  $D_S$  of the rotation group of the three dimensional ordinary space. Since they are symmetric spinors of  $2T$  rank with respect to  $Q$ , they transforms according to the irreducible representation  $D_T$  of the  $\omega$ -rotation group. Thus the equations (2.12) describe the fields of particles with ordinary spin  $S$  and  $\omega$ -spin  $T$ . Clearly the Pais' equations are involved in them as special cases. The ordinary general equations of Dirac-Fierz's type are involved in them as special cases of scalars with respect to  $Q$  and  $m_\rho^\sigma = i\mu_0 \partial_\rho^\sigma$ .

The second rank wave equation can be easily derived from (2.12)

$$\begin{aligned} \{\square - M^2\} \psi(x, \omega) &= 0, \\ \{\square - M^2\} \varphi(x, \omega) &= 0, \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} \square &= 1/2 \partial_{rs} \partial^{rs}, \\ M^2 &= 1/2 m_{\rho\sigma} m^{\rho\sigma} \end{aligned} \quad (2.14)$$

operating on the arguments  $x$  and  $\omega$  respectively in the wave functions.

Since subsidiary conditions are derived by considering the symmetry character with respect to the Latin indices :

$$\begin{aligned} \partial_r^s \psi_{s, \dots, \rho, \dots}^{\dots} &= 0, \\ \partial_r^\mu \varphi_{\mu, \dots, \sigma, \dots}^{\dots} &= 0. \end{aligned} \quad (2.15)$$

Furthermore in our case the same circumstances arise for Greek indices also and we have in the same method subsidiary conditions with respect to the Greek letters :

$$\begin{aligned} m^{\rho\sigma} \psi_{s, \dots, \rho\sigma, \dots}^{\dots} &= 0 \\ m^{\rho\sigma} \varphi_{\mu, \dots, \rho\sigma, \dots}^{\dots} &= 0 \end{aligned} \quad (2.16)$$

Also in this case the field equations (2.7) can be expressed in tensor forms or in mixed (spinor-tensor) forms. They are given for integral or half odd integral values of spin  $S$  or spin  $T$  as follows where  $n, m$  indicate arbitrary integers.

(i) *integral ordinary spin and  $\omega$ -spin* ( $S=n, T=m$ )

The wave function is a symmetric tensor of the  $n$ -th rank in the ordinary space time and a symmetric tensor of the  $m$ -th rank in the  $\omega$ -space,  $r_1, \dots, r_n$  and  $\rho_1, \dots, \rho_m$  being tensor indices with respect to  $L$  and  $Q$  respectively.

$$\begin{aligned} \{\square - M^2\} \psi_{r_1 r_2 \dots, \rho_1 \dots, \rho_m} &= 0, \\ \psi_{r_1 r_2 \dots, r_m, \rho_1 \dots, \rho_m} &= 0, \\ \partial_r \psi_{r_2 \dots r_m, \rho_1 \dots, \rho_m} &= 0, \end{aligned}$$

$$M_\rho \psi_{r_1 \dots r_n, \rho_1 \dots \rho_m} = 0.$$

( $M_\rho$  is a vector operator corresponding to  $m_\rho^a$ )

(ii) *integral ordinary spin and half odd integral  $\omega$ -spin* ( $S=n, T=m+1/2$ )

The wave function is a symmetric tensor of the  $n$ -th rank in the ordinary space time and a  $2m+1$ -th rank symmetric spinor in the  $\omega$ -space, and we can write it as a symmetric tensor with two spinor components with respect to  $\Omega$  in the following way.

$$\Psi_{r_1 \dots r_n, \rho_1 \dots \rho_m} = \begin{pmatrix} \psi_{r_1 \dots r_n, 1 \rho_1 \dots \rho_m} \\ \psi_{r_1 \dots r_n, 2 \rho_1 \dots \rho_m} \end{pmatrix}$$

where 1, 2 are spinor indices and others are tensor indices in the same sense as in (i).

$$\{\gamma^\lambda \partial_\lambda - \tau^\lambda M_\lambda\} \Psi_{r_1 \dots r_n, \rho_1 \dots \rho_m} = 0,$$

$$\gamma^r \Psi_{r r_2 \dots r_n, \rho_1 \dots \rho_m} = 0,$$

$$\partial_r \Psi_{r r_2 \dots r_n, \rho_1 \dots \rho_m} = 0,$$

$$\tau^\rho \Psi_{r_1 \dots r_n, \rho \rho_2 \dots \rho_m} = 0.$$

( $\gamma^\lambda$  is the Dirac matrix,  $\tau^\rho$  is the isotopic spin matrix)

(iii) *half odd integral ordinary spin and integral  $\omega$ -spin* ( $S=n+1/2, T=m$ )

The wave function can be expressed as a symmetric tensor in the sense in (i) with four components with respect to the ordinary space time in the following way.

$$\Psi_{r_1 \dots r_n, \rho_1 \dots \rho_m} = \begin{pmatrix} \psi_{r_1 \dots r_n, \rho_1 \dots \rho_m}^1 \\ \psi_{r_1 \dots r_n, \rho_1 \dots \rho_m}^2 \\ \psi_{r_1 \dots r_n, \rho_1 \dots \rho_m}^3 \\ \psi_{r_1 \dots r_n, \rho_1 \dots \rho_m}^4 \end{pmatrix},$$

where 1, 2, 3, 4 are spinor indices and others are tensor indices.

$$\{\gamma^\lambda \partial_\lambda - \tau^\lambda M_\lambda\} \Psi_{r_1 \dots r_n, \rho_1 \dots \rho_m} = 0,$$

$$\gamma^r \Psi_{r r_2 \dots r_n, \rho_1 \dots \rho_m} = 0,$$

$$\Psi_{r_1 \dots r_n, \rho \rho \rho_3 \dots \rho_m} = 0,$$

$$M_\rho \Psi_{r_1 \dots r_n, \rho \rho_2 \dots \rho_m} = 0.$$

(iv) *half odd integral ordinary spin and  $\omega$ -spin* ( $S=n+1/2, T=m+1/2$ )

The wave function can be expressed as a symmetric tensor in the sense of (i) with eight components which forms a spinor of the first rank with respect both to  $L$  and to  $\Omega$ . For example with respect to  $L$  it can be arranged as

$$\Psi_{r_1 \dots r_n, \rho_1 \dots \rho_m} = \begin{pmatrix} \psi_{r_1 \dots r_n, \lambda \rho_1 \dots \rho_m}^1 \\ \psi_{r_1 \dots r_n, \lambda \rho_1 \dots \rho_m}^2 \\ \psi_{r_1 \dots r_n, \lambda \rho_1 \dots \rho_m}^3 \\ \psi_{r_1 \dots r_n, \lambda \rho_1 \dots \rho_m}^4 \end{pmatrix},$$

where  $\lambda$  is a spinor index ( $\lambda=1, 2$ ) with respect to  $\Omega$ .

$$\{\gamma^t \partial_t - \tau^\lambda M_\lambda\} \Psi_{r_1 \dots r_n, \rho_1 \dots \rho_m} = 0,$$

$$\gamma^r \Psi_{r r_2 \dots r_n, \rho_1 \rho_2 \dots \rho_m} = 0,$$

$$\tau^p \Psi_{r_1 \dots r_n, \rho \rho_2 \dots \rho_m} = 0.$$

Now let us consider the solutions of wave equations. All wave functions satisfy the second rank wave equation (2.13). This equation can be solved by separating variables, or assuming

$$\psi(x, \omega) = \phi(x) \chi(\omega). \quad (2.17)$$

So that we have

$$\{\square - \lambda\} \phi(x) = 0, \quad (2.18)$$

$$\{M^2 - \lambda\} \chi(\omega) = 0. \quad (2.19)$$

The operator  $M^2$  is an invariant operator in the  $\omega$ -space, which is compact, so the eigenvalue problem of the latter equation yields the discrete eigen values  $\lambda_1, \lambda_2, \dots$ , and we have an orthonormal set of solutions  $\chi_l$  corresponding to  $\lambda_l$ . Writing the normal solutions of the former equations with respect to  $\lambda_l$  as  $\phi_k^l(x)$ , we have the normal solutions of (2.13):

$$\psi_k^l(x, \omega) = \phi_k^l(x) \chi_l(\omega) \quad (2.20)$$

by means of which we can expand any solutions of the wave equation.

The discrete eigen values  $\lambda_l$  play the role of the rest mass in the usual theory. In the case of (2.8) we can easily show

$$M^2 = \Delta(\omega) / A^2 + \mu_0^2, \quad (2.21)$$

where  $\Delta(\omega)$  is the Laplacian in the  $\omega$ -space. So the equation (2.19) becomes

$$\{\Delta(\omega) - A^2(-\mu_0^2 + \lambda)\} \chi(\omega) = 0. \quad (2.22)$$

As well known the eigen values of this equation are

$$A^2(-\mu_0^2 + \lambda_l) = l(l+1),$$

which determine the eigen value  $\lambda_l$ .

$$\lambda_l = l(l+1) / A^2 + \mu_0^2. \quad (2.23)$$

The wave functions  $\chi_{lm}$  are determined by the eigenvalues  $l, m$  of  $\Delta(\omega)$  and  $K_3$ . In the case of  $\omega$ -spin  $T$  the dependence of the proper function  $\chi(\omega)$  on the variable  $\omega_3$  is  $e^{im\omega_3}$ ,  $m = \mu + \mu', \mu' = T, T-1, \dots, 1-T, -T$ .

Concluding this section we must notice that above developed formalism is concerned with the proper Lorentz group and the proper  $\omega$ -rotation group. In order to gain a covariant theory with respect to the whole Lorentz group, we must add the equation arising after exchanging the number of undotted suffices with that of dotted suffices. Or in representation theoretical words we must add the equation corresponding to  $D_{l,k}$  to the equation (2.7), if the latter corresponds to  $D_{k,l}$ . Further, in order to gain a covariant theory with respect to the whole  $\omega$ -orthogonal group, we must add one more equation of

the same type where the wave functions are to be exchanged in the case of improper  $\omega$ -rotations. This is the standard method in the spinor analysis in the three dimensional Euclidean space and corresponds to the use of the four dimensional  $\tau$  spin matrices. The added equation is

$$\begin{aligned}\partial_{rs} \psi_{u\dots, \rho\nu\dots}^{\dot{r}\dot{s}\dots} &= m_{\rho}^{\sigma} \psi_{su\dots, \sigma\nu\dots}^{\dot{r}\dots} \\ \partial_{rs} \varphi_{u\dots, \rho\nu\dots}^{\dot{r}\dots} &= m_{\sigma}^{\rho} \psi_{su\dots, \rho\nu\dots}^{\dot{r}\dots}\end{aligned}\quad (2.24)$$

Introducing the fields

$$\psi^{\pm} = \psi \pm \psi', \quad \varphi^{\pm} = \varphi \pm \varphi', \quad (2.25)$$

we have

$$\begin{aligned}\partial_{rs} \psi_{u\dots, \rho\nu\dots}^{\pm\dot{r}\dot{s}\dots} &= m_{\rho}^{\sigma} \psi_{su\dots, \sigma\nu\dots}^{\pm\dot{r}\dots} \\ \partial_{rs} \varphi_{u\dots, \rho\nu\dots}^{\pm\dot{r}\dots} &= m_{\sigma}^{\rho} \psi_{su\dots, \rho\nu\dots}^{\pm\dot{r}\dots}\end{aligned}\quad (2.26)$$

In the case of the  $\omega$ -inversion, the wave functions transform as

$$\psi \leftrightarrow \psi', \quad \varphi \leftrightarrow \varphi', \quad (2.27)$$

which means

$$\psi^{+} \rightarrow \psi^{+}, \quad \psi^{-} \rightarrow -\psi^{-}, \quad \varphi^{+} \rightarrow \varphi^{+}, \quad \varphi^{-} \rightarrow -\varphi^{-}. \quad (2.28)$$

In this way the particle described with (+) field and the one described with (-) field behave in different ways under  $\omega$ -inversion.

We want also to notice that there is another possibility of generalization which did not occur in the ordinary case. Let us consider the equations of the following type

$$\begin{aligned}\partial \psi_{\dots, \rho\dots}^{\dot{r}\dots} &= m'_{\rho}{}^{\sigma} \varphi_{\dots, \sigma\dots} \\ \partial \varphi_{\dots, \sigma\dots} &= m''^{\rho}{}_{\sigma} \psi_{\dots, \rho\dots}\end{aligned}\quad (2.29)$$

In the ordinary case the choice

$$m'_{\rho}{}^{\sigma} = i\mu_0' \delta_{\rho}^{\sigma}, \quad m''^{\rho}{}_{\sigma} = i\mu_0'' \delta_{\sigma}^{\rho}, \quad \mu_0' \mu_0'' = \mu_0^2 \quad (2.30)$$

did not cause any new affair physically important. In our case, however, somewhat notable circumstances arise, namely the choice, for example,

$$m'_{\rho}{}^{\sigma} = i\mu_0' \delta_{\rho}^{\sigma} + k'_{\rho}{}^{\sigma}, \quad m''^{\rho}{}_{\sigma} = i\mu_0'' \delta_{\sigma}^{\rho} + k'^{\rho}{}_{\sigma}, \quad \mu_0' \mu_0'' = \mu_0^2 \quad (2.31)$$

alters the operator  $M^2$  in the second order wave equation as

$$\mu_0^2 + i/2(\mu_0' - \mu_0'') k'_{\rho}{}^{\rho} + 1/2 k'_{\rho\sigma} k'^{\rho\sigma}. \quad (2.32)$$

The equation (2.29) is invariant merely under the orthochronous proper Lorentz transformation. For the full Lorentz group we must add to it

$$\begin{aligned}\partial \psi_{\dots, \rho\dots}^{*\dot{r}\dots} &= m'_{\rho}{}^{\sigma} \varphi_{\dots, \sigma\dots}^{*} \\ \partial \varphi_{\dots, \sigma\dots}^{*} &= m''^{\rho}{}_{\sigma} \psi_{\dots, \rho\dots}^{*}\end{aligned}\quad (2.33)$$

even if the case  $D_{k_i}$  of  $k=l$ , which was not necessary in the case  $m'_{\rho}{}^{\sigma}=m''_{\rho}{}^{\sigma}$ .

### § 3. The conservation laws

The wave equation introduced in the preceding section does not describe only one particle in the sense of the usual theory. The wave equation with certain  $S$  and  $T$  describes a certain particle-family. Thus physical quantities such as currents and energy-momenta have somewhat wider meanings than usual. So we must notice such circumstances in considering physical laws such as the conservation laws.

In this section we use, for simplicity, the special description of the function where the numbers of the dotted and undotted suffices are convenient to handle with.

#### (1) Family conservation

Firstly let us consider the case of half odd integral ordinary spins  $(n+1/2)$ . The wave equation can be written with the wave functions in which the number of dotted suffices and that of the undotted ones differ by only one.

$$\begin{aligned}\partial_{\dot{r}\dot{s}} \dot{\varphi}_{\dot{r}\dot{r}_1\cdots\dot{r}_n, \dot{p}\cdots}^{\dot{s}_1\cdots\dot{s}_n} &= m_{\dot{p}}^{\sigma} \dot{\varphi}_{\dot{t}_1\cdots\dot{t}_n, \sigma}^{\dot{s}_1\cdots\dot{s}_n} \\ \partial_{\dot{r}\dot{s}} \dot{\varphi}_{\dot{r}\dot{r}_1\cdots\dot{r}_n, \sigma\cdots}^{\dot{s}_1\cdots\dot{s}_n} &= m_{\sigma}^{\dot{p}} \dot{\varphi}_{\dot{t}_1\cdots\dot{t}_n, \dot{p}\cdots}^{\dot{s}_1\cdots\dot{s}_n}\end{aligned}\quad (3.1)$$

The conjugate equations are

$$\begin{aligned}\partial_{\dot{s}\dot{r}} \dot{\varphi}_{\dot{r}_1\cdots\dot{r}_n, \dot{p}\cdots}^{*\dot{s}_1\cdots\dot{s}_n} &= m_{\dot{p}}^{*\sigma} \dot{\varphi}_{\dot{r}_1\cdots\dot{r}_n, \sigma\cdots}^{*\dot{s}_1\cdots\dot{s}_n} \\ \partial_{\dot{s}\dot{r}} \dot{\varphi}_{\dot{r}_1\cdots\dot{r}_n, \sigma\cdots}^{*\dot{s}_1\cdots\dot{s}_n} &= m_{\sigma}^{*\dot{p}} \dot{\varphi}_{\dot{t}_1\cdots\dot{t}_n, \dot{p}\cdots}^{*\dot{s}_1\cdots\dot{s}_n}\end{aligned}\quad (3.2)$$

where  $m_{\dot{p}\sigma}^{*\sigma}$  is the complex conjugate operator of  $m_{\dot{p}\sigma}$ . The operator  $m_{\dot{p}\sigma}$  of (2.8) has the form

$$m_{\dot{p}}^{\sigma} = k_{\dot{p}}^{\sigma} \cdot m + i m'_{\dot{p}}{}^{\sigma} \quad (3.3)$$

where  $m$  is a real operator and  $m'_{\dot{p}}{}^{\sigma}$  is a real number. When the operator  $m_{\dot{p}\sigma}$  has the form of (3.3),  $m_{\dot{p}}^{*\sigma}$  is

$$m_{\dot{p}}^{*\sigma} = k_{\dot{p}}^{\sigma} \cdot m - i m'_{\dot{p}}{}^{\sigma}$$

and we can derive the conservation law

$$\partial_{\dot{r}\dot{s}} \dot{s}_{\dot{r}\dot{s}} + k^{\dot{p}\sigma} u_{\dot{p}\sigma} = 0, \quad (3.4)$$

where

$$\dot{s}_{\dot{r}\dot{s}} = \dot{\varphi}_{\dot{t}_1\cdots\dot{t}_n, \dot{p}\cdots}^{*\dot{s}_1\cdots\dot{s}_n} \dot{\varphi}_{\dot{r}_1\cdots\dot{r}_n, \dot{p}\cdots}^{\dot{s}_1\cdots\dot{s}_n} + \dot{\varphi}_{\dot{r}_1\cdots\dot{r}_n, \sigma\cdots}^{*\dot{s}_1\cdots\dot{s}_n} \dot{\varphi}_{\dot{t}_1\cdots\dot{t}_n, \sigma\cdots}^{\dot{s}_1\cdots\dot{s}_n}$$



$$u_{\sigma}^p = m \left( \psi_{s_1 \dots s_n}^{* \ell_1 \dots \ell_n \ p \nu \dots} \psi_{\ell_1 \dots \ell_n \ s_1 \dots s_n}^{\dot{s}_1 \dots \dot{s}_n} + \varphi_{r s_1 \dots s_n}^{* \ell_1 \dots \ell_n \ p \nu \dots} \varphi_{\ell_1 \dots \ell_n \ s_1 \dots s_n}^{\dot{s}_1 \dots \dot{s}_n} \right). \quad (3.6)$$

If the wave functions are one valued with respect to  $\omega$ -variables, the integral

$$\int k^{\rho\sigma} u_{\rho\sigma} d\omega$$

over the whole  $\omega$ -space vanishes. Thus in this case for the average of  $S_{rt}$  over  $\omega$ -space

$$S_{rt}(x) = \int S_{rt} d\omega \quad (3.7)$$

satisfies the conservation law with respect to the Minkowski space

$$\partial^{\dot{r}\ell} S_{rt}(x) = 0. \quad (3.8)$$

The same considerations about the energy momentum tensor are possible. If we introduce

$$\begin{aligned} t_{kl,rs} = \frac{1}{2} \bigg( & \psi_{l s_1 \dots s_n}^{* \ell_1 \dots \ell_n \ p \dots} \partial_{rs} \psi_{k \ell_1 \dots \ell_n \ p \dots}^{\dot{s}_1 \dots \dot{s}_n} - \psi_{k \dots}^{\ell_1 \dots \ell_n \ p \dots} \partial_{rs} \psi_{l \dots}^{* \dots} \\ & + \varphi_{k s_1 \dots s_n}^{* \ell_1 \dots \ell_n \ \sigma \dots} \partial_{rs} \varphi_{l \ell_1 \dots \ell_n \ p \dots}^{\dot{s}_1 \dots \dot{s}_n} - \varphi_{l \dots}^{\dots \sigma \dots} \partial_{rs} \varphi_{k \dots}^{* \dots} \bigg) \end{aligned} \quad (3.9)$$

and make the symmetrized average

$$T_{kl,rs}(x) = 1/2 \int (t_{kl,rs} + t_{rs,kl}) d\omega. \quad (3.10)$$

The tensor  $T_{ij}$  corresponding to  $T_{kl,rs}$  is a symmetric tensor satisfying the conservation law

$$\partial T_{ij}(x) / \partial x_j = 0. \quad (3.11)$$

This quantity can be interpreted as the energy momentum tensor in the usual sense.

The meaning of (3.4) is that  $s_{rt}^*$  does not conserve in the Minkowski space but an unified quantity  $(S_{rt}, u_{\rho\sigma})$  conserves in a sense with respect to the whole direct product space of the Minkowski space and the  $\omega$ -space. Only the averaged quantities over the whole  $\omega$ -space conserves in the Minkowski space in the sense of the usual theory. These conservation laws are concerned with the particle-family. (3.8) is the particle conservation of the family and (3.11) is the total energy momentum conservation of the family in the usual sense.

The same considerations can be made also in the case of integer ordinary spins. As the forms of  $S$  and  $T$  are given in the following way.

$$S_t(x) = \frac{1}{2i} \int d\omega \left\{ \Psi^{* \dots j m \dots \rho \dots} \Phi_{\dots [j] m \dots \rho \dots} - \Psi^{\dots j m \dots \rho \dots} \Phi_{\dots [j] m \dots \rho \dots}^{* \dots} \right\}, \quad (3.12)$$

$$\begin{aligned} T_i^j(x) = \frac{1}{2} \int d\omega \bigg\{ & M^2 (\Psi^{* \dots j \dots \rho \dots} \Psi_{\dots i \dots \rho \dots} + \Psi_{\dots i \dots \rho \dots}^{* \dots} \Psi^{\dots j \dots \rho \dots}) \\ & + (\Phi^{* \dots [mj] \dots \rho \dots} \Phi_{\dots [mi] \dots \rho \dots} + \Phi_{\dots [mi] \dots \rho \dots}^{* \dots} \Phi^{\dots [mj] \dots \rho \dots}) \end{aligned}$$

$$-\partial_t^j (M^2 \Psi^{* \dots m \dots, \rho \dots} \Psi^{\dots m \dots, \rho \dots} + 1/2 \Phi^{\dots [mn] \dots, \rho \dots} \Phi^{* \dots [mn] \dots, \rho \dots}) \} \quad (3.13)$$

where  $\Phi$  fields are defined as

$$\Phi_{[jl]m \dots, \rho \dots} = \partial_j \Psi_{lm \dots, \rho \dots} - \partial_l \Psi_{jm \dots, \rho \dots} \quad (3.13)$$

and satisfy the equation

$$\partial^j \Phi_{[jl]m \dots, \rho \dots} = M^2 \Psi_{lm \dots, \rho \dots}$$

all Latin indices being tensor indices.

## (2) Charge conservation

Introduction of the interaction with the electromagnetic field is not simple in general also in the ordinary theory. The method of taking into account the effect of the electromagnetic field, proposed by Dirac for the electron field interacting with the electromagnetic field, by the substitution

$$\partial_k \rightarrow \partial_k - ie A_k \quad (3.14)$$

is allowed only for the case of lower spin values 0, 1/2, 1. In the case of higher spin values this method leads to inconsistencies with subsidiary conditions, but we have a way of relief found by Fierz and Pauli<sup>(6)</sup>, though it forces somewhat complicated treatments. The method is to introduce a Lagrangian of the free field involving auxiliary fields of the lower ranks and parameters, which is to be determined so that the auxiliary fields and the subsidiary conditions vanishes. With this Lagrangian the method of (3.14) introducing the electromagnetic interaction can be performed consistently. In this treatment the free field is the same as usual, but the interacting field involves the auxiliary field and differs from those which arises after the substitution (3.14) in the free field.

Pais' interactions for the case of lower  $S$  and  $T$  spins are introduced by the substitutions

$$\partial_k \rightarrow \partial_k - ie (I_3 + 1/2) A_k \quad (T=1/2), \quad (3.15)$$

$$\partial_k \rightarrow \partial_k - ie I_3 A_k \quad (T=1), \quad (3.16)$$

where  $I_3$  is the third component of the operator

$$I = K.$$

In general we assume (3.15) for half odd integer  $\omega$ -spins and (3.16) for integer  $\omega$ -spins.

Also in the general case the substitution concerns with  $\partial_k$  only,  $m_{\rho\sigma}$  is leaved as it is and  $A_i$  is a constant with respect to  $\omega$ , so the confusions occur only with respect to the subsidiary condition (2.15). Thus the method of Fierz and Pauli can be performed without great alterations. We can proceed with the following Lagrangian in the same way, where the auxiliary field  $(\chi, \phi), \dots$  are fields of irreducible tensors of rank  $S-2, \dots, 1, 0$  in the case of integer ordinary spins and irreducible spinors of rank  $2S-2, \dots, 1$  in the case of half odd integer ordinary spins.

$$\begin{aligned}
L(x, \omega) = & \left\{ \varphi_{\dots}^{*} \dots \varphi_{\dots} m_{\sigma}^{\nu} \phi_{\dots}^{i\dots} + \phi_{\dots}^{*} \dots \varphi_{\dots} m_{\rho}^{\sigma} \varphi_{\dots} \dots \right\} \\
& - \left\{ \phi_{\dots}^{*} \dots \varphi_{\dots} \partial_{\tau s} \phi_{\dots}^{r\dots} + \varphi_{\dots}^{*} \dots \partial_{\tau s} \varphi_{\dots} \dots \right\} \\
& + \text{const.} \left\{ \phi_{\dots}^{*} \dots \varphi_{\dots} \partial_{\ell}^{\dot{s}} \phi_{\dots}^{u\dots} + \varphi_{\dots}^{*} \dots \partial_{\ell}^{\dot{s}} \chi_{\dots}^{u\dots} + \text{conj.} \right\} \\
& + \text{const.} \left\{ \chi_{\dots}^{*} \dots \varphi_{\dots} \partial_{\tau s} \chi_{\dots}^{s\dots} + \phi_{\dots}^{*} \dots \partial_{\tau s} \phi_{\dots}^{s\dots} \right\} \\
& + \text{const.} \left\{ \chi_{\dots}^{*} \dots \varphi_{\dots} m_{\sigma}^{\rho} \phi_{\dots}^{s\dots} + \phi_{\dots}^{*} \dots \varphi_{\dots} m_{\sigma}^{\rho} \chi_{\dots}^{s\dots} \right\} \\
& + \dots \dots \dots
\end{aligned}$$

The electric charge current can be derived

$$J_i = \frac{\partial}{\partial A_i} \int L(x, \omega) d\omega,$$

and it satisfies the conservation law

$$\partial J_i / \partial x_i = 0.$$

This means the total charge conservation of the family in the Minkowski space in the usual sense.

#### § 4. Quantization of the fields

The wave functions can be expanded as (2.20) with the normal solutions of (2.18) and (2.19). Since the function  $\phi_k^i(x)$  is the same as the wave function of a particle with the rest mass  $\lambda_i$  and the spin  $S$ , we can quantize this according to Fierz's method of quantization<sup>3)</sup> of the free field with the rest mass  $\lambda_i$  and the spin  $S$ . We may be allowed not to enter into details in this problem, for the commutation relations are completely the same about  $\phi_k^i(x)$  as the ordinary theory.

We can also quantize the function  $\chi_i(\tau v)$  by the corresponding way of the above stated quantization method of the rest mass  $\lambda_i$  and the spin  $T$ , and the only alteration is the substitution of the operator  $m_{\rho}^{\sigma}$  for the operator  $\partial_{\rho}^{\sigma}$ . The generalization is straightforward, so we do not enter into details.

The field quantized in the above method obeys the statistics due to Bose or Fermi in the Minkowski space according to the integer or half odd integer ordinary spins, and it obeys independently Bose or Fermi statistics in the  $\omega$ -space according to the integer or half odd integer  $\omega$ -spins. However, is it possible to quantize according to the opposite statistics? Does the results of the Pauli's discussions about the connection between spin and statistics hold in this case also? This depends on the interpretation of the physical quantities. If we define the observable quantities such as the current vector and

the energy momentum tensor with the averaged values over the whole  $\omega$ -space, Pauli's discussions hold only for the Minkowski space. We cannot say then that the above stated quantization method with respect to the  $\omega$ -part is unique, and it may be also possible to quantize according to opposite statistics. This is the reason, I think, Pais left the  $\omega$ -part unquantized. In this stage of the theory this method seems to be most reasonable, in this case the field obeys the classical statistics, concerning the  $\omega$ -space.

### § 5. Remarks on another generalization

There is another way of generalization concerning the spin, namely the method due to Bhabha and Chandra. Bhabha started with the first rank wave equation of the same type as Dirac's electron wave equation having another type matrices and gave up the subsidiary conditions. Without subsidiary conditions wave functions do not satisfy in general ordinary second order wave equations which mean one fixed mass, but the matrix commutation relations which yields the ordinary second order wave equations were obtained by Chandra. We can also apply this method to our generalization of  $S$  and  $T$  spins.

The wave equation is given in the following way

$$\{\alpha^i \partial_i + M(\mu_0, \sigma, \mathbf{K})\} \Psi = 0,$$

$$\sum \alpha^{l_1} \alpha^{l_2} \dots \alpha^{l_n} = \sum \sigma^{l_1 l_2} \alpha^{l_3} \dots \alpha^{l_n} \quad (l_i = 1, 2, 3, 4), \quad (5.2)$$

$$\sum \sigma^{\rho_1} \sigma^{\rho_2} \dots \sigma^{\rho_m} = \sum \delta^{\nu_1 \rho_2} \sigma^{\rho_3} \dots \sigma^{\rho_m} \quad (\rho_i = 1, 2, 3), \quad (5.3)$$

where the symbol  $\sum$  means the summation with respect to all permutations of indices  $l_i$  or  $\rho_i$  and the operator  $M$  has the same dimensions as the matrix  $\sigma$ . The matrices  $\alpha$  and  $\sigma$  are concerned with the Minkowski space and the  $\omega$ -space respectively. The integers  $n$  and  $m$  are the algebraic degrees of the commutation relations and in general  $n=2S+1$ ,  $m=2T+1$ . The matrices  $\alpha$  and  $\sigma$  are appropriate ones of irreducible representations of these commutation relations.

In this method the circumstances are quite similar to the Dirac's electron theory, so we can proceed in the same way. For example the introduction of the interaction with the electromagnetic field can be performed by the previously stated substitution method only, without any inconsistency. Physical quantities can be defined in the similar form as Dirac's theory. However the irreducible representations of the commutation relations are not yet obtained for general cases and it is not so easy to obtain them even for a fixed degree  $n, m$ . Moreover the meaning of the spins is not so clear, or in the rest system the wave functions separate into several sorts of components transforming according to the several irreducible representations of the three dimensional orthogonal group of the ordinary space. This means that a rest particle has several angular momenta, corresponding to spins. Bhabha named as the spin with the maximum momenta in the rest system. Therefore in this procedure it will be not so easy to obtain selection rules in the method used by Peaslee<sup>7)</sup> and Yang<sup>8)</sup>, for in the rest system the spin angular momenta are not unique.

However this method coincides with the previous method for lower spin values  $S$ ,



$T=0, 1/2, 1$ , where the equations become of the Dirac electron type and of Kemmer meson type.

*Part II. Possibilities of the Space Time Approach to the New Degree of Freedom*

## § 6. The full Lorentz group

The physical theories are compelled with the requirement of the relativistic invariance with respect to the orthochronous Lorentz group. But there are such theories that are invariant also under the full Lorentz group. In this case the time reversal may impose selection rules, whose results depend on transformation characters of the quantities with respect to the time inversion.

The full Lorentz group consists of the not connected four classes, within any one of which all elements are connected with each other:

(i) The orthochronous proper Lorentz transformation  $L_+^+ (J=1, \sigma_t=1)$ ; every element connected with the identity.

(ii) The orthochronous improper Lorentz transformation  $L_-^+ (J=-1, \sigma_t=1)$ ; every element connected with the space inversion.

(iii) The antichronous proper Lorentz transformation  $L_+^- (J=1, \sigma_t=-1)$ ; every element connected with the space-time inversion.

(iv) The antichronous improper Lorentz transformation  $L_-^- (J=-1, \sigma_t=-1)$ ; every element connected with the time inversion.

Here  $J$  is the determinant of the transformation coefficients ( $a_{ij}$ ) and  $\sigma_t$  is the sign of  $a_{44}$ .

The ordinary pseudo quantities are concerned with the determinant  $J$ , where the meaning is the same as for the space inversion in the case of orthochronous Lorentz transformations. There are also other pseudo quantities which are concerned with the time sense  $\sigma_t$ . Thus we have the three kinds of pseudo quantities, transforming according to  $JA$ ,  $\sigma_t A$ ,  $J\sigma_t A$ , for the quantity transforming according to  $A$ . The third of these three can be composed of the first two kinds.

The usual parity in the ordinary theory is concerned with the pseudo character with respect to the determinant  $J$ . We can assign to the parity concerned with the Pais' even odd rule the one concerned with the pseudo character with respect to the time sense  $\sigma_t$ . The even particle is described with an ordinary quantity and the odd particle is described with a pseudo quantity with respect to the time sense  $\sigma_t$ . This implies that the fields of the ordinary stable particles have the ordinary transformation character and those of the new unstable particle have the pseudo transformation character with respect to  $\sigma_t$  in the completely relativistic theories invariant under the whole Lorentz group.

## § 7. The refinement of the Lorentz group.

The orthochronous proper Lorentz group is a connected group but not simply connected. The covering group is simply connected and introduced by separating every element of the Lorentz group  $L$  into two pieces  $L_+$  and  $L_-$ . In the case of the two dimensional rotation



group the covering group is obtained by distinguishing between the rotations from the left and from the right. In the case of the Lorentz group the same sort of separation of  $L$  into  $L_+$  and  $L_-$  can be performed by taking into account the path from the unit element in the parameter space. In general the following rule holds

$$L'_\pm L''_\pm = L'''_\pm, \quad L'_\pm L''_\mp = L'''_\mp.$$

Let us call the above stated procedure as the refinement of the Lorentz group, corresponding to which we must perform the refinement of the frame of reference. We distinguish one coordinate in the ordinary sense into two ones according to the method whether the coordinate is reached from the original observer by the transformation  $L_+$  or  $L_-$ . By taking all the coordinates connected by  $L_+$  into one class, we have two classes; coordinates of different classes being connected by  $L_-$ .

The measurements of all quantities as well as of the position are performed in a frame of reference. So the wave functions are described as  $\Phi_+(x)$ ,  $\Phi_-(x)$ , the indices  $+$ ,  $-$  denoting the two classes of coordinates. This refinement is faithful and natural for the "Zweideutigkeit" of the spin representation<sup>9)</sup>. Since spinors are necessary and unavoidable in the present theory the above stated interpretation may be not so extravagant. The separation into  $L_+$  and  $L_-$  corresponds to the distinction of the two valued spinor representation  $S_+$ ,  $S_-$ ; where  $S_- = -S_+$ . In the case of the separated Lorentz group tensors are also made as two valued, by introducing quantities transforming as  $A_- = -A_+$ ,  $A_+$  being the ordinary tensor transformation. For example the two sorts of vectors are constructed

$$A_+^i = \phi_+ \gamma^i \phi_+ \text{ or } \phi_+ \gamma^i \phi_-, \\ A_-^i = \phi_+ \gamma^i \phi_- \text{ or } \phi_- \gamma^i \phi_+.$$

In this refined theory Pais' "even odd rule" holds quite naturally. The even particles are described with  $A_+$ ,  $\phi_+$ , ... and the odd particles with  $A_-$ ,  $\phi_-$ , ... The external form of the present physical theory is not necessary to be amended at all. In order to introduce other effects such as mass differences of the family we need only to amend the equations slightly by taking into account the components indicated with  $+$ ,  $-$ . In order to involve many other components we need only to take up pseudo quantities of three kinds<sup>9)</sup>.

## § 8. The curved space time

The Minkowski's flat space is a good approximation for our present theory. But in general the curved space time can not be denied. The curvature may yield a new degree of freedom. The quantum theory in general curved space time is difficult to be constructed and to be solved. However a quantum theory in the de Sitter space can be introduced and solved<sup>10)</sup>, by virtue of the higher degree of symmetry of that space time. At the present stage of the field theory we believe that the flat space time is a good approximation. But we can not deny that the de Sitter space is a better approximation for certain cases, particularly in small regions where masses are concentrated extraordinarily strongly.

Detailed theory for particles in the de Sitter space may be given elsewhere by us<sup>11)</sup>. Here we give several indications of possibilities. The reason why the de Sitter space can be treated without much difficulty is that space can be embedded in a five dimensional flat space as a four dimensional hyperboloidal surface in it. The group of the five dimensional rotation induces one sort of transformation group in the de Sitter surface, which we may call the de Sitter group and interpret it as the group of the special relativity in that space. This group corresponds to the inhomogeneous Lorentz group (homogeneous Lorentz transformation + translation) in the Minkowski space.

In the usual theory the group is the homogeneous Lorentz group and so the group in a rest system becomes the rotation group of the three dimensional Euclidean space. In the case of the de Sitter space the group is the five dimensional transformation group and the group in the rest system becomes the rotation group of the four dimensional Euclidean space. As well known the irreducible representation of the four dimensional rotation group can be indicated by the symbol  $D_{3/2}$ , which has the same character as the direct product  $D_2 \times D_1$  of the three dimensional rotation group. This implies that particles in de Sitter space has two sorts of spin-like quantum numbers. This yields a new degree of freedom with structure of a new spin.

### § 9. The inhomogeneous Lorentz group

In the present theory the covariance is under the homogeneous Lorentz group and the inhomogeneous Lorentz group is not yet taken into account fully. The structure of the inhomogeneous Lorentz group is quite similar to those of the de Sitter group. They have the same number of parameters 10 as Lie groups in the neighbourhood of the unity they correspond to each other one to one. As shown in the preceding section the particles in the de Sitter space have the desired new degree of freedom. So it is supposed that the theory in the flat space in which the inhomogeneous Lorentz group is taken into account suitably may be also promising.

It is not so easy to obtain the general invariant theory with respect to the inhomogeneous Lorentz group. The representation of this group are determined<sup>12)</sup> but they are in general of infinitely many dimensions. The field corresponding to such representation seems not to be so adequate to our purpose<sup>13)</sup>, since in this case only slight amendment of the usual theory is required. However we must remember that also the representation of the homogeneous Lorentz group is in general of infinitely many dimensions and we selected suitable representations of finite dimensions to describe particles with discrete spins. This method of selection in the so-called unitary trick, namely to determine the representation corresponding directly to the representations of four dimensional rotation group. Is there similar way also in our present case?

Dirac treated formerly the wave equation in the conformal space<sup>14)</sup>. The group corresponds to the inhomogeneous group and the dilatation. This can be expressed as the six dimensional group which leave the form

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2$$

and the point  $(0, 0, 0, 0, 1, i)$  invariant. The four dimensional inhomogeneous group corresponds to the subgroup which leave  $x_5 + x_6$  invariant. In the six dimensional space we can construct wave equations and impose this restriction. In the rest system, if it exists, the group may become the five dimensional group, whose irreducible representations can be expressed with the symbol  $D_{j,j'}$  with two indices. These indices may play the roles of the two sorts of spins. The details will be given elsewhere. The wave equations given by Dirac and Fierz for the conformal space permit the dilatation. But the dilatation seems to us unimportant physically. However it is worth noting that the Maxwell field is invariant under the conformal group and is not necessary to be amended, this fact corresponds to the circumstance that the electromagnetic field is not necessary to be considered specially in the  $\omega$ -space.

### § 10. The semi-local field

In the Yukawa's non-local field theory<sup>15)</sup> the field  $U$  is described concerning to two points in the ordinary space time concept. Using a linear coordinate it is written as  $(x_1, x_2, x_3, x_4 | U | x_1', x_2', x_3', x_4')$ , where the representation in which the linear coordinates  $x_i$ 's are diagonal is chosen. This can be done on the assumption that all the linear coordinates can be diagonalized at the same time and the proper values can take all the real numbers.

We assume the same thing can be done in any curved coordinates  $(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \sigma)$ , where  $\sigma$  defines a space like surface and  $\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3$  a curved coordinate in the surface  $\sigma$ . Namely we assume that  $\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \sigma$  can be diagonalized at the same time and the proper values can take all the real numbers. Then the field can be expressed as  $(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \sigma | U | \hat{\xi}_1', \hat{\xi}_2', \hat{\xi}_3', \sigma')$ . This means a somewhat enlarged interpretation of the Yukawa's non local concept.

With this enlarged non local idea we can give the assumption of the semi local field as its special case. We can define a field which is local in the timelike direction and non local in the spacelike direction and call it a semi local field in the time like direction. This field can be expressed as  $(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3 | U(\sigma) | \hat{\xi}_1', \hat{\xi}_2', \hat{\xi}_3')$ , since all the components for which  $\sigma \neq \sigma'$  vanish. Correspondingly we can also define a field which is non local in the space like direction and call it a semi local field in the space like direction. This field can be expressed as  $(\sigma | U(\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3) | \sigma')$ .

In the semi local field in the time like direction the point  $(\hat{\xi}_1 + \hat{\xi}_1'/2, \hat{\xi}_2 + \hat{\xi}_2'/2, \hat{\xi}_3 + \hat{\xi}_3'/2, \sigma)$  may behave according to the ordinary field theory in the limit of the local field. On the contrary the point  $(\hat{\xi}_1 - \hat{\xi}_2'/2, \hat{\xi}_2 - \hat{\xi}_2'/2, \hat{\xi}_3 - \hat{\xi}_3'/2)$  forms a three dimensional Riemannian manifold whose local metric is positive definite everywhere, since it is a point on a space like surface. We can impose on this manifold the operation which is induced on the surface by the four dimensional rotation. This degree of freedom will yield the role of the Pais'  $\omega$  space, in a somewhat generalized way. Indeed in the special case where all the  $\sigma$  surfaces are flat spaces  $x_4 = \text{const.}$  the above stated three dimensional manifold becomes a three dimensional Euclidean space and the operation becomes the three dimensional rotation, thus the results coincide with the Pais' theory completely. The

detailed theory will be given elsewhere.

By the way we will give some discussions about this line of development. The semi local field in the space like direction has two sorts of time like parameters. This corresponds to the Dysons' field in the intermediate representation, and it may play an important role in the divergence problems. Such seems to be the roles of the two types of the semi local fields. More general non local field may be composed from these two sorts semi local fields. Let us call such fields that is expressible with the two sorts of semi local fields a "separable non local field", though this may be impossible in the case of the most general non local field. The above stated method of using the separated forms seems to be very much promising for dealing with the problems of the new freedom and the divergencies at the same time.

In the Yukawa's non local field theory the space time are not yet quantized completely since all the components of positions can be diagonalized at the same time and their proper values can take all the real values. This implies the space time position can be measured precisely and may be treated as non commutable quantities only with regard to other physical quantities. The method of the quantized space time seems to be very much promising. The quantized space time is not so extraordinary comparing to the Pais'  $\omega$ -space. The Pais' method is to consider every ordinary space time point as a three dimensional  $\omega$ -space, which can be not explained directly. The quantized space time method is to describe the ordinary space time point, which is described with numbers in usual, with non commutative algebras.

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## Upper and Lower Bounds of Born Approximation, I

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The properties of one dimensional Born approximation are investigated. It is shown that upper and lower bounds for any order Born approximation can be given by a single parameter (namely, by the convergence radius of Born expansion). In the convergency region of Born expansion it has been proved that, if the interaction force is (positive or negative) definite, the Born approximation of even order (including second) always gives a lower bound of the absolute value of  $\tan \delta$  ( $\delta$ : phase shift).

In appendix we also prove that, under the same conditions as above, the approximate method based on the Schwinger's variational principle which involves just the same computations as the second Born approximation, is always superior to the second Born approximation and gives a lower (upper) bound of the absolute value of  $\tan \delta$ .

## § 1. Introduction and summary

The Born approximation is the most important tool<sup>1)</sup> for the analysis of high energy scattering problems. Recently the nature of Born expansion has been studied by Jost and Pais<sup>2)</sup> by observing the behavior of the solution of scattering integral equation as a function of the potential strength  $\lambda$ . This wave function is usually a meromorphic function of  $\lambda$  with poles at the zeros ( $\lambda = \nu^{(n)}$ ) of the Fredholm determinant. Hence the zero with the smallest absolute value gives the radius of convergence for the Born expansion. W. Kohn<sup>3)</sup> has estimated this convergence radius  $\lambda_c$  in the case of central potential. However, it should be noticed that, for any meromorphic function of  $\lambda$  it is impossible to estimate by only the knowledge of the radius of convergence how large errors are introduced by neglecting the terms higher than  $O(\lambda^n)$  in the power series expansion of this function in  $\lambda$ . Therefore it seems somewhat worth while to investigate that, 1). whether we can obtain definite informations for the errors in  $n$ -th Born approximation from the convergence radius  $\lambda_c$  only, 2). inversely if the computations of the initial several approximations (for example, the first and second Born approximations) are performed, whether we can obtain definite informations for the convergence radius  $\lambda_c$ .

Consider the elastic scattering of a particle by a central potential  $V(r)$ . If we decompose the system into partial waves, the radial part of the wave function for angular momentum  $l$ :  $u_l(r)$  satisfies the integral equation ( $k > 0$ )

$$u_l(r) = j_l(kr) + \lambda \int_0^\infty g_l(r, r') V(r') u_l(r') dr' \rightarrow j_l(kr) + \tan \delta_l \hat{n}_l(kr), \quad (1)$$



where

$$\begin{aligned} g_l(r, r') &= (1/k) j_l(kr) u_l(kr'), \quad (r \rightarrow \infty), \\ j_l(kr) &= \sqrt{\pi kr/2} J_{l+1/2}(kr) \rightarrow \sin(kr - l\pi/2), \\ u_l(kr) &= (-1)^l \sqrt{\pi kr/2} J_{-l-1/2}(kr) \rightarrow \cos(kr - l\pi/2), \end{aligned}$$

$J_p(x)$  being  $p$ -th order Bessel function of first kind. In such one dimensional case the quantity corresponding to the scattering amplitude is given by (reactance matrix element)

$$\omega_l = (\lambda/k) \int_0^\infty j_l(kr) V(r) u_l(r) dr = \tan \delta_l.$$

In the  $n$ -th Born approximation  $\omega_l$  is replaced by

$$\omega_l^{(n)} = (\lambda/k) \int_0^\infty j_l(kr) V(r) u_l^{(n-1)}(r) dr,$$

where

$$\begin{aligned} u_l^{(n)}(r) &= j_l(kr) + \lambda \int_0^\infty g_l(r, r^{(1)}) V(r^{(1)}) j_l(kr^{(1)}) dr^{(1)} + \dots \\ &\dots + \lambda^n \int_0^\infty g_l(r, r^{(1)}) V(r^{(1)}) \dots g_l(r^{(n-1)}, r^{(n)}) V(r^{(n)}) j_l(kr^{(n)}) dr^{(1)} \dots dr^{(n)}. \end{aligned}$$

Our discussions are confined to such one dimensional Born approximation in the present paper, and the properties of three dimensional Born approximation will be investigated in the subsequent paper. In sec. 2 the errors of wave functions themselves are studied, and the following relations are derived,

$$|Ju_l^{(1)}|/|u_l| \leq |\lambda|/\lambda_{l,c}, \quad |Ju_l^{(n+1)}|/|Ju_l^{(n)}| \leq |\lambda|/\lambda_{l,c},$$

and hence

$$|Ju_l^{(n+1)}|/|u_l| \leq (|\lambda|/\lambda_{l,c})^{n+1},$$

where the errors of wave functions are defined as

$$Ju_l^{(n)}(r) \equiv u_l(r) - u_l^{(n)}(r),$$

and the "norm" of any function  $\Psi(r)$  is defined as

$$|\Psi(r)| \equiv \sqrt{\int_0^\infty \Psi^*(r) V(r) \Psi(r) dr},$$

for  $V(r) \neq 0$ . These formulas indicate that the more  $|\mu_l \equiv \lambda_{l,c}/|\lambda||$  is large, the more Born approximation is effective. We shall call  $\mu_l$  the coefficient of convergence. The necessary and sufficient condition for the convergence of Born expansion for angular momentum  $l$  is of course  $\mu_l > 1$ . The arguments in sec. 3 concern the errors of  $\omega_l$  in the first and the second Born approximations, and the following results are obtained. An upper bound and a lower bound of the first Born approximation are given by

$$-1/(\mu_l + 1) < \Delta\omega_l^{(1)}/\omega_l^{(1)} < 1/(\mu_l - 1), \quad \mu_l \equiv \lambda_{l,c}/|\lambda|,$$

if  $\mu_l > 1$ , where  $\Delta\omega_l^{(n)} \equiv \omega_l - \omega_l^{(n)}$ , (hereafter  $\lambda \neq 0$  is assumed),

and furthermore in the limiting case  $\hbar^2 \rightarrow 0$ ,

$$0 < \Delta\omega_l^{(1)}/\omega_l^{(1)} < 1/(\mu_l - 1), \quad (\mu_l > 1).$$

For the second Born approximation, it follows that for all the energy range

$$0 < \Delta\omega_l^{(2)}/\omega_l^{(1)} < 1/\mu_l(\mu_l - 1), \quad (\mu_l > 1).$$

Since the sign of  $\omega_l^{(1)}$  is same as that of  $\lambda$ , when the force is attractive ( $\lambda > 0$ )  $\omega^{(2)}$  gives a lower bound for correct  $\omega$  and in the repulsive force ( $\lambda < 0$ )  $\omega^{(2)}$  gives an upper bound for  $\omega$ . These situations are illustrated by a typical example. The similar discussions for higher Born approximations are found in sec. 4, together with some other remarks. For example, the following relation is derived

$$|\omega_l^{(1)}|/|\omega_l^{(2)} - \omega_l^{(1)}| > \mu_l > 0.$$

An upper bound of  $\mu \equiv \lambda_c/|\lambda|$  is obtained from the computations of the second Born approximation as above. This formula indicates quantitatively the fact that when  $|\omega^{(2)} - \omega^{(1)}|$  is not small compared with  $|\omega^{(1)}|$ , the higher Born approximations do not give good results. The converse of this proposition is not always true. Finally an approximate method based on the Schwinger's variational principle is analysed in Appendix. Schwinger's method which involves the same computations as the second Born approximation is

$$\omega_{S,l}^{(2)} = \omega_l^{(1)} / \{1 - (\omega_l^{(2)} - \omega_l^{(1)})/\omega_l^{(1)}\}.$$

Concerning this approximation the following relations hold:

$$0 < \Delta\omega_{S,l}^{(2)}/\omega_l^{(1)} < \Delta\omega_l^{(2)}/\omega_l^{(1)}; \nu^{(-)} < \lambda < \nu^{(+)}, \quad \Delta\omega_{S,l}^{(2)} \equiv \omega_l - \omega_{S,l}^{(2)}.$$

This shows Schwinger's  $\omega_S^{(2)}$  is always superior to Born's  $\omega^{(2)}$  and also gives a lower (an upper) bound for  $\omega$  in the convergent region of Born expansion if  $\lambda > 0$  ( $\lambda < 0$ ).

## § 2. Formal considerations

In the present paragraph no special assumption for  $V$  is made.

The fundamental equations (1) can be written in general form as

$$\phi = j + \lambda G V \phi, \quad (2)$$

where  $GV$  and  $V$  are short range (not too singular) operators, and we assume that the solution exists with the following properties. The eigenfunctions  $\varphi^{(m)}$  for the discrete eigenvalues  $\nu^{(m)}$  satisfying

$$\varphi^{(m)} = \nu^{(m)} G V \varphi^{(m)}, \quad (3)$$

constitute a complete set in the sense of mean convergence for  $V$ . Then the relations

$$(\phi V j) = \sum_m A_m (\phi V \varphi^{(m)}), \quad A_m = (\varphi^{(m)} V j) / \varepsilon_m, \quad (4a)$$

$$(\phi V \psi) = \sum_m a_m (\phi V \varphi^{(m)}), \quad a_m = (\varphi^{(m)} V \psi) / \varepsilon_m, \quad (4b)$$

$$(\phi V \psi^{(n)}) = \sum_m \alpha_m^{(n)} (\phi V \varphi^{(m)}), \quad \alpha_m^{(n)} = (\varphi^{(m)} V \psi^{(n)}) / \varepsilon_m, \quad (4c)$$

hold for any continuous function  $\phi$  which is restricted by suitable conditions (for example : if  $V$  is positive definite,  $(\phi^* V \phi) < \infty$ ), where

$$(\phi_1 V \phi_2) \equiv \int \phi_1 V \phi_2, \quad (\varphi^{(n)} V \varphi^{(p)}) = \varepsilon_n \delta_{np}.$$

In the  $n$ -th order Born approximation, we have

$$\psi^{(n)} = j + \lambda G V j + \dots + \lambda^n (G V)^n j. \quad (5)$$

Putting (4a) and (4b) into (2) we get

$$\sum a_m (\phi V \varphi^{(m)}) = \sum A_m (\phi V \varphi^{(m)}) + \sum (\lambda / \nu^{(m)}) a_m (\phi V \varphi^{(m)}),$$

hence

$$a_m = \nu^{(m)} A_m / (\nu^{(m)} - \lambda). \quad (6a)$$

It follows similarly from (5) that

$$\alpha_m^{(n)} = \{1 + \lambda / \nu^{(m)} + (\lambda / \nu^{(m)})^2 + \dots + (\lambda / \nu^{(m)})^n\} A_m. \quad (6b)$$

The Jost and Pais condition<sup>2)</sup> for convergence of Born expansion is  $\mu \equiv \lambda_c / |\lambda| \sim 1$ ,  $\lambda_c \equiv |\nu^{(m)}|_{\text{Min}}$ , which is seen in (6) clearly.

By noticing the relation

$$(\phi_1 V \phi_2) = \sum (\phi_1 V \varphi^{(m)}) (\varphi^{(m)} V \phi_2) / \varepsilon_m \quad (7)$$

we now define the "norm" of any function based on the  $\varphi^{(m)}$  series as

$$|\phi| = \sqrt{\sum_m |(\varphi^{(m)} V \phi)|^2 / \varepsilon_m}, \quad (8)$$

which accords with the norm defined in sec. 1 if  $V$  is positive definite\* and if  $\varphi^{(m)}$  is essentially real. Using (6) and (8) we obtain

$$\begin{aligned} |A \psi^{(0)}| / |\psi| &= |\psi - j| / |\psi| = \sqrt{\sum_m |\lambda a_m / \nu^{(m)}|^2 \varepsilon_m} / \sqrt{\sum_m |a_m|^2 \varepsilon_m} \\ &\leq |\lambda| / |\nu^{(m)}|_{\text{Min}} = 1 / \mu, \end{aligned}$$

\* The relations in this paragraph are all valid even if we take the more general definition of norm as

$$|\phi| = \sqrt{\sum_m \sigma_m |(\varphi^{(m)} V \phi)|^2},$$

where  $\sigma_m \neq 0$  are arbitrary constant.

and similarly

$$|\Delta\psi^{(n+1)}|/|\Delta\psi^{(n)}| \leq 1/\mu, \quad n \geq 0,$$

Hence

$$|\Delta\psi^{(n)}|/|\psi| \leq 1/\mu^{n+1}.$$

If  $k^2 \neq 0$ , and  $\lambda \neq \nu^{(m)}$  for all  $m$ , we have\* also from (6)

$$\lim_{n \rightarrow \infty} \mu^{n+1} |\Delta\psi^{(n)}|/|\psi| = 1.$$

Thus the coefficient of convergency  $\mu$  indicates the accuracy of Born approximation for any order. This situation will be shown clearly in sec.3~sec.4.

### § 3. First and second Born approximation

The object to be discussed in this paragraph concerns the errors of first and second Born approximation  $\omega^{(1)}$  and  $\omega^{(2)}$ . All the quantities appeared in this sec. are real.

As stated in sec. 2 the following relations hold for  $u = j + \lambda g V u$  and  $\phi^{(m)} = \nu^{(m)} g V \phi^{(m)}$ .

$$(\phi V u) = \sum \alpha_m (\phi V \phi^{(m)}), \quad \alpha_m = (\phi^{(m)} V u) / (\phi^{(m)} V \phi^{(m)}),$$

$$(\phi V j) = \sum A_m (\phi V \phi^{(m)}), \quad A_m = (\phi^{(m)} V j) / (\phi^{(m)} V \phi^{(m)}).$$

Replacing  $\phi$  by  $j$

$$(k/\lambda) \omega \equiv (j V u) = \sum_m \alpha_m (j V \phi^{(m)}) = \sum \alpha_m A_m (\phi^{(m)} V \phi^{(m)}),$$

$$(k/\lambda) \omega^{(1)} \equiv (f V j) = \sum_m A_m (j V \phi^{(m)}) = \sum A_m^2 (\phi^{(m)} V \phi^{(m)}).$$

According to (6) one gets

$$(k/\lambda) \omega^{(1)} = \sum c_m, \quad (k/\lambda) \omega = \sum c_m \nu^{(m)} / (\nu^{(m)} - \lambda), \quad (9)$$

$$(k/\lambda) \Delta \omega^{(1)} = \sum c_m \lambda / (\nu^{(m)} - \lambda).$$

where  $c_m \equiv A_m^2 (\phi^{(m)} V \phi^{(m)})$ . Let  $\nu^{(+)}$  and  $\nu^{(-)}$  be the nearest to zero in (+) and (-) regions respectively. ( $\nu^{(-)} < 0 < \nu^{(+)}$ ). Since  $c_m$  are all positive if  $V$  is positive definite (This is assumed in the present paper.), it follows that

$$\nu^{(-)} \omega^{(1)} / (\nu^{(-)} - \lambda) < \omega < \nu^{(+)} \omega^{(1)} / (\nu^{(+)} - \lambda) : \nu^{(-)} < \lambda < \nu^{(+)}, \quad (10)$$

It gives an upper and a lower bounds of the first Born approximation. When the energy

\* By specifying the normalizations of  $\psi$  and  $\varphi^{(m)}$ , the following relations hold:

$$\alpha_m e_m = (\varphi^{(m)} V \psi) = k \cot \delta / (\nu^{(m)} - \lambda) : \varphi^{(m)} \rightarrow \cos k r,$$

$$= k e^{-i\delta} / (\nu^{(m)} - \lambda) : \varphi^{(m)} \rightarrow e^{i k r},$$

$$\psi \rightarrow \sin(kr + \delta), \quad (r \rightarrow \infty).$$

These are derived directly from the fundamental differential equations.

$k^2$  approaches to zero,  $\nu^{(+)}$  approaches to  $-\infty$ . Hence at the limiting case  $k^2 \rightarrow 0$ , (10) is reduced to

$$\omega^{(1)} < \omega < \nu^{(+)} \omega^{(1)} / (\nu^{(+)} - \lambda), \quad -\infty < \lambda < \nu^{(+)}, \quad \lambda \neq 0$$

This indicates that the first Born approximation gives a lower bound of the correct value in some low energy region.

If  $|\lambda| < \lambda_c$ , we can derive the following relation from (10) in terms of the coefficient of convergency  $\mu \equiv \lambda_c / |\lambda|$ .

$$\mu \omega^{(1)} / (\mu + 1) \leq \omega \leq \mu \omega^{(1)} / (\mu - 1) : \lambda \geq 0, \quad \mu > 1.$$

Using (4) and (6), it is possible to observe the properties of the second Born approximation in similar way

$$\begin{aligned} (k/\lambda) \omega^{(2)} &= \sum c_m (1 + \lambda/\nu^{(m)}), \\ (k/\lambda) \Delta \omega^{(2)} &= \sum c_m \lambda^2 / \{(\nu^{(m)} - \lambda) \nu^{(m)}\}. \end{aligned} \quad (11)$$

Hence

$$0 < \Delta \omega^{(2)} / \omega^{(1)} < \lambda^2 / \{(\nu^{(+)} - \lambda) \nu^{(+)}\}, \quad \nu^{(+)} > \lambda \geq \nu^{(+)} + \nu^{(-)}, \quad (12a)$$

$$0 < \Delta \omega^{(2)} / \omega^{(1)} < \lambda^2 / \{(\nu^{(-)} - \lambda) \nu^{(-)}\}, \quad \nu^{(+)} + \nu^{(-)} \geq \lambda > \nu^{(-)}. \quad (12b)$$

These inequalities give an upper and a lower bounds of second Born approximation. In particular if  $|\lambda| < \lambda_c$  is satisfied, we find

$$\omega^{(2)} \leq \omega \leq \omega^{(2)} + \omega^{(1)} / \{\mu(\mu - 1)\}. \quad \lambda \geq 0, \quad \mu > 1.$$

This shows that, in the convergent region of Born expansion, the quantity  $\tan \delta^{(2)} \equiv \omega^{(2)}$  computed by second Born approximation gives a lower (an upper) bound for the true  $\tan \delta = \omega$  for all energies when  $\lambda$  is positive (negative).

(Example)

Neutron-proton S-wave scattering by a square well potential.

$$V(r) = 1 : r < 1, \quad = 0 : r > 1.$$

The fundamental equation is

$$u(r) = \sin kr + \lambda \int_0^1 \sin kr' \cos kr'' u(r') dr'.$$

The correct solution is

$$u(r) = (k) \sin(\sqrt{k^2 + \lambda} r), \quad r < 1.$$

From the Born approximations we have

$$\omega^{(1)} = \sin kr + \lambda(kr \cos kr - \cos^2 k \sin kr) / 2k^2, \quad r < 1.$$

The correct and approximate values for  $\omega = \tan \delta$  are

$$\omega = (k \tan \sqrt{k^2 + \lambda} - \sqrt{k^2 + \lambda} \tan k) / (k \tan k \tan \sqrt{k^2 + \lambda} + \sqrt{k^2 + \lambda}),$$

$$\omega^{(1)} = \lambda(k - \cos k \sin k) / 2k^2,$$

$$\omega^{(2)} = \omega^{(1)} + \lambda^2 \{k(1 - 4 \cos^2 k) + \sin k \cos k(1 + 2 \cos^2 k)\} / 8k^4.$$



To verify the preceding arguments we shall take  $\lambda=3.5$  which corresponds to the spin triplet scattering and  $k^2=10$  and  $k^2=50$ . From Table 3 in the subsequent paper, we find

$k^2=10 :$      $\nu^{(+)}=12.3,$      $\nu^{(-)}=-7.42,$      $\mu=2.12,$   
 $k^2=50 :$      $\nu^{(+)}=22.9,$      $\nu^{(-)}=-18.5,$      $\mu=5.3.$

In the analysis of nucleon-nucleon scattering the Born approximation is therefore not so good\* for  $k^2 \gtrsim 10$  ( $\gtrsim 170$  Mev).

This fact is also seen from Table 1 and 2 in which numerical values are calculated by the formulas (10), (12) and (15). It is also obvious from these tables that even if  $\mu$  is considerably large (namely, the convergency of Born approximation is good) the second Born's result does not always give a better approximation than the first Born's. This fact will be due to the situation that in the second Born approximation the errors from the expansion coefficients corresponding to each eigenfunctions are accumulated in the same direction so that the error thus accumulated becomes large enough to ensure to give a lower (an upper) bound of the correct solution.

	$\omega_0^{(2)}$	Lower bound	Correct value	Upper bound
first Born	$\omega_0^{(1)}=0.2300$	0.1934	0.2320	0.2715
second Born	$\omega_0^{(2)}=0.2263$	0.2263		0.2332
Schwinger's	$\omega_{S,0}^{(2)}=0.2264$	0.2264		

Table 1. The values of  $\omega_0^{(2)}$  (=tangent of S-wave phase shift) which are calculated from the formulae (10), (12) and (15) in each step of approximation for  $k^2=50$  (Incident energy of neutron is about 835 Mev.)

	$\omega_0^{(2)}$	Lower bound	Correct value	Upper bound
first Born	$\omega_0^{(1)}=0.550$	0.374	0.481	0.769
second Born	$\omega_0^{(2)}=0.405$	0.405		0.488
Schwinger's	$\omega_{S,0}^{(2)}=0.435$	0.435		

Table 2. The values of  $\omega_0^{(2)}$  corresponding to those of in Table 1 for  $k^2=10$ .

§ 4. Various discussions (higher Born approximation, etc)

In the preceding paragraphs we have derived several results, many of which might look quite plausible. It is not however possible without such detailed investigation to assert that the second Born's result gives a lower (an upper) bound for the true value. This statement can be generalized for the higher approximations. It is proved that

$(k/\lambda) \Delta \omega^{(n)} = \sum c_m \lambda^n / \{ (\nu^{(m)} - \lambda) \nu^{(m)n-1} \},$

and hence when n is "even" it follows that

$(k/\lambda) \Delta \omega^{(n)} > 0 : \nu^{(-)} < \lambda < \nu^{(+)}.$

\* In the case of elastic scattering of an electron by a hydrogen (helium) atom, the depth of potential corresponds to about  $\lambda \sim 2.3$  (2.7), and  $k^2=1$  corresponds to about 15eV (50eV.)

Namely, the Born approximation of even order gives a lower (an upper) limit in general when  $\lambda$  is positive (negative). It is not the case for approximations of odd order. To show this, the explicit formulas also are derived in the region  $\mu > 1$ .

$$\begin{aligned} n : \text{even}, \quad & 0 < \Delta\omega^{(n)}/\omega^{(1)} < 1/(\mu-1)\mu^{n-1}, \\ n : \text{odd}, \quad & -1/(\mu+1)\mu^{n-1} < \Delta\omega^{(n)}/\omega^{(1)} < 1/(\mu-1)\mu^{n-1}. \end{aligned}$$

We shall add one more remarkable property of  $\omega^{(n)}$ .

$$(k/\lambda)(\omega^{(n+1)} - \omega^{(n)}) = \sum_m c_m \lambda^n / \nu^{(m)n},$$

It is seen from the above formula that any approximation of odd order gives in general larger (smaller) value than that of the next lower order if  $\lambda > 0$  ( $\lambda < 0$ ).

We have observed how the Born approximation is suitable when  $\mu$  is large. The contraposition of this preposition can be expressed quantitatively. From (9) and (11),

$$(k/\lambda)\omega^{(1)} = \sum c_m, \quad (k/\lambda)(\omega^{(2)} - \omega^{(1)}) = \sum \lambda c_m / \nu^{(m)},$$

hence for any  $\lambda$

$$\lambda/\nu^{(-)} \leq (\omega^{(2)} - \omega^{(1)})/\omega^{(1)} \leq \lambda/\nu^{(+)}, \quad \lambda \leq 0, \quad (13)$$

or

$$|\omega^{(1)}|/|\omega^{(2)} - \omega^{(1)}| > \mu > 0. \quad (14)$$

(14) gives an upper bound of  $\mu$  by only  $\omega^{(1)}$  and  $\omega^{(2)}$ , and indicates that the more  $|\omega^{(2)} - \omega^{(1)}|$  grows large, the more Born's approximation is unsuitable.

In conclusion I wish to thank Professors T. Kato and T. Yamanouchi for their kind advices and interests in this work.

### Appendix. Schwinger approximation

We shall take up the Schwinger's method. The quantity  $f_{ab} = (\lambda/k)(j_a^* V j_b)$  is rewritten by using the equations  $\psi_c = \hat{j}_c + \lambda G V \psi_c$ , ( $c=a$  or  $b$ ), in sec. 2 as follows<sup>4)</sup>

$$f_{ab} = \lambda(\psi_a^* V j_b)(j_a^* V \psi_b)/k \{ (\psi_a^* V \psi_b) - \lambda(\psi_a^* V G V \psi_b) \},$$

which have stationary property for independent variations of  $\psi_a^*$  and  $\psi_b$ . Schwinger's approximation which involves the same computations as  $f^{(p+q+2)}$  is

$$f_S^{(p+q+2)} = (\lambda/k) \{ (\psi_a^{(p)*} V j_b) (j_a^* V \psi_b^{(q)}) / \{ (\psi_a^{(p)*} V \psi_b^{(q)}) - \lambda(\psi_a^{(p)*} V G V \psi_b^{(q)}) \} \}.$$

Putting  $p=q=0$ ,

$$f_S^{(2)} = (\lambda/k) (j_a^* V j_b)^2 / \{ (j_a^* V j_a) - \lambda(j_a^* V G V j_b) \}.$$

Note that  $(\lambda^2/k)(j_a^* V G V j_b) = f_{a,b}^{(2)} - f_{a,b}^{(1)}$ , hence  $f_S^{(2)}$  is written as

$$f_S^{(2)} = f_{a,b}^{(1)} \{ 1 - (f_{a,b}^{(2)} - f_{a,b}^{(1)})/f_{a,b}^{(1)} \}.$$

In the case of one dimensional analysis, this becomes

$$\omega_S^{(2)} = \omega^{(1)} / \{ 1 - (\omega^{(2)} - \omega^{(1)})/\omega^{(1)} \}, \quad (15)$$

From (15) we find

$$\omega_S^{(2)}/\omega^{(1)} \geq \omega^{(2)}/\omega^{(1)} : \omega^{(2)}/\omega^{(1)} \leq 2, \quad (16)$$

Further by (9) and (11)

$$\begin{aligned} \Delta\omega_S^{(2)} \equiv \omega - \omega_S^{(2)} &= (\lambda/k) [\sum \{c_m \nu^{(m)} / (\nu^{(m)} - \lambda)\} \cdot \sum \{c_p (\nu^{(p)} - \lambda) / \nu^{(p)}\} - (\sum c_m)^2] \\ &\div \{\sum c_m (\nu^{(m)} - \lambda) / \nu^{(m)}\}. \end{aligned}$$

If  $\nu^{(-)} < \lambda < \nu^{(+)}$ , we observe that  $\nu^{(m)} / (\nu^{(m)} - \lambda) \equiv \alpha_m > 0$ , and

$$(\sum c_m \alpha_m) (\sum c_p / \alpha_p) - (\sum c_m)^2 = \sum \{(a_m / \alpha_p + a_p / \alpha_m - 2\delta_{mp}) c_m c_p\} / 2 > 0,$$

so that

$$0 < \Delta\omega_S^{(2)} / \omega^{(1)} : \nu^{(-)} < \lambda < \nu^{(+)}. \quad (17)$$

Since the quantity  $\omega_S^{(2)}$  is derived from the Schwinger's variational method, the relation (17) accords with the result proved by T. Kato<sup>5)</sup>, though his proof loses its validity when the trial function is  $j_i(kr)$ . Combining (12), (16), (17) and (13), we get

$$\omega^{(2)} < \omega_S^{(2)} < \omega : 0 < \lambda < \nu^{(+)},$$

$$\omega < \omega_S^{(2)} < \omega^{(2)} : \nu^{(-)} < \lambda < 0.$$

In this way we find that the Schwinger's method is superior<sup>6)</sup> to the corresponding second Born approximation for all the energy range and gives a lower (an upper) bound for  $\lambda > 0$  ( $\lambda < 0$ ).

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**Note added in proof.** The error in the  $n$ -th Born approximation can not be estimated definitely from the following two knowledges, that is 1). radius of convergence of Born expansion  $\lambda_c$ , 2), that the wave function is a meromorphic functions of  $\lambda$ . Because there exist certainly meromorphic functions of  $\lambda$  having the given  $\lambda_c$  such that, when the series expansion of these functions in  $\lambda$  are cut to the order  $(\lambda^{(n)})$ , any large errors are introduced. Thus the relations obtained in § 2~§ 4 (which ensure the availability of the Born approximation when  $|\lambda|/\lambda_c$  is small.) due to the following facts, a). the wave function is subject to the physical law as (2), b).  $\varphi^{(m)}$  in (3) constitute a complete set in the sense of (4). The inequality equations in § 2~§ 4 can not be replaced by more accurate ones so long as we handle  $\lambda_c$  alone, for the region permitted in the inequality equation is filled everywhere by taking  $V$  suitably.

## Upper and Lower Bounds of Born Approximation, II

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The properties of three dimensional Born approximation are investigated along similar lines as Part I. Relations between one dimensional and three dimensional Born approximations are elucidated. We also compare the convergence radii for both approximations for typical examples.

## § 1. Introduction

In the preceding paper we have investigated the properties of one dimensional Born approximation. The object of this paper is to analyze the three dimensional Born approximation by comparing closely with the one dimensional.

In the elastic scattering of a particle by a central potential, the equation to be solved is

$$\{p^2 + k^2 + \lambda V(r)\} \psi(r) = 0, \quad (1a)$$

with the boundary conditions

$$\psi(r) \rightarrow e^{ikz} + f(\theta) e^{ikr}/r : |r| \equiv r \rightarrow \infty, \quad \psi(r) : \text{everywhere finite}, \quad (1b)$$

where  $\lambda$  is given and is a real number. Rewriting (1) into the integral equation, we obtain

$$\psi(r) = e^{ikz} + \lambda \int G(r, r') V(r') \psi(r') dr', \quad (2a)$$

$$G(r, r') = e^{ik|r-r'|} / 4\pi |r-r'|, \quad (2b)$$

$$= \sum_{l=0}^{\infty} (2l+1) G_l(r, r') P_l(\cos(\mathbf{r} \cdot \mathbf{r}')) / 4\pi r r',$$

$$G_l(r, r') = (1/k) j_l(kr_<) / h_l(kr_>),$$

where  $r_<$  and  $r_>$  are the smaller and the greater of  $r, r'$  respectively, and  $h_l(kr) = n_l(kr) + i j_l(kr)$ . The scattering amplitude is given by

$$f(\theta) = (\lambda/4\pi) \int e^{-ik\mathbf{n} \cdot \mathbf{r}} V(r) \psi(r) d\mathbf{r}, \quad \theta = (\mathbf{z}\text{-axis} \vee \mathbf{n}), \quad (3)$$

in which  $\mathbf{n}$  is a unit vector in the direction of the scattered particle. In the  $n$ -th order Born approximation,  $f(\theta)$  is replaced by

$$f^{(n)}(\theta) = (\lambda/4\pi) \int e^{-ik\mathbf{n} \cdot \mathbf{r}} V(r) \psi^{(n-1)}(r) d\mathbf{r}, \quad n \geq 1, \quad (4)$$

where

$$\begin{aligned}\phi^{(n)}(\mathbf{r}) = & e^{ikz} + \lambda \int \mathbf{G}(\mathbf{r}, \mathbf{r}^{(1)}) V(\mathbf{r}^{(1)}) e^{ikz^{(1)}} d\mathbf{r}^{(1)} + \dots \\ & \dots + \lambda^n \int \mathbf{G}(\mathbf{r}, \mathbf{r}^{(1)}) V(\mathbf{r}^{(1)}) \dots \mathbf{G}(\mathbf{r}^{(n-1)}, \mathbf{r}^{(n)}) V(\mathbf{r}^{(n)}) e^{ikz^{(n)}} d\mathbf{r}^{(1)} \dots d\mathbf{r}^{(n)}. \quad (5)\end{aligned}$$

In particular

$$\phi^{(0)}(\mathbf{r}) = e^{ikz}.$$

These quantities are decomposed into the partial waves in the following way

$$\begin{aligned}e^{ikz} &= \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta) / kr, \\ \phi(\mathbf{r}) &= \sum_{l=0}^{\infty} (2l+1) i^l \psi_l(r) P_l(\cos \theta) / kr,\end{aligned}$$

where  $\psi_l(r)$  satisfies the following integral equation by (2),

$$\psi_l(r) = j_l(kr) + \lambda \int_0^{\infty} G_l(r, r') V(r') \psi_l(r') dr' \rightarrow j_l(kr) + \sin \partial_l e^{i\delta_l} h_l(kr). \quad (r \rightarrow \infty). \quad (2')$$

The scattered amplitude is expressed by means of  $\psi_l(r)$  as follows:

$$\begin{aligned}f(\theta) &= \sum_{l=0}^{\infty} (2l+1) f_l P_l(\cos \theta) / k, \quad (\partial_l: \text{phase shift}, \quad S_l: \text{S-matrix element}), \\ f_l &= (\lambda/k) \int_0^{\infty} j_l(kr) V(r) \psi_l(r) dr = \sin \partial_l e^{i\delta_l} = (S_l - 1) / 2i. \quad (3')\end{aligned}$$

In the  $n$ -th order Born approximation,

$$\begin{aligned}f^{(n)}(\theta) &= \sum_{l=0}^{\infty} (2l+1) f_l^{(n)} P_l(\cos \theta) / k, \quad n \geq 1, \\ f_l^{(n)} &= (\lambda/k) \int_0^{\infty} j_l(kr) V(r) \psi_l^{(n-1)}(r) dr, \quad (4')\end{aligned}$$

where, from (2b), (4) and (5),

$$\begin{aligned}\psi_l^{(n)}(r) = & j_l(kr) + \lambda \int_0^{\infty} G_l(r, r^{(1)}) V(r^{(1)}) j_l(kr^{(1)}) dr^{(1)} + \dots \\ & \dots + \lambda^n \int_0^{\infty} G_l(r, r^{(1)}) V(r^{(1)}) \dots G_l(r^{(n-1)}, r^{(n)}) V(r^{(n)}) j_l(kr^{(n)}) dr^{(1)} \dots dr^{(n)}.\end{aligned} \quad (5')$$

Particularly

$$\psi_l^{(0)}(r) = j_l(kr).$$

We shall confine our attention to  $f_l^{(n)}$ , which is more conveniently investigated than  $f^{(n)}(\theta)$ .

We see from above descriptions that, if the three dimensional approximation is decomposed to partial waves, the integral equation (2') for this radial wave function  $\psi_l(r)$



is different from the integral equation (1) in Part I which is satisfied by  $u_l(r)$  in one dimensional treatment. This  $\phi_l(r)$  accords with  $u_l(r)$  except some constant factor, however, in finite approximations these two methods are different substantially to each other. The close relations between the two methods are discussed in sec. 2. In this paragraph we see that it is possible to give an upper and a lower bound of three dimensional Born approximation. In sec. 3 the convergence radii for both one and three dimensional Born approximations are calculated for S and P waves in the case of square well potential.

## § 2. Relation to one dimensional approximation

In sec. 2 and sec. 3 of Part I we have investigated the mathematical structure of the one dimensional Born approximation. The discussions given there also hold for the three dimensional case\* excepting that the eigenvalues associated with the equations (3) are not real, however, it is impossible to find an upper (a lower) bound of scattering amplitude  $f$  due to this fact of non-reality of eigenvalues, so long as one proceeds in the same line. We must find the relations between  $f^{(n)}$  and  $\omega^{(n)}$  by noticing that  $f_l^{(n)}$  is the Taylor expansion of  $\sin \partial_l e^{i\delta_l} = (S_l - 1)/2i$  for  $\lambda$  up to the order  $\lambda^n$ , while  $\omega_l^{(n)}$  is that of  $\tan \partial_l$ . This is done simply, first by noticing

$$G_l(r, r') = g_l(r, r') + (i/k) j_l(r) j_l(r'),$$

which is abbreviated in

$$G = g + (i/k) \cdot j \cdot j,$$

and setting

$$\begin{aligned} f^{(n)} &= (1/k) \sum_{p=1}^n A_p \lambda^p, \quad A_p = (j V G V \cdots G V j), \\ \omega^{(n)} &= (1/k) \sum_{p=1}^n \alpha_p \lambda^p, \quad \alpha_p = (j V g V \cdots g V j). \end{aligned} \quad (\text{number of } V : p)$$

Then we have the relations,

$$\begin{aligned} A_1 &= (j V j) = \alpha_1, \\ A_2 &= (j V g V j) + (i/k) (j V j) (j V j) = \alpha_2 + (i/k) \alpha_1^2, \\ &\vdots \\ A_p &= \alpha_p + \cdots + (i/k)^m \binom{p-1}{m} \alpha_1^m \alpha_{p-m} + \cdots + (i/k)^{p-1} \alpha_1^p, \end{aligned}$$

and similarly

$$\alpha_p = A_p + \cdots + (-i/k)^m \binom{p-1}{m} A_1^m A_{p-m} + \cdots + (-i/k)^{p-1} A_1^p.$$

\* For example, the following relations are valid:

$$|\Delta \psi^{(n+1)}|, |\Delta \psi^{(n)}| \leq |\lambda|/\lambda_c, \quad |\Delta \psi^{(n)}|/|\psi| \leq |\lambda|/\lambda_c.$$

For  $f$  and  $\omega$ , it follows that

$$f = (S-1)/2i = \sin \delta e^{i\delta} = \omega/(1+\omega^2) + i\omega^2/(1+\omega^2),$$

and

$$\omega = \tan \delta = f(1+f^2) - if^2/(1+f^2).$$

If  $f^{(n)}$  is once calculated,<sup>1)</sup> we can thus get all  $\omega^{(i)}$ . ( $1 \leq i \leq n$ ). As described in sec. 3 of Part I, since it is possible to give an upper (and also a lower) bound for  $\omega = \tan \delta$  by  $\omega^{(i)}$ , we can get an upper (a lower) bound for  $R_e[f] = \sin \delta \cos \delta$ ,  $I_m[f] = \sin^2 \delta$  and  $|S - S^{(n)}|$ .

### § 3. Convergence radii of Born approximation

In order to understand the properties of Born approximation it turns out necessary to investigate the properties of eigenvalues  $\nu^{(m)}$ . For that, more detailed discussions for these eigenvalues appearing in (3, I) must be given.

First the energy dependence of these quantities will be briefly studied. By the equation (3) in Part I,  $\nu$  is expressed by normalizing  $\varphi^{(m)}$  and  $\phi^{(m)}$  as

$$\varphi_l^{(m)}(r) \rightarrow e^{i(kr-l\pi/2)}, \quad \nu_l^{(m)} = k / \int_0^\infty j_l(kr) V(r) \varphi_l^{(m)}(r) dr, \quad (\text{three dimensional analysis}),$$

$$\phi_l^{(m)}(r) \rightarrow \cos(kr-l\pi/2), \quad \nu_l^{(m)} = k / \int_0^\infty j_l(kr) l^*(r) \phi_l^{(m)}(r) dr, \quad (\text{one dimensional analysis}).$$

For  $k$  increasing endlessly, the denominators remain finite if  $\int_0^\infty |V(r)| dr < \infty$ , and increase at most<sup>\*2)</sup> as  $O(\log k)$  if  $\int_0^\infty r|V(r)| dr < \infty$ .

Therefore we can see the reason why the Born approximation is good for high energies. When the energy approaches to zero,  $|j_l(kr) \varphi_l^{(m)}(r)|$  and  $|j_l(kr) \phi_l^{(m)}(r)|$  may also decrease like as  $O(k)$ , hence in this case the eigenvalues do not always tend to zero.<sup>2),3)</sup>

We now take a typical example (namely, square well potential) to understand the general situations.

$$V(r) = 1 : r < 1, \quad = 0 : r > 1.$$

#### A) Three dimensional case

To solve the equation ((3), I), we may also consider the following equation

$$\{d^2/dr^2 + k^2 - l(l+1)/r^2 + \nu_l^{(m)} l^*(r)\} \varphi_l^{(m)}(r) = 0,$$

with boundary conditions

$$\varphi_l^{(m)}(0) = 0, \quad \varphi_l^{(m)}(r) \rightarrow \text{const} \times e^{i(kr-l\pi/2)}, \quad r \rightarrow \infty,$$

\* For positive constants  $A, B, C, D, E$  and  $b$ , we may set  $|\varphi_l^{(m)}| \leq B$ ,  $|V(r)| \leq A/r$ ,  $\int_0^\infty r|V(r)| dr < \infty$ ,  $|j_l(r)| \leq C(kr)^{l+1}$ :  $kr < b$ ,  $\leq D$ :  $kr \geq b$ . Then it follows directly that

$$|\int_0^\infty j_l(kr) V(r) \varphi_l^{(m)}(r) dr| \leq D \log k + E.$$

which correspond to that  $\varphi_l^{(m)}(r)$  are pure outgoing waves spouting from the region<sup>1)</sup> of potential. Hence the eigenvalues  $\nu_l^{(m)}$  can not be real if the velocity of the relative motion does not approaches to zero. The eigenfunctions are

$$\varphi_l^{(m)} = j_l((k^2 + \nu_l^{(m)})^{1/2}r) : 0 \leq r < 1, \quad = j_l((k^2 + \nu_l^{(m)})^{1/2}) h_l(kr) / h_l(k) : 1 < r.$$

These two solutions should be combined smoothly at  $r=1$ . We can then determine  $\nu_l^{(m)}$  from (\*: complex conjugate)

$$(k^2 + \nu_l^{(m)})^{1/2} j_{l-1}((k^2 + \nu_l^{(m)})^{1/2}) / j_l((k^2 + \nu_l^{(m)})^{1/2}) = -k h_{l-1}^*(k) / h_l(k), \quad m \geq 1.$$

For S-wave<sup>3)</sup>

$$\tan(k^2 + \nu_0^{(m)})^{1/2} / (k^2 + \nu_0^{(m)})^{1/2} = -i/k,$$

$$\nu_0^{(m)}(k) = (m-1/2)^2 \pi^2 + \{1/(m-1/2)^2 \pi^2 - 1\} k^2 + O(k^4) + i\{-2k + O(k^3)\}, \quad m \geq 1.$$

For P-wave

$$1/(k^2 + \nu_1^{(m)}) - \cot(k^2 + \nu_1^{(m)})^{1/2} / (k^2 + \nu_1^{(m)})^{1/2} = 1/k^2 + i/k,$$

$$\nu_1^{(m)}(k) = m\pi^2 - 3k^2 + O(k^4) + i\{2k^3 + O(k^5)\}, \quad m \geq 1.$$

Values of  $\nu_0^{(m)}$  and  $\nu_1^{(m)}$  are tabulated for several energy values. Only two sets of eigenvalues are given in the Table 1 and 2, because these two are sufficient to determine  $\lambda_c$  for  $k^2 \leq 50$ . The superfix m has no essential meaning excepting that  $\nu_l^{(m)}$  having same m are continuous from  $k=0$  to  $k=\infty$ . The convergence radius of the three dimensional Born approximation without decomposing to partial waves is the smallest of all  $\lambda_c$ .

$k^2 \backslash$	$\nu_0^{(1)}$			$\nu_0^{(2)}$			$\lambda_c$
	$Re[\nu_0^{(1)}]$	$Im[\nu_0^{(1)}]$	$ \nu_0^{(1)} $	$Re[\nu_0^{(2)}]$	$Im[\nu_0^{(2)}]$	$ \nu_0^{(2)} $	
0	2.467	0.000	2.467	22.21	0.00	22.21	2.467
1	1.87	-1.94	2.70	21.3	-2.0	21.4	2.70
10	-3.9	-4.5	6.0	13.0	-7.2	14.9	6.0
50	-40.9	-2.8	41.0	-16.6	-12.5	20.8	20.8
$\infty$	$\pi^2 - k^2$	0.000	$k^2 - \pi^2$	$4\pi^2 - k^2$	0.00	$k^2 - 4\pi^2$	$\infty$

Table 1. The eigenvalues  $\nu_0^{(m)}$  associated with square well potential for S-wave. If we assume the range of square well is  $2.2 \times 10^{-13}$  cm and consider neutron-proton scattering,  $k^2=1$  corresponds to 17.1 Mev of incident neutron energy. (Three dimensional analysis).  $\lambda_c$  denotes the convergence radius of Born expansion.

$k^2 \backslash$	$\nu_1^{(1)}$			$\nu_1^{(2)}$			$\lambda_c$
	$Re[\nu_1^{(1)}]$	$Im[\nu_1^{(1)}]$	$ \nu_1^{(1)} $	$Re[\nu_1^{(2)}]$	$Im[\nu_1^{(2)}]$	$ \nu_1^{(2)} $	
0	9.87	0.0	9.87	39.48	0.0	39.48	9.87
1	8.38	1.5	8.51	37.7	4.	37.9	8.51
10	0.8	5.2	5.2	27.9	7.	28.7	5.2
50	-25.	9.	27	-3.	17.	18.	18
$\infty$	$20.2 - k^2$	0.	$k^2 - 20.2$	$59.7 - k^2$	0.	$k^2 - 59.7$	$\infty$

Table 2. The eigenvalues  $\nu_1^{(m)}$  associated with square well potential for P-wave. (Three dimensional analysis)

## B) One dimensional case

In this case consider the following equation

$$\{d^2/dx^2 + k^2 - l(l+1)/x^2 + \nu_l^{(m)} V(x)\} \phi_l^{(m)}(x) = 0,$$

with boundary conditions

$$\phi_l^{(m)}(0) = 0, \quad \phi_l^{(m)}(r) \rightarrow \text{const} \times \cos(kr - l\pi/2), \quad r \rightarrow \infty,$$

which are equivalent to (3) of Part 1. Each of these eigenfunctions correspond to the actual states which give rise to the largest scattering for the given incident energy. The eigenfunctions are

$$\phi_l^{(m)}(r) = j_l((k^2 + \nu_l^{(m)})^{1/2}r) : 0 \leq r < 1, \quad = j_l((k^2 + \nu_l^{(m)})^{1/2})n_l(kr)/n_l(k) : 1 < r.$$

The eigenvalues are real and determined from

$$(k^2 + \nu_l^{(m)})^{1/2} j_{l-1}((k^2 + \nu_l^{(m)})^{1/2}) / j_l((k^2 + \nu_l^{(m)})^{1/2}) = -k n_{l-1}(k) / n_l(k).$$

For S-wave<sup>5)</sup>

$$\tan(k^2 + \nu_0^{(m)})^{1/2} / (k^2 + \nu_0^{(m)})^{1/2} = -\cot k/k,$$

$$\nu_0^{(m)} = (m-1/2)^2 \pi^2 + k^2 + O(k^4), \quad m \geq 1.$$

For P-wave

$$1/(k^2 + \nu_1^{(m)}) - \cot((k^2 + \nu_1^{(m)})^{1/2}) / (k^2 + \nu_1^{(m)})^{1/2} = 1/k^2 + \tan k/k,$$

$$\nu_1^{(m)} = m^2 \pi^2 - 3k^2 + O(k^4), \quad m \geq 1.$$

The results are summarized in the Table 3 and 4 in a similar manner as in the Table 1 and 2.

$k^2$	$\nu_0^{(m)}$	$\nu_0^{(-)}$	$\nu_0^{(+)}$	$\nu_0^{(+,+)}$	$\lambda_0$
0	$-\infty$	$-\infty$	2.467	22.21	2.467
1	$-\infty$	$-\infty$	3.79	24.1	3.79
10	$-\infty$	-7.42	12.3	51.7	7.42
50	-42.3	-18.5	22.9	84.	18.5
$\infty$	$\sim -3\pi k$	$\sim -\pi k$	$\sim \pi k$	$\sim 3\pi k$	$\sim \pi k$

Table 3. The eigenvalues  $\nu_0^{(m)}$  associated with square well potential for S-wave. (One dimensional analysis).

$k^2$	$\nu_1^{(m)}$	$\nu_1^{(-)}$	$\nu_1^{(+)}$	$\nu_1^{(+,+)}$	$\lambda_0$
0	$-\infty$	$-\infty$	9.87	39.5	9.87
1	$-\infty$	$-\infty$	8.0	37.6	8.0
10	$-\infty$	$-\infty$	17.6	67	17.6
50	-22.3	$-\infty$	26.8	94	22.3
$\infty$	$\sim -\pi k$	$\sim -\pi k$	$\sim \pi k$	$\sim 3\pi k$	$\sim \pi k$

Table 4. The eigenvalues  $\nu_1^{(m)}$  associated with square well potential for P-wave. (One dimensional analysis).

From the data of neutron-proton interaction we may take  $\lambda$  about 3.5 and 2.3 for spin triplet and singlet states of S-wave respectively. For P-waves it has been inferred that  $\lambda=0 \sim -1$ . The critical values of  $\lambda$  to exist a bound state in the S and P states are  $\lambda=(\pi/2)^2=2.467$ , and  $\lambda=\pi^2=9.87$  respectively. We may see many remarkable features

from these tables. For example: 1) The convergence properties for both one and three dimensional Born expansions have somewhat similar nature, and the one is not always superior to the other. 2) For low energies the availability of Born approximation is restricted only by S-wave, while in higher energies the restriction is not necessarily due to S-wave etc.

The author would like to express his gratitude to Prof. T. Kato and Prof. T. Yamanouchi for their kind guidances and interests to this work. He is also indebted to the Yukawa Yomiuri Fellowship for the financial aid.

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**Note added in proof.** The alternative expressions for the total cross section,  $\int |f(\theta)|^2 d\Omega$  and  $(4\pi/k) I_m[f(0)]$  are different to each other in the finite approximations. However we can derive the following relation between the two as an immediate application of the formulas in § 2:

$$\int |f^{<1>}(\theta)|^2 d\Omega = (4\pi/k) I_m[f^{<2>}(0)].$$



## Letters to the Editor

### Effect of the Pauli Moment on the $\gamma$ -decay of Neutral Meson

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July 28, 1954

As is well known, the observed mean life for the  $\gamma$ -decay of neutral meson is about  $10^{-14}$  sec., while the theoretical value is  $7.1 \times 10^{-16} (g^2/4\pi\hbar c)^{-1}$  sec. which is calculated from the lowest order perturbation theory ( $e^2g$ ) assuming a pseudoscalar meson field with pseudoscalar coupling. In order to explain this discrepancy we should search for some causes by which the matrix element of this process would be made smaller at least by the factor of an order  $10^{-1}$ . Hayakawa and Mugibayashi<sup>1)</sup> have already shown that the inclusion of the higher order correction gives a good tendency but the correction up to  $e^2g^3$  is not enough to explain the discrepancy.

In this note we examine what an effect the life time is subject to when we assume the additive interaction term of the phenomenological Pauli moment. Of course, in this case, unrenormalizable divergent terms appear and we must resort to some cut-off method, which contains many arbitrariness and gives the results which is sensitive to the cut-off value. However, judging from the calculations of the nucleon mass difference as the effect of the Pauli moment in the cases of various cut-off methods<sup>2)</sup>, only the qualitative tendency seems not to depend on the details of the cut-off techniques. Then we attempt to see the qualitative feature of the present problem by means of a simple cut-off method.

The calculation is carried out after the Schwinger's proper time method<sup>3)</sup>. A Green function for the Dirac field  $G(x, x')$  satisfies the following equation in the presence of the Pauli term,

$$\left[ \gamma_\mu (-i\partial_\mu - eA_\mu(x)) + M - \frac{e\mu}{4M} \sigma_{\mu\nu} F_{\mu\nu}(x) \right] G(x, x') = \delta(x, x') \quad (1)$$

where the Schwinger's notations are used and  $\mu$  represents the anomalous magnetic moment in nuclear magnetons. Eq. (1) is considered to be a matrix

element of the operator equation

$$(\gamma \Pi + \sigma A + M)G = 1 \quad (2)$$

where

$$\Pi_\mu = \hat{p}_\mu - eA_\mu \quad \text{and} \quad A_{\mu\nu} = -\frac{e\mu}{4M} F_{\mu\nu}. \quad (3)$$

Then  $G$  is calculated by the representation

$$G = (\gamma \Pi + \sigma A + M)^{-1} \\ = (-\gamma \Pi + \sigma A + M) i \int_0^\infty ds \exp \{ -i[(M + \sigma A)^2 - (\gamma \Pi)^2]s \}. \quad (4)$$

In case of a scalar neutral meson field in scalar coupling with the nucleon field, the effective Lagrange function of coupling term between the neutral meson and the electromagnetic field is given by

$$\mathcal{L}'_s(x) = -g\phi(x) \left\langle \frac{1}{2} [\bar{\psi}(x), \psi(x)] \right\rangle \\ = -ig\phi(x) \text{tr} G(x, x) \\ = -g\phi M \int_0^\infty ds \exp(-iM^2s) \\ \times \left[ \text{tr}(x(s)'|x(0)'') \right]_{x', x'' \rightarrow x} \quad (5)$$

where

$$(x(s)')|x(0)'') = (x'| \exp(-iHs) |x'') \quad (6)$$

and

$$\mathcal{H} = \Pi^2 - \frac{e}{2}(1+\mu)\sigma F + \frac{e\mu}{2M} \left( \frac{1}{2}\sigma F \right)^2. \quad (7)$$

The approximation used by Schwinger that the electromagnetic field is homogeneous is also assumed here. Therefore the transformation function is calculated in the same manner as in the section III of Schwinger's paper. That is,

$$(x(s)')|x(0)'') = -i(4\pi)^{-2} \mathcal{D}(x', x'') e^{-L(s)} s^{-2} \\ \times \exp \left[ \frac{i}{4} (x' - x'') eF \coth(eFs) (x' - x'') \right] \\ \times \exp \left\{ i \left[ \frac{e}{2} (1+\mu)\sigma F - \left( \frac{e\mu}{2M} \right)^2 \left( \frac{1}{2}\sigma F \right)^2 \right] s \right\} \quad (8)$$

where

$$\mathcal{D}(x', x'') = \exp \left[ ie \int_{x''}^{x'} dx A(x) \right]$$

and

$$L(s) = \frac{1}{2} \text{tr} \log [(iFs)^{-1} \sinh(iFs)] .$$

Inserting (8) into (5) and substituting  $s \rightarrow -is$ , we obtain

$$\begin{aligned} \mathcal{L}'_s(x) = & \frac{1}{4\pi^2} g\phi M \int_0^\infty ds s^{-2} \exp(-M^2 s) \left[ (es)^2 \mathcal{C} \right. \\ & \times \left. \frac{Re\{\cosh[(1+\mu)sX] \cdot \exp[-(e\mu/2M)^2 sX^2]\}}{I_m\{\cosh(esX)\}} - 1 \right] \end{aligned} \quad (9)$$

where Schwinger's notations  $X = (2(+i))^{1/2}$ ,  $\mathcal{C} = 1/2(H^2 - E^2)$ ,  $\mathcal{C} = E \cdot H$  are used. The first term in the expansion of  $\mathcal{L}'_s$  for weak fields is

$$\begin{aligned} \mathcal{L}'_s(x) \simeq & \frac{e^2}{6\pi^2} \frac{g}{M} \left[ \left(1 + 3\mu + \frac{3}{2}\mu^2\right) M^2 \int_0^\infty ds \right. \\ & \times \exp(-M^2 s) - \frac{3}{4}\mu^2 \int_0^\infty ds s^{-1} \exp(-M^2 s) \left. \right] \phi \mathcal{F} . \end{aligned} \quad (10)$$

In case of a pseudoscalar meson field with pseudoscalar coupling, the calculation of  $\mathcal{L}'_{ps}$  is similarly done.

$$\begin{aligned} \mathcal{L}'_{ps}(x) = & g\phi(x) \left\langle \frac{1}{2} [\bar{\psi}(x), \gamma_5 \psi(x)] \right\rangle \\ = & ig\phi(x) \text{tr} \gamma_5 G(x, x) \\ = & \frac{e^2}{4\pi^2} g\phi M \int_0^\infty ds \exp(-M^2 s) \mathcal{C} \\ & \times \frac{I_m\{\cosh[e(1+\mu)sX] \cdot \exp[-(e\mu/2M)^2 sX^2]\}}{I_m\{\cosh(esX)\}} \end{aligned} \quad (11)$$

$$\begin{aligned} \simeq & \frac{e^2}{4\pi^2} \frac{g}{M} \left[ (1 + 2\mu + \mu^2) M^2 \int_0^\infty ds \exp(-M^2 s) \right. \\ & \left. - \frac{1}{2}\mu^2 \int_0^\infty ds s^{-1} \exp(-M^2 s) \right] \phi \mathcal{C} . \end{aligned} \quad (12)$$

The integral in the second term of eqs. (10) and (13) is logarithmically divergent at  $s=0$ . We may suppose that this divergence is originated from ignoring of the charge spreading responsible for the moment, so we take  $s_0$  as the lower limit of the integral in the terms involving  $\mu$ .

Putting together the transition matrix elements for the decay through the proton-antiproton pair and

through the neutron-antineutron pair, we get the life time as follows

$$\begin{aligned} s(s) : \quad 1/\tau_s = & (a^2/36\pi^2) (g^2/4\pi\hbar c) (m/M)^2 (mc^2/\hbar) \\ & \times \left[ 1 + \left( 3\mu_P + \frac{3}{2}\mu_P^2 + \frac{3}{2}\mu_N^2 \right) e^{-s_0 M^2} \right. \\ & \left. - \frac{3}{4}(\mu_P^2 + \mu_N^2) \left( \log \frac{1}{s_0 M^2} - \frac{C}{2} \right) \right] , \end{aligned} \quad (14)$$

$$\begin{aligned} p(s) : \quad 1/\tau_{ps} = & (a^2/16\pi^2) (g^2/4\pi\hbar c) (m/M)^2 \\ & \times (mc^2/\hbar) \times \left[ 1 + (2\mu_P + \mu_P^2 + \mu_N^2) e^{-s_0 M^2} \right. \\ & \left. - \frac{1}{2}(\mu_P^2 + \mu_N^2) \left( \log \frac{1}{s_0 M^2} - \frac{C}{2} \right) \right] \end{aligned} \quad (15)$$

where  $m$  is the meson mass,  $\mu_P = 1.79$ ,  $\mu_N = -1.91$ , and  $C = 0.5772 \dots$  (Euler's constant). The quantities within the parentheses [ ] in (14) and (15) may have the absolute value larger or smaller than unity according to the cut-off value  $s_0$ . Fig. 1 shows the relation between the numerical value within the parentheses of (15) ( $\equiv f(s_0)$ ) and  $s_0$ . Taking

$$6.7 M \leq \frac{1}{\sqrt{s_0}} \leq 7.0 M ,$$

we find  $|f(s_0)| \leq 10^{-1}$ , and so the life  $\tau_{ps}$  becomes  $\leq 10^2$  times longer.

The writer wishes to express his sincere thanks to Prof. S. Hayakawa for his kind advice.

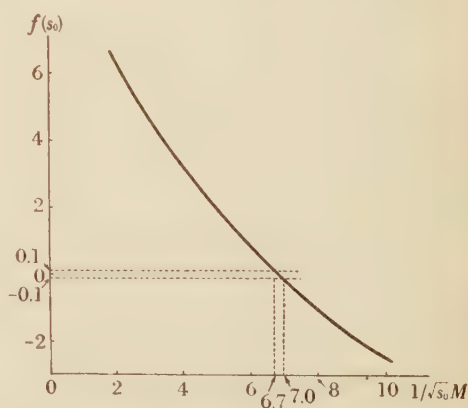


Fig. 1

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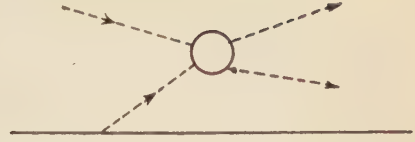


Fig. 2

Collisions in the case of strong meson-meson coupling.

## Meson Production in Meson-nucleon Collisions

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July 30, 1954

Recent developments in high energy accelerators have brought into light many properties of  $\pi$  mesons. The pseudoscalar nature of  $\pi$  mesons thus seems to be established but the interaction between pions and nucleons has not been definitely determined, the  $\not{p}s$ -coupling or the  $\not{p}v$ -coupling having its own advantages and disadvantages. In order to clarify this point many works have been done theoretically from various aspects<sup>1)</sup>.

In this short note we consider the process  $\pi^- + P \rightarrow \pi^- + \pi^+ + N$  because this process will be accessible experimentally<sup>2)</sup> and briefly discuss the differences between the  $\not{p}s$ -coupling and the  $\not{p}v$ -coupling.

Our basic assumptions are as follows:

- (a) The meson-nucleon coupling is stronger than the meson-meson coupling<sup>3)</sup>.
  - (b) Nucleons can be treated non-relativistically<sup>4)</sup>.
- The assumption (a) excludes the  $\pi$  meson emission mechanism of Fig. 2 and we here consider the one of Fig. 1 only. Under these two assumptions we first investigated the angular distributions of the emitted  $\pi^+$  meson in the lowest order approximation of the perturbation calculations.

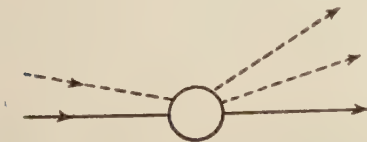


Fig. 1

Collisions in the case of strong meson-nucleon coupling.

According to Drell and Henley<sup>5)</sup>, the velocity independent feature ( $\sigma$ ,  $\tau$ ) of the nucleon can readily be identified for the pseudoscalar meson theory with  $\not{p}s$ -coupling by applying the Dyson transformation<sup>6)</sup>. The meson-nucleon interaction term that is relevant for  $\pi^- + P \rightarrow \pi^- + \pi^+ + N$  in the transformed Hamiltonian is written as

$$\begin{aligned}
 H' = & (f/2M) \int \psi^* (\sigma \nabla) \phi \tau \phi d\mathbf{r} \\
 & + (f^2/2M) \int \bar{\psi} \psi \phi^2 d\mathbf{r} \\
 & + (f/2M)^2 \int \psi^\dagger \pi \times \phi \psi d\mathbf{r} \\
 & + (2/3) (f/2M)^3 \int \psi^\dagger [(1/2) (\sigma \nabla) \phi \times \phi \cdot \phi \times \tau \\
 & + (1/2) \phi \times \tau \cdot (\sigma \nabla) \phi \times \phi] \psi d\mathbf{r}.
 \end{aligned}$$

Thus the angular distribution of the emitted meson for  $\not{p}s$ -coupling contains, in addition to the contribution from the equivalent  $\not{p}v$ -coupling, those from the second, the third and the fourth terms. Therefore, we can expect rather different results for the  $\not{p}s$ - and  $\not{p}v$ -coupling.

For the incident  $\pi^-$  meson of energy 210 Mev (threshold energy of this process being about 170 Mev) the angular distributions of the emitted  $\pi^+$  meson are given in Fig. 3. The result for the  $\not{p}s$ -coupling is mainly backward while that for the  $\not{p}v$ -coupling is  $2 + \cos^2 \theta$ , where  $\theta$  is the angle between incident  $\pi^-$  meson and the emitted  $\pi^+$  meson.

Near the threshold total cross sections for this process are approximately

$$\begin{aligned}
 \sigma_{ps} = & (f_{ps}^2/4\pi)^3 (16\pi/3) (1/\mu^2) (k_0/k_0) [(w_0/\mu) - 2]^2 \\
 & \times [4(w_0/\mu)^2 + 5(w_0/\mu) - 11], \\
 \sigma_{pv} = & (g^2/4\pi)^3 (16\pi/3) (1/\mu^2) (k_0/\mu) (\mu^2/w_0^2) \\
 & \times [(w_0/\mu) - 2]^4,
 \end{aligned}$$

where  $f_{ps} = \mu f/2M$  and  $(k_0, w_0)$  is the momentum energy four-vector of the incident  $\pi^-$  meson,

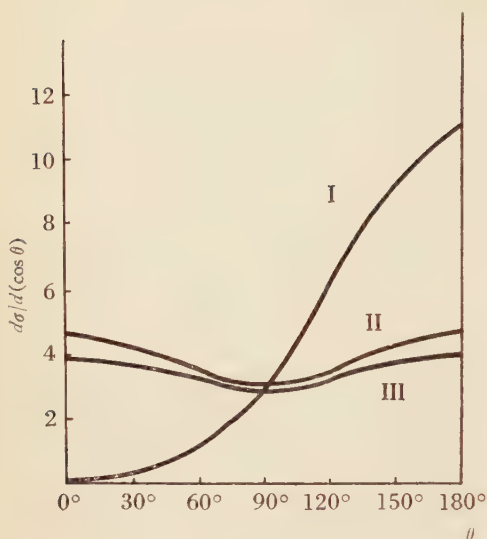


Fig. 3

Angular distribution of  $\pi^+$  meson.

I.  $\hat{p}s$  in units of  $(f_{ps}^2/4\pi)^3 \cdot 10^{-26}$ .

II.  $\hat{p}v$  in units of  $(g^2/4\pi)^3 \cdot 10^{-28}$ .

III. 2nd order in units of  $(g^2/4\pi) \cdot 10^{-28}$ .

It is well known that the lowest order results of the perturbation theory are not of good approximation and further we treated nucleons non-relativistically, it is desired to investigate the effect of higher order corrections. For this purpose we next considered this problem as the second order process, which consists of the following two steps:

- (a) Incident  $\pi^-$  meson is scattered by nucleon via the "meson-nucleon potential".
- (b)  $\pi^+$  meson is emitted by nucleon.

As Chew<sup>7)</sup> has pointed out, the experimental data on meson-nucleon scattering can be fitted if we take into account higher order effects of one-pion states. Using the model of Chew, Henley and Ruderman<sup>8)</sup> have calculated for  $\hat{p}$ -wave mesons the real or virtual meson-nucleon scattering matrix elements, which are almost the same as those of Chew for the scattering of real mesons for  $\mu \leq \nu_0 \leq \nu_{\max}$  ( $\nu_{\max}$  being the cutoff energy). Adopting this meson-nucleon scattering matrix elements for the step (a) and  $\hat{p}v$ -coupling for the step (b), the angular distribution for the emitted  $\pi^+$  meson was calculated for the incident  $\pi^-$  meson of energy 210 Mev and is also shown in Fig. 3<sup>9)</sup>.

The above results show that the angular distribution of the emitted meson is markedly different for the  $\hat{p}s$ -coupling and for the  $\hat{p}v$ -coupling and that the higher order effects may not alter the qualitative features of the lowest order approximation (at least for the  $\hat{p}v$ -coupling). Thus the measurement of  $\pi^+$  meson production by  $\pi^-$  meson on proton serves as a direct test of the meson-nucleon interaction.

The author wishes to express his heartily thanks to Mr. T. Kimura and to Mr. H. Wakita for their valuable discussions.

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- 3) W. Fry, *Phys. Rev.* **91** (1953), 1576.
- 4) Strictly speaking, the cross section for the  $\hat{p}s$ -coupling vanishes in the limit of  $M \rightarrow \infty$ , but we regard the common factor  $(\mu f/2M)^6$  as a finite constant.
- 5) S. D. Drell and E. M. Henley, *Phys. Rev.* **88** (1952), 1053.
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- 9) In the actual calculations  $\nu_1^{-1}$  is approximated by  $\nu_1^{-1} \sim -(8ig^2/3\mu^2\nu_0) [1 + 4A'(\nu_0)]^{-1}$ .

Numerical values of the coupling constant and the cut-off energy in the meson-nucleon scattering matrix elements are taken to be  $g^2/4\pi = 0.2$  and  $\nu_{\max} = 3.2\mu$  (see reference 5).

## Electronic Structure of Graphite and Boron Nitride

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July 30, 1954

Electronic structure of graphite was studied by several workers.<sup>1) 2) 3)</sup> They all explained some parts of the physical properties of graphite qualitatively or in some cases quantitatively, but did not give unified



interpretation of the quite different temperature dependency of electronic conductivity in single crystals and polycrystals and the relation between the diamagnetic anisotropy and the conductivity anisotropy. Because of it, by supposing overlapping bands in some cases or splitting bands in other cases, they tried to interpret the experiments. We, however, have obtained a little different conclusion in many respects from these workers' by studying the electronic structure of graphites of various types to be considered.

As Löwdin<sup>4)</sup> showed in the calculation of cohesive energy of sodium in metallic crystals, it could not be admitted to neglect the overlapping of the atomic functions between neighbours what is usually done. For valence crystals, having very short atomic distance as diamond or graphite, the circumstances are quite equal to metals, so we use the Löwdin's orthogonal atomic functions as the starting point and consider the resonance integrals of higher orders.

First we consider graphite lattices. For two dimensional lattice, valence and conduction bands touch at the corner of lowest Brillouin zone independently on the order of approximation, and the state density is symmetric in both sides near the corner. If we neglect the resonance integrals except those between the nearest neighbours, the widths of both bands become equal. Coulson and Taylor obtained the result that valence band becomes narrow and conversely conduction band broad in taking account of the overlappings between nearest neighbours; it should be mentioned, however, that in the case where these give rise to the large effect, second nearest neighbour overlappings are also important and thus the symmetry of both bands may again be maintained by considering the overlappings of all orders. Never-the-less, in actuality, since we must consider the resonance integrals of higher orders as the overlappings, they are not equal exactly, but the ratio of the widths of conduction band and valence band is obtained to be fairly smaller than unity. In uniformly displaced lattice the splitting of band occurs at the corner in certain cases whereas the overlapping doesn't; the splitting is at best of the order of  $\delta x/a\gamma$ , where  $\delta x$  is the displacement of atoms,  $a$  the atomic distance,  $\gamma$  the resonance integral between nearest neighbours.

For three dimensional lattice several variations may occur in stacking because of very weak interlayer binding. In usual *ab* type lattice atom A and B in a unit cell are not equivalent as to the *c*-axis, so the crystal orbitals of type A and B are different.

Solving the secular equation<sup>5)</sup>

$$\begin{vmatrix} H_{AA}-E & H_{AB} & H_{AB'} & H_{AA'} \\ & H_{BB}-E & H_{AA'} & H_{BA'} \\ \text{compl. conj.} & & H_{AA}-E & H_{BA} \\ & & & H_{BB}-E \end{vmatrix} = 0 \quad (1)$$

one finds that once again there is no energy gap between conduction band and valence band, independently on the order of approximation and at the point of Wallace's argument an energy difference of 0.02ev between both bands. In *aa* type the both bands touch at the point discussed by Wallace, on the other hand in *abc* type lattice, structure factor disappears for (110) (020), unlike in *ab* type. The number of electrons per atom permissible to the lowest Brillouin zone (Fig. 1) surrounded by (003) (111) (022) planes is

$$n=1+\frac{7}{48}\left(\frac{a}{c}\right)^4\left\{1-\frac{1}{7}\left(\frac{a}{c}\right)^2+\dots\right\}=1.0046. \quad (2)$$

Thus at the absolute zero of temperature there exist holes of about 0.005 per atom at the top of valence band. From this it follows that state densities near the top of the band are extremely different according to the type. As the number of electrons, however, contributing to the difference is very small, the energy difference for type is also very small, so it may be difficult to discuss the stability of each type.

We have calculated the energy band of  $\sigma$ -electrons by the tight binding approximation. The results are given in Fig. 2. The most remarkable difference from that of Hund and Mrowka<sup>6)</sup> is that the bottom states of conduction band is degenerated and the third and the fourth bands are a little dependent on wave number vector. The general features are similar to that of diamond calculated by Morita<sup>7)</sup> and Herman<sup>8)</sup>. Further the  $\sigma$ -bands of two dimensional boron nitride is given in Fig. 3. The essential difference of graphite and boron nitride is that in the latter there exists an energy gap at the point  $k=(2/3\sqrt{3}a, 0)$ .



Fig. 1

Lowest Brillouin zone of *abc* type, graphite.



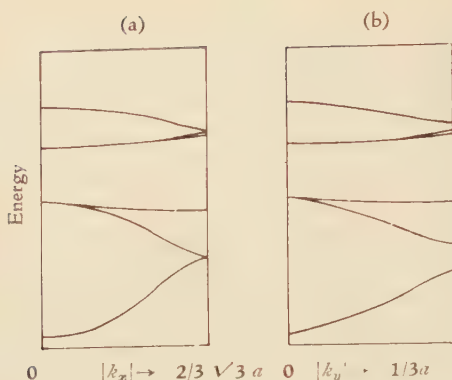


Fig. 2 Schematic  $E(K)$  curves of graphite. (a)  $k_x$  direction (b)  $k_y$  direction.



Fig. 3 Schematic  $E(K)$  curve for  $k_x$  direction of boron nitride.

Studies of boron nitride are very interesting from the points of view that its constituent elements lie on both sides of carbon in periodic table and the crystal structure is similar to graphite except the stacking. We can explain the atomic distance of boron nitride by the double bond model, similarly as in graphite following Pauling's formula.<sup>9)</sup> A few years ago we showed that boron nitride is insulator and is expected to have only small diamagnetic susceptibility.<sup>10)</sup> Now we have found that the soft X-ray emission bands of boron and nitrogen expected from the double bond model look like that given in Fig. 4. This is similar to the experimental results of O'Bryan and Skinner<sup>11)</sup> for boron, but not for nitrogen. From the interpretation of the emission band boron nitride seems to be rather an ionic crystal, but from this side it would be difficult to explain the very short atomic distance.



Fig. 4

Schematic X-ray emission band of boron nitride expected from the double bond model.

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- 11) H. M. O'Bryan and H. W. B. Skinner, Proc. Roy. Soc. A **176** (1940), 229.

## An Interpretation for the Electronic Conductivity and Diamagnetic Susceptibility of Graphite

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It may be considered that although graphite is a mixture of crystals of certain variant types, it

mainly consists of ab type, since this type may be energetically more stable than the other types<sup>1)</sup>. Therefore, zero energy gap between the valence band and the conduction band being calculated for this type, a perfect crystal may behave like metal or semiconductor. However, as the number of excited electrons in low temperatures is very few, if extremely minute quantity of impurities be contained, occupation of states near the top of the lower band, i.e. the bottom of the higher band changes drastically. It is likely that nitrogen and boron atoms, having a small radius, take the lattice point substitutionally.

When we consider the electronic conductivity of graphite, we must take into account the fact that many degenerated electrons exist at the bottom of the conduction band even at the absolute zero of temperature. The number of electrons excited from the full band to the conduction band is given by  $pT$ , where  $p$  is of the order of  $0.25 \times 10^{-19}$  hence in the temperature region below  $n_0/p$ , where  $n_0$  is the number of degenerated electrons, the number of conduction electrons  $n$  becomes almost constant. In usual natural graphite such a temperature may be comparatively high.

First we consider the resistivity  $\rho_{\parallel}$  in the vertical direction to the hexagonal plane. Because the energies of the lattice of ab type, aa type and abc type are fairly near, stacking disorder may often appear. Bacon<sup>2)</sup> found that displacement of layers increases by grinding and its probability  $p$  is about 0.02 in some samples. We suppose that in single crystal  $p$  is about 0.011 like boron nitride.<sup>3)</sup> If a diffuse scattering occurs on this plane, we must take about  $300 \text{ \AA}$  for the mean free path in this direction. Because this is independent of temperature,  $\rho_{\parallel}$  becomes temperature independent below  $n_0/p$ . However, above this temperature  $\rho_{\parallel}$  decreases with temperatures since the number of conduction electrons increases rapidly.

Let us next consider  $\rho_{\perp}$ . Following to Kinchin's analysis,<sup>4)</sup> the thermal mean free path for the conduction decreases exponentially with temperatures. Although the mean free path effect by slip boundary scattering decreases with temperature and this causes decreasing of the resistivity,  $\rho_{\perp}$  increases rapidly till  $n_0/p$ , above this temperature, however, its increment becomes somewhat slow because of canceling out by the increment of  $n$ . These predictions agree well with the results of Dutta<sup>5)</sup> and Kinchin's experiments. The rapid changes of  $\rho_{\parallel}$  and  $\rho_{\perp}$  bordered by about  $100^\circ\text{K}$  relate intimately to the degeneracy. In Wallace's calculation the anisotropy factor  $\rho_{\perp}/\rho_{\parallel}$  is

of the order of  $10^{-2}$ , which is different from the experimental value by a factor  $10^{-2}$ . However, since the ratio of both mean free paths  $l_{\perp}/l_{\parallel}$  is about 6.6 at room temperature, we could explain satisfactorily the disagreement by considering this fact and assuming for the resonance integrals  $\gamma_0$  3.6 eV which is later deduced. The more samples become pure, the more  $\rho_{\parallel}$  will increase till the low temperatures. Near the absolute zero of temperature  $\rho_{\perp}$  becomes independent of temperatures by the same reason as in metals. The rapid decrease of  $\rho_{\parallel}$  in high temperatures above  $400^\circ\text{K}$  may not be related to the eigenproperty of the energy bands, but to the transformation to the stacking ordered state. In general Wallace's calculation is not incorrect, but the samples used in observation for comparison could be imperfect crystals.

On the other hand in the case of polycrystals, the conductivity in the direction to the c-axis is out of the question and it appears as the mean value of  $\sigma_{\perp}$  in the plane. However, since the crystal size is small, we must also consider scattering at the crystal boundary unlike in single crystal. We denote the corresponding relaxation time with  $\tau_b$  and the relaxation time by lattice scattering with  $\tau_g$ , so the resistivity is approximately expressed by

$$\rho \propto \frac{m}{n} \left( \frac{1}{\langle \tau_b \rangle} + \frac{1}{\langle \tau_g \rangle} \right) \approx \frac{mv}{n} \left( \frac{1}{l_b} + \frac{1}{l_g} \right). \quad (2)$$

By using the result of Kinchin's analysis, we know that in a crystal of diameter  $1,000 \text{ \AA}$ , there exists below  $500^\circ\text{K}$  a relation,  $1/l_b > 1/l_g$ . Therefore  $\rho$  is determined by  $mv/n$ . The more samples are pure, the more the resistivity increases exponentially until low temperatures because of decrease of  $n$ . But at the temperature where the number of excited electrons are comparable to that of degenerated electrons,  $\rho$  levels off and maintains a constant value. In higher temperature region above  $500^\circ\text{K}$ , because one has  $1/l_b < 1/l_g$  and decrease of  $l_g$  prevails over the increase of  $n$ , polycrystals also behave like metals. The more crystal size becomes small the temperature of this transfer shifts to higher temperature. On the contrary even in polycrystals, if impurity increases, so the thermal excitation becomes immaterial and the resistivity becomes constant in wide range of temperatures. This agrees with Kinchin's result. We do not yet obtain quantitative interpretation why  $l_g$  changes exponentially or with a higher power of  $T$ .

We can interpret the diamagnetic anisotropy as well as the conductivity. In this case, however, the susceptibility  $\chi_{\perp}$  in a magnetic field parallel to the

hexagonal plane, is constant in all temperature regions, since the susceptibility of electrons in the filled region, which amounts to  $-0.5 \times 10^{-6}$  is larger than that of the conduction electrons, (the maximum value of this is  $-1.8 \times 10^{-8}$ , which value is derived from the expression

$$\chi_{\perp} = -\frac{4\pi e (c/2hc)^2 \gamma_1}{3A} \left[ 1 - \frac{\pi^2}{4} \left( \frac{kT}{\gamma_1} \right)^2 \ln \frac{4}{kT/\gamma_1} + \frac{11}{24} \pi^2 \left( \frac{kT}{\gamma_1} \right)^2 \right], \quad (2)$$

where  $c$ ,  $\gamma_0$  and  $\gamma_1$  are those of Wallace's paper and  $A$  is the density). On the other hand the component parallel to the  $c$ -axis, i.e.  $\chi_{\parallel}$  may take a large constant value below the temperature at which  $\rho_{\parallel}$  becomes constant, since in such a temperature energy surface is temperature independent and the excentricity in  $k$ -space is very large. But the anisotropy  $\chi_{\parallel}/\chi_{\perp}$  is 220 times smaller than the conductivity anisotropy  $\sigma_{\perp}/\sigma_{\parallel}$ , since both factors are not dependent only on the anisotropy of energy surface but the former denominator is about 33 times of the contribution from the conduction electrons and the latter denominator is 1/6.6 of that of perfect condition at room temperature. Moreover it becomes constant from a comparatively high temperature. Above  $n_0/p$  the energy gap does not become essential and the energy surface becomes more spherically symmetrical, so the anisotropy may begin to decrease somewhat rapidly.<sup>(5)</sup> Below room temperature (but  $T' \gtrsim 150^\circ\text{K}$ ) we obtained

$$\chi_{\parallel} = -\frac{4\pi e^2 \gamma_0^2 (c/2hc)^2}{J_0 \gamma_1} \times \left[ \ln \frac{\gamma_1}{kT'} + \ln \left( 4 - \frac{9}{4} \left( \frac{kT'}{\gamma_1} \right)^2 \right) - 2 + \pi^2 \left\{ \ln \frac{\gamma_1}{kT'} + \ln \left( 4 - \frac{9}{4} \left( \frac{kT'}{\gamma_1} \right)^2 \right) - \frac{19}{12} \right\} \left( \frac{kT'}{\gamma_1} \right)^2 \right]. \quad (3)$$

We must assume for  $\gamma_0$  four times of argument to obtain the agreement with the experimental value, (the logarithmic divergence of the expressions is (2) and (3) gives no difficulty, when we take into account the degeneracy of states). We find that degeneracy temperature is  $80^\circ\text{K}$  assuming the value  $2 \times 10^{-18}$  for the number of donors. Temperature dependency of mean free path may be related to this comparatively high degeneracy temperature.

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- 5) A. K. Dutta, Phys. Rev. **90** (1953), 187.
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## On the Excited States of Even-even Nuclei

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Scharff-Goldhaber<sup>(1)</sup> and Preiswerk and Stählin<sup>(2)</sup> have given general survey on the second as well as the first excited states of even-even nuclei. A marked magic number dependence of the excitation energies of the first excited states, and the predominance of the  $0^+ \rightarrow 2^+ \rightarrow 4^+$  and the  $0^+ \rightarrow 2^+ \rightarrow 2^+$  schemes, have been pointed out by them. After the publication of these works, experimental knowledges of the spin and parity of the second excited states have been almost doubled, and they are summarized in Fig. 1.

From this figure we see that the excitation energies of the second excited states have similar magic number dependence as that of the first excited states. This is more clearly seen from Fig. 2, where the ratio,  $\rho$ , of these two excitation energies are plotted against neutron numbers. The almost constancy of these ratios confirms the statement.<sup>(3)</sup>

The dotted line in Fig. 2 corresponds to  $\rho=3.3$ , a value which is expected if these two levels are considered to be the rotational levels in the sense of Bohr and Mottelson,<sup>(4), 5)</sup> and there are several elements which come just on this line. For other elements  $\rho$ -values are close to two and Bohr model can give also such values, if we take into account the rotation-vibration interaction; while, so long as the published calculations show,<sup>(6)</sup> it is difficult to obtain such large values on the shell model assumption.

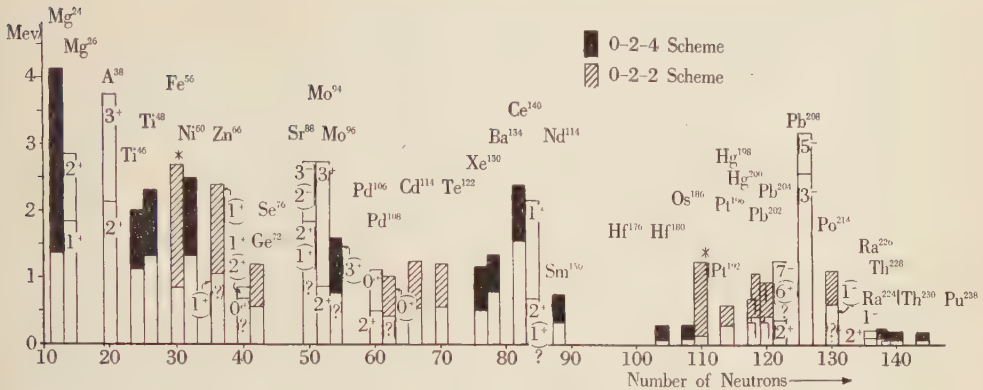


Fig. 1

First and Second Excitation Energies versus Neutron Numbers. References from which these data are taken are, for each element, as follow;  $Mg^{24}$ , May *et al.*, Phys. Rev. **93**, 246;  $Fe^{56}$ , Metzger *et al.*, Phys. Rev. **92**, 904;  $Ni^{60}$ , Kloepper, Phys. Rev. **91**, 1026;  $Zn^{66}$ , Mann *et al.*, Phys. Rev. **92**, 1481;  $Pd^{108}$ , Perlman *et al.*, Phys. Rev. **92**, 1236;  $Xe^{130}$ , Caird *et al.*, Phys. Rev. **94**, 412;  $Hf^{176}$  and  $Hf^{180}$ , Bohr *et al.*, Phys. Rev. **90**, 717;  $Pt^{192}$ , Gillon *et al.*, Phys. Rev. **93**, 124;  $Hg^{200}$ , Bergström *et al.*, Phys. Rev. **92**,

918;  $Pb^{202}$ , Maeder *et al.*, Phys. Rev. **93**, 1433;  $Pb^{204}$ , Frauenfelder *et al.*, Phys. Rev. **93**, 1126;  $Pb^{208}$ , Elliott *et al.*, Phys. Rev. **93**, 356;  $Po^{214}$ , Demichells *et al.*, Nuovo Cimento **10**, 1359;  $Ra^{224}$ , Asaro *et al.*, Phys. Rev. **92** 1495;  $Ra^{226}$ ,  $Th^{228}$  and  $Th^{230}$ , Bohr *et al.*, Phys. Rev. **90**, 717;  $Pu^{238}$ , Asaro *et al.*, Phys. Rev. **92**, 694. For other elements see G. Scharff-Goldhaber, Phys. Rev. **90**, 587. Data with \* and ? are somewhat ambiguous concerning energy, and spin and parity, respectively.

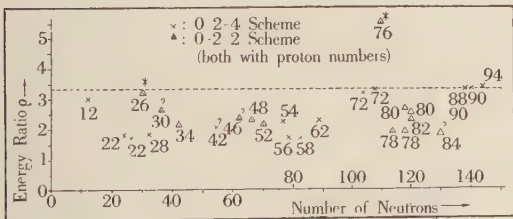


Fig. 2

Ratios of the second and the first excitation energies versus neutron.

On the other hand Bohr model seems somewhat difficult to give 0-2-0 scheme,<sup>7)</sup> while Preiwerk and Stählin<sup>2)</sup> and de-Shalit and Goldhaber<sup>8)</sup> have given interesting explanation of this scheme on the assumption of the configurational interaction.

In the region with  $N > 100$ , 0-2-4 scheme nuclei appear in the region near double magic  $Pb^{208}$  while 0-2-2 scheme nuclei appear outside of this region. On the other hand in the region  $N < 100$ , 0-2-2 and 0-2-4 scheme nuclei appear mixed. For lighter nuclei it is expected from shell model calculation that 0-2-4 scheme could appear, because

there the neutrons and the protons occupy equivalent orbits, and  $Mg^{24}$ ,  $Ti^{46}$  and  $Ti^{48}$  may be considered as such examples. On the other hand  $Ni^{60}$  and  $Ce^{140}$ , although they have 0-2-4 scheme, do not belong to this category. However, it is interesting to note that they are both single magic nuclei, and in such cases we can expect that the level scheme of the non-magic group mainly determines the lowest level scheme of the nucleus as a whole, and gives 0-2-4 scheme, but with  $\rho$ -value close to one. However, if we take into account the perturbational interaction between this non-magic group and the counter magic group, as was considered in reference 2 and 8, the first excited  $2^+$  level will be lowered and we will obtain  $\rho \approx 2$ , while preserving 0-2-4 scheme.  $Ba^{134}$  and  $Xe^{150}$  might also be considered similarly, i.e. their levels might not be rotational ones, because they, though are not magic nuclei, lie rather near to  $Ce^{140}$ , and because  $Cs^{133}$  which lies near to them has very small quadrupole moment, inspite of its having spin  $5/2$ , showing small deformability of the core in this region of the periodic table.

On the other hand large  $\rho$ -values of  $Ti^{46}$  and  $Ti^{48}$  might be explained using similar reasoning



proposed by Flowers<sup>9)</sup> to explain the anomalous spin of  $Ti^{47}$ , because as is seen from Table 6 of the paper of Flowers<sup>10)</sup>,  $Ti^{46}$  and  $Ti^{48}$  with configurations  $(1f_{7/2})^6$  and  $(1f_{7/2})^8$ , respectively, both have  $2^+$  and  $4^+$  levels which are degenerate concerning the partition numbers of both the unitary and the symplectic groups, and so there is possibility that the large non-diagonal terms between these levels make the  $\rho$ -values come close to two. On this connection the  $\rho$ -value for  $Ti^{44}$  may be interesting to obtain experimentally, because there exists no such degeneracy, in this case.

As the preceding discussions show, the Bohr model and the shell model may be considered as mutually complementally. One of the most powerful clue to discriminate, into which of these two categories some nuclei should be classified, would be the ratio,  $F$ , of the experimental and the theoretical transition probabilities<sup>5)</sup>. For example  $Sm^{152}$ , of which the spin and parity of the second excited state are unknown yet, has  $F \approx 100^{11)}$ , which permits us to conceive the level scheme of  $Sm^{150}$  to be interpreted by Bohr model. It is hoped that more informations on this quantity will be obtained experimentally.

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- 7) K. W. Ford, Phys. Rev. **90** (1953), 29.
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- 11) Computed from the data of reference 1.

## Corrigenda: Radiative Corrections to the Anomalous Magnetic Moment of the Nucleon in Pseudoscalar Meson Theory

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In the previous paper<sup>1)</sup> there were some mistakes from carelessness which have been crept into the numerical evaluation of the parameter integrals occurring in the 4-th order meson contributions to the anomalous magnetic moment of the nucleon. The values cited there should be revised as follows:

$$\begin{aligned}\mu_{IIIa} &= 0.178 \cdot \frac{1}{4} \beta (\beta_3 - \beta_4) (\tau_p - \tau_n), \\ \mu_{IIIb} &= -0.200 \cdot \frac{1}{8} \beta (2\beta + \beta_3 + \beta_4) (\tau_p - \tau_n), \\ \mu_{IIIc} &= 0.215 \cdot \beta^2 (\tau_p - \tau_n), \\ \mu_{IIId} &= -0.178 \cdot \beta^2 (\tau_p - \tau_n).\end{aligned}\tag{1}$$

With these values the anomalous magnetic moment in symmetrical theory ( $\beta = \beta_3$ ,  $\beta_4 = 0$ ) amounts to  $\mu_p = 0.111\beta + 0.313\beta^2$  and  $\mu_n = -0.822\beta - 0.058\beta^2$  and in order to reproduce the experimental values the coupling constant must be taken as  $\beta = 2.22$  and 2.03 (or in a more conventional symbol  $(f/\sqrt{2})^2 (4\pi\hbar c)^{-1} = 7.0$  and 6.4) for proton and neutron respectively.

It is interesting that the meson contributions largely cancel in all, i.e.  $\mu_{III} = 0.007\beta^2(\tau_p - \tau_n)$ , although each part is rather large. The same result is expected in a non-relativistic calculation<sup>2)</sup> up to the same order, for instance, using the Hamiltonian<sup>3)</sup> transformed by Dyson transformation and cutting off meson momenta larger than  $Mc$ . For there the nucleon is treated as fixed, so that the core term which is most significant in the interaction Hamiltonian can produce virtual mesons only in the  $S$ -state, giving of course no contribution to the magnetic moment. Other terms are not effective in view of smallness of the effective coupling constant,  $(f/\sqrt{2})^2 \cdot (4\pi\hbar c)^{-1} \cdot (\mu/M)^2$ .

To compare this situation with our case, we examine the effect of  $\Delta_{p'}$  (the modified meson propagator) to the magnetic moment, which gives



the largest meson contribution, rather in detail. The modified part of  $\Delta_{F'}$  up to the second order turns out

$$\bar{\Delta}_{F'}(k) = 2\beta \left[ \frac{\theta/\tan\theta - 1 + \varepsilon_1}{k^2 + x^2} - x^2 \frac{\theta/\tan\theta - 1 + \varepsilon_2}{(k^2 + x^2)^2} \right], \quad (2)$$

where  $-k^2 = 4K^2 \sin^2\theta$ ,  $\varepsilon_1 \approx 2\varepsilon_2 \approx x^2/6K^2$ ,

$K$ ,  $k$  being reciprocal Compton wave length of nucleon and meson.

The second term of (2) contributes to the a.m.m. only about 2% relative to the first term and is therefore neglected. Thus the modified meson propagator is reduced to the non-modified one except for the factor  $2\beta(\theta/\tan\theta - 1)$ . To see the effects of the factor on the a.m.m., we first calculated the second order meson contributions from the two kinds of processes separately, classified according as whether or not nucleon pair is created in the intermediate states<sup>4)</sup>. The results were this: they contribute the same magnitude; while in the processes not involving nucleon pair only virtual mesons with the magnitude of momenta between  $2x$  and  $K$  play the dominant role, virtual mesons associated with nucleon pair are essentially of momenta  $> 0.5K$  and therefore spread only over the range of order of  $\hbar/Mc$  around the nucleon, thus showing that nucleon recoil effect can never be neglected. The effects of the extra factor introduced in the fourth order are then almost evident, on rewriting the first term of (2) in the form

$$2\beta \left[ \frac{\theta/\tan\theta - 1}{k^2 + x^2} \right] \sim 2\beta \int \frac{dz z^2}{1 - z^2} \frac{1}{k^2 + (4K^2/1 - z^2)}, \quad (3)$$

where the terms multiplied by  $x^2/4K^2$  are discarded. This suggests that virtual mesons here behave having large mass  $\geq 2M$  instead of  $\mu$ . This change of effective meson mass results in the effects of lower energy virtual mesons being depressed compared with those of higher energy mesons. The detailed calculation verifies this conclusion and the contribution from the processes involving nucleon pair is considerably enhanced (about 2 times) relative to the other processes.

The above conclusion indicates that  $\mu_{IIIc}$  can be hardly evaluated by the non-relativistic method or the cut off procedure. (This will be the case also for meson-nucleon scattering and we expect that the cut-off procedure adopted by Fubini<sup>5)</sup> in Tamm-Dancoff approximation for the scattering may be inadequate for renormalized radiative corrections.<sup>6)</sup>)

The same holds for other radiative corrections. This fact may be expected from general consideration that radiative corrections obtained after renormalization become predominant only in high energy-momentum transfer.<sup>7)</sup> These radiative corrections for the a.m.m., however, vanish by cancellation, as stated before. This is considered to be due to the fact that the radiative corrections for the propagators are depressed by those for the vertex functions, as was recently stated by Kroll and Ruderman.<sup>8)</sup> The comparatively good result above obtained is thus due to the 4-th order nucleon contributions alone.

It is noted that modified meson propagator,  $\Delta_{F'}$ , which summarizes the effects from all the diagrams obtained by repeating the lowest order proper meson self energy part, is

$$\Delta_{F'}(k) = \Delta_F(k) \left/ \left[ 1 - \frac{1}{2\pi i} \Delta_c(k) \right] \right., \quad (4)$$

where

$$\frac{1}{2\pi i} \Delta_c(k) = \frac{2\beta}{k^2 + x^2} [k^2(\theta/\tan\theta - 1 + \varepsilon_1) + (\varepsilon_1 - \varepsilon_2)x^2].$$

Here the factor,  $\Delta_c(k)/2\pi i$ , contributes effectively the order of magnitude  $\sim 1$  to the a.m.m., employing the aforementioned value of the coupling constant. Evidently the perturbation treatment of this can't be trusted on in any case. Thus one might be tempted to re-calculate the second order a.m.m. replacing  $\Delta_{F'}(k)$  with (4), as Feldman<sup>9)</sup> has done for the nucleon propagator. Especially for the nucleon contribution the result will be different in sign from the proper second order one for suitable values of the coupling constant. However, it is unfit to take into account only special subset of radiative corrections, because the effect of  $\Delta_{F'}(k)$  might be suppressed if higher order radiative corrections for  $\partial_\mu$ -vertex are considered in the similar manner. This is indeed the case for the fourth order calculation. (The analogous argument is applicable to the effect of  $S_{F'}(p)$ . It must be noted, in the fourth order result, that the contribution to the a.m.m. from the radiative correction for  $\partial_\mu$ -vertex has a sign opposite to the one from the modified part of nucleon propagator,  $\bar{S}_{F'}(p)$  and that both of them are of the same order of magnitude. This is true for both nucleon- and meson-contributions. Therefore to lay much stress only on the radiative corrections for nucleon propagator is not admissible.) Moreover, it is entirely doubtful whether such approximate propagators can be good counterparts of the ones.<sup>10)</sup>

Thus much scrutiny seems to be required before the modified propagators can be used in practice.

Finally we would remark that  $\mu_{IIa}$ , which had been in disagreement with the result by A. Thellung, was re-examined and that no error could be found.  $\mu_{IIa}$  was also evaluated analytically in the limit of  $(\mu/M)^2 \rightarrow 0$ , obtaining

$$\mu_{IIa} = -\frac{1}{4} \beta^2 (\tau_p + 2\tau_n) \left( \frac{\pi^2}{36} - \frac{1}{2} \right).$$

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- 4) Each kind of processes splits again into two subclasses, one containing virtual meson simply scattered, and the other containing meson pair created or destructed by the external magnetic field. The latter amounts about twice larger than the former. The more detailed and associated arguments will be published in another letter by Hasegawa, Matsuyama and one of us.
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## A Formulation of the Theory of Alpha-particle Decay from Time-independent Equations\*

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The theory of alpha-particle decay is formulated from the time-independent  $R$ -matrix theory of nuclear reactions. It is shown that the most appropriate proper solutions are the ones with radial nodes near the channel radii, and therefore the Teichmann-Wigner sum rules involving the alpha-particle kinetic energy may be applied. Exact and approximate relations are given for the dependence on the angular momentum of the decay rate for the square well model. Barrier penetration formulas are given which include estimates of the effects of an external nuclear exponential potential and the atomic electron screening.

### § 1. Introduction

There are so many theoretical papers in the literature on alpha-particle decay<sup>1)</sup> that it is difficult to decide which theory or what combinations of theories should be used for the most significant interpretation of the experimental data. For example, in a recent paper<sup>2)</sup> it was considered desirable to use five different formalisms for the interpretations. It is therefore with reluctance that we present here yet another approach. However, in the opinion of the writer it is not only simpler but also more general than the previous ones. Moreover, it is based on the methods which have been found to be particularly satisfactory for the interpretation of nuclear resonance reactions, of which alpha-particle decay may be considered as a special case. Although the application of resonance theory to alpha-decay is not new, some of the previous attempts have overlooked the important effect of the boundary condition on the decay rate<sup>3)</sup>.

In § 2 it is shown that the generalized one-level collision matrix of Wigner and Eisenbud<sup>4)</sup> may be applied, the decay rate being related to the imaginary part of the energy of the pole of the resonance denominator. It is then shown that due to the intense alpha-decay barrier, the proper boundary condition for the radial derivative of the eigen-solutions is one which is essentially infinite, rather than close to zero as in the familiar applications of the resonance theory to light nuclear interactions where the barriers are relatively weak. Consequently, the sum rules to be considered are those given by Teichmann and Wigner<sup>5)</sup> which compare the kinetic energy of the alpha-particle on the nuclear

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surface to its average value within the surface, rather than the rules which compare the squares of the amplitudes of the wave function. The square well model is treated in detail in § 3 where the dependence of its reduced widths on the angular momentum  $L$ , as first obtained by Preston,<sup>6)</sup> is derived in a simple manner. The effect of an exponentially decaying external nuclear potential on barrier penetration is estimated in § 4. The JWKB formulas for the barrier penetration and the level shift factors and a detailed consideration of the corrections for atomic electron screening are given in the Appendices. Throughout this account it is assumed that the nucleus is spherically symmetric. It is hoped that the generality of the method may lend itself to the interpretation of the recent evidences for the non-sphericity of the nucleus.<sup>7)</sup> Although the present formalism is applicable to a many-body description of alpha decay,<sup>8,9)</sup> it is beyond its scope to provide specific reasons for the irregularity and the hindrance of alpha emission from odd- $A$  nuclei,<sup>8)</sup> in contrast with the apparent regularity of emission from the even-even nuclei to the ground and first excited states.<sup>10)</sup>

## § 2. General Theory

It can be shown by means of appropriate wave packets constructed with the time-independent solutions of the wave equation that the decay constants of radioactive states are  $\hbar^{-1}$  times the imaginary parts of the energies of the poles of the collision matrix.<sup>11)</sup> The poles of the collision matrix may be expressed in terms of the significant parameters  $E_\lambda$  and  $\gamma_\lambda$  of the nuclear system by means of the  $R$ -matrix formalism of Wigner and collaborators.<sup>4,5)</sup> If one level  $\lambda$  of the  $R$  matrix\*

$$R = \sum_\mu (\gamma_\mu \times \gamma_\mu) / (E_\mu - E) \quad (1a)$$

is explicitly separated from the rest in the manner

$$R = R^\infty + (\gamma_\lambda \times \gamma_\lambda) / (E_\lambda - E), \quad (1b)$$

it is found that the resonance denominator of the generalized collision matrix of Wigner and Eisenbud becomes

$$E_\lambda + \mathcal{A}_\lambda - E - \frac{1}{2}i\Gamma'_\lambda, \quad (2)$$

the total level width  $\Gamma'_\lambda$  and total level shift  $\mathcal{A}_\lambda$  being expressed as scalar products over both positive and negative energy channels:

$$\begin{aligned} \frac{1}{2}\Gamma'_\lambda &= (\alpha_\lambda^*, P\alpha_\lambda), \\ \mathcal{A}_\lambda &= -(\alpha_\lambda^*, \bar{S}\alpha_\lambda) + (\beta_\lambda^*, R^\infty\beta_\lambda), \\ \alpha_\lambda &= (1 - R^\infty\bar{\mathcal{L}})^{-1}\gamma_\lambda, \quad \beta_\lambda = \bar{\mathcal{L}}\alpha_\lambda, \end{aligned} \quad (3)$$

\* The notation used here differs in some respects from that of Teichmann and Wigner.<sup>5)</sup> Our reduced width  $\gamma_\lambda c^2$  is  $a_c^{-1}$  times theirs and therefore has the dimension of energy, and our  $R$  matrix is dimensionless. The relation between the boundary conditions is  $-B_c = a_c \bar{b}_c$ . Our  $F$  and  $G$  functions are  $\hbar^{1/2}$  times theirs.



$$\begin{aligned}\bar{\mathcal{L}} &= \bar{S} + i\bar{P} = \mathcal{L} - B \\ \mathcal{L} &= S + iP, \quad \bar{S} = S - B, \quad \bar{P} = P.\end{aligned}$$

Here the components of the real, constant, diagonal matrix  $B$  are the radial boundary conditions satisfied by the normalized proper solutions  $X_\lambda$  of the wave equation  $\Delta X_\lambda = E_\lambda X_\lambda$  on the various channel entrances. The components of the diagonal matrix  $\mathcal{L}$  are the products of the channel radii  $a$  and the logarithmic derivatives at  $r=a$  of  $r$ -times the radial parts of the emerging wave functions of the form  $(G + iF)r^{-1}$ ,  $F$  and  $G$  being the regular and irregular coulomb wave functions of the various channels. The real parts  $S$  and the imaginary parts  $P$  of the components of the matrix  $\mathcal{L}$  are referred to as the shift and penetration factors, respectively, and in the standard notation for coulomb wave functions<sup>12)</sup> where  $\rho = kr$  and a prime denotes differentiation with respect to  $\rho$ , they are given by

$$S = \rho \frac{FF' + GG'}{F^2 + G^2}, \quad P = \frac{\rho}{F^2 + G^2}, \quad (4)$$

the various quantities being evaluated at  $r=a$ . The  $\gamma_{\lambda c}$  are the probability amplitudes at the channel entrances  $c$  for the  $X_\lambda$ :

$$\gamma_{\lambda c} = (\hbar^2/2M_c a_c)^{1/2} \int_S \phi_c^* X_\lambda dS \quad (5)$$

where  $\phi_c$  is the product wave function for the internal coordinates of the alpha particle, the residual nucleus, and the spherical harmonic, as described by Teichmann and Wigner, and  $S$  is the surface surrounding the internal region;  $\gamma_{\lambda c}$  is referred to as the reduced width at the level  $\lambda$  for the channel  $c$ .

In the study of an isolated resonance level, it is usually assumed that the contribution  $R^\infty$  from the other levels is constant in the vicinity of the level. In the absence of such an assumption, the collision matrix with (2) as resonance denominator is exact in  $R$ -matrix theory. It will now be shown that in alpha-decay applications where the penetration factor is extremely small, the effect of the other levels may be made negligible by suitable choice of the boundary-condition matrix components. This result is in conformity with the well-known fact that the complex wave functions describing alpha decay may be closely approximated by real standing waves.<sup>6)</sup>

The components of  $B$  are set so that  $B_c = S_c(E_\alpha)$ , where  $E_\alpha = E_\lambda$  is the real energy of the emitted alpha particle, and therefore at complex energies  $E$  in the neighborhood of  $E_\alpha$  the components of  $\bar{S}$  may be closely approximated by the first non-vanishing term of the series expansion,

$$\bar{S}(E) = (E - E_\alpha) \dot{S}(E_\alpha) \quad (6)$$

where the dot denotes differentiation with respect to energy. It is now assumed that the complex energy  $E$ , which is the zero of (2), is so close to  $E_\alpha$  that the components of the matrix  $R^\infty \mathcal{L}$  are negligible, and the width and shift matrices may therefore be approximated by



$$\begin{aligned}\frac{1}{2}I'_{\lambda}(E) &= \sum_c P_c(E_{\alpha}) \gamma_{\lambda c}^2, \\ \dot{A}_{\lambda}(E) &= (E - E_{\alpha}) \dot{A}_{\lambda}(E_{\alpha}), \\ \dot{A}_{\lambda}(E_{\alpha}) &= -\sum_c \dot{S}(E_{\alpha}) \gamma_{\lambda c}^2.\end{aligned}\tag{7}$$

The results obtained confirm the assumption. The pole of the collision matrix is then the energy  $E$  which satisfies the equation

$$E_{\alpha} - E + (E - E_{\alpha}) \dot{A}_{\lambda} - \frac{1}{2} i I'_{\lambda}(E_{\alpha}) = 0$$

or

$$E = E_{\alpha} - \frac{1}{2} i I'_{\lambda}(E_{\alpha}) / (1 - \dot{A}_{\lambda})^{-1}.\tag{8}$$

It is shown in Appendix III that the quantity  $\dot{A}_{\lambda}$  is less than  $10^{-2}$  in alpha decay and therefore may be neglected. The disintegration rate  $\lambda$  then reduces to the familiar expression

$$\lambda = \hbar^{-1} I'_{\lambda}(E_{\alpha}).\tag{9}$$

The presence of the term  $\dot{A}_{\lambda}$  in (8), which may be unfamiliar, essentially normalizes the wave function of the system within the volume bounded in the vicinity of the classical turning points rather than within the volume bounded by the channel radii  $a_c$ , as in the normalization of the  $X_{\lambda}$  and  $\gamma_{\lambda c}$ . As a result of the strong alpha-decay barriers, the difference between these two normalizations is very small. However, it may be of importance in the study of resonances in light nuclear collisions where the barriers are weak.<sup>13)</sup> This distinction has been discussed by Breit<sup>11)</sup> and by Teichmann and Wigner.<sup>5)</sup>

Breit and his collaborators have found both the JWKB and steepest-descent approximations to the irregular function  $G$  to be accurate within a percent in a number of typical alpha-decay evaluations.<sup>12)</sup> The various formulas of the JWKB approximation which are used are given in Appendix I. The partial width for the decay through an alpha-emitting channel of relative angular momentum  $L$  may be written as

$$\frac{1}{2} I'_{\lambda L} = P_L \gamma_{\lambda L}^2 \approx \zeta_L \exp(-2C_L) \gamma_{\lambda L}^2.\tag{10}$$

By means of Eq. (29) of Appendix I, one finds that in typical alpha decays that the dimensionless value of the shift factor  $S$ , and likewise the boundary condition  $B$ , for the alpha-decay channel lies between  $-17$  and  $-21$ . This shows that at  $r=a$  the radial part of the wave function is close to a node,\* and it is therefore more convenient to use a reduced width  $\partial_{\lambda L}$  for the derivative of the radial wave function rather than the reduced width  $\gamma_{\lambda L}^2$  for the value; the relation is

$$\partial_{\lambda L}^2 = B_L^2 \gamma_{\lambda L}^2 \approx \zeta_L^2 \gamma_{\lambda L}^2\tag{11}$$

if we use the approximate relation (29b) for  $B_L = S_L$ , and the width therefore becomes

\* This distinctive feature of alpha decay was noted in a personal communication from G. Breit which appears in the paper by Kaplan.<sup>10)</sup>

$$\begin{aligned}\frac{1}{2}\Gamma_{\lambda L} &= \zeta_L S_L^{-2} \exp(-2C_L) \delta_{\lambda L}^2 \\ &\approx \zeta_L^{-1} \exp(-2C_L) \delta_{\lambda L}^2.\end{aligned}\quad (12)$$

In terms of the product function  $\phi_c$ , the formal definition is

$$\delta_{\lambda c} = \gamma_{\lambda c} + (a_c \hbar^2 / 2M_c)^{1/2} \int_S \phi_c^* \text{grad}_n X_\lambda dS. \quad (13)$$

The numerical value of the dimensionless boundary condition is so large that little error is made by considering it to be infinite for the determination of  $\delta_{\lambda c}$ , in which case the  $X_\lambda$  become the solutions having a radial node at the channel surface where  $r=a$ , that is  $\gamma_{\lambda c}=0$ . Eq. (12) indicates that the decay width is proportional to the square of the radial derivative at the node. This approximation permits us to apply the rule of Teichmann and Wigner<sup>5)</sup> which assumes that the kinetic energy of the alpha particle on the nuclear surface is equal to or less than its average  $T_\alpha$  within the nucleus,

$$\delta_{\lambda \alpha}^2 \lesssim T_\alpha, \quad (14)$$

the approximate equality being used if a two-body description is valid and the inequality being applicable to the many-body case. Although it is beyond the scope of the present theory to specify precisely the value of  $T_\alpha$ , it is expected to be of the order of magnitude of a few Mev, as indicated below.

The question of a many-body description for alpha decay has been discussed by a number of writers.<sup>8,9)</sup> For any model that is assumed for the determination of the  $X_\lambda$  and  $\delta_{\lambda c}$ , the shell model for example, Eq. (13) gives the reduced width. Its general evaluation is unfortunately rather difficult. As a result the usual procedure in the study of alpha-decay data has been to assume that the two-body description applies, specifically the square well model for which

$$\delta_{\lambda c}^2 = \pi^2 \hbar^2 / M_c a_c^2 \approx 1 \text{ Mev} \quad (15)$$

as shown in the next section, and to use the width formula to determine the channel radius. It is well known that this procedure probably gives a lower limit of  $a$ . It would be desirable to be able to determine the  $\delta_{\lambda c}^2$  in some other way so that the decay rates could be used to give the actual radii. In view of the general usage of the square well as a model for the alpha-particle-residual-nucleus interaction, it is treated in some detail in the next section, in particular with regard to the determination of the dependence of the reduced width on  $L$ . More complicated two-body potentials, which usually require detailed numerical evaluations, will not be discussed.

From the theories of Feshbach, Peaslee and Weisskopf,<sup>14)</sup> of Teichmann and Wigner,<sup>5)</sup> and of Bohr and Mottelson<sup>7)</sup> one may infer that if the many-body description applies, the average reduced width  $\delta_{\lambda \alpha}^2$  is obtained by multiplying the upper-limit value  $T_\alpha$  by the ratio of the actual mean level spacing  $D$  to the characteristic single-particle level spacing  $W = \pi \hbar^2 K / M a$  where  $K \sim 10^{13} \text{ cm}^{-1}$  is the wave number characteristic of a nucleon of mass  $M$  within the nucleus,

$$\langle \delta_{\lambda\alpha}^2 \rangle_{AV} \approx q_\alpha (D/W) T_\alpha; \quad (16)$$

$q_\alpha \sim 1$  is a factor for the probability of the existence of alpha particles on the nuclear surface. From the presumption that the cross section for the absorption of fast alpha particles in collisions with heavy nuclei is about equal to the cross sectional area of the heavy nucleus, it is usually inferred that  $q_\alpha \approx 1$ . The fallacy of this argument is that part of the absorption cross section may be due to non-compound-nucleus processes to which the usual argument that the compound nucleus decays independently of its mode of formation does not apply. However, some experimental data appear to indicate that the intrinsic emission probabilities  $q$  do not differ significantly from unity for alpha particles and tritons as well.<sup>17)</sup>

### § 3. The square well model

The wave function of a particle with angular momentum  $L$  interacting with a square well potential  $V_0$  is the half-integer Bessel function

$$F_L(z) \equiv (\pi z/2)^{1/2} J_{L+1/2}(z) \quad (17)$$

where

$$z = [2M(E - V_0)/\hbar^2]^{1/2} r.$$

For these functions the boundary condition  $B_L$ , which is  $\alpha$ -times the logarithmic derivative, is

$$B_L = z F_L(z)' / F_L(z) \quad (18)$$

at  $r = a$ ; the prime denotes differentiation with respect to  $z$ . It is convenient to introduce a dimensionless reduced width

$$\theta_{\lambda L}^2 = (Ma^2/\hbar^2) \gamma_{\lambda L}^2. \quad (19)$$

If  $B_L$  is now regarded as the energy-dependent logarithmic derivative, the Green's-theorem relation<sup>18)</sup> gives

$$-\theta_{\lambda L}^{-2} = (\hbar^2/Ma^2) (dB_L/dE) = z^{-1} B_L(z)'. \quad (20)$$

By means of the Bessel equation

$$F_L'' + [1 - L(L+1)z^{-2}] F_L = 0, \quad (21)$$

one finds that

$$\theta_{\lambda L}^{-2} = [(B_L + L)(B_L - L - 1) + z^2]/z^2. \quad (22a)$$

Therefore,

$$\delta_{\lambda L}^2 = \frac{\hbar^2}{Ma^2} \cdot \frac{z_L^2 B_L^2}{(B_L + L)(B_L - L - 1) + z_L^2} \quad (22b)$$

where  $z_L$  is the solution to the transcendental equation (18), which may be expressed alternatively as

$$B_L = z_L F_{L-1}(z_L) F_L(z_L)^{-1} - L \quad (23)$$

by means of the recursion relation for  $F_L'$ . As indicated in § 2, in alpha decay the boundary condition is

$$B_L = \rho(G_L'/G_L) \approx -\zeta_L, \quad (24)$$

a large quantity. As an approximation one may therefore assume it to be infinite, obtaining

$$\delta_{\lambda L}^2 \approx (\hbar^2/Ma^2) z_L^2 \quad (25)$$

where  $z_L$  is the first zero of the tabulated function  $j_{L+1/2}(z) = z^{-1} F_L(z)$ .<sup>16)</sup> The accuracy of this approximation is easily assessed. With  $\mu=9.42$ ,  $\eta=25.4$ , which corresponds to the emission of a 5-Mev alpha particle by uranium from a radius  $a=9.6 \times 10^{-13}$  cm, one finds that  $B_0=-18.96$ ,  $\Gamma_0(\text{approx.})/\Gamma_0(\text{exact})=1.16$ , and  $B_5=-20.20$ ,  $\Gamma_5(\text{approx.})/\Gamma_5(\text{exact})=1.25$ . In view of the approximate nature of the square well model, Eq. (25) would appear to be sufficiently accurate.

The approximation (25) indicates that the reduced widths  $\delta_{\lambda L}^2$  are simply proportional to the square of the roots of the half-integer Bessel functions. In the case  $L=0$ , the first zero is  $z_0=\pi$  so that the decay rate becomes

$$\lambda = \hbar^{-1} \Gamma = (2\hbar/Ma^2) \pi^2 \zeta_0^{-1} \exp(-2C_0), \quad (26)$$

which is identical to Bethe's Eq. (587)<sup>11)</sup> and to the approximation to Preston's exact expressions, as given by Rasmussen's Eq. (24).<sup>21)</sup> Table I lists the squares of the first zeros for  $L$  in the range 0(1)5.

Table I

The squares of the zeros  $z_L$  of the Bessel functions  $J_{L+1/2}(z)$  are listed. The square well alpha-decay reduced widths for the various angular momenta  $L$  are very nearly proportional to them.

$L$	0	1	2	3	4	5
$z_L^2$	9.9	20.2	33.2	48.7	66.9	87.8

The reduced width  $\delta_{\lambda L}^2$  thus increases by a factor of nine as  $L$  increases from zero to five. This increase will partially compensate for the decrease of the decay rate due to the increase of the centrifugal barrier. Table II lists for the same example  $\zeta_L^{-2}$  times the penetration factor for angular momentum  $L$  divided by that for  $L=0$ , and the corresponding ratios of decay rates. The decay rate increases with  $L$  with a maximum at  $L=2$  and then slowly decreases. Such a behavior was first noticed by Preston,<sup>6)</sup> whose results are presumably the same as the exact treatment given here, although they appear in a considerably more complicated form. An interpretation of the increase of the reduced widths with  $L$  has also been given by Preston. The larger is  $L$ , the larger is the centrifugal repulsion near the origin. This repulsion must be compensated by an increase of the well depth  $|V_0|$ , and as a result the kinetic energy of the particle is larger at the surface where the centrifugal repulsion is not so large. However, this effect has not been confirmed experimentally.<sup>17)</sup>

Table II

The ratios of the penetration factor  $\zeta_L^{-1}\exp(-2C_L)$  for angular momentum  $L$  to  $\zeta_0^{-1}\exp(-2C_0)$  for  $L=0$  and the ratios of the corresponding decay rates are listed for the case  $\rho=9.42$ ,  $\eta=25.4$ , corresponding to the emission of a 5-Mev alpha particle by uranium from a radius  $a=9.6\times 10^{-13}\text{cm}$ .

$L$	0	1	2	3	4	5
$\frac{\zeta_L^{-1}\exp(-2C_L)}{\zeta_0^{-1}\exp(-2C_0)}$	1.000	.854	.588	.354	.181	.082
$\frac{\Gamma_{\lambda L}}{\Gamma_{\lambda 0}}$	1.00	1.75	1.98	1.75	1.21	0.73

#### § 4. The effect of an external nuclear potential on barrier penetration

It is of interest to consider the effect of an extended nuclear force potential on the usual barrier penetration formula. An estimate may easily be made for an exponential potential. The modified irregular wave function  $g(r)$  may be expressed in terms of the unmodified function  $G(r)$  and the additional potential  $V(r)=\hbar^2 P(r)/2M$  as<sup>18)</sup>

$$g(r) = G(r)[1 + \mathcal{A}(r)] \quad (27)$$

where

$$\mathcal{A}(r) = \int_r^\infty G(r')^{-2} dr' \int_{r'}^\infty \bar{P}(r'') G(r'')^2 dr'',$$

$$P(r) = [g(r)/G(r)]P(r).$$

In the case of an attractive potential,  $P$  is negative, and if it does not completely annul the barrier potential,  $0 < g(r) < G(r)$ . To obtain an upper bound for  $|\mathcal{A}|$  one may therefore replace  $\bar{P}$  in (27) by  $P$ .

We consider the exponential with the range  $r_0$ :

$$V = V_0 \exp(-r/r_0).$$

In the range of radii wherein the main contributions to  $\mathcal{A}$  occur,  $G(r)$  may be replaced by an exponential  $\exp[-(\zeta/\rho)kr]$  having an appropriate mean  $(\zeta/\rho)$ . By carrying out the integrations one finds that

$$\mathcal{A}(r) = [2M_1'(r)r^2/\hbar^2]/[2(\zeta/\rho)\rho - (r/r_0)](r/r_0).$$

Inserting typical values,  $\bar{\zeta}=24$ ,  $(\rho/\bar{\rho})=0.7$ ,  $r=10^{-12}\text{cm}$ ,  $r_0=1/3\cdot r$ , one finds  $\mathcal{A}(r) \approx 1/4 \cdot P(r)$ ,  $P(r)$  being measured in Mev. Thus for the particular range value used, the potential must be less than about 1 Mev if its effect is to be unimportant. The statement made in § 2 concerning the determination of a lower limit for the alpha-decay radius may now be made somewhat more precise: It is the lower limit beyond which the nuclear forces are reduced to the order of magnitude of 1 Mev or less.



## Appendix I

### The JWKB formulas

Although the JWKB penetration formula has been given many times in the literature, it appears desirable to give it here in the standard notation for coulomb wave functions.<sup>12)</sup> For angular momentum  $L$  it is\*

$$P_L \approx \rho / G_L^2 = \zeta_L \exp(-2C_L) \quad (28)$$

where

$$\zeta_L = (l^2 + \frac{1}{4}x^2 - \rho^2)^{1/2},$$

$$C_L = \eta \left[ \frac{\pi}{2} + \sin^{-1} \frac{(\eta - \rho)}{(\eta^2 + l^2)^{1/2}} \right] - \zeta_L + l \cdot \log \frac{l\zeta_L + l^2 + \frac{1}{8}x^2}{\rho(\eta^2 + l^2)^{1/2}},$$

$$l = L + \frac{1}{2}, \quad x^2 = 8\rho\eta,$$

$$\eta = Z_\alpha Z_r e^2 / \hbar v, \quad \rho = Mva / \hbar,$$

$v$  being the relative velocity and  $M$  the reduced mass; the  $\sin^{-1}$  lies between  $-\pi/2$  and  $\pi/2$ . The Langer modification of replacing  $L$  by  $L + \frac{1}{2}$  has been used.<sup>12,19),†</sup> In the same approximation the shift factor is given by

$$S_L \approx \rho(G_L'/G_L) = -\zeta_L + \frac{1}{2}\zeta_L^{-2}(\frac{1}{8}x^2 + l^2), \quad (29a)$$

and its rate of change with respect to energy can be obtained from

$$(\partial S_L / \partial \rho^2)_{\rho\eta} \approx \frac{1}{2}\zeta_L^{-4}(\zeta_L^3 + \frac{1}{8}x^2 + l^2). \quad (30)$$

In alpha-decay applications, the second term on the right side of (29a) is small compared with the first so that

$$S_L \approx -\zeta_L. \quad (29b)$$

From the recent compilation of atomic constants by DuMond and Cohen<sup>20)</sup> the following numerical evaluations are obtained:

$$x^2 = 0.2776 MZ_\alpha Z_r a,$$

$$\eta = 0.15805 Z_\alpha Z_r M^{1/2} E_\alpha^{-1/2} (M_r / M_r + M_\alpha)^{1/2},$$

$$\rho = 0.2195 M^{1/2} E_\alpha^{1/2} a (M_r / M_r + M_\alpha)^{-1/2}$$

where

$M_\alpha$  is the alpha-particle mass in a unit of the proton mass,

$M_r$  is the mass of the recoil system in the same unit,

\* Our penetration factor differs by the factor  $\zeta_L$  from Bethe's<sup>9)</sup> barrier penetrability  $P_L$ , as given by his eq. (632a).

† The Langer modification is actually not significant in alpha-decay applications. It is included in the present JWKB formulas so that they may also be used in applications involving relatively weak coulomb barriers where it may be significant. In the unmodified form,  $l = [L(L+1)]^{1/2}$ .

$M = M_\alpha M_r / (M_\alpha + M_r)$  is the reduced mass in the same unit,

$Z_\alpha = 2$  is the charge number of the alpha particle,

$Z_r$  is the charge number of the recoil nucleus,

$a$  is the channel radius in units of  $10^{-13}$  cm,

and  $E_\alpha$  is the energy in Mev of the alpha particle in the laboratory system, corrected for atomic electron screening and interaction effects, as indicated in Appendix II.

## Appendix II

### The electron screening and interaction corrections

The coulomb field of the nucleus is partially screened by the atomic electrons. According to the observation by Foldy,<sup>(21)</sup> in the vicinity of the nucleus the potential energy of an alpha particle in the atomic electron field of the initial nucleus of charge number  $Z_i = Z_\alpha + Z_r$  is given by  $65.3 Z_i^{7/5}$  e.v. A very fast alpha particle would lose this much kinetic energy in passing through the electron cloud. However, in alpha-decay applications the alpha-particle velocity is slower than most of the atomic orbital velocities. Therefore, an adiabatic correction term should be included for the fraction  $f$  of principal atomic shells, the electron orbital velocities of which are greater than the alpha-particle velocity. According to the results of Serber and Snyder,<sup>(22)</sup> the above potential energy should accordingly be reduced by an amount which is approximately given by  $91.4 Z_i^{2/5} f$  e.v. As the effective alpha-particle-electron interactions occur beyond the classical turning point in most applications, the JWKB penetration formulas as given in Appendix I may be used by simply substituting for  $E_\alpha$  the corrected quantity

$$E_\alpha + (M_r / M_r + M_\alpha) (65.3 Z_i^{7/5} - 91.4 Z_i^{2/5} f) \cdot 10^{-6} \text{Mev.} \quad (31)$$

If the alpha-particle energy is very small so that the classical turning point (in the case of  $l=0$  it is the value of the radius where  $\mu=2\epsilon_l$ ) occurs in the vicinity of some of the electron orbits, then a more detailed evaluation of the penetration is required.

It is possible to make a sufficiently accurate estimate of the fraction  $f$  by using Slater's atomic shielding constants.<sup>(23)</sup> The electron orbital velocity is usually considered as given by  $(Z-s)e^2/\hbar n^*$  where  $s$  is the screening constant and  $n^*$  is the effective quantum number. As a typical example, a 4.5-Mev Th alpha particle would have a velocity equal to that of an orbital electron for which  $(Z-s)/n^* \approx 7$ . For the outer orbits  $n^* \approx 4.0$  so that  $s \approx 93-28$ , which is only slightly more than the number of electrons filling the first four principal shells. Since the effective number of filled principal shells in Th is 4.6, the fraction  $f \approx 0.87$ . The adiabatic correction term, 480 e.v., is always small compared with the main screening term, 36.2 kev, both for the emission of an alpha particle from Th, and consequently it is frequently ignored.<sup>(21)</sup> It should also be mentioned that the adiabatic correction term is the average value for a distribution function; the strong dependence of the barrier penetration on the decay energy may require an effective value differing somewhat from the average.

### Appendix III

#### Upper-limit estimate of the correction term $-\dot{J}_{\lambda L}(E_\alpha)$ of eq. (8)

The correction term for the case of only one channel of angular momentum  $L$  is given by

$$-\dot{J}_{\lambda L}(E_\alpha) = \dot{S}_L(E_\alpha) \gamma_{\lambda L}^2 = (\dot{S}_L/S_L^2) \delta_{\lambda L}^2. \quad (32)$$

According to (30) and (29b)

$$\dot{S}_L \approx (Mr^2/\hbar^2) \zeta_L^{-1}$$

since  $\zeta_L^3$  is large compared with  $(\frac{1}{8}A^2 + l^2)$ . For an upper-limit estimate of  $-\dot{J}_{\lambda L}$  one may use the results of § 3 for the square well where  $(Mr^2/\hbar^2) \delta_{\lambda L}^2 \approx \varepsilon_L^2$ ,  $\varepsilon_L^2$  being listed in Table I, with the result

$$-\dot{J}_{\lambda L}(E_\alpha) \approx \varepsilon_L^2 \zeta_L^{-3}. \quad (33)$$

Since  $\zeta_L \approx 20$ , this term is negligible even though  $\varepsilon_L^2$  becomes almost as large as 100 when  $L=5$ .

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# A Hydrodynamical Description of Many Bose Particle Systems

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The hydrodynamical description of an interacting many boson system given in the foregoing papers is studied more carefully by making use of the so-called hydrodynamical approximation in the lattice space that the expectation value of the particle number in a cell is very large. We get the commutation relation of the velocity operator due to Landau and Thellung and the roton spectrum is computed. The phonon spectrum is obtained in the relation to the ground state of the system and the sound speed is evaluated for the rigid sphere problem. Further the relationship to the collective description of Bohm and Pines is discussed.

## § 1. Introduction

It has been confirmed by the recent experiments that Landau's theory<sup>1)</sup> of Helium II is successful in attacking both the equilibrium and the dynamical properties at very low temperatures such as the heat capacity, the entropy, the concentration of the normal fluid, the viscosity and the heat conductivity. His method consists in dealing with hydrodynamics to make the quantization analogously with quantum electrodynamics. The most satisfactory account of it has been given by several authors<sup>2,3,4)</sup> by applying Clebsch's transformation. This classical standpoint has been examined from the view point of the quantized wave theory (second quantization) in the foregoing papers referred to as I.<sup>4)</sup>

In Landau's theory there appear two kinds of excitations, the phonon and the roton. Their spectra, however, are determined by experimental data. It is expected that the scheme of the quantized wave theory can, in principle, give these spectra from the true interaction energy of atoms.

In the next section we derive the commutation relation of the velocity operator given by Landau and Thellung from the approximate commutation relation of the phase operator. This approximation is called "hydrodynamical". In Section 3 the incompressible case is considered and it is shown that the free particle energy is expressed by the hydrodynamical hamiltonian given in respect to the solenoidal component of the velocity operator and it corresponds to the energy of rotons which may appear as the wave number of the excitation becomes large. In Section 4 the compressible case is considered. The phonon hamiltonian is obtained in relation to the ground state of the system and it serves a good approximation of the total hamiltonian for small wave numbers of the excitation. In this case the velocity field becomes irrotational. The phonon represents the longitudinal motion of the assembly, while the roton represents the transverse motion of it. In Section 5 we consider the case that the velocity operator has two components of the solenoidal and



the irrotational part. It is shown that the field variables  $\alpha$  and  $\beta$  appearing in Clebsch's transformation to represent the vortex line are given by the phase operator or the wave function. In Section 6 the relationship between the hydrodynamical description and the collective description, which has been formulated by Bohm and Pines and applied to boson assemblies in Paper I, is computed.

## § 2. Hydrodynamical approximation

We try to investigate the commutation relations between the phase operators obtained from those of the wave functions. Following the conventional treatment of the field quantization we consider a lattice space and the field quantities become the functions of the lattice point  $X$ . The lattice wave functions are written as

$$\varphi_L(X) = \sqrt{\Delta} \varphi(X), \quad \varphi_L^*(X) = \sqrt{\Delta} \varphi^*(X), \quad (2.1)$$

where  $\Delta$  is the cell volume of the lattice space and we get

$$\left. \begin{aligned} [\varphi_L(X), \varphi_L(X')] &= [\varphi_L^*(X), \varphi_L^*(X')] = 0, \\ [\varphi_L(X), \varphi_L^*(X')] &= \delta_{XX'}, \end{aligned} \right\} \quad (2.2)$$

$$\varphi_L^*(X) \varphi_L(X) = N(X), \quad (2.3)$$

$$[N(X), \varphi_L(X')] = -\varphi_L(X) \delta_{XX'}, \quad (2.4)$$

$$[N(X), \varphi_L^*(X')] = \varphi_L^*(X) \delta_{XX'}. \quad (2.5)$$

The lattice phase operators are defined by  $\psi_L(X) = \varphi_L(X) / \sqrt{N(X)}$ ,  $\psi_L^*(X) = \sqrt{N(X)} \varphi_L^*(X)$  and the commutation relations of the phase operators become

$$[\psi_L(X), \psi_L(X')] = [\psi_L^*(X), \psi_L^*(X')] = 0, \quad (2.7)$$

$$[\psi_L(X), \psi_L^*(X')] = \varepsilon(X) \Delta \delta_{XX'}, \quad (2.8)$$

$$\psi_L(X) \psi_L^*(X) = \Delta,$$

where  $\varepsilon(X)$  is the projection operator which is zero if  $N(X)$  is not zero and is one if  $N(X)$  is zero as mentioned in I. Here we introduce an operator defined by

$$v_L(X) = (\hbar/2mi) \{ \psi_L^*(X) d(\psi_L(X)) - d(\psi_L^*(X)) \psi_L(X) \}, \quad (2.9)$$

$$d(\psi_L(X)) = \{ \psi_L(X+d) - \psi_L(X) \} / d \text{ etc.} \quad (2.10)$$

in which  $X+d$  is one of the nearest neighboring lattice points to  $X$ . With the aid of the identity:

$$\begin{aligned} \Delta d(\varepsilon(X)) &= -d(\psi_L^*(X)) \psi_L^*(X+d) - \psi_L^*(X) d(\psi_L(X)) \\ &= -d(\psi_L^*(X)) \psi_L^*(X) - \psi_L^*(X+d) d(\psi_L(X)) \end{aligned}$$

we get the commutation relation of  $v_L$ .

$$[v_L(X), v_L(X')]$$

$$\begin{aligned} &= (\hbar^2/4m^2) [ \{ d(\phi_L^*) \phi_L(X+d) + d(\phi_L^*) \phi_L \} \{ \phi_L^* d'(\phi_L) + \phi_L^*(X+d') d'(\phi_L) \} \\ &- \{ d'(\phi_L^*) \phi_L(X+d') + d'(\phi_L^*) \phi_L \} \{ \phi_L^*(X+d) d(\phi_L) + \phi_L^* d(\phi_L) \} \\ &+ \Delta d(\varepsilon) \{ \phi_L^* d'(\phi_L) - d'(\phi_L^*) \phi_L \} + \{ \phi_L^* d'(\phi_L) - d'(\phi_L^*) \phi_L \} \Delta d(\varepsilon) \\ &- \Delta d'(\varepsilon) \{ \phi_L^* d(\phi_L) - d(\phi_L^*) \phi_L \} - \{ \phi_L^* d(\phi_L) - d(\phi_L^*) \phi_L \} \Delta d'(\varepsilon) ] \partial_{XX'}. \quad (2.11) \end{aligned}$$

When the volume of the cell is so large that it involves at least one particle the value of  $\varepsilon(X)$  is set to be zero, but on taking the limit of  $v_L$ ;  $d \rightarrow 0$ , there appear several cells in which  $\varepsilon$  has value one. Let the 'atomic volume' be  $\Delta_0 = d_0^3$ . And if  $d$  becomes smaller than  $d_0$  we replace  $\varepsilon(X)$  and  $N(X)$  by the average ones. Then  $\varepsilon$  is set everywhere to be zero and the limit of the commutation relation becomes

$$[v_j(x), v_k(x')] = (\hbar/m)^2 [\text{grad } \phi^*(x) \times \text{grad } \phi(x)]_{jk} \partial(x-x') / \rho_0(x) \quad j, k = 1, 2, 3, \quad (2.12)$$

where  $\rho_0(x) = 1/\Delta_0$ . This is identical with Landau's expression except that  $\rho_0$  is a c-number here.  $1/\Delta_0$  may be replaced by  $N/\Delta = \rho$  if the particle distribution is nearly uniform. Further we get the following commutation relation:

$$[\rho(x), v(x')] = \frac{i\hbar}{m} \text{grad } \partial(x-x'), \quad (2.13)$$

which is just identical to the exact one except for the projection operator (see (3.8) in I). As in the exact case, it is shown that the rotation of the velocity operator is commutative with the density operator:

$$\text{rot } v(x) = (\hbar/mi) \text{grad } \phi^*(x) \times \text{grad } \phi(x), \quad (2.14)$$

$$[\rho(x), \text{rot } v(x')] = 0. \quad (2.15)$$

Introducing the current operator of the form (see Appendix 1)

$$J_k(x) = \frac{1}{2} (\rho(x) v_k(x) + v_k(x) \rho(x)), \quad (2.16)$$

we get

$$\begin{aligned} [J_k(x), J_l(x')] &= i\hbar/m \cdot [\text{rot } J]_{kl} \\ &+ \frac{i\hbar}{m} \left\{ J_k(x) \frac{\partial}{\partial x'_l} \partial(x'-x) - J_l(x') \frac{\partial}{\partial x_k} \partial(x-x') \right\} \\ &= \frac{\hbar}{mi} \left\{ J_k(x') \frac{\partial}{\partial x_l} \partial(x-x') - J_l(x') \frac{\partial}{\partial x'_k} \partial(x'-x) \right\}, \quad (2.17) \end{aligned}$$

which is identical with the foregoing result.<sup>5)</sup>

The physical meaning of this approximation can be interpreted as follows. We consider a considerably large cell to allow to confine several atoms, say  $N$ , and take the limit that  $\Delta$  becomes infinitely small keeping  $N/\Delta$  unchanged. Such a limiting process

is possible only if  $N$  is very large and the particle distribution in the cell is uniform. In fact, if  $N$  is small or the distribution is not uniform, there appear regions such that the particle is not contained in them and the meaning of the limit becomes indeterminate. This is the reason why this approximation is called "hydrodynamical", since the objective of hydrodynamics is *a priori* restricted to the cases mentioned above.

### § 3. Incompressible case

We consider the case that the expectation value of the density is constant everywhere and the kinetic energy of free particles  $K$  is given in terms of the phase operator :

$$K = D \frac{\hbar^2}{2m} \int \text{grad } \psi^*(x) \cdot \text{grad } \psi(x) dx. \quad (3.1)$$

Hence the equation of continuity becomes

$$\frac{\partial \psi^* \psi}{\partial t} + \text{div } v = \text{div } v = \text{div } J / D = 0, \quad (3.2)$$

and

$$\text{grad } \psi^* \psi = 0, \quad \psi^* \psi = 1, \quad (3.3)$$

since  $\psi^* \psi$  is a constant of motion. The equation (3.2) means that the velocity is solenoidal and making use of (3.3) we find that (3.1) is expressed by the hydrodynamical hamiltonian of the form

$$K = \frac{mD}{2} \int v^2 dx. \quad (3.4)$$

Letting the fourier coefficients of the wave functions be

$$a(k) = \frac{1}{\sqrt{V}} \int \varphi(x) \exp(-ikx) dx, \quad (3.5)$$

$$a^*(k) = \frac{1}{\sqrt{V}} \int \varphi^*(x) \exp(ikx) dx, \quad (3.6)$$

(3.4) becomes

$$K = \frac{mN}{2} \sum_k v(k) v(-k), \quad N = DV, \quad (3.7)$$

$$v(k) = \frac{1}{V} \int v(x) \exp(-ikx) dx = \frac{\hbar}{mN} \sum_l l a^* \left( l - \frac{k}{2} \right) a \left( l + \frac{k}{2} \right), \quad (3.8)$$

From (3.2) we get

$$\frac{k}{|k|} \cdot v(k) = \frac{\hbar}{mN} \sum_l \frac{(l \cdot k)}{|k|} a^* \left( l - \frac{k}{2} \right) a \left( l + \frac{k}{2} \right) = 0. \quad (3.9)$$

Therefore (3.7) takes the form

$$K = \frac{\hbar^2}{2mN} \sum_k \left\{ \sum_l \left( l - \frac{(l \cdot k)}{|k|^2} k \right) a^* \left( l - \frac{k}{2} \right) a \left( l - \frac{k}{2} \right) + \sum_l \sum_{l'} \left( l - \frac{(l \cdot k)}{|k|^2} k \right) \cdot \left( l' - \frac{(l' \cdot k)}{|k|^2} k \right) a^* \left( l - \frac{k}{2} \right) a^* \left( l' + \frac{k}{2} \right) a \left( l + \frac{k}{2} \right) a \left( l' - \frac{k}{2} \right) \right\}. \quad (3.10)$$

Since  $K$  is commutative with  $N(k) = a^*(k)a(k)$ , the non-diagonal terms in (3.10) ought to vanish and (3.10) turns out into

$$K = \frac{\hbar^2}{2mN} \sum_{l,k} \left[ l^2 - \frac{(l \cdot k)^2}{k^2} \right] N(l) + \frac{\hbar^2}{2mN} \sum_{l,k} \left[ l^2 - \frac{(l \cdot k)^2}{k^2} \right] N(l) N(l+k). \quad (3.11)$$

The eigenvalue of (3.11) becomes infinity, since it involves the sum over the momentum space;  $\sum_k$ , but the eigenvalue of (3.11) must be identical with that of (3.1), hence the integration should be done within the limit  $k=k_0$  which depends on the distribution of particles in the momentum space.

#### § 4. Compressible and irrotational case

As mentioned in I, the hamiltonian of sound waves serves a good approximation for long wave lengths in dealing with a weak short range repulsive interaction energy. This method of the linearization to ignore the higher order unharmonic terms than the third order in the density fluctuation is extensible to the more general case of strong repulsive forces, *i. e.* nearly the rigid sphere. In our foregoing treatment as well as Bogolubov's<sup>2)</sup> the difficulty in treating the rigid sphere appears connected with the region that the expectation value of the density becomes nearly equal to zero. We avoid this difficulty by assuming the ground-state distribution of particles and considering small variations of the density about the ground-state distribution.

Here we consider excited states being orthogonal to the wave function  $\chi_0$  of the ground state :

$$H\chi_0 = E_0\chi_0. \quad (4.1)$$

The approximation consists in expanding the functionals of the density quantity in power series of the density fluctuation  $\rho'$ . For instance

$$\rho = \rho_0 + \rho', \quad (4.2)$$

$$\rho_0 = (\chi_0, \rho \chi_0), \quad (4.3)$$

$$\nabla \rho = (\nabla \rho)_0 + \nabla \rho', \quad (4.4)$$

$$(\nabla \rho)_0 = (\chi_0, \nabla \rho \chi_0) = -(\chi_0, \rho \nabla \chi_0), \quad (4.5)$$

where  $(\nabla \rho)_0$  is not identical to  $\nabla \rho_0$  of course. First we expand the energy density written in terms of the amplitude and the velocity operator in the power series of  $\rho'/\rho_0$ . Especially, the quantum mechanical potential is expressed by

$$\hbar^2/2m \cdot |\nabla R|^2 = (\hbar^2/8m\rho_0) \cdot |\nabla \rho'|^2 - (\hbar^2/8m\rho_0^2) \cdot \rho' \cdot |\nabla \rho'|^2 + \dots, \quad (4.6)$$

and the total energy is given in terms of the fourier coefficients as follows :

$$H = \frac{1}{2m} \sum_{k,l} (k \cdot l) \rho(k-l) \phi(k) \phi(-l) + \frac{m}{2N} \sum_k \left\{ \frac{\hbar^2 k^2}{4m^2} + \frac{N}{m} G(k) \right\} \rho(k) \rho(-k) \\ + \frac{\hbar^2}{8mN^2} \sum_{k,l} (k \cdot l) \rho(k) \rho(l) \rho(-k-l) + \frac{N^2}{2} G(0) - \frac{N}{2} G_0 - \frac{\hbar^2}{4m} \sum k^2, \quad (4.7)$$

up to the third order terms, and

$$\rho(k) = \int_V \rho(x) \exp(-ikx) dx, \quad (4.8)$$

$$\phi(k) = \frac{1}{V} \int_V \phi(x) \exp(ikx) dx, \quad (4.9)$$

$$[\rho(k), \phi(l)] = i\hbar \delta_{kl}. \quad (4.10)$$

If the interaction is weak the eigenfrequency is given by  $\omega(k) = |k| [\hbar^2 k^2 / 4m^2 + N/m \cdot G(k)]^{1/2}$  which is identical with Bogolubov's. But in cases that the interaction is so strong that atoms tend to keep separate with one another, the contribution of the quantum mechanical potential to the eigenfrequency ought to be different from Bogolubov's which is given in Paper I by our independent calculation. Here we go over to the  $N$ -th configuration space in which the representation of the fourier coefficients  $\rho(k)$  and  $\phi(k)$  are composed and we restrict ourselves to the density fluctuation about the ground state. Let the coordinate of the  $j$ -th particle in the ground state be  $x_{0j}$ . After receiving perturbations it becomes  $x_{0j} + \xi_j$  and  $\rho(k)$  is expressed by

$$\rho(k) = \sum_{j=1}^N \exp(-ikx_{0j}) \exp(-ik\xi_j) = \rho_0'(k) + q(k), \\ \rho_0'(k) = \sum_{j=1}^N \exp(-ikx_{0j}), \quad q(k) = -ik \sum_{j=1}^N \xi_j \exp(-ikx_{0j}). \quad (4.11)$$

In the classical sense of words the  $j$ -th particle moves along the path described by  $x_{0j}$ , but in our case  $x_{0j}$  is meant for a parameter of the prescribed 'path-ensemble' which is given by the ground state wave function. We try to calculate the expectation value of the quantum mechanical potential taking the expectation value with respect to  $x_{0j}$ . From the fourth order term in the expansion (4.6) one gets the coefficient of  $|q|^2$  in the form

$$I'_0 = (\hbar^2/8m) \cdot |F| \rho_0'^2 / \rho_0^3, \quad (4.12)$$

$$\int (x_0, I'_0 \chi_0) q^2 dx = \hbar^2/8N^3m \sum_k \sum_l \sum_p (k \cdot l) (\chi_0, \rho_0'(k) \rho_0'(-l) \chi_0) q(l-k-p) q(p) \\ = \hbar^2/8N^3m \sum_k \sum_p (\chi_0, k^2 |\rho_0'(k)|^2 \chi_0) q(-p) q(p). \quad (4.13)$$

It deserves to notice that this factor results from the classical definition of the sound speed which is given by the energy density  $f(m\rho)$  letting  $f(m\rho) = \hbar^2 |FR|^2/2m$  and one gets



$$c^2 = \partial P / \partial (m\rho) = m \rho f''(m\rho) = (\hbar^2 / 4m^2) |\nabla \rho|^2 / \rho^2 = 2\rho_0 \Gamma / m, \quad (4.14)$$

from the equation of state :

$$P = m\rho f' - f. \quad (4.15)$$

The difference comes from the denominator of  $l'$  which is not constant in this case and it is readily shown by the expansion that this expression involves the coefficients of  $|q|^2$  which result from the higher order terms than the fourth order in the expansion (4.6).

In general the sound speed  $c$  depends on the wave number. The expression given by Landau and Khalatnikov<sup>9)</sup> takes the form

$$c(k) = c_0 (1 - \gamma/2 \cdot k^2 + \dots), \quad (4.16)$$

$$\gamma = 6.2 \times 10^{-17} cm^2.$$

### § 5. Phonons and rotons

When the velocity operator has the solenoidal and the irrotational component, the total hamiltonian takes the following form

$$\left. \begin{aligned} H &= H_r + H_{\text{INT.}} + H_p \\ H_r &= \frac{m}{2} \int \rho v_r^2 dx, \quad H_{\text{INT.}} = \int J_r v_p dx, \\ H_p &= \int \left\{ \frac{m}{4} (\rho v_p^2 + v_p^2 \rho) + \frac{m}{2\rho_0} c^2 \rho^2 - \frac{\hbar^2}{4m} \sum k^2 \right\} dx, \end{aligned} \right\} \quad (5.1)$$

where

$$v_r = \hbar / 2mi \cdot [\psi^* \text{grad } \psi - \text{grad } \psi^* \cdot \psi], \quad (5.2)$$

$$J_r = \rho v_r, \quad (5.3)$$

$$v_p = 1/m \cdot \text{grad } \Phi, \quad (5.4)$$

$$\text{div } v_r = 0, \quad (5.5)$$

$$\text{rot } v_p = 0, \quad (5.6)$$

in which  $\psi$  and  $\psi^*$  satisfy  $[\psi(x), \psi^*(x')] = \delta(x-x')/\rho(x)$  and they are commutative with the density. The commutator  $[v_r, v_r]$  becomes

$$\begin{aligned} [v_r(x), v_r(x')] &= \frac{\hbar}{im} \left\{ (v_r(x')/\rho(x')) \cdot \frac{\partial}{\partial x} \delta(x-x') - (v_r(x)/\rho(x)) \cdot \frac{\partial}{\partial x'} \right. \\ &\quad \left. \cdot \delta(x'-x) \right\} + \frac{i\hbar}{m} (\text{rot } v_r(x) \delta(x-x')/\rho(x)), \end{aligned} \quad (5.7)$$

and  $[v_r, v_p]$  must be

$$[v_r(x), v_p(x')] = \frac{\hbar}{im} (v_r(x)/\rho(x)) \frac{\partial}{\partial x'} \delta(x'-x), \quad (5.8)$$

so that [C. R.]'s (2.12) and (2.13) remain valid and we are led to the incompressible case by setting  $\rho$  to be constant and  $v_p$  to be zero. These conditions are sufficient for (2.12), (2.13) and (2.17), and it proves to be necessary, since  $v_p$  is commutative with  $\rho$  and the functional form of (5.7) ought to remain valid in the incompressible case. In fact, if the expression  $[v_p, v_p]$  is different from (5.7) as a functional of  $\rho$  and  $v_p$ , the commutation relation of the incompressible case becomes different from  $[J_r(x), J_r(x')]/D^2$  which is a special case of (5.7).

Now we pursue the relationship of the present description to Clebsch's transformation. He introduced the canonical variables  $\mu, \beta$  which are subject to the Poisson bracket  $\{\mu(x), \beta(x)\} = \delta(x-x')$  and the velocity operator is expressed by  $v = (\mu/\rho) \text{grad } \beta + v_p$ . The line determined by  $(\mu/\rho) = \text{const.}, \beta = \text{const.}$  is the vortex line. In our treatment the quantity  $\alpha = (\mu/\rho)$  corresponds to  $\psi^* \psi$ . As a matter of fact, if one sets

$$\psi = (\exp i/\hbar \cdot \beta) \alpha^{1/2}, \quad \psi^* = \alpha^{1/2} (\exp -i/\hbar \cdot \beta), \quad (5.9)$$

one gets Clebsch's expression of the velocity operator except for the symmetrization.

From the hamiltonian we get

$$\partial \alpha / \partial t + 1/2 \cdot (v \cdot \text{grad } \alpha + \text{grad } \alpha \cdot v) = 0 \quad (5.10)$$

which shows that  $\alpha$  is constant along a stream line in the classical sense of words. The quantity  $\beta$  has also the similar property.

Now we consider the spectrum of the roton. Since the operator  $J_r = \rho v_r$  is no more solenoidal, the eigenvalue is more complicated than that discussed in Section 3. We write the roton energy  $II_r = \int m \rho_0 v_r^2 / 2 dx$  in terms of the wave function of the roton which is given by  $\varphi_r = \psi_r \rho^{1/2}$ ,  $a(k) = \int \varphi_r(x) \exp(-ikx) dx / V^{1/2}$  and we get

$$H_r = H_{r0} + H_{rr} + H_r', \quad (5.11)$$

$$H_{r0} = \frac{\hbar^2}{2mNV} \sum_k \sum_l \left( l + \frac{k}{2} \right) a^*(l) a(l), \quad (5.12)$$

$$H_{rr} = \frac{\hbar^2}{2mNV} \sum_k \sum_l \sum_{l'} \left( l + \frac{k}{2} \right) \cdot \left( l' + \frac{k}{2} \right) a^*(l) a^*(l' + k) a(l + k) a(l'). \quad (5.13)$$

Here we pay attention to  $II_{r0}$  involving an integro over the wave number space which amounts to the infinity. This situation is discussed by Ziman<sup>1)</sup> who proposed to cut the integro within a certain wave number  $k_0$ . As mentioned in Section 3 this wave number depends on the particle distribution in a very complicated way; it, however, is interesting enough, as pointed out by Ziman, that the infinity term  $\hbar^2 \int k^4 dk / 16\pi^2 m \rho_0$  is added to the incompressible energy given by (3.11). From our point of view it is not yet decisive to conclude that this term is attributed to the excitation energy of the roton, because we must obtain more complete account of the roton-roton interaction and the roton-phonon interaction noting that  $\text{div } v_r = 0$ .

## § 6. Collective description

Apart from the hydrodynamical approximation we like to study on the roton-phonon

model of Landau in the more convincing manner of quantum mechanics. In Paper I we have applied the collective description of Bohm and Pines<sup>7)</sup> to consider the phonon field of boson assemblies as the collective oscillation field such as the plasma oscillation. The scheme of the treatment of I is briefly mentioned as follows :

The hamiltonian of  $N$  Bose particles interacting with the energy  $G(x, x')$  is given by

$$H = \frac{1}{2m} \sum_{j=1}^N P_j^2 + \frac{1}{2} \sum_{i \neq j}^N G(x_i, x_j). \quad (6.1)$$

After subtracting the interaction energy from this we add the phonon energy given in Section 4 together with an interaction energy  $H'$ ,

$$H_{ph.} = -\frac{N}{2m} \sum_k k^2 \Phi(k) \Phi(-k) + \frac{m}{2N} \sum_k c^2(k) q(k) q(-k) - \frac{\hbar^2}{4m} \sum_k k^2, \quad (6.2)$$

$$H' = \sum_j \sum_k G(k) q(k) \exp(ikx_j), \quad (6.3)$$

and further we impose the following subsidiary condition :

$$q(k) = 0. \quad (6.4)$$

Hence the many body problem is reduced to the interaction of an assembly of free particle with the phonon field, if the kinetic energy of phonons is equal to the expectation value of the mutual potential and the subsidiary condition is fulfilled. In Appendix 3 one will see that the dispersion relation in Paper I is derived by a renormalization method so that the first condition that the difference between the true hamiltonian and ours is zero must be satisfied.

Now we go over to the alternative expression of our hamiltonian. Under the subsidiary condition (6.4) we set the total hamiltonian in the form

$$\mathfrak{H} = H + H' + m/2N \cdot \sum_k c(k)^2 q(k) q(-k) - \hbar^2/4m \cdot \sum_k k^2, \quad (6.5)$$

and then the canonical transformation of the form

$$U = \exp \left\{ i/\hbar \cdot \sum_k \sum_j \Phi(k) \exp(-ikx_j) \right\}, \quad (6.6)$$

sends  $\mathfrak{H}$  into

$$\begin{aligned} \mathfrak{H}_{new} = U^{-1} H U = & 1/2m \cdot \sum_k \{ P_k - \sum_j ik \Phi(k) \exp(-ikx_j) \}^2 \\ & + \frac{1}{2} \sum_k G_{new}(k) \sum_{i \neq j} \exp ik(x_i - x_j) + \frac{m}{2N} \sum_k c(k)^2 q(k) q(-k) - \frac{\hbar^2}{4m} \sum_k k^2, \end{aligned} \quad (6.7)$$

$$G_{new}(k) = G(k) - m/N \cdot c(k)^2, \quad (6.8)$$

which resembles the electromagnetic interaction and can be written in the following quantized form

$$\mathfrak{H}_{new} = \frac{\hbar^2}{2m} \int V \left\{ \varphi^* \exp \left( -\frac{i}{\hbar} \cdot \Phi \right) \right\} \cdot V \left\{ \varphi \exp \left( \frac{i}{\hbar} \cdot \Phi \right) \right\} dX$$

$$\begin{aligned}
& + \frac{1}{2} \int \left\{ G_{new}(x, x') \varphi^*(x) \varphi^*(x') \varphi(x') \varphi(x) dx dx' \right. \\
& \left. + \frac{m}{2N} \sum_k c(k)^2 q(k) q(-k) - \frac{\hbar^2}{4m} \sum^2 k \right\}, \quad (6.9)
\end{aligned}$$

in which  $\varphi$  and  $\varphi^*$  are the quantized wave functions of particles and

$$G_{new}(x, x') = \sum_k \left\{ G(k) - \frac{m}{N} c(k)^2 \right\} \exp ik(x - x'). \quad (6.10)$$

Owing to this transformation the subsidiary condition turns out into

$$q(k) - \int \varphi^*(x) \varphi(x) \exp(-ikx) dx = 0. \quad (6.11)$$

In paper I the whole interaction energy is absorbed by the phonon energy but here the interaction energy is partially absorbed by the phonon energy and the remainder acts as a new interaction energy of particles, which are considered as moving in a Hartree field of this new interaction potential  $G_{new}(x, x')$ . Hence the total interaction energy becomes

$$H_{INT.} = \int J(x) \cdot \Gamma \Phi(x) dx + \frac{1}{2} \iint G_{new}(x, x') \varphi^*(x) \varphi^*(x') \varphi(x) \varphi(x') dx dx', \quad (6.12)$$

$$J(x) = \hbar/2mi \cdot \{ \varphi^*(x) \Gamma \varphi(x) - \Gamma \varphi^*(x) \varphi(x) \}. \quad (6.13)$$

Here we apply the interaction representation and the first term in (6.12) can be eliminated. In this description the individual particle plays the part of the roton in the hydrodynamical description.

## § 7. Conclusion ... The ground state

If the interaction energy  $H_{INT.}$  involved in the hamiltonian (6.9) is considered as the small perturbation, the unperturbed wave function is given by

$$\chi_n = \psi_n(x_1, \dots, x_N) \prod_k \exp(-\gamma_k |\rho(k)|^2/2) H_n(\sqrt{\gamma_k} |\rho(k)|), \quad (7.1)$$

$$\gamma_k = \frac{m}{\hbar N |k|} c(k) \quad (7.2)$$

$$H_n(x) = (-1)^n \exp(x^2) \frac{\partial^n}{\partial x^n} \exp(-x^2). \quad (7.3)$$

in the configuration space. This is expressed by a product of the particle wave function  $\psi_n$ , given by the Slater permanant and the phonon wave function which is the Hermite polynomial and shows the dynamical correlation of atoms to keep separate with one another. Hence the ground-state wave function is expressed by

$$\chi_0 = \psi_{n0}(x_1, \dots, x_N) \prod_{k=-K}^K \exp(-\gamma_k |\rho(k)|^2/2). \quad (7.4)$$

where  $K$  is the cut-off wave number. If the interaction energy is very small, the ground

state is represented solely by the phonon wave function which is shifted slightly by the individual particle wave function. Due to the phonon energy the attractive potential of the individual particle becomes larger than the real potential. As a simple example, we consider the two body problem with a hard core potential such as

$$\begin{aligned} V(r) &= V_0, \quad r \leq r_0, \\ V(r) &= 0, \quad r > r_0. \end{aligned} \quad (7.5)$$

In this case the interaction potential can be set to be zero letting

$$V_0 = mc^2, \quad (7.6)$$

and we can find the cut-off wave number  $K = 3.1 \times 10^7 \text{ cm}^{-1}$  from the relation

$$|\partial/\partial r \cdot \log \psi_{\text{pot}}|_{r=r_0} = |\partial/\partial r \cdot \log \chi_0|_{r=r_0}, \quad (7.7)$$

$$\rho_0^{-1} = 1/6 \cdot \pi r_0^3, \quad r_0 = \hbar/mc = 2 \times 10^{-8} \text{ cm}, \quad (7.8)$$

where  $\psi_{\text{pot}}$  is the wave function within the potential and  $\chi_0$  is the phonon wave function :

$$\psi_{\text{pot}} = \sinh(\hbar c/m \cdot r)/r, \quad (7.9)$$

$$\chi_0 = \exp\{-mc/\pi^2 \hbar \rho_0 \cdot \sin^2(Kr/2)/r^2\}. \quad (7.10)$$

On dealing with the rigid sphere of radius  $r_0/2$ , the wave function is given by

$$\begin{aligned} \psi_{\text{rig}} &= \sin \eta / \eta, \quad \eta = \pi r / r_0, \quad r \geq r_0, \\ \psi_{\text{rig}} &= 0, \quad r < r_0. \end{aligned} \quad (7.11)$$

As done in the first example the phonon wave function is continued to (7.11) and we get the cut-off wave number for the given particle density and  $K = 2 \times 10^8 \text{ cm}^{-1}$  for  $r_0 = 2 \times 10^{-8} \text{ cm}$ . The sound speed is solved from (4.13) as follows :

$$c^2 = \frac{\hbar}{4N^2 m^2} \sum_k (\chi_0, k^2 | \rho_0'(k) |^2 \chi_0) = \frac{N'}{16N} \frac{\hbar^3 K^3}{m^3 c}, \quad (7.12)$$

$$c = (N'/16N)^{1/3} \hbar K/m, \quad N' = 1/6\pi^2 \cdot K^3. \quad (7.13)$$

From these two body problems we see that the energy gap between the  $S$  state and the  $P$  state, the first excited state is very large compared with the spacing of the phonon spectrum. If the states with non-zero angular momenta are attributed to the roton states, we may say that the phonon is the unique excitation at very low temperatures.

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## Appendix I

The current operator defined in Paper I is

$$J(x) = R(x)v(x)R(x) \quad (\text{I} \cdot 1)$$

Since the commutation relation of the density and the velocity does not involve the velocity but only the density, one may write the current operator in the more familiar form

$$J(x) = \lim_{\delta \rightarrow 0} R(x)v(x+\delta)R(x) = 1/2 \cdot \{\rho(x)v(x) + v(x)\rho(x)\}. \quad (\text{I} \cdot 2)$$

## Appendix II. Second sound speed and pressure

Assuming that the energy spectrum is expressed by

$$E(k) = \hbar |k|, \quad c(k) = \hbar |k| (c_0 - \gamma/2 \cdot c_0 k^2 + \dots), \quad (\text{II} \cdot 1)$$

we get the thermodynamical quantities such as the heat capacity, the entropy and the normal density at very low temperatures:

$$\begin{aligned} c_v &= \frac{\partial E}{\partial T} = 4\alpha T^3 + 6\beta T^5, \\ S &= \int \frac{c_v}{T^2} dT = \frac{4}{3} \alpha T^3 + \frac{6}{5} \beta T^5, \\ \rho_{\text{ph.}} &= \frac{4}{3} \frac{\alpha}{c_0^2} T^3 + \frac{14}{5} \frac{\beta}{c_0^2} T^5, \\ \alpha &= \frac{\pi^2}{30} \frac{\hbar^4}{\hbar^3 c_0^3}, \quad \beta = \frac{10}{63} \pi^4 \frac{\hbar^6}{\hbar^5 c_0^5}. \end{aligned}$$

Hence the second sound speed of Landau turns out to be

$$c_2 = (TS^2/\rho_0/c_v \rho_{\text{ph.}})^{1/2} = \frac{c_0}{\sqrt{3}} \left(1 - \frac{5}{4} \frac{\beta}{\alpha} T^2 + \dots\right), \quad (\text{II} \cdot 2)$$

therefore the deviation which shows the temperature dependency of the velocity comes from the dispersion term of the first sound speed.

From the spectrum given by (II.1) we can readily find the expression for the pressure. If it were not for the dispersion term the pressure would be given by the average value of the quantity

$$-V \partial E_0(k) / \partial V = \frac{5}{6} \hbar c_0 |k|, \quad c^0 = c_0 (V^{-1/2}), \quad |k| = |k| (V^{-1/3})$$

and introducing the dispersion term it becomes

$$-V \frac{\partial E(k)}{\partial V} = \hbar |k| \left( \frac{5}{6} c_0 - \frac{3}{4} \gamma c_0 k^2 - \frac{\hbar^2}{8m^2 c_0} k^2 + \dots \right)$$

and the average value reads

$$P = \frac{5}{6} \alpha T^4 + \frac{7}{10} \beta' T^6, \quad (\text{II} \cdot 3)$$

$$\beta' = \beta - \frac{5}{441} \pi^4 \frac{\kappa^6}{m^2 \hbar^3 c_0^7}.$$

$\kappa$  : Boltzmann coefficient.

### Appendix III. Renormalization

In Paper I we have obtained the dispersion equation of the form

$$1 = \frac{k^2 G(k)}{m} \sum_{j=1}^N \frac{1}{[\omega(k) - (k \cdot P_j/m)]^2 - (\hbar k^2/2m)^2}, \quad (\text{III} \cdot 1)$$

which is identical with the recent result of Bohm and Pines,<sup>(7)</sup> though the canonical transformation applied is different from theirs. Here we try to find this dispersion equation applying our transformation function.

The total hamiltonian takes the following quantized form

$$H = \frac{\hbar^2}{2m} \sum_k k^2 a^*(k) a(k) + \frac{N^2}{2} G(0) - \frac{N}{2} G_0 \\ + H_{\text{ph.}} + H' + H_{\text{res.}} + \frac{1}{2} \sum_{k \neq 0} G(k) \rho(k) \rho(-k), \quad (\text{III} \cdot 2)$$

$$H' = \sum_h G(k) q(k) \rho(-k), \quad (\text{III} \cdot 3)$$

$$H_{\text{res.}} = -\frac{N}{2m} \sum_k k^2 \phi(k) \phi(-k) - \frac{1}{2} \sum_{k \neq 0} \left\{ G(k) - \frac{m}{Nk^2} \omega(k)^2 \right\} q(k) q(-k), \quad (\text{III} \cdot 4)$$

where  $H_{\text{res.}}$  is the rest part which cancels the variation of the phonon frequency caused by the second-order perturbation effect, if we assume the dispersion formula in the form

$$1 = \frac{k^2 G(k)}{m} \sum_K \frac{a^*(K) a(K)}{\{\omega(k) - \hbar k \cdot K/m\}^2 - (\hbar k^2/2m)^2}. \quad (\text{III} \cdot 5)$$

In this case subsidiary condition (6.4) becomes

$$U^{-1} q(k) U = q(k) - [k^2 G(k) N/m] \sum_l \frac{a^*(l + (k/2)) a(l - (k/2))}{\omega(k)^2 - (\hbar^2/m^2)(l \cdot k)^2} = 0, \quad (\text{III} \cdot 6) \\ q(k) = |k| [\hbar N/2m\omega(k)]^{1/2} [b^*(-k) + b(k)]$$

using our transformation function :

$$U = 1 - \sum_l \sum_k |k| G(k) [N/2m\hbar\omega(k)]^{1/2} a^*(l + (k/2)) a(l - (k/2)) \\ \times \left[ \frac{b^*(-k)}{\omega(k) + (\hbar/m)(l \cdot k)} - \frac{b(k)}{\omega(k) - (\hbar/m)(l \cdot k)} \right]. \quad (\text{III} \cdot 7)$$

Owing to this subsidiary condition the last term in (III·2), which gives an interaction energy of particles, becomes very small.

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## Many-body Problem in Quantum Field Theory, II

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In view of the fact that really observable particles are not bare but dressed with their self field, the procedures to treat rigorously the dressed particles are presented. For this purpose, the Green function formalism by Schwinger and covariant equations by Bethe and Salpeter are useful.

In this paper, we first discuss the relations among various covariant formulations. Then it is investigated how to normalize the Bethe-Salpeter wave functions and how to express the expectation values of given observables in terms of Bethe-Salpeter wave functions. Finally the difference of treatments between dressed and bare particles are shown in connection with the scattering problems.

### § 1. Introduction

The covariant formulation of the quantum field theory has enabled us to get insight into the deep structure of the quantum electrodynamics in which brilliant successes have been achieved in eliminating divergences consistently and in improving the agreement of the theory with experiments.

There are, however, many problems left unsolved within the framework of the current field theory. The problem of bound states would be one of the most important subjects. The basic equations that govern bound states as well as scattering states have been derived by several authors<sup>1)-4)</sup> and known by the name of Bethe-Salpeter equation<sup>2)</sup>. However, it still remains to be clarified how the Bethe-Salpeter wave functions are related to the probability amplitude. In a previous paper<sup>5)</sup> (to be referred to as I hereafter), this problem was investigated in detail, but some results seem to necessitate further elucidation. In this paper, therefore, we shall deal with the same problem more completely in a simpler form. Through this work, one will see that the chronological ordering operators play an essential rôle in the theory.

In § 2, the equivalence of the theory developed in I to the Schwinger theory of Green functions<sup>6)</sup> is proved in the interaction representation. The utility of this representation consists in the fact that we can put many independent equations into a single one by introducing external sources as generating parameters. It will be worth while mentioning that the external sources are hardly introduced in the Heisenberg representation because of the difficulty arising in the definition of the true vacuum.

In § 3, the relation of the Green functions in the Heisenberg representation to those in the interaction representation is studied with the aid of the Gell-Mann-Low theory.<sup>3)</sup> The boundary conditions for Green functions are more clearly imposed in the Heisenberg

representation than in the interaction representation. With these preliminary investigations on Green functions, we shall proceed to the discussions about the Bethe-Salpeter wave functions. The information on the energy levels of bound states can be obtained from the Bethe-Salpeter equation as is obviously seen in the derivation of this equation by Gell-Mann and Low. However, it is not immediately clear how other quantities are related to this equation. Concretely speaking, the following problems are left to be solved:

- (a) how to compute expectation values of given observables in a given bound state,
- (b) how to normalize the Bethe-Salpeter wave functions,
- (c) how to deal with scattering problems involving composite particles.

For these purposes, we first derive the Bethe-Salpeter equation from those of Green functions thereby introducing a new kind of chronological operators for the later convenience. (§ 4).

In I, we have dealt with the problems (a) and (b) introducing co- and contra-variant components of state vectors. The former is familiar in the sense that they are nothing but the Bethe-Salpeter wave functions, while the latter is rather unfamiliar and hardly determined, and the elimination of such an unacquainted concept would much facilitate the theory to be conceived. Hence we reformulate the theory in terms of covariant components alone as has been suggested in the appendix of I. One will find that the normalization problem (b) is intimately connected with the renormalization problem. (§ 5)

In the last section, the problem (c) is investigated. (§ 6) Although this problem has already been treated in I, we shall repeat some points concentrating our attention to the hypothesis of adiabatic switching since it seems very instructive for understanding the physical meaning of the present theory.

The differences between the concepts of "dressed" and "bare" particles would be most plainly recognized in this section.

## § 2. Theory of Green functions in the interaction representation

In this section, the equivalence of the theory developed in I to the Schwinger theory of Green functions is proved.

### (a) *Functional derivatives with respect to external sources*

Throughout the discussions of this section, we employ the interaction representation. The interaction Hamiltonian assumes the following form:

$$H_{int} = i\gamma : \bar{\psi} O_a \psi : \phi_a + Q_a \phi_a - H_{self}, \quad (2.1)$$

and

$$\gamma = \begin{cases} f \\ -e \end{cases}, \quad O_a = \begin{cases} \gamma_5 \tau_a \\ \gamma_\mu \end{cases}, \quad \phi_a = \begin{cases} \phi_a & \text{Sym. } Ps(p_s) \\ A_\mu & \text{Q.E.D.} \end{cases} \quad (2.2)$$

where  $\gamma$  is the coupling constant,  $O_a$  the coupling matrix, and  $\psi$  and  $\phi$  are the wave functions of a Fermi field and a Bose field respectively.  $H_{self}$  is the counter term to renormalize the theory, and  $Q$  is the c-number external source. The colons are the Wick's S-product<sup>7)</sup>, and another notation  $N$  will also be used in some cases.



In the interaction representation, the propagation functions of the Fermion and Boson which we shall call nucleon and meson hereafter are given respectively by

$$\langle T[\psi(1)\bar{\psi}(2)] \rangle_{\text{vac}} = S_F(1-2) = \frac{i}{(2\pi)^4} \int \frac{i\not{p}\gamma - M}{p^2 + M^2 - i_\epsilon} e^{ip(\tau_1 - \tau_2)}(d\mathbf{p}), \quad (2.3)$$

$$\langle T[\phi_a(1)\phi_b(2)] \rangle_{\text{vac}} = \delta_{ab} \mathcal{J}_F(1-2) = \delta_{ab} \frac{-i}{(2\pi)^4} \int \frac{1}{q^2 + \mu^2 - i_\epsilon} e^{iq(x_1 - x_2)}(d\mathbf{q}),$$

where  $T$  is Wick's  $T$ -product,  $\text{vac.}$  is the abbreviation of the free vacuum, and  $M$  and  $\mu$  are the rest masses of a nucleon and a meson respectively.

The propagation functions satisfy the following equations :

$$\begin{aligned} (\gamma\partial + M)S_F(x) &= -i\delta(x), \\ (\square - \mu^2)\mathcal{J}_F(x) &= i\delta(x). \end{aligned} \quad (2.4)$$

The transformation function  $U$  is defined by the Tomonaga-Schwinger equation

$$i \frac{\delta}{\delta\sigma(x)} U(\sigma, \sigma_0) = H_{\text{int}}(x) U(\sigma, \sigma_0), \quad \text{with } U(\sigma_0, \sigma_0) = 1. \quad (2.5)$$

Using the above transformation function, we introduce the following notation<sup>\*)</sup>

$$\langle AB \dots Z \rangle = \frac{\langle T[U(\infty, -\infty) AB \dots Z] \rangle_{\text{vac}}}{\langle U(\infty, -\infty) \rangle_{\text{vac}}}. \quad (2.6)$$

The usefulness of introducing external sources will be exhibited by the formula

$$i \frac{\delta}{\delta Q_a(x)} T[U(\infty, -\infty) AB \dots Z] = T[U(\infty, -\infty) \phi_a(x) AB \dots Z], \quad (2.7)$$

which can readily be proved if one expresses the solution of eq. (2.5) in the form

$$U(\infty, -\infty) = T[\exp(-i \int_{-\infty}^{\infty} H_{\text{int}}(x) (dx))].$$

We define the Green functions under the influence of the external source  $Q$  by

$$\begin{aligned} S'_F(1, 2) &= \langle \psi(1) \bar{\psi}(2) \rangle = \frac{\langle T[\psi(1) \bar{\psi}(2)] \rangle_{\text{vac}}}{\langle U \rangle_{\text{vac}}}, \\ \mathcal{J}'_F(1, 2) &= \langle \phi(1) \phi(2) \rangle - \langle \phi(1) \rangle \langle \phi(2) \rangle = i \frac{\delta \langle \phi(1) \rangle}{\delta Q(2)} = i \frac{\delta \langle \phi(2) \rangle}{\delta Q(1)}, \end{aligned} \quad (2.8)$$

where  $U = U(\infty, -\infty)$ , and we understand in these formulas that the isotopic spin indices as well as spin indices are suppressed in the arguments "1" and "2". For simplicity we employ the following abbreviation

$$\langle U(\infty, -\infty) \rangle_{\text{vac}} = \langle U \rangle_{\text{vac}} = C = e^{-I}. \quad (2.9)$$

\* In I, thick letters are used, but thin letters seem to be more suitable.

Combining (2.7) and (2.9), we have useful formulas

$$i \frac{\partial}{\partial Q(1)} C = \langle \phi(1) | C \rangle, \quad i \frac{\partial}{\partial Q(1)} L = -\langle \phi(1) |. \quad (2.10)$$

(b) *Equations for the chronologically ordered products of operators*

From the definition of the  $T$ -product, it can readily be rewritten as

$$T[\phi(1)\phi(2)] = \frac{1}{2} \{\phi(1)\phi(2)\} + \frac{1}{2} \varepsilon(1, 2) [\phi(1)\phi(2)],$$

where

$$\varepsilon(1, 2) = \begin{cases} 1, & \text{for } t_1 > t_2 \\ -1, & \text{for } t_1 < t_2. \end{cases}$$

Hence we have the equation

$$(\square | - t^2)_1 T[\phi(1)\phi(2)] = i\partial(1-2), \quad (2.11a)$$

the subscript "1" indicates the argument that  $\square$  operates.

In a similar way, we also have

$$(\gamma\partial + M)_1 T[\psi(1)\bar{\psi}(2)] = -i\partial(1-2). \quad (2.11b)$$

The  $\partial$ -functions appear from the contracted parts of the  $T$ -product as is seen from above, so that we have generally

$$\begin{aligned} (\square | - t^2)_1 N[\dots\phi(1)\dots] &= 0, \\ (\gamma\partial + M)_1 N[\dots\psi(1)\dots] &= 0. \end{aligned} \quad (2.12)$$

$N$  is the normal product being equivalent to the  $S$ -product, which was expressed by colons in (2.1). Both kinds of notations will be used case by case.

Generalization of (2.11a) gives

$$(\square | - t^2)_1 T[\phi(1).AB\dots Z] = i \frac{\partial}{\partial \phi(1)} T[AB\dots Z], \quad (2.13a)$$

where we have utilized the definition of the functional derivatives

$$\frac{\partial \phi(2)}{\partial \phi(1)} = \partial(1-2).$$

In a similar way, (2.11b) can also be generalized to yield

$$(\gamma\partial + M)_1 T[\psi(1).AB\dots Z] = -i \frac{\partial}{\partial \psi(1)} T[AB\dots Z]. \quad (2.13b)$$

In this case we must be cautious of the order of operators since  $\psi$ 's and  $\bar{\psi}$ 's anti-commute in the  $T$ -product, otherwise the sign of the differentiated expression cannot be fixed. Hence we define the functional derivative  $\partial/\partial\bar{\psi}$  (or  $\partial/\partial\psi$ ) in such a way that it is performed after bringing the  $\bar{\psi}$  (or  $\psi$ ) to be differentiated to the left edge of the operand. For instance in (2.13b),  $\psi$  to be differentiated should be brought to the position of  $A$ .

Then, from the definition follows the commutation relations

$$\{\partial/\partial\phi(1), \partial/\partial\phi(2)\} = \{\partial/\partial\bar{\phi}(1), \partial/\partial\phi(2)\} = \{\partial/\partial\bar{\phi}(1), \partial/\partial\bar{\phi}(2)\} = 0,$$

namely they anticommute with each other.

Equations like (2.13a) and (2.13b) are already utilized by Matthews and Salam<sup>8)</sup> and by Freese and Zimmermann<sup>9)</sup> in the Heisenberg representation.

(c) *Schwinger equations for Green functions*<sup>6)</sup>

Schwinger developed the theory of Green functions from his own theory of quantum dynamical principle. We shall show that his equations can also be derived from the theory in I with the aid of the above techniques. This proof of equivalence will be instructive for understanding various theories from a unified point of view.

First we shall calculate the following quantity :

$$\begin{aligned} (\gamma\partial + M)_1 T[U\psi(1)\bar{\psi}(2)] &= -i(\partial/\partial\bar{\psi}(1)) \cdot T[U\bar{\psi}(2)] \\ &= -i\partial(1-2)U - iT[\partial U/\partial\bar{\psi}(1) \cdot \bar{\psi}(2)] \\ &= -i\partial(1-2)U - i\eta O(1)T[U\psi(1)\phi(1)\bar{\psi}(2)] \\ &= -i\partial(1-2)U - i\eta O(1)i(\partial/\partial Q(1)) \cdot T[U\psi(1)\bar{\psi}(2)], \end{aligned}$$

since

$$\begin{aligned} i(\partial U/\partial\bar{\psi}(1)) &= i(\partial/\partial\bar{\psi}(1)) \cdot T[\exp(-i\int H_{\text{int}}(x)(dx))] \\ &= \int (dx_2) T\left[\frac{\partial H_{\text{int}}(2)}{\partial\bar{\psi}(1)}U\right] = i\eta O(1)T[\psi(1)\phi(1)U]. \end{aligned}$$

This result will also be expressed with reference to eq. (2.7) as

$$(\gamma\partial_1 + M - \eta O(1) \cdot \partial/\partial Q(1)) T[\psi(1)\bar{\psi}(2)] = -i\partial(1-2)U. \quad (2.14)$$

This is an equation for a chronologically ordered product, and we can derive c-number equations by taking its matrix elements. For instance, taking  $r, s$ -elements of (2.14), we have

$$\left(\gamma\partial_1 + M + i\eta O(1)\langle\phi(1)\rangle_{rs} - \eta O(1)\frac{\partial}{\partial Q(1)}\right)\langle\psi(1)\bar{\psi}(2)\rangle_{rs} = -i\partial(1-2),$$

where

$$\langle AB \dots Z \rangle_{rs} = \frac{(\phi_r, T[UAB \dots Z]\phi_s)}{(\phi_r, U\phi_s)}. \quad (2.6')$$

If we consider only the case in which both the states  $r$  and  $s$  coincide with the free vacuum, we get

$$\left(\gamma\partial_1 + M + i\eta O(1)\langle\phi(1)\rangle - \eta O(1)\frac{\partial}{\partial Q(1)}\right)S'_F(1, 2) = -i\partial(1-2). \quad (2.15)$$

Since this equation contains  $\langle\phi(1)\rangle$ , we need another equation for this quantity. This can be done as follows :

$$(\square - \mu^2)_1 T[U\phi(1)] = i \frac{\partial U}{\partial \phi(1)} = i\eta T[U : \bar{\psi} O \psi(1) : ] + Q(1) U.$$

Taking the free vacuum equation, the desired equation is obtained

$$(\square - \mu^2)_1 \langle \phi(1) \rangle = -i\eta S\rho(O(1) S'_F(1, 1'))_{1' \rightarrow \pm 0} + Q(1), \quad (2.16)$$

where  $1' \rightarrow 1 \pm 0$  means to take the average of the two limiting values from future and past to "1".<sup>6)</sup>

Combining (2.15) and (2.16), we have a complete set of equations for  $S'_F$  and  $\langle \phi \rangle$ .

It is sometimes convenient to write eqs. (2.15) and (2.16) in the integral form

$$S'_F(1, 2) = S_F(1-2) + \eta \int d\omega_3 S_F(1-3) O(3) \langle \phi(3) \rangle S'_F(3, 2) \\ + \eta \int d\omega_3 S_F(1-3) O(3) (i \cdot \partial / \partial Q(3)) S'_F(3, 2), \quad (2.15')$$

$$\langle \phi(1) \rangle = -i \int d\omega_2 J_F(1-2) Q(2) - \eta \int d\omega_2 J_F(1-2) S\rho(O(2) S'_F(2, 2'))_{2' \rightarrow 2 \pm 0}. \quad (2.16')$$

In the perturbation theory,  $S'_F(1, 2)$  and  $\langle \phi(1) \rangle$  are obtained as functionals of  $Q$  by successive substitutions. In these equations  $\int d\omega$  means the integration over whole space-time region and the summation over spin and isotopic spin indices.

By differentiating eqs. (2.16) and (2.16') with respect to the external source  $Q$ , we have from (2.8) eqs. for  $J'_F$

$$(\square - \mu^2)_1 J'_F(1, 2) = i\partial(1-2) + \eta S\rho[O(1) i \cdot \partial / \partial Q(2) \cdot S'_F(1, 1')]_{1' \rightarrow 1 \pm 0}, \quad (2.17)$$

or in the integral form

$$J'_F(1, 2) = J_F(1-2) - \eta \int d\omega_3 J_F(1-3) S\rho \left[ O(3) i \frac{\partial}{\partial Q(2)} S'_F(3, 3') \right]_{3' \rightarrow 3 \pm 0}. \quad (2.17')$$

In order to relate the present theory to the Dyson theory<sup>10)</sup>, we introduce the proper self energy  $\Sigma^*$  as follows:

$$S'_F(1, 2) = S_F(1-2) + \eta \int d\omega_3 S_F(1-3) O(3) \langle \phi(3) \rangle S'_F(3, 2) \\ + \int d\omega_3 d\omega_4 S_F(1-3) \Sigma^*(3, 4) S'_F(4, 2). \quad (2.18)$$

Then comparison of the above eq. with (2.15') yields the eq.

$$i\eta O(1) \frac{\partial}{\partial Q(1)} S'_F(1, 2) = \int d\omega_3 \Sigma^*(1, 3) S'_F(3, 2) \equiv \Sigma^*(1) S'_F(1, 2). \quad (2.19)$$

Using  $\Sigma^*$ , the eq. (2.15) can be written as

$$(\gamma \partial_1 + M + i\eta O(1) \langle \phi(1) \rangle + i \Sigma^*(1)) S'_F(1, 2) = -i\partial(1-2). \quad (2.20)$$

For the sake of simplicity, we shall write eqs. (2.15) and (2.20) symbolically as

$$\mathcal{D}_Q(1) S'_F(1, 2) = -i\delta(1-2),$$

$$\text{and} \quad \mathcal{D}_N(1) S'_F(1, 2) = -i\delta(1-2),$$

respectively.

Differentiating (2.20) with respect to  $Q(3)$ , we get

$$\mathcal{D}_N(1) \left( i \frac{\partial}{\partial Q(3)} S'_F(1, 2) \right) = -i \left[ \eta O(1) \mathcal{A}'_F(1, 3) + \left( i \frac{\partial}{\partial Q(3)} \Sigma^*(1) \right) \right] S'_F(1, 2),$$

from which we have with the aid of the Green function  $S'_F$

$$i \frac{\partial}{\partial Q(3)} S'_F(1, 2) = \int d\omega_4 S'_F(1, 4) \left[ \eta O(4) \mathcal{A}'_F(4, 3) + \left( i \frac{\partial}{\partial Q(3)} \Sigma^*(4) \right) \right] S'_F(4, 2). \quad (2.21)$$

We define the vertex operator  $\mathcal{O}$  by

$$i \frac{\partial}{\partial Q(3)} S'_F(1, 2) = \eta \int d\omega_4 d\omega_5 d\omega_6 S'_F(1, 4) \mathcal{O}(45; 6) S'_F(5, 2) \mathcal{A}'_F(6, 3). \quad (2.22)$$

By comparing eq. (2.22) with (2.21), we see

$$i \frac{\partial}{\partial Q(3)} \Sigma^*(1, 2) = \eta \int d\omega_4 \mathcal{O}(12; 4) \mathcal{A}'_F(4, 3) - \eta O(1) \mathcal{A}'_F(3, 1) \delta(1-2). \quad (2.23)$$

Further from eqs. (2.19) and (2.22), we arrive at a relation

$$\Sigma^*(1, 2) = \eta^2 \int d\omega_3 d\omega_4 O(1) S'_F(1, 3) \mathcal{O}(32; 4) \mathcal{A}'_F(4, 1). \quad (2.24)$$

The relations (2.23) and (2.24) constitute a set of defining equations of  $\Sigma^*$  and  $\mathcal{O}$ .

In a similar way, we shall introduce the proper self energy operator or the polarization operator  $\Pi^*$  for the meson field, by

$$\mathcal{A}'_F(1, 2) = \mathcal{A}_F(1-2) + \int d\omega_3 d\omega_4 \mathcal{A}_F(1-3) \Pi^*(3, 4) \mathcal{A}'_F(4, 2), \quad (2.25)$$

then from (2.17') we readily see

$$\int d\omega_4 \Pi^*(3, 4) \mathcal{A}'_F(4, 2) = -\eta S \rho \left[ O(3) i \frac{\partial}{\partial Q(2)} S'_F(3, 3') \right]_{3' \rightarrow 2; \pm 0}.$$

Inserting the expression (2.22) into the above equation, we find

$$\Pi^*(1, 2) = -\eta^2 \int d\omega_3 d\omega_4 S \rho [O(1) S'_F(1, 3) \mathcal{O}(34; 2) S'_F(4, 1)]. \quad (2.26)$$

The equation (2.25) becomes in the differential form as

$$(\square - \mu^2 - i\Pi^*) \mathcal{A}'_F(1, 2) = i\delta(1-2), \quad (2.27)$$

or symbolically

$$\mathcal{D}_M(1) \mathcal{A}'_F(1, 2) = i\delta(1-2).$$

The readers will find that the equations derived here are completely the same with Schwinger's ones, and the proof of the equivalence of two theories is accomplished.



For the sake of completeness, we shall further discuss the two-body Green functions. In the case of a two-nucleon system, we start from the relation

$$(\gamma\partial + M)_1 T[\psi(1)\psi(2)\bar{\psi}(3)\bar{\psi}(4)] = -i \frac{\partial}{\partial\bar{\psi}(1)} T[\psi(2)\bar{\psi}(3)\bar{\psi}(4)]. \quad (2.28)$$

The two-body Green function defined by

$$K(12; 34) = -\langle \psi(1)\psi(2)\bar{\psi}(3)\bar{\psi}(4) \rangle,$$

then, satisfies the equation

$$\mathcal{D}_Q(1)K(12; 34) = -i\partial(1-3)S'_F(2, 4) + i\partial(1-4)S'_F(2, 3), \quad (2.29)$$

which follows immediately from eq. (2.28)

The function  $K$  must satisfy the Bethe-Salpeter equation

$$K(12; 34) = S'_F(12; 34) + \int d\omega_5 \cdots d\omega_8 S'_F(1, 5)S'_F(2, 6)G(56; 78)K(78; 34), \quad (2.30)$$

where

$$S'_F(12; 34) = S'_F(1, 3)S'_F(2, 4) - S'_F(1, 4)S'_F(2, 3).$$

From these eqs. (2.29) and (2.30), two quantities  $K$  and  $G$  can be determined provided that  $S'_F$  is already known.

Inserting (2.30) into (2.29), and utilizing eq. (2.22) and the formula

$$\mathcal{D}_Q(1)[S'_F(1, 2)A] = -i\partial(1-2)A - \eta O(1)S'_F(1, 2)\frac{\partial A}{\partial Q(1)},$$

we have

$$\begin{aligned} \int d\omega_5 d\omega_6 G(12; 56)K(56; 34) &= \eta^2 \int d\omega_5 d\omega_6 O(1)\mathcal{O}(25; 6)\mathcal{A}'_F(6, 1)K(15; 34) \\ &+ i\gamma \int d\omega_5 \cdots d\omega_8 O(1)S'_F(1, 5)\partial(2-6)\frac{\partial}{\partial Q(1)}[G(56; 78)K(78; 34)]. \end{aligned} \quad (2.31)$$

Two equations (2.30) and (2.31) can be utilized to fix  $K$  and  $G$  in place of (2.29) and (2.30).

The Green function for the meson-nucleon scattering can be obtained by differentiating  $S'_F$  twice with respect to  $Q^{(1)}$ . However, it is sometimes more useful to treat the problem upon the Bethe-Salpeter equation. We start in this case from the equation

$$(\gamma\partial + M)_1 T[\psi(1)\bar{\psi}(3)\phi(2)\phi(4)] = -i \frac{\partial}{\partial\bar{\psi}(1)} T[\bar{\psi}(3)\phi(2)\phi(4)],$$

from which follows the eq.

$$\mathcal{D}_Q(1)K_{MN}(12; 34) = -i\partial(1-3)\mathcal{A}'_F(2, 4), \quad (2.32)$$

where

$$K_{MN}(12; 34) = \langle \psi(1)\bar{\psi}(3)\phi(2)\phi(4) \rangle.$$

$K_{MN}$  will also satisfy a Bethe-Salpeter eq.

$$\begin{aligned} K_{MN}(12; 34) &= S'_F(1, 3)\mathcal{A}'_F(2, 4) + \int d\omega_5 \cdots d\omega_8 S'_F(1, 5)\mathcal{A}'_F(2, 6) \\ &\times G_{MN}(56; 78)K_{MN}(78; 34). \end{aligned} \quad (2.33)$$

The eq. which corresponds to (2.31) can be derived in a similar way.<sup>11)</sup> In the symmetrical pseudoscalar meson theory, we must further supplement a counter term of the form  $\lambda\phi_a^\dagger\phi_a^\dagger$  for renormalization. The introduction of this term, however, complicates the procedure<sup>12)</sup> and we do not discuss it here.

### § 3. Theory of Green functions in the Heisenberg representation

The discussions in the previous section are concerned with the Green functions in the interaction representation with the presence of the external source. Such a treatment is convenient mathematically.

For the discussions on the physical quantities such as the  $S$  matrix, expectation values, etc., however, we have better to employ the Heisenberg representation. Hence, we shall devote this section to explore the relation between these two representations.

For this purpose, let us first expand an arbitrary Green function  $G[Q]$  in the interaction representation in powers of the external source  $Q$

$$G[Q] = G[0] + (-i) \int d\omega_1 Q(1) \left( i \frac{\partial G[0]}{\partial Q(1)} \right) + \frac{(-i)^2}{2!} \int d\omega_1 d\omega_2 Q(1) Q(2) \left( - \frac{\partial^2 G[0]}{\partial Q(1) \partial Q(2)} \right) + \dots \quad (3.1)$$

The differentiation with respect to the external source  $Q$  can easily be carried out in reference to the formula (2.7)

In the above expansion, the expansion coefficients are Green functions in the interaction representation with the absence of the external source. As is well known such quantities are connected with the Heisenberg operators by the Gell-Mann-Low relation

$$\langle T[\mathbf{A}\mathbf{B}\dots\mathbf{Z}] \rangle_{vac} = \frac{\langle T[U\mathbf{A}\mathbf{B}\dots\mathbf{Z}] \rangle_{vac}}{\langle U \rangle_{vac}} = \langle \mathbf{A}\mathbf{B}\dots\mathbf{Z} \rangle \quad (3.2)$$

provided that  $Q=0$ .

In this formula, thick letters represent Heisenberg operators and  $Vac.$  is the true vacuum.

Especially combining (2.7) and (2.9), we have the following expression

$$C[Q] = C[0] \left[ 1 + (-i) \int d\omega_1 \langle \phi(1) \rangle_{vac} Q(1) + \frac{(-i)^2}{2!} \int d\omega_1 d\omega_2 \langle T[\phi(1)\phi(2)] \rangle_{vac} Q(1) Q(2) + \dots \right]. \quad (3.3)$$

It must be noticed that in quantum electrodynamics or symmetrical pseudoscalar meson theory, odd order terms vanish due to Furry's theorem

$$\langle \phi(1) \rangle_{vac} = \langle T[\phi(1)\phi(2)\phi(3)] \rangle_{vac} = \dots = 0. \quad (3.4)$$

The formula (2.7) may be expressed as

$$i^n \frac{\partial^n}{\partial Q(1) \dots \partial Q(n)} C \langle \mathbf{A}\mathbf{B}\dots\mathbf{Z} \rangle = C \langle \mathbf{A}\mathbf{B}\dots\mathbf{Z} \phi(1) \dots \phi(n) \rangle. \quad (3.5)$$

Combining (3.3), (3.4) and (3.5), we have

$$\begin{aligned} & \left[ 1 + \frac{(-i)^2}{2!} \int d\omega_1 d\omega_2 \langle T[\phi(1)\phi(2)] \rangle_{vac} Q(1)Q(2) + \dots \right] \langle AB \dots Z \rangle \\ &= \langle T[AB \dots Z] \rangle_{vac} + (-i) \int d\omega_1 \langle T[AB \dots Z\phi(1)] \rangle_{vac} Q(1) \\ &+ \frac{(-i)^2}{2!} \int d\omega_1 d\omega_2 \langle T[AB \dots Z\phi(1)\phi(2)] \rangle_{vac} Q(1)Q(2) + \dots \end{aligned} \quad (3.6)$$

By this formula, the quantity  $\langle AB \dots Z \rangle$  in the interaction representation can be expressed in terms of the true vacuum expectation values of Heisenberg operators and the external source  $Q$ . Inserting this expression for  $\langle AB \dots Z \rangle$  into the equations in the previous section we have many independent equations by observing the coefficients of all powers of  $Q$ . Hence the functional of  $Q$  plays the rôle of the generating function of many independent equations.

As an illustration of this statement, we shall consider the problem of a one-nucleon system.

The basic equations that govern the one-nucleon system are (2.15) and (2.16) or (2.15') and (2.16').

For the sake of simplicity, we shall assume without violating the essential features of the theory that the contributions from nucleon closed loops can be neglected. This approximation makes the basic equations linear as we shall see soon.

This approximation means to neglect  $H^*$  in (2.26) and hence the second term in (2.16').

Then we get

$$\langle \phi(1) \rangle \approx -i \int d\omega_2 J_F(1-2) Q(2) \quad (3.7)$$

and the eq. (2.10) is readily integrated to give

$$\begin{aligned} L &\approx \frac{1}{2} \int d\omega_1 d\omega_2 J_F(1-2) Q(1)Q(2), \\ C &\approx \exp\left[-\frac{1}{2} \int d\omega_1 d\omega_2 J_F(1-2) Q(1)Q(2)\right]. \end{aligned} \quad (3.8)$$

Inserting the expression (3.7) into (2.15'), we have a linear equation for  $S'_F$

$$\begin{aligned} S'_F(1, 2) &\approx S_F(1-2) + \gamma_f \int d\omega_3 S_F(1-3) Q(3) \\ &\times \left[ -i \int d\omega_4 J_F(3-4) Q(4) + i \frac{\partial}{\partial Q(3)} \right] S'_F(3, 2). \end{aligned} \quad (3.9)$$

We expand  $S'_F$  as the power series in the external source  $Q$

$$S'_F(1, 2) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d\hat{\xi}_1 \dots d\hat{\xi}_n K(1, 2; \hat{\xi}_1, \dots, \hat{\xi}_n) Q(\hat{\xi}_1) \dots Q(\hat{\xi}_n). \quad (3.10)$$

Then we know in reference to (3.6)

$$K(1, 2) = \langle T[\phi(1)\bar{\phi}(2)] \rangle_{vac} = S'_F(1-2),$$

$$K(1, 2; \hat{\xi}_1) = \langle T[\phi(1)\bar{\psi}(2)\phi(\hat{\xi}_1)] \rangle_{vac},$$

$$\begin{aligned} K(1, 2; \hat{\xi}_1, \hat{\xi}_2) &= \langle T[\phi(1)\bar{\psi}(2)\phi(\hat{\xi}_1)\phi(\hat{\xi}_2)] \rangle_{vac} - \langle T[\phi(1)\bar{\psi}(2)] \rangle_{vac} \langle T[\phi(\hat{\xi}_1)\phi(\hat{\xi}_2)] \rangle_{vac} \\ &= \langle T[\phi(1)\bar{\psi}(2)\phi(\hat{\xi}_1)\phi(\hat{\xi}_2)] \rangle_{vac} - S'_F(1-2) \mathcal{A}'_F(\hat{\xi}_1 - \hat{\xi}_2), \\ &\text{etc.} \end{aligned}$$

If we write  $K$ 's in the interaction representation, they express the connected parts of  $\langle \phi(1)\bar{\psi}(2)\phi(\hat{\xi}_1)\cdots\phi(\hat{\xi}_n) \rangle$  in the absence of  $Q$

$$K(1, 2; \hat{\xi}_1, \dots, \hat{\xi}_n) = \langle \phi(1)\bar{\psi}(2)\phi(\hat{\xi}_1)\cdots\phi(\hat{\xi}_n) \rangle_{conn.} \text{ for } Q=0.$$

Substituting the expression (3.10) for  $S'_F$  in (3.9), and comparing the coefficients of each power of  $Q$  between both sides, we arrive at a set of coupled equations for  $K$ 's:<sup>13)</sup>

$$\begin{aligned} K(1, 2; \hat{\xi}_1, \dots, \hat{\xi}_n) &= \partial_{0n} \cdot S_F(1-2) + \eta \sum_k \{ d\hat{\xi}_0 d\hat{\xi}_k S_F(1-\hat{\xi}_0) O(\hat{\xi}_0) \mathcal{A}_F(\hat{\xi}_0 - \hat{\xi}_k) K(\hat{\xi}_0 2; \hat{\xi}_1 \dots \hat{\xi}_{\check{k}} \dots \hat{\xi}_n) \\ &\quad + \eta \int d\hat{\xi}_0 S_F(1-\hat{\xi}_0) O(\hat{\xi}_0) K(\hat{\xi}_0 2; \hat{\xi}_0 \hat{\xi}_1 \dots \hat{\xi}_n), \end{aligned} \quad (3.11)$$

where  $\check{k}$  means to omit  $\hat{\xi}_k$ . These equations constitute the basis of the covariant Tamm-Dancoff approximation.

So far we have discussed the combination of the Schwinger theory with the Gell-Mann-Low relation. There is another method by Matthews and Salam<sup>14)</sup> and by Freese and Zimmermann<sup>15)</sup>. This is to apply the method in the previous section—which we owe these authors—to Heisenberg operators.

Then we have the following equations:

$$\begin{aligned} (\gamma\partial + M)_1 T[\phi(1)AB\cdots Z] + i\eta O(1) T[\phi(1)\phi(1)AB\cdots Z] \\ = -i \frac{\partial}{\partial \bar{\psi}(1)} T[AB\cdots Z], \end{aligned} \quad (3.12)$$

$$\begin{aligned} (\square - \mu^2)_1 T[\phi(1)AB\cdots Z] - i\eta T[:\bar{\psi}(1)O(1)\phi(1):AB\cdots Z] \\ = i \frac{\partial}{\partial \phi(1)} T[AB\cdots Z]. \end{aligned} \quad (3.13)$$

We need the same caution of the functional derivative  $\partial/\partial\bar{\psi}$  as in the previous section.

Taking the true vacuum expectation values of these equations, we find coupled equations for the Green functions. On decoupling these equations, we obtain the same equations in the previous section. The connection will be easily seen with the aid of eq. (3.2).

Here we shall introduce the normal product or the  $S$ -product for Heisenberg operators.<sup>\*)</sup> The definition is<sup>16)17)</sup>

\*) Recently possible definitions of the  $S$ -product are discussed by E. Freese, Nuovo Cimento 11 (1954), 312.

$$\begin{aligned}
N[ABC \cdots Z] &= T[ABC \cdots Z] - \sum \langle T[AB] \rangle_{vac} T[C \cdots Z] \\
&\quad + \sum \langle T[AB] \rangle_{vac} \langle T[CD] \rangle_{vac} T[E \cdots Z] \\
&\quad - \cdots
\end{aligned}
\tag{3.14}^*$$

The summations are extended over all possible combinations of operators. The usefulness of this product will be recognized in the definition of the  $S$  matrix.

#### Boundary conditions

The superiority of the Heisenberg representation over the interaction representation lies in its direct connection to physically observable quantities or conditions. One of them is the boundary condition for the Green functions.

From the definition of the  $T$ -product, we have

$$\begin{aligned}
&(Vac|T[A(1)B(2) \cdots A'(1')B'(2') \cdots]|Vac) \\
&= \sum (Vac|T[A(1)B(2) \cdots]|s) (s|T[A'(1')B'(2') \cdots]|Vac) \tag{3.15} \\
&\quad \text{for } t_1=t_2=\cdots=t > t'_1=t'_2=\cdots=t',
\end{aligned}$$

where the summation should be extended over all stationary states.

Let  $\Psi_s$  be the state vector for a stationary state  $s$ , being an eigenstate of the total energy momentum  $P_\mu$

$$(P_\mu - p_{s,\mu})\Psi_s = 0,$$

and put

$$(p_s - p_{vac})_0 = E_s \geq 0 \quad (\text{Equality holds only for } s = Vac.)$$

Then in the eq. (3.15),  $T[A(1)B(2) \cdots]$  and  $T[A'(1')B'(2') \cdots]$  should behave as destruction and creation operators respectively. Hence the  $t$ -dependence of the summand in the expression (3.15) must be as

$$\propto e^{-iE_s(t-t')}, \quad \text{for } t > t',$$

and similarly

$$\propto e^{-iE_s(t'-t)}, \quad \text{for } t < t'. \tag{3.16}$$

This is nothing but the Feynman's positive frequency condition<sup>14)</sup>. This fact has already been stated and utilized in I, but because of its fundamental importance we have repeated it here.

It should further be mentioned that this condition holds for an arbitrary division of operators in the operand of  $T$  into two groups.

\* Colons will also be used for short expressions. The definition of the normal products in references 8) and 9) is not identical with the present one.



#### § 4. Equations for the covariant components and the introduction of a new kind of chronological operators

We shall begin with this section by presenting some nomenclatures. Quantities like

$$(Vac|T[AB\cdots Z]|Vac) = (0|T[AB\cdots Z]|0) \quad (4.1)$$

are called Green functions in general.

We call

$$(0|T[AB\cdots Z]|s) \text{ or } (s|T[AB\cdots Z]|0) \quad (4.2)$$

the B-S (Bethe-Salpeter) wave function or the B-S component, and

$$(r|T[AB\cdots Z]|s) \quad (4.3)$$

the expectation value.

Further we shall introduce the covariant component or the Feynman amplitude by

$$(0|N[AB\cdots Z]|s) \quad (4.4)$$

which is similar to but more convenient than the B-S component.

Then for a given stationary state  $s$ , various covariant components of this state form a covariant Fock space as a whole.

$$\Psi_s = \begin{pmatrix} (0|A|s) \\ (0|T[AB]|s) \\ (0|T[ABC]|s) \end{pmatrix}. \quad (4.5)$$

The coupled equations for these components are derived by Matthews and Salam, but we shall discuss the decoupled form of these equations as has been done in I. We deduce the equations for the covariant components from the decoupled equations for Green functions following the method by Gell-Mann and Low. This has already been discussed in I, however, our present purpose is to abstract some chronological operation from this procedure and apply it to other problems.

The two-nucleon Green functions satisfy the eq. (2.30) or

$$K(12; 34) = S'_F(12; 34) + \int d\omega_5 \cdots d\omega_8 S'_F(1-5) S'_F(2-6) G(56; 78) K(78; 34) \quad (4.6)$$

for the vanishing external source.

We now let  $t_3$  and  $t_4$  tend to  $-\infty$ , then follows from (3.15) the factorized expression for  $K$

$$K(12; 34) = (0|T[\phi(1)\phi(2)\bar{\psi}(4)\bar{\psi}(3)]|0) \rightarrow \sum_s g_s(12)\bar{g}_s(34), \quad (4.7)$$

where

$$\begin{aligned} g_s(12) &= (0|T[\phi(1)\phi(2)]|s) = (0|N[\phi(1)\phi(2)]|s), \\ \bar{g}_s(34) &= (s|T[\bar{\psi}(4)\bar{\psi}(3)]|0) = (s|N[\bar{\psi}(4)\bar{\psi}(3)]|0). \end{aligned}$$

In a similar way, we have

$$S_F'(12; 34) \rightarrow \sum_s g_s^0(12) \bar{g}_s^0(34), \quad (4.8)$$

where

$$g_s^0(12) = g_{s'}(1) g_{s''}(2) - g_{s''}(1) g_{s'}(2),$$

$$g_{s'}(1) = (0 | \phi(1) | s'),$$

$s'$  and  $s''$  being the one nucleon states of the two incident particles. The superscript 0 denotes the antisymmetrized product of the covariant components of the individual particles.

The function  $K$  in the integrand of (4.6) cannot be separated as (4.7) or (4.8) since  $t_7$  and  $t_8$  are integration parameters and can assume from  $-\infty$  to  $+\infty$ . However, for fixed values of  $t_1$  and  $t_2$ , contributions from the region  $t_7, t_8 < t_3$ ,  $t_4$  would finally be neglected in the limit  $t_3, t_4 \rightarrow -\infty$ . This is correct only when  $G$  corresponds to a short range force.

Hence for  $t_3, t_4 \rightarrow -\infty$ , we have

$$\sum_s g_s(12) \bar{g}_s(34) \rightarrow \sum_s g_s^0(12) \bar{g}_s^0(34) + \sum_s \int d\omega_5 \cdots d\omega_8 S_F'(1-5) S_F'(2-6) \\ \times G(56; 78) g_s(78) \cdot \bar{g}_s(34). \quad (4.9)^*$$

Let  $s$  be a bound state and compare the Fourier components of the both sides with respect to  $x_1$  and  $x_2$ .

It is obvious that the  $g^0$  term does not contain such component that corresponds to the bound state  $s$ .

Hence for degenerating bound states, we have

$$\sum_{\text{bound}} g_s(12) \bar{g}_s(34) = \sum_{\text{bound}} \int d\omega_5 \cdots d\omega_8 S_F'(1-5) S_F'(2-6) G(56; 78) g_s(78) \cdot \bar{g}_s(34),$$

the summation is extended over all bound states with the same total energy and momentum. If the degeneracy is got rid of by using suitable quantum numbers, the above equation will be reduced to

$$g_s(12) = \int d\omega_5 \cdots d\omega_8 S_F'(1-5) S_F'(2-6) G(56; 78) g_s(78). \quad (4.10a)$$

This is the well-known Bethe-Salpeter equation for bound states.

Next we shall study the equation for scattering states. In this case the factorized equation (4.9) follows only under the assumption that the contributions from the region  $t_7, t_8 \rightarrow -\infty$  can be neglected, and this assumption should consistently be taken into account for the derivation of the scattering equation.

Let us consider such scattering problem that the interaction is switched on adiabatically. We replace  $G$  by

$$G(56; 78) \rightarrow e^{\varepsilon t} G(56; 78), \quad t = \frac{1}{4}(t_5 + t_6 + t_7 + t_8), \quad \text{for } t < 0 \quad (4.11)$$

where  $\varepsilon$  is a positive infinitesimal quantity.

\*) Rigorously speaking the limiting procedures must be performed as  $\lim_{t_3 \rightarrow -\infty} \lim_{t_4 \rightarrow -\infty}$  for bound states, and  $\lim_{t_3 \rightarrow -\infty} \lim_{t_4 \rightarrow -\infty}$  for scattering states.

This is the hypothesis of adiabatic switching for the interaction between dressed particles, and justifies the eq. (4.9).

It is worth while noticing that we have not assumed the hypothesis for bare particles since it requires to substitute

$$H_{int}(t) \rightarrow e^{\epsilon t} H_{int}(t), \quad (4.12)$$

by which not only the interaction between dressed particles but also the interaction of a particle with its self field is adiabatically switched on. This difference is especially important in treating scattering problems involving composite particles.

The substitution (4.11) requires at remote past

$$\lim_{t_1 \rightarrow -\infty} \lim_{t_2 \rightarrow -\infty} (g_s(12) - g_s^0(12)) = 0 \quad (4.13)$$

for scattering states.

Hence combining (4.13) with (4.9), we have the scattering equation

$$g_s(12) = g_s^0(12) + \int d\omega_5 \cdots d\omega_s S'_F(1-5) S'_F(2-6) G(56; 78) g_s(78). \quad (4.10b)$$

The hypothesis assumed here must be proved on the wave packet formalism.

In the above discussion, the essential point is the factorization of the kernels from which we have derived the equations for the covariant components.

We shall abstract this procedure by introducing a new kind of chronological operators and apply the above method to the derivation of the  $S$  matrix and the expectation values.

#### *Chronological operators "in" and "out"*

In the above derivation, we have utilized the limiting procedure  $t \rightarrow -\infty$ . In order to exploit this device in a wider class of problems, we shall introduce an operational superscript "in".

This superscript is applied to quantized wave functions whose time arguments are to be brought to  $-\infty$ .

Then from the definition of the  $T$ -product, we see

$$T[AB \cdots C^{\text{in}} D^{\text{in}} \cdots] = T[AB \cdots] T[C^{\text{in}} D^{\text{in}} \cdots]. \quad (4.14)$$

The time arguments of  $C, D, \cdots$  are not necessarily equal to  $-\infty$ , however, we treat them as if they were brought to remote past. This concept is more or less similar to that of incoming field of Yang and Feldman<sup>15)</sup> and we have employed the notation "in".

The self consistency of such operations are justified provided that the limiting procedures  $\lim_{t_C \rightarrow -\infty}, \lim_{t_D \rightarrow -\infty}, \cdots$  are commutative with each other. This point will be discussed in later sections.

In some cases, we need to fix the means of limiting procedures, e.g., when it is required to take the following limit

$$\lim_{t_C = t_D \rightarrow -\infty}$$

we symbolize the operation by writing it in place of (4.14) as,

$$T[AB \cdots (CD)^{\text{in}} \cdots]. \quad (4.15)$$

This operation is necessary in the treatment of scattering problems involving composite particles.

Quite similarly to the operator "in", we introduce another superscript "out" corresponding to the limiting procedure  $t \rightarrow +\infty$ . Then corresponding to (4.14), we have

$$T[A^{\text{out}} B^{\text{out}} \dots CD \dots] = T[A^{\text{out}} B^{\text{out}} \dots] T[CD \dots]. \quad (4.16)$$

Though most of the properties of these operators are readily derived from eqs. (4.14) and (4.16), there is an additional important formula which enables us to eliminate the superscripts. For the sole occurrence of the superscript in the matrix elements, we have

$$\langle r | A^{\text{out}} | s \rangle = \langle r | A^{\text{in}} | s \rangle = \langle r | A | s \rangle, \quad (4.17)$$

or  $\langle r | T[(AB)^{\text{out}}] | s \rangle = \langle r | T[(AB)^{\text{in}}] | s \rangle = \langle r | T[AB] | s \rangle$ ,

since there is no other time argument to be compared to  $t_A$  in the above expressions.

With these properties the readers may easily verify

$$g_s(1^{\text{in}}, 2^{\text{in}}) = g_s^0(1, 2),$$

which is necessary for the consistency of these operations. Cf. (4.13)

$$\text{ex. 1} \quad \langle T[\psi^{\text{out}}(1) \bar{\psi}(2)] \rangle_{vac}.$$

$$\langle 0 | T[\psi^{\text{out}}(1) \bar{\psi}(2)] | 0 \rangle = \sum_s \langle 0 | \psi^{\text{out}}(1) | s \rangle \langle s | \bar{\psi}(2) | 0 \rangle \quad \text{from (4.16)}$$

$$= \sum_s \langle 0 | \psi(1) | s \rangle \langle s | \bar{\psi}(2) | 0 \rangle. \quad \text{from (4.17)}$$

In general the matrix elements of chronologically ordered products of operators involving "in" or "out" are computed by using (4.14), (4.16) and (4.17). Thus they can be expressed in terms of Green functions and covariant components.

It must be noticed, however, that quantities like

$$\langle 0 | T[A^{\text{in}} B^{\text{in}}] | 0 \rangle \quad \text{or} \quad \langle 0 | T[A^{\text{out}} B^{\text{out}}] | 0 \rangle$$

cannot be defined uniquely since  $\lim_{t_A \rightarrow \pm\infty}$  and  $\lim_{t_B \rightarrow \pm\infty}$  do not commute with each other.

In this connection, the normal product is more convenient than the  $T$ -product since quantities like above are subtracted in the normal product. Cf. (3.14)

$$\text{ex. 2.} \quad \langle T[\psi(1) \bar{\psi}^{\text{in}}(2) \phi(3)] \rangle_{vac}.$$

By letting  $Q$  vanish in (2.22), we have with the aid of (3.2)

$$\langle 0 | T[\psi(1) \bar{\psi}(2) \phi(3)] | 0 \rangle = \eta \int d\omega_1 d\omega_3 d\omega_6 S_F'(1-4) \odot (45; 6) S_F'(5-2) J_F'(6-3).$$

Hence we have utilizing the result of ex. 1

$$\begin{aligned} \langle 0 | T[\psi(1) \bar{\psi}^{\text{in}}(2) \phi(3)] | 0 \rangle &= \eta \int d\omega_1 d\omega_3 d\omega_6 S_F'(1-4) \odot (45; 6) J_F'(6-3) \\ &\quad \times [0 | T(\psi(5) \bar{\psi}^{\text{in}}(2)) | 0] \end{aligned}$$

or

$$\sum_s g_s(1; 3) \cdot \bar{g}_s(2) = \eta \sum_s \int d\omega_1 d\omega_3 d\omega_6 S_F'(1-4) \odot (45; 6) J_F'(6-3) g_s(5) \cdot \bar{g}_s(2),$$

where

$$g_s(1; 3) = (0|T[\phi(1)\phi(3)]|s), \quad g_s(5) = (0|\phi(5)|s).$$

From the above factorized equation, we obtain

$$g_s(1; 3) = \gamma \int d\omega_4 d\omega_5 d\omega_6 S'_F(1-4) \odot (45; 6) g_s(5) \cdot A'_F(6-3),$$

provided that  $\bar{g}_s(2) \neq 0$ .

This result has already been given in I.

## § 5. Expectation values and normalization of covariant components

At present the central problem in the Bethe-Salpeter theory will be the exposition of the physical meaning of the wave functions.

This is achieved by exhibiting the relation of the B-S wave functions or synonymously the covariant components to the probability amplitudes.

In I, this problem was investigated by introducing the contravariant components together with the covariant components. However, the computation of the former quantities is too complicated for practical purposes. Besides its introduction seems to be redundant as suggested in the appendix of I. Hence we shall reformulate the theory developed in I without any use of the contravariant components.

Then the problem to be settled in this section is to express

$$(r|W|s) = (\Psi_r, W \Psi_s) \quad (5.1)$$

in terms of covariant components.  $W$  is an arbitrary observable whose expectation value is in question.

Without loss of generality we assume that  $W$  is a point function, i.e., a local quantity. Then we can define

$$g_{s,W}(12\cdots; 0) = (0|T[:A(1)B(2)\cdots:W(0)]|s),$$

where colons denote the  $S$ - or normal product, written as  $N$  before.

If we further assume  $t_1, t_2, \cdots > t_0$ , then the above expression can be factorized

$$g_{s,W}(12\cdots; 0) = \sum_r g_r(12\cdots) W_{rs}(0), \quad (5.2)$$

$$g_r(12\cdots) = (0|N[A(1)B(2)\cdots]|r), \quad (5.3)$$

$$\text{and} \quad W_{rs}(0) = (r|W(0)|s).$$

The relation (5.2) can also be expressed using "in" as

$$\begin{aligned} & (0|T[:A(1)B(2)\cdots:W^{\text{in}}(0)]|s) \\ &= \sum_r (0|N[A(1)B(2)\cdots]|r) (r|W(0)|s), \end{aligned} \quad (5.4)^*$$

\* Alternatively we can write (5.4) as

$$(0|T[:A(1)B(2)\cdots:]^{\text{out}}W(0)]|s).$$



upon which we need not impose the temporal condition  $t_1, t_2, \dots > t_0$ .

For simplicity, we shall write (5.4) as

$$g_{sW}(1, 2, \dots; 0^{\text{int}}) = \sum_r g_r(1, 2, \dots) \mathbf{H}_{rs}^*(0). \quad (5.5)$$

This is nothing but the equation for  $\mathbf{H}_{rs}^*(0)$  provided that the covariant components  $g_{sW}$  and  $g_r$  are known.

It is interesting that this equation can formally be solved with the aid of Green functions, and we shall illustrate the solution of this equation by an example.

ex. 1. *The expectation value of  $\bar{\psi}\psi$  : in one nucleon state.*

This problem has already been solved in the appendix of I, however, we shall repeat it as an illustration of the new technique.

$$g_{sW}(1; 2) = (0|T[\psi(1) : \bar{\psi}(2)\psi(2)]|s).$$

As given in I,  $g_{sW}$  can be converted into

$$g_{sW}(1; 2) = S_F'(1-2)g_s(2) + \int d\omega_3 \dots d\omega_8 S_F'(1-5)S_F'(2-6)G(56; 78)\chi_s(78; 2), \quad (5.6)$$

where

$$\begin{aligned} \chi_s(12; 3) &= (0|T[\psi(1)\psi(2)\bar{\psi}(3)]|s) \\ &= S_F'(2-3)g_s(1) - S_F'(1-3)g_s(2) \\ &\quad - \int d\omega_3 \dots d\omega_8 K(12; 56)G(56; 78)S_F'(7-3)g_s(8), \\ g_s(1) &= (0|\psi(1)|s). \end{aligned}$$

Substituting  $g_{sW}(1^{\text{int}}; 2)$  for  $g_{sW}(1; 2)$  in (5.6), one obtains

$$\begin{aligned} g_{sW}(1; 2^{\text{int}}) &= \sum_r g_r(1)\mathbf{H}_{rs}^*(2) \\ &= \sum_r g_r(1)[\bar{r}(2)g_s(2) + \int d\omega_3 \dots d\omega_8 \bar{r}(5)S_F'(2-6)G(56; 78)\chi_s(78; 2)], \end{aligned} \quad (5.7)$$

where use has been made of the relation

$$\begin{aligned} S_F'(1^{\text{int}}-2) &= (0|T[\psi^{\text{int}}(1)\bar{\psi}(2)]|0) \\ &= \sum_r g_r(1)\bar{r}(2). \end{aligned} \quad (5.8)$$

Hence we have the solution

$$\mathbf{H}_{rs}^*(2) = g_r(2)g_s(2) + \int d\omega_3 \dots d\omega_8 \bar{r}(5)S_F'(2-6)G(56; 78)\chi_s(78; 2). \quad (5.9)$$

An alternative form of this expression will be found in the appendix.

From the above consideration, one will easily see the general means to obtain the solutions of such problems. A more complicated example is given in the appendix.

The next question is how to normalize the covariant components. For this solution we utilize the inverse procedure of the foregoing problem.

A stationary state is determined by fixing a set of quantum numbers involving the total energy-momentum, and the expectation values of these quantum numbers are known before the normalization is settled. While they can be expressed in terms of the covariant components as we have seen before. Hence by equating them we can fix the normalization of the covariant components.

Let  $\Psi_s$  and  $Q$  be such a state and such a conservative quantity satisfying

$$Q\Psi_s = Q_s\Psi_s, \quad (\Psi_s, \Psi_s) = \delta(r, s).$$

then we may write

$$(r|Q|s) = Q_s\delta(r, s).$$

The quantum numbers available for this purpose are such ones as the total energy-momentum, total charge, isotopic spin and number of nucleons of the system.

In general these quantities are expressed as the spatial integrals of densities defined by

$$Q_A = \int_{\sigma} q_{A\mu}(x) d\sigma_{\mu} (= \int q_{A0}(x) d^3x \text{ for a flat } \sigma). \quad (5.10)$$

The quantum number density  $q$  is a local quantity, so that its expectation value and therefore that of  $Q$ , the spatial integral of the former, can be expressed in terms of the covariant components. We can choose the canonical energy momentum tensor, charge current density etc. as  $q$ .

The condition which manifests that  $Q$  is a quantum number is

$$\frac{\partial q_{A\mu}}{\partial x_{\mu}} = 0. \quad (5.11)$$

The present idea will be understood in the following example.

ex. 2. *The normalization of the covariant component of a one nucleon state.*

In this case it will be most convenient to choose the nucleon number as  $Q$ . Then the normalization is determined from

$$i \int_{\sigma} (r|N[\bar{\psi}\gamma_{\mu}\psi]|s) d\sigma_{\mu} = \delta(r, s). \quad (5.12)$$

The integrand has been expressed in terms of covariant components in the foregoing example. Hence (5.12) assumes the form

$$i \int d\sigma_{\mu}(2) [\bar{g}_{\mu}(2) \gamma_{\mu} g_s(2) + \int d\omega_5 \cdots d\omega_s \bar{g}_{\mu}(5) S'_F(2-6) G(56; 78) \chi_s(78; 2) \gamma_{\mu}(2)] \\ = \delta(r, s), \quad (5.13)$$

in which the last  $\gamma_{\mu}(2)$  combines the spinor indices of  $S'_F(2-6)$  with those of  $\chi_s(78; 2)$ .

The function  $\chi_s$  is known to be expressible in  $g$  so that the normalization of  $g$  is fixed by this equation.

Let us study this problem from a more general point of view without being bothered by the concrete expression (5.13).

We define the "bare particle probability of a nucleon"  $\phi_N$  by

$$b_N = \frac{i \int d\sigma_\mu \bar{g}_\mu \gamma_\mu g}{i \int d\sigma_\mu [g_\mu \gamma_\mu g + \dots]}, \quad (5.14)$$

in which the expression (5.13) should be substituted for the denominator.

Then we have from eqs. (5.13) and (5.14)

$$i \int d\sigma_\mu g_\mu \gamma_\mu g_s = b_N \partial(r, s). \quad (5.15)$$

The free particle wave function  $g_{s1}(1)$  is defined by

$$(\phi_0, \psi(1) \phi_s) = g_{s1}(1) \quad (5.16)$$

where  $\phi_0$  is the free vacuum and  $\phi_s$  is the free nucleon state corresponding to the dressed nucleon state  $\Psi_s$ .

Then it satisfies

$$(\gamma \partial + M) g_{s1} = 0. \quad (5.17a)$$

On the other hand  $g_s$  satisfies

$$\mathcal{D}_N g_s = (\gamma \partial + M + i \sum^*) g_s = 0 \text{ for } +\infty > t > -\infty \quad (5.17b)$$

in which we understand that  $H_{\text{self}}$  in (2.1) is already taken into account, i.e.  $\partial M$  is subtracted in the above  $\sum^*$ . Then as is well known eq. (5.17b) has only such a solution as to satisfy

$$(\gamma \partial + M) g_s = \sum^* g_s = 0 \quad (5.18)$$

provided that there is no stable nucleon isobar with spin  $\frac{1}{2}$  and isotopic spin  $\frac{1}{2}$ .

Hence  $g_s$  differs from  $g_{s1}$  only by a constant factor

$$g_s \propto g_{s1}. \quad (5.19)$$

The normalization of  $g_s$  is given by

$$i \int d\sigma_\mu \bar{g}_\mu \gamma_\mu g_{s1} = \partial(r, s), \quad (5.20)$$

so that the normalization of  $g_s$  is fixed with recourse to eqs. (5.15), (5.19) and (5.20) by

$$g_s = b_N^{1/2} g_{s1}. \quad (5.21)$$

It will be instructive to express  $b_N$  in terms of familiar quantities.

$$\begin{aligned} S'_F(1^{\text{out}} - 2) &= (0 | T[\phi^{\text{out}}(1) \bar{\psi}(2)] | 0) \\ &= \sum_s g_s(1) \bar{g}_s(2) \\ &= b_N \sum_s g_{s1}(1) \bar{g}_{s1}(2) \\ &= b_N S_F(1^{\text{out}} - 2). \end{aligned}$$

In a similar way we have

$$S'_F(1^{\text{in}} - 2) = b_N S_F(1^{\text{in}} - 2).$$

From the above equations we see

$$S'_F(p) \rightarrow b_N S_F(p) \quad \text{for} \quad i\hat{p}\gamma + M \rightarrow 0 \quad (5.22)$$

in the momentum representation.

According to the general procedure in the renormalization theory,<sup>16)</sup> we must put

$$\begin{aligned} S'_F &= Z_2 S'_F, \\ \Delta'_F &= Z_3 \Delta'_{F1}, \\ \mathcal{O} &= Z_1^{-1} \mathcal{O}_1, \\ \gamma_i &= Z_1 (Z_2 Z_3^{1/2})^{-1} \gamma_{i1}, \end{aligned} \quad (5.23)$$

where  $Z_2$  and  $Z_3$  are so chosen as to yield

$$\begin{aligned} S'_{F1}(p) &\rightarrow S_F(p) \quad \text{for} \quad i\hat{p}\gamma + M \rightarrow 0, \\ \Delta'_{F1}(q) &\rightarrow \Delta_F(q) \quad \text{for} \quad q^2 + \mu^2 \rightarrow 0. \end{aligned} \quad (5.24)$$

Hence by comparing eqs. (5.22), (5.23) and (5.24) we have

$$b_N = Z_2. \quad (5.25)$$

Similarly we have

$$b_M = Z_3, \quad (5.26)$$

where  $b_M$  is the "bare particle probability of a meson". The equation (5.21) is written as

$$g_s = Z_2^{1/2} g_{s1}. \quad (5.27)$$

We see that Dyson's renormalization of wave functions is needed for reasons of the normalization of covariant components in our language. Such a consideration has previously been reached by Karplus and Kroll.<sup>17)</sup>

The normalization problem for a single particle state is equivalent to the renormalization problem.

## § 6. Theory of scattering—The S matrix

In the previous section the normalization of covariant components was settled. Hence we shall investigate how the S matrix is expressed in terms of these normalized covariant components. Especially the differences between the scattering matrices for dressed particles and for bare particles will be accounted for.

One will again find that the chronological operator "in" or "out" is useful in this problem.

(a) *The S matrix describing the scattering of dressed particles.*

In the conventional Feynman-Dyson theory a scattering process consists of three steps. The bare particles are first adiabatically dressed; then interact with each other so that the scattering takes place and finally they are undressed to be free again.

It will become clear later that the  $U$  matrix is obtained in this way as scattering matrix.

So long as we are concerned with such processes that involve no composite particle this method gives a correct result provided that the renormalization procedure is properly taken into account.

However it is clear that this method fails to treat scattering problems involving composite particles since the bound state cannot be formed by introducing the field-particle interaction adiabatically. For this reason we shall investigate the scattering theory in which particles are dressed from the very beginning. In the present treatment, the interaction between dressed particles is switched on and off instead of the field-particle interaction  $H_{\text{int}}$ . Cf. (4.11) and (4.12).

We shall illustrate the treatment by the nucleon-nucleon scattering

$$g_a(12) = g_a^0(12) + \int d\omega_1 \cdots d\omega_n S'_F(1-5) S'_F(2-6) G(56; 78) g_a(78). \quad (6.1)$$

The  $S$  matrix is obtained from the asymptotic form of  $g_a^{(s)}$

$$g_a(12) \sim \sum_b S_{ba} g_b^0(12) = \sum_b S_{ba} g_b(1^{\text{in}}, 2^{\text{in}}) \quad (6.2)$$

for the elastic scattering and

$$g_a(12; 1'2'\cdots) \sim \sum_b S_{ba} g_b^0(12; 1'2'\cdots) \quad (6.2')$$

for the inelastic scattering.  $g_a$  is defined by

$$g_a(12; 1'2'\cdots) = (0|V[\phi(1)\phi(2)\phi(1')\phi(2')\cdots]|s),$$

and  $g_b^0$  is the symmetrized (for Bosons) or antisymmetrized (for Fermions) product of the covariant components of individual dressed particles as given by (4.8).

Next we shall construct the asymptotic forms of the covariant components. In the elastic nucleon-nucleon collisions nucleons are separated far away from each other in the final states. In I the asymptotic form corresponding to such a boundary condition has been given. In the present work we shall give a more convenient and compact means to take account of the boundary condition.

As seen from § 3, the Green function satisfies the positive frequency condition so that a part of its Fourier transform corresponding to the particle propagation from past to future should involve a factor  $\partial_+(E_i - E_f)$ . In order to construct the asymptotic form, we must retain this factor replacing it by  $\partial(E_i - E_f)$ .

This can be done utilizing the relation.

$$\lim_{t \rightarrow +\infty} e^{-i\omega t} \partial_+(\omega) = \partial(\omega). \quad (6.3)$$

Hence the desired asymptotic form is obtained as



$$\lim_{t_1 \rightarrow +\infty} \lim_{t_2 \rightarrow +\infty} g_a(12), \quad (6.4)$$

where  $\lim_{t_1 \rightarrow +\infty}$  plays the rôle of carrying the nucleon "1" away and  $\lim_{t_2 \rightarrow +\infty}$  has a similar meaning. It is worthy to be noticed that these two limiting operations commute with each other.

Indeed we see in this case

$$\lim_{t_1 \rightarrow +\infty} \lim_{t_2 \rightarrow +\infty} S'_F(1-5) S'_F(2-6) \rightarrow \sum_{b', b''} g_{b'}(1) \bar{g}_{b'}(5) \cdot g_{b''}(2) \bar{g}_{b''}(6),$$

and hence the asymptotic form of  $g_a$  is given by

$$g_a(12) \sim g_a^0(12) + \frac{1}{2} \sum_b g_b^0(12) \int d\omega_5 \cdots d\omega_s \bar{g}_b^0(56) G(56; 78) g_a(78). \quad (6.5)$$

The above asymptotic form can be written as

$$g_a(12) \sim g_a(1^{\text{out}}, 2^{\text{out}}) = (0 | \mathcal{N} [\phi^{\text{out}}(1) \phi^{\text{out}}(2)] | s).$$

This result can further be generalized:

*Theorem.* The asymptotic form of  $g(1, 2 \cdots)$  is given by

$$g(1^{\text{out}}, 2^{\text{out}}, \cdots)$$

when particles 1 and 2 are not bound in the final state, and

$$g((1, 2)^{\text{out}}, \cdots)$$

when particles 1 and 2 are bound to form a composite particle in the final state.

For instance, the asymptotic form of  $g_s(12; 3)$  corresponding to

$$p + p \rightarrow d + \pi^+$$

is given by

$$g_s(12; 3) \sim (0 | \mathcal{N} [(\phi(1) \phi(2))^{\text{out}} \phi^{\text{out}}(3)] | s).$$

In this case we must express  $g_s(12; 3)$  in terms of  $K(12; \cdots)$  so that we can utilize the formula

$$\begin{aligned} & (0 | T [(\phi(1) \phi(2))^{\text{out}} \bar{\phi}(3) \bar{\phi}(4)] | 0) \\ &= \sum_s (0 | \mathcal{N} [\phi(1) \phi(2)] | s) (s | \mathcal{N} [\bar{\phi}(3) \bar{\phi}(4)] | 0). \end{aligned}$$

In this way the use of the chronological operator "out" enables one to readily write down the asymptotic forms of the covariant components for given boundary conditions.

On comparing (6.5) with the definition of  $S$ , (6.2), we have

$$S_{ba} = \partial_{ba} + \frac{1}{2} \int d\omega_5 \cdots d\omega_s \bar{g}_b^0(56) G(56; 78) g_a(78). \quad (6.6a)$$

The eq. (2.30) can be written in the reciprocal form

$$K(12; 34) = S'_F(12; 34) + \int d\omega_5 \cdots d\omega_s K(12; 56) G(56; 78) S'_F(7-3) S'_F(8-4). \quad (2.30')$$

This can be proved by expanding  $K$  in powers of  $G$  or more rigorously by the operation of charge conjugation.

With the aid of (2.30'), (6.6a) can also be written in the following reciprocal form

$$S_{ba} = \delta_{ba} + \frac{1}{2} \{ d\omega_5 \cdots d\omega_s g_b(56) G(56; 78) g_a^0(78) \}. \quad (6.6b)$$

Here we shall stress the apparent affinity of the present theory to the Lippmann-Schwinger theory.<sup>18)</sup> The following formulas correspond to (6.6a) and (6.6b)

$$S_{ba} = \delta_{ba} - 2\pi i \delta(E_b - E_a) (\phi_b, H_1 \Psi_a^{(+)} ) \quad (6.6a')$$

$$= \delta_{ba} - 2\pi i \delta(E_b - E_a) (\Psi_b^{(-)}, H_1 \phi_a) . \quad (6.6b')$$

The equations that correspond to (2.30) and (2.30') are given by

$$\begin{aligned} \frac{1}{E + i\varepsilon - H_0 - H_1} &= \frac{1}{E + i\varepsilon - H_0} + \frac{1}{E + i\varepsilon - H_0} H_1 \frac{1}{E + i\varepsilon - H_0 - H_1} \\ &= \frac{1}{E + i\varepsilon - H_0} + \frac{1}{E + i\varepsilon - H_0 - H_1} H_1 \frac{1}{E + i\varepsilon - H_0} . \end{aligned}$$

The equation (6.1) is obviously the analogue of the Lippmann-Schwinger equation

$$\Psi_a^{(+)} = \phi_a + \frac{1}{E + i\varepsilon - H_0} H_1 \Psi_a^{(+)} .$$

The equation that corresponds to the Chew-Goldberger's formal solution<sup>19)</sup>

$$\Psi_a^{(+)} = \phi_a + \frac{1}{E + i\varepsilon - H_0 - H_1} H_1 \phi_a$$

is readily found to be

$$g_a(12) = g_a^0(12) + \frac{1}{2} \{ d\omega_5 \cdots d\omega_s K(12; 56) G(56; 78) g_a^0(78) \} \quad (6.7)$$

which follows from (2.30').

A deeper examination shows that the correspondence of these theories are not only apparent but also essential. Hence we can easily translate the results in one of these two theories into another.

For instance, we know the following stationary expression of the transition matrix  $R = S - 1$

$$\begin{aligned} \langle \langle R_{ba} \rangle \rangle &= (\Psi_b^{(-)}, H_1 \phi_a) + (\phi_b, H_1 \Psi_a^{(+)} ) - (\Psi_b^{(-)}, H_1 \Psi_a^{(+)} ) \\ &\quad + \left( \Psi_b^{(-)}, H_1 \frac{1}{E + i\varepsilon - H_0} H_1 \Psi_a^{(+)} \right) , \end{aligned}$$

where  $R_{ba} = -2\pi i \delta(E_b - E_a) R_{ba}$ , and this expression can be translated into the language of the present formalism.

The expression

$$\langle \langle R_{ba} \rangle \rangle = \{ \bar{g}_b G g_a + \{ \bar{g}_b^0 G g_a - \{ \bar{g}_b G g_a + \{ \bar{g}_b G S_F' G g_a \} \} \} \quad (6.8)$$

is stationary for arbitrary variations of  $g_a$  and  $g_b$  about the solutions of (6.1) and the stationary value of " $R_{ba}$ " is  $R_{ba}$ .

For simplicity we have dropped arguments in (6.8), and  $S'_F$  is the abbreviation of the product of two  $S'_F$  functions.

Indeed the variation of " $R_{ba}$ " is given by

$$\delta \langle R_{ba} \rangle = \int \partial \bar{g}_b \cdot G (g_a^0 - g_a + \{ S'_F G g_a \} + \{ (\bar{g}_b^0 - \bar{g}_b + \{ \bar{g}_b G S'_F \}) G \delta g_a,$$

and the first term vanishes due to (6.1) and the second term due to the equation reciprocal to (6.1).

The reason why we employ the present formalism in spite of the apparent equivalence to the simpler Lippmann-Schwinger formalism consists in the fact that the separation of the interaction into those between bare particles and their self fields *e.g.*  $\sum^*$  in (2.18) and those between dressed particles *e.g.*  $G$  in (2.30) cannot rigorously be realized by splitting the interaction Hamiltonian  $H_{int}$ . More plainly speaking a bare particle cannot be dressed by means of canonical transformations or synonymously unitary transformations since quantities like  $\sum^*$  and  $G$  are not necessarily real.

The most characteristic feature of the present formalism will be exhibited in the treatment of the scattering processes involving composite particles which had not been dealt with in the original Feynman-Dyson theory.

The general ideas are already given in I and in the present work, however, for the sake of completeness we shall give an example

$$p + p \rightarrow d + \pi^+ \quad (6.9)$$

The matrix element of this process was derived in I

$$\begin{aligned} S_{ba} = & \frac{\eta}{2} \int d\omega_0 \cdots d\omega_4 \bar{g}_d(12) g_m^*(0) \mathcal{O}(12; 34; 0) g_a(34) \\ & - \frac{i\eta}{2} \int d\omega_0 \cdots d\omega_8 \bar{g}_d(12) g_m^*(0) \mathcal{O}(23; 0) \mathcal{D}(1, 4) \\ & \times S'_F(34; 56) G(56; 78) g_a(78), \end{aligned} \quad (6.10)$$

where  $g_d$  and  $g_m$  are the covariant components for the final deuteron state and the final pion state respectively, and

$$\begin{aligned} \mathcal{D}(1, 2) &= \mathcal{D}_N(1) \delta(1-2), \\ i \frac{\partial}{\partial Q(0)} G(12; 34) &= \eta \int d\omega_5 \mathcal{O}(12; 34; 5) A'_F(5, 0). \end{aligned} \quad (6.11)$$

Finally we shall give some remarks.

For a stable one particle state  $a$  either elementary or composite we have

$$g_a(\cdots) = g_a(\cdots), \quad (6.12)$$

or more precisely  $g_a^0(1) = g_a(1)$ , for an elementary particle,  
 $g_a^0(1, 2, \dots) = g_a(1, 2, \dots)$ , for a composite particle,

from which we see

$$g_a((\dots)^{\text{int}}) = g_a(\dots) = g_a((\dots)^{\text{in}}) = g_a^0(\dots),$$

and hence

$$S_{aa} = 1, \quad R_{aa} = 0. \quad (6 \cdot 13)$$

Another remark is concerned with the reason for the necessity of the normal product.

The expression

$$(0 | T[A^{\text{int}}(1) B^{\text{int}}(2)] | 0)$$

has not definite meaning since two limiting procedures  $\lim_{t_1 \rightarrow \infty}$  and  $\lim_{t_2 \rightarrow \infty}$  do not commute with each other as seen from

$$\begin{aligned} (0 | T[A(2) B(2)] | 0) &= \sum_s (0 | A(1) | s) (s | B(2) | 0) \quad \text{for } t_1 > t_2, \\ &= \sum_s (0 | B(2) | s) (s | A(1) | 0) \quad \text{for } t_2 > t_1. \end{aligned}$$

Such terms as above do not appear in the covariant components since they are defined by using the normal product but not the T-product. (§ 4)

(b) *The S matrix describing the scattering of bare particles.*

In order to understand that the foregoing treatment really corresponds to the description of the scattering of dressed particles it seems instructive to compare the theory with the scattering of bare particles.

In the former treatment the interaction of particles with their self-field has been taken into account over the whole stage and the interaction between dressed particles is adiabatically switched on and off. In addition, if there were composite particles in the initial or final states some of the interaction between dressed particles was retained from the very beginning.

While in the latter treatment the whole interaction  $H_{\text{int}}$  is adiabatically switched on and off.

*Hypothesis of adiabatic switching*

That we introduce  $H_{\text{int}}$  adiabatically means to assume as in I

$$\begin{aligned} &\lim_{\substack{t \rightarrow +\infty \\ t' \rightarrow -\infty}} \lim_{\substack{\tau \rightarrow +\infty \\ \tau' \rightarrow -\infty}} T[U(\tau, \tau') A(t) \cdots A'(t') \cdots] \\ &= \lim_{\substack{\tau \rightarrow +\infty \\ \tau' \rightarrow -\infty}} \lim_{\substack{t \rightarrow +\infty \\ t' \rightarrow -\infty}} T[U(\tau, \tau') A(t) \cdots A'(t') \cdots]. \end{aligned} \quad (6 \cdot 14)$$

The first term describes the scattering of dressed particles and the second one the scattering of bare particles, and they are equated upon the adiabatic hypothesis.

Then we may write

$$\begin{aligned}
 & (\Psi_0, T[A^{\text{out}}(1)B^{\text{out}}(2)\cdots C(3)D(4)\cdots]\Psi_0) \\
 &= \frac{(\Phi_0, T[A(1)B(2)\cdots]T[U(\infty, -\infty)C(3)D(4)\cdots]\Phi_0)}{(\Phi_0, U(\infty, -\infty)\Phi_0)} \\
 &= \sum_s (\Phi_0, T[A(1)B(2)\cdots]\Phi_s) \cdot (\Phi_s, T[U(\infty, -\infty)C(3)D(4)\cdots]\Phi_0) / \langle U \rangle_{\text{vac}}
 \end{aligned} \tag{6.15}$$

and we have similar expressions for the use of “in”. Clearly there is no bound state involved in the summation. Let us substitute  $3^{\text{in}}$  and  $4^{\text{in}}$  for 3 and 4 in the following expression:

$$K(12; 34) = S'_F(12; 34) + \int d\omega_5 \cdots d\omega_n S'_F(1-5) S'_F(2-6) G(56; 78) K(78; 34).$$

We may write

$$\begin{aligned}
 K(12; 3^{\text{in}} 4^{\text{in}}) &= - \sum_a (\Phi_0, T[U\psi(1)\psi(2)]\Phi_a) (\Phi_a, \bar{\psi}(3)\bar{\psi}(4)\Phi_0) / \langle U \rangle_{\text{vac}}, \\
 S'_F(12; 3^{\text{in}} 4^{\text{in}}) &= - \sum_{a', a''} [(\Phi_0, T[U\psi(1)]\Phi_{a'}) (\Phi_{a'} \bar{\psi}(3)\Phi_0) \\
 &\quad \cdot (\Phi_0, T[U\psi(2)]\Phi_{a''}) (\Phi_{a''} \bar{\psi}(4)\Phi_0) - (3 \leftrightarrow 4)] / \langle U \rangle_{\text{vac}}^2.
 \end{aligned}$$

Inserting the above into the integral equation for  $K$  and dividing it by  $(\Phi_a, \bar{\psi}\bar{\psi}\Phi_0)$ , we have

$$f_a(12) = f_a^0(12) + \int d\omega_3 \cdots d\omega_n S'_F(1-5) S'_F(2-6) G(56; 78) f_a(78) \tag{6.16}$$

provided that

$$(\Phi_a, \bar{\psi}(3)\bar{\psi}(4)\Phi_0) = (\Phi_a, \bar{\psi}(3)\Phi_0) (\Phi_{a'}, \bar{\psi}(4)\Phi_0) - (3 \leftrightarrow 4)$$

does never vanish.  $f$ 's are defined by

$$\begin{aligned}
 f_a(12) &= (\Phi_0, T[U\psi(1)\psi(2)]\Phi_s) / \langle U \rangle_{\text{vac}}, \\
 f_a^0(12) &= f_{a'}(1) f_{a''}(2) - (1 \leftrightarrow 2), \\
 f_{a'}(1) &= (\Phi_0, T[U\psi(1)]\Phi_{a'}) / \langle U \rangle_{\text{vac}} = Z_2(\Phi_0, \psi(1)\Phi_{a'}),
 \end{aligned} \tag{6.17}$$

where the final expression is found in reference to § 5.

The equation (6.16) holds only when  $f_a^0(12) \neq 0$  and hence only for scattering states, so that such a treatment can never give rise to bound states.

The asymptotic form is found to be

$$\begin{aligned}
 f_a(12) &\sim f_a(1^{\text{out}}, 2^{\text{out}}) = \sum_b (\Phi_0, \psi(1)\psi(2)\Phi_b) (\Phi_b U \Phi_a) / \langle U \rangle_{\text{vac}}, \\
 &= (Z_2^2 \langle U \rangle_{\text{vac}})^{-1} \sum_b f_b^0(12) U_{ba},
 \end{aligned}$$

and hence

$$S_{ba} = Z_2^{-2} U_{ba} / \langle U \rangle_{\text{vac}}. \tag{6.18}^*)$$

Thus we see the adiabatic switching hypothesis leads to the  $U$  matrix.

\*) The present normalization for the derivation of the  $S$  matrix can be applied only to the elastic scattering.



The conventional Feynman-Dyson theory in which one computes  $U$  as  $S$  gives the correct results provided that renormalization of wave functions

$$\psi \rightarrow Z_2^{1/2} \psi_1, \quad \bar{\psi} \rightarrow Z_2^{1/2} \bar{\psi}_1, \quad \phi \rightarrow Z_3^{1/2} \phi_1 \quad (6.19)$$

has been performed and no bound state appears.

This conclusion is directly verified by expanding (6.6) and (6.18) in powers of the coupling constant.

The author thanks Prof. S. Hayakawa for his interest taken in this work and for careful reading of the manuscript.

*Note added after the completion of the manuscript*

Let  $r$  and  $s$  be two single electron states, then we have in quantum electrodynamics

$$(r|N[\bar{\psi}(1)\gamma_\mu\psi(1)]|s) = \int d\omega_2 d\omega_3 \bar{g}_r(2) \Gamma_\mu(23; 1) g_s(3)$$

in virtue of the equation (A.24). This is an alternative expression of (5.9).

From the general arguments on the normalization of the covariant components we can write

$$i \int d\sigma_\mu(1) (r|N[\bar{\psi}(1)\gamma_\mu\psi(1)]|s) = \delta(r, s) = i \int d\sigma_\mu(1) \bar{g}_{r1}(1) \gamma_\mu g_{s1}(1),$$

since this equation represents the conservation of charge or the number of electrons minus the number of positrons.

Combining these two equations and (5.23), we readily see

$$\begin{aligned} i \int d\sigma_\mu(1) \int d\omega_2 d\omega_3 \bar{g}_r(2) \Gamma_\mu(23; 1) g_s(3) &= i Z_2 Z_1^{-1} \int d\sigma_\mu(1) \bar{g}_{r1}(1) \gamma_\mu g_{s1}(1) \\ &= i \int d\sigma_\mu(1) \bar{g}_{r1}(1) \gamma_\mu g_{s1}(1). \end{aligned}$$

Hence we have Ward's identity

$$Z_1 = Z_2.$$

The author is indebted to Mr. Z. Maki for pointing out this derivation.

## Appendix

### *The Charge Current Density in a Deuteron.*

As an application of the general theory developed in the text we shall calculate the charge current density in a deuteron. The complete answer is obtained only after the solution of the Bethe-Salpeter equation is known, of course. In the present work we assume that this solution is already known and express the current density in terms of the B-S wave functions.

The charge current density for the meson-nucleon system is given in unit of  $e$  by

$$\begin{aligned} j_\mu &= i : \bar{\psi} \gamma_\mu \psi : + \left( \phi_1 \frac{\partial \phi_2}{\partial x_\mu} - \frac{\partial \phi_1}{\partial x_\mu} \phi_2 \right) \\ &= \frac{1}{2} i : \bar{\psi} \gamma_\mu \psi : + i_{\mu\tau}, \quad \left( \tau_\tau = \frac{1 + \tau_3}{2} \right) \end{aligned} \quad (A.1)$$

where

$$i_{\mu\alpha} = \frac{1}{2} i : \bar{\psi} \gamma_{\mu} \tau_{\alpha} \psi : + \left( \phi \times \frac{\partial \phi}{\partial x_{\mu}} \right)_{\alpha} \quad (\text{A} \cdot 2)$$

is the  $\alpha$ -th component of the isotopic spin density and  $\times$  denotes the vector product in isotopic space.

Since a deuteron is a charge singlet state, the expectation value of  $i_{\mu\alpha}$  vanishes in this state. Hence we have

$$\langle d' | j_{\mu} | d \rangle = \frac{1}{2} i \langle d' | : \bar{\psi} \gamma_{\mu} \psi : | d \rangle. \quad (\text{A} \cdot 3)$$

According to the general procedure of § 5, we must first calculate

$$\langle 0 | T [ (\psi(1) \psi(2))^{out} : \bar{\psi}(3) \gamma_{\mu} \psi(3) : ] | d \rangle.$$

This can be done as in I or in § 5, and we must start from the equation for the three-body Green function.

However we shall exhibit another means which is physically more accessible. This method is more or less similar to Deser's one.<sup>20)</sup>

Let us consider the Hamiltonian

$$H_{int} = i \gamma : \bar{\psi} O_a \psi : \phi_a + Q_a \phi_a + W_b \phi_b - H_{s:lf}, \quad (\text{A} \cdot 4)$$

which is the sum of  $H_{int}$  in (2.1) and  $W_b \phi_b$ ,  $W$  being the quantized quantity whose expectation value is now in question and  $q$  a c-number external field.

Then we have

$$i \frac{\partial}{\partial q(0)} T[U A B \cdots Z] = T[U W(0) A B \cdots Z] \quad (\text{A} \cdot 5)$$

corresponding to (2.7). With the notation of (2.6) we may also write

$$i \frac{\partial}{\partial q(0)} \langle A B \cdots Z \rangle = \langle W(0) A B \cdots Z \rangle - \langle W(0) \rangle \langle A B \cdots Z \rangle. \quad (\text{A} \cdot 6)$$

We now put

$$W_{\mu} = i : \bar{\psi} \gamma_{\mu} \psi :, \quad (\text{A} \cdot 7)$$

then the equation for  $S'_F(1, 2)$  is changed by  $i \gamma q$  to yield

$$\left( \gamma \partial_1 + M + i \gamma q(1) + i \eta O(1) \langle \phi(1) \rangle - \eta O(1) \frac{\partial}{\partial Q(1)} \right) S'_F(1, 2) = -i \delta(1-2), \quad (\text{A} \cdot 8)$$

the equation for  $\langle \phi(1) \rangle$ , however, does not change its form

$$(\square - \mu^2)_1 \langle \phi(1) \rangle = i \gamma S \phi(O(1) S'_F(1, 1'))_{1 \rightarrow 1 \pm 0} + Q(1). \quad (\text{A} \cdot 9)$$

From these equations we have easily

$$\left( i \frac{\partial}{\partial q(0)} S'_F(1, 2) \right)_{Q, q \rightarrow 0} = \langle T[ : i \bar{\psi}(0) \gamma \psi(0) : \psi(1) \bar{\psi}(2) : ] \rangle_{vac.} \\ = i S'_F(1-0) \gamma(0) S'_F(0-2) + \int d\omega_3 d\omega_4 S'_F(1-3) \Sigma_q^*(34; 0) S'_F(4-2), \quad (A \cdot 10)$$

where

$$\Sigma_q^*(34; 0) = \left( i \frac{\partial}{\partial q(0)} \Sigma^*(3, 4) \right)_{Q, q \rightarrow 0}. \quad (A \cdot 11)$$

We put (A·10) simply as

$$i \int S'_F(1-3) I'(34; 0) S'_F(4-2) d\omega_3 d\omega_4, \quad (A \cdot 12)$$

where  $I'_\mu$  is readily understood to express the vertex operator for  $\gamma_\mu$  with mesonic radiative corrections.

Next we consider the two-nucleon equation

$$K(12; 34) = S'_F(12; 34) + \int d\omega_5 \cdots d\omega_8 S'_F(1, 5) S'_F(2, 6) G(56; 78) K(78; 34) \quad (A \cdot 13)$$

in which the external fields  $Q$  and  $q$  are retained.

We write this equation symbolically as

$$(1 - \mathcal{K}(12)) K(12; 34) = S'_F(12; 34). \quad (A \cdot 14)$$

Differentiating the above equation with respect to  $q$  and putting  $Q=q=0$ , the following equation is arrived at

$$(1 - \mathcal{K}(12)) K(12; 34; 0) = \left( i \frac{\partial}{\partial q(0)} S'_F(12; 34) \right)_0 + i \left( \frac{\partial \mathcal{K}(12)}{\partial q(0)} \right)_0 K(12; 34), \quad (A \cdot 15)$$

where

$$K(12; 34; 0) = -(0| T[ \psi(1) \psi(2) \bar{\psi}(3) \bar{\psi}(4) : i \bar{\psi}(0) \gamma \psi(0) : ] | 0).$$

Substituting (3, 4)<sup>in</sup> for 3 and 4 in the above equation, we have

$$(1 - \mathcal{K}(12)) g_{a,W}(12; 0) = i \left( \frac{\partial \mathcal{K}(12)}{\partial q(0)} \right)_0 g_a(12), \quad (A \cdot 16)$$

where as in § 5 we have put

$$g_{a,W}(12; 0) = (0| T[ \psi(1) \psi(2) : i \bar{\psi}(0) \gamma \psi(0) : ] | d). \quad (A \cdot 17)$$

Utilizing (A·12),  $\partial \mathcal{K} / \partial q$  is found to be

$$\left( i \frac{\partial \mathcal{K}(12)}{\partial q(0)} \right)_0 = i \int d\omega_3 \cdots d\omega_8 [ S'_F(1-3) \Gamma(34; 0) S'_F(4-5) \cdot S'_F(2-6) \\ + S'_F(1-5) \cdot S'_F(2-3) \Gamma(34; 0) S'_F(4-6) ] G(56; 78) \cdots \\ + \int d\omega_5 \cdots d\omega_8 S'_F(1-5) S'_F(2-6) G_q(56; 78; 0) \cdots \quad (A \cdot 18)$$

where ... means the functions of arguments 7 and 8 to be integrated and

$$G_q(56; 78; 0) = i \left( \frac{\partial}{\partial q(0)} G(56; 78) \right)_0. \quad (\text{A} \cdot 19)$$

$G_q$  is obtained by inserting  $i\gamma$  in the nucleon lines of the graph for  $G$ . Inserting (A·18) into (A·16), one obtains

$$\begin{aligned} (1 - \mathcal{H}(12)) g_{a,W}(12; 0) &= i \int d\omega_3 d\omega_4 (S'_F(1-3) \Gamma(34; 0) g_a(42) - (1 \rightleftharpoons 2)) \\ &+ \int d\omega_5 \cdots d\omega_8 S'_F(1-5) S'_F(2-6) G_q(56; 78; 0) g_a(78), \end{aligned} \quad (\text{A} \cdot 20)$$

with the aid of the relation

$$\int d\omega_5 \cdots d\omega_8 S'_F(1-5) S'_F(2-6) G(56; 78) g_a(78) = g_a(12).$$

The first term in (A·20) represents the contribution of the nucleon moments, while the second term stands for the effect of the exchange moments.

The solution of (A·20) can be utilized to compute the magnetic dipole and electric quadrupole moments of a deuteron as well as the photo-disintegration of a deuteron.

In the latter case, however, the calculation is limited to charge singlet final states since  $\hat{t}_{\mu 3}$  is dropped from  $\hat{J}_\mu$  in the present calculation.

The equation (A·20) can be solved in reference to the method presented in I.

$$\text{Put} \quad \mathcal{D}_N(1) \delta(1-2) = \mathcal{D}(1, 2)$$

as in (6·11), then we can write

$$\int d\omega_3 \mathcal{D}(1, 3) S'_F(3, 2) = \int d\omega_2 S'_F(1, 3) \mathcal{D}(3, 2) = -i \delta(1-2).$$

The first term in (A·20) is then

$$- \int d\omega_3 \cdots d\omega_6 S'_F(12; 35) \mathcal{D}(5, 6) \Gamma(34; 0) g_a(4, 6).$$

The solution of eq. (A·20) is obtained by inserting the above expression

$$\begin{aligned} g_{a,W}(12; 0) &= - \int d\omega_3 \cdots d\omega_6 K(12; 35) \mathcal{D}(5, 6) \Gamma(34; 0) g_a(4, 6) \\ &+ \frac{1}{2} \int d\omega_3 \cdots d\omega_6 K(12; 34) G_q(34; 56; 0) g_a(78). \end{aligned} \quad (\text{A} \cdot 21)$$

Substituting  $(1, 2)^{\text{out}}$  for 1, 2 in (A·21), we find

$$\begin{aligned} \langle d' | : i\bar{\psi}(0) \gamma \psi(0) : | d \rangle &= - \int d\omega_3 \cdots d\omega_6 \bar{g}_{a'}(35) \mathcal{D}(5, 6) \Gamma(34; 0) g_a(46) \\ &+ \frac{1}{2} \int d\omega_3 \cdots d\omega_6 \bar{g}_{a'}(34) G_q(34; 56; 0) g_a(78). \end{aligned} \quad (\text{A} \cdot 22)$$

The equation for  $g_a$  is of help to eliminate  $\mathcal{D}(5, 6)$  from (A·22) and we find for the first term of the right hand side of (A·22)

$$i \int d\omega_1 \cdots d\omega_8 \bar{g}_{a'}(12) \Gamma(13; 0) S'_F(3-5) G(52; 78) g_a(78). \quad (\text{A} \cdot 23)$$

The application of the present method to ex. 1 in § 5 yields

$$(0|T[:i\bar{\psi}(0)\gamma\psi(0):\psi(1)\bar{\psi}(2)]|0)=i\{S'_F(1-3)I'(34;0)S'_F(4-2)d\omega_3d\omega_4$$

and hence

$$(\nu|N[i\bar{\psi}(0)\gamma\psi(0)]|s)=i\{\bar{g}_\tau(3)I'(34;0)g_s(4)d\omega_3d\omega_4. \quad (\text{A}\cdot 24)$$

This expression is much simpler than (5.9).

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**Note added in proof.** Recently similar theories are extensively developed by German authors.  
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## Wave Fields in de Sitter Space

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General wave equations for particles with arbitrary spin values in de Sitter space are investigated. The particles have generally two sorts of spin like quantum numbers. Solutions of the second order wave equations, which are to be satisfied by each component of wave functions, are obtained. They are composed of only discrete modes of wave oscillations. Behaviours of particles and their potentials in spatially small regions are approximately the same as those of particles with certain other corresponding masses in the flat space.

### § 1. Introduction

In the ordinary theory we use the Minkowski space, which seems to give a good approximation to the natural phenomena. However, the curved space time can not be denied in general. We can not deny also that there might be such regions of phenomena that the space time curvature give a good approximation. For the events involving the new unstable particles the ordinary wave equations of elementary particles in the flat space seem to be not so satisfactory. A new degree of freedom is necessary. The curved space time seems to give some generalization of wave equations. The general curved space is not so easy to treat. The de Sitter space seems to be the only one which can be treated in the almost same details as the Minkowski space, even if much complications occur. So we investigated the de Sitter space as a model.

De Sitter space possesses an excellent degree of space time symmetry, by virtue of which we can formulate a covariant quantum theory under the group of special relativistic rotations and translations, even if it is difficult to construct quantum theory in general curved spaces. As well known, the de Sitter space can be described as a four dimensional surface of a hyperhyperboloid embedded in a five dimensional flat space. Five dimensional pseudo-rotations around the center of the hyperboloid leave the hyperboloid invariant and induce in it the de Sitter transformations which construct a group. This can be interpreted as the group of the special relativity in the de Sitter space in somewhat wider meaning than the one in the Minkowski space, and so a covariant theory under this group can be interpreted as a special relativistic theory in de Sitter space in such wider meaning. Rotations in the five dimensional flat space approach, in small regions on the surface, to homogeneous Lorentz transformations and translations in the tangential Minkowski space, which construct the inhomogeneous Lorentz group. In the Minkowski space it is not easy to develop a relativistic quantum theory in which the covariance under the inhomogeneous Lorentz group is taken into account fully. Since the de Sitter group corresponds

to this wider group rather than to the homogeneous Lorentz group, the quantum theory in de Sitter space may be offer a foothold for constructing such wider theory in the Minkowski space.

Wave equations for the electron field and for the electromagnetic field were discussed for the first time by Dirac,<sup>1)</sup> after which several authors, such as Prof. Husimi,<sup>2)</sup> Prof. Watanabe<sup>3)</sup> and Dr. S. A. Raju<sup>4)</sup>, gave interesting investigations about quantum theories in the de Sitter space. Among them Husimi's investigation is the most fundamental, even if it is not yet published and was given only in a form of a lecture. He gave a very natural generalization of physical conceptions and laws, although it is confined to the classical theories such as the point dynamics and the electrodynamics. His method is based upon the variation principle. The variational method seems to be the most natural and unambiguous method for generalization to the de Sitter space from the Minkowski space, since it generalizes the invariant action integral. The tensor indices of the quantities takes the values from 0 to 4, on the contrary integrations and variations are confined on the hyperboloidal surface. The classical particle dynamics is the most fundamental basis for generalizing dynamical conceptions. All sorts of linear conceptions and the corresponding rotational conceptions in the four dimensional flat space are combined into rotational conceptions in five dimensions. For example translations and rotations are combined into five dimensional rotations, and linear momenta and angular momenta are combined into five dimensional angular momenta. As the result, for example, the conservation laws of linear momenta and angular momenta are combined into a law of conservation of five dimensional angular momenta. Another sort of important rotational conception is the five dimensional moment. In the theory of the electromagnetic field introduced by him all the physical quantities such as current densities and energy momenta, are derived from the Lagrangian function by noting on the restriction of regions of integrations and variations. These quantities are described unambiguously as a moment of a vector or a double moment of a tensor.

As mathematical investigations, the representations of the de Sitter group were given by Thomas<sup>5)</sup> and the useful form of five dimensional spinor analysis was given by us.<sup>6)</sup> In this paper we will introduce general theories for particles with arbitrary spin values, by using there mathematical tools. We discussed<sup>7)</sup> about general equations for arbitrary spins which approach to the Bhabha's wave equations<sup>8)</sup> in the Minkowski space when the space becomes flat. Since further rules for five dimensional spinors were developed, it became possible to establish general wave equations which approach to the Dirac-Fierz-Pauli's general equations<sup>9)</sup> in the Minkowski space when the space becomes flat. Our two sorts general equations coincide with each other for lower spin values 0,  $1/2$ , 1, this is the same circumstance as between the Dirac's equations and the Bhabha's equations in the flat space. The theories for these three cases, corresponding to already observed particles, are discussed somewhat in detail.

For any particle the second order wave equation, which must be satisfied by each component of the wave function, is derived. The solutions of this wave equation are obtained. They are composed of only discrete modes of wave oscillations. Static solutions

with spherical symmetry are also obtained. Their behaviors in spatially small regions are discussed, where the Yukawa potentials are the same as that of particles with some other masses in the flat space and the behaviors of particles are also so.

The pseudo de Sitter space, whose tangential space is the Minkowski space in the same way as the de Sitter space, is also investigated, where the hermitian character of the operator is more comfortable.

## § 2. The de Sitter space time

(i) *The de Sitter world embedded in a five dimensional flat space*

The de Sitter world is a four dimensional curved space time whose metric is

$$ds^2 = -\frac{dr^2}{1-r^2/R^2} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + \left(1 - \frac{r^2}{R^2}\right)c^2 dt^2, \quad (2.1)$$

where  $(r, \theta, \varphi)$  is the spherical coordinate of  $(x, y, z)$  or

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta \quad (2.2)$$

and  $R$  is a constant whose inverse is characteristic of the curvature of the world and the cosmological constant of the world is  $\lambda = 3/R^2$ . As Robertson showed, if we put

$$\begin{aligned} x^0 &= R \sqrt{1-r^2/R^2} \cosh(ct/R) \\ x^1 &= x, \quad x^2 = y, \quad x^3 = z \\ x^4 &= R \sqrt{1-r^2/R^2} \sinh(ct/R), \end{aligned} \quad (2.3)$$

(2.1) becomes

$$ds^2 = -(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2 \quad (2.4)$$

and the next relation holds identically.

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = R^2. \quad (2.5)$$

Thus the de Sitter world has been described as a four dimensional hyper-hyperboloidal surface (2.5) of radius  $R$  in the five dimensional pseudo-Euclidean flat space whose metric is (2.4). The connexion between this space and the Minkowski space of special relativity is that in a small region about the point  $P(R, 0, 0, 0, 0)$  the coordinates  $x^1, x^2, x^3, x^4$  becomes the  $x, y, z, ct$  and the de Sitter space becomes the Minkowski space when  $R$  tends to infinity. The intersection of the de Sitter space with hyper-plane  $x^4 = T$  (const.) is the three dimensional surface of the four dimensional sphere of radius  $\sqrt{R^2 + T^2}$

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = R^2 + T^2, \quad (2.6)$$

This section is a three dimensional closed world and its volume is  $2\pi^2(R^2 + T^2)^{3/2}$ .

In this five dimensional space we can take various sorts of coordinates similar to the three dimensional spherical coordinate and the points of the de Sitter space are expressed as follows.

(a) *Coordinate system (A)*

$$\begin{aligned}
 x^0 &= R \cos \theta_1 \cosh \theta_4, \\
 x^1 &= R \sin \theta_1 \sin \theta_2 \cos \theta_3, \\
 x^2 &= R \sin \theta_1 \sin \theta_2 \sin \theta_3, \\
 x^3 &= R \sin \theta_1 \cos \theta_2, \\
 x^4 &= R \cos \theta_1 \sinh \theta_4.
 \end{aligned} \tag{2.7}$$

The connexion between these angles and the original space time coordinate is

$$\begin{aligned}
 \theta_1 &= \sin^{-1}(r/R), \quad (r = R \sin \theta_1) \\
 \theta_2 &= \theta, \quad \theta_3 = \varphi, \\
 \theta_4 &= ct/R.
 \end{aligned} \tag{2.8}$$

The region for small  $\theta_1$  is that for small values of  $r$ . The neighbourhood of the point  $P$  is indicated analytically with small  $\theta_1, \theta_4$ .

The good point of this coordinate is its direct correspondence to the original space time coordinate system in four dimensions with curvature, and so the direct physical interpretation of arguments is possible.

(b) *Coordinate system (B)*

$$\begin{aligned}
 x^0 &= R \cos \omega_1 \cos \omega_3, \\
 x^1 &= R \cos \omega_1 \sin \omega_3, \\
 x^2 &= R \sin \omega_1 \cos \omega_2, \\
 x^3 &= R \sin \omega_1 \sin \omega_2 \cosh \omega_4, \\
 x_4 &= R \sin \omega_1 \sin \omega_2 \sinh \omega_4,
 \end{aligned} \tag{2.9}$$

The neighbourhood of the point  $P$  is indicated with small  $\omega_1, \omega_3$ . This coordinate system (B) is closely connected with the system (A) formally.

The good point of this coordinate system is the formal regularity, and it is convenient for considering motions along the great circle on the  $x_0 x_1$  plane.

(c) *Coordinate system (C)*

$$\begin{aligned}
 x^0 &= \cosh \varphi_1 \cos \varphi_2, \\
 x^1 &= \cosh \varphi_1 \sin \varphi_2 \cos \varphi_3, \\
 x^2 &= \cosh \varphi_1 \sin \varphi_2 \sin \varphi_3 \cos \varphi_4, \\
 x^3 &= \cosh \varphi_1 \sin \varphi_2 \sin \varphi_3 \sin \varphi_4, \\
 x^4 &= \sinh \varphi_1.
 \end{aligned} \tag{2.10}$$

The neighbourhood of the point  $P$  is indicated with small  $\varphi_1, \varphi_2$ . This coordinate system



is the natural generalization of the ordinary three dimensional case.

The good point of this coordinate system is also the formal regularity, and it is convenient for considering circumstances from the view point of the time or of the separated space and time. The spatial part of this coordinate may be altered as the next one.

(d) *Coordinate system (D)*

$$\begin{aligned}x^0 &= R \cosh \phi_1 \cos \phi_2 \cos \phi_3, \\x^1 &= R \cosh \phi_1 \cos \phi_2 \sin \phi_3, \\x^2 &= R \cosh \phi_1 \sin \phi_2 \cos \phi_3, \\x^3 &= R \cosh \phi_1 \sin \phi_2 \sin \phi_3, \\x^4 &= R \sinh \phi_1.\end{aligned}\tag{2.11}$$

The neighbourhood of the point  $P$  is indicated with small  $\phi_1, \phi_2, \phi_3$ . This coordinate (D) is connected with (C) in the point that  $\varphi_1$  and  $\phi_1$  have the same meaning.

The strong point of this coordinate system is the mathematical symmetry, and it is excellent for the group theoretical considerations. Indeed, it has some direct connections to the representing coordinates of the Thomas<sup>(5)</sup> work about the representations of the de Sitter group.

(ii) *The de Sitter group*

The fundamental metric tensor, in the normal coordinate, of the above introduced five dimensional pseudo-Euclidean space is

$$\begin{aligned}g_{00} &= g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = 1, \quad g^{\lambda\mu} = 0 \quad (\lambda \neq \mu), \\g^{\lambda\mu} &= g_{\lambda\mu}, \quad (\lambda, \mu = 0, 1, 2, 3, 4)\end{aligned}\tag{2.12}$$

and

$$g = \det(g_{\lambda\mu}) = 1.\tag{2.13}$$

A linear transformation

$$x'^{\lambda} = a^{\lambda}_{\mu} x^{\mu}\tag{2.14}$$

will let the equation of the world (2.5) invariant, if and only if

$$g_{\lambda\mu} a^{\lambda}_{\nu} a^{\mu}_{\rho} = g_{\nu\rho}.\tag{2.15}$$

In this case the coordinate

$$x_{\lambda} = g_{\lambda\mu} x^{\mu}\tag{2.16}$$

transforms according to

$$x'_{\lambda} = b_{\lambda}^{\mu} x_{\mu} \quad (b_{\lambda}^{\mu} = g_{\mu\nu} a^{\nu}_{\rho} g^{\rho\mu}).\tag{2.17}$$

The linear transformation among  $x^0, x^1, x^2, x^3, x^4$  imposed with the condition (2.11) means a transformation in the de Sitter space. So we shall call this transformation as a de Sitter transformation. The whole of the de Sitter transformations builds up a transformation group. This can be easily shown in the same way as in the case of the rotation. Let us call this group as the de Sitter group. The de Sitter group can be interpreted as wider



one than the homogeneous Lorentz group in the ordinary special relativity, for it includes not only the transformations which correspond to homogeneous Lorentz transformations (the ones among  $x^1, x^2, x^3, x^4$ ) but also those which correspond to translations (the one between,  $x^0, x^4$  corresponds to a translation concerning  $x^0$ ) and, hence, it corresponds rather to the inhomogeneous Lorentz group.

From (2.13) follows

$$[\det(a^\lambda_\mu)]^2 = 1, \text{ or } \det(a^\lambda_\mu) = \pm 1. \quad (2.18)$$

The transformation will then be called as proper or improper according as  $\det(a^\lambda_\mu) = +1$  or  $-1$ . The 4-4 element of (2.13) gives

$$(a^4_4)^2 = 1 + (a^0_4)^2 + (a^1_4)^2 + (a^2_4)^2 + (a^3_4)^2 \geq 1, \quad (2.19)$$

so that either  $a^4_4 \geq 1$  or  $a^4_4 \leq -1$ . The transformation will then be called as orthochronous or antichronous according as  $a^4_4 \geq 1$  or  $a^4_4 \leq -1$ . The de Sitter transformations are therefore classified into four disconnected sets :

- (I) orthochronous proper transformation,
- (II) antichronous proper transformation,
- (III) orthochronous improper transformation,
- (IV) antichronous improper transformation. (2.20)

In the same way as in the case of the Lorentz group, it will be shown easily that transformations among each of these four sets can be connected continuously. The combinations of these sets (I), (I) (II), (I) (III) and (I) (II) (III) (IV) build up groups which are called as the orthochronous proper de Sitter group, the proper de Sitter group, the orthochronous de Sitter group and the whole de Sitter group respectively. The first one of them forms a continuous Lie group.

In the usual way we can introduce vectors which transform in the same way as the de Sitter coordinates  $x^0, x^1, x^2, x^3, x^4$ . The tensors can be also introduced as usual as the quantities which transform in the same way as products of vectors. Three kinds of pseudo tensor can be also introduced by imposing the negative signs to the tensor transformations about the space inversion and the time inversion. When these tensors and pseudotensors lie in the de Sitter space, they become de Sitter tensors and pseudotensors. These quantities have a wider meaning than the quantities in the theory of the special relativity, for they correspond to those which transform according to the inhomogeneous Lorentz group rather than to the restricted homogeneous Lorentz group.

### (iii) *Differential operations*

All meaningful operations in our case must be concerned with the quantities lying in the de Sitter space. Hence it has in general no meaning to differentiate a function with respect to one of  $x^\lambda$ 's, such as  $\partial/\partial x^\lambda$ , for this implies to depart from the de Sitter space. The only process of differentiation which have a meaning is those referring to differentiations along directions in the de Sitter space. The most fundamental of them are

$$R_{\lambda\mu} = x_\lambda \cdot \partial / \partial x^\mu - x_\mu \cdot \partial / \partial x^\lambda, \quad (2 \cdot 21)$$

which may be called as the infinitesimal rotations of the de Sitter space.

All functions lying in the de Sitter space can be expressed as functions with respect to the angles of the spherical coordinates. To differentiate them with respect to one of the angles has a meaning always. The rotation operators  $R^{\lambda\mu}$  can be expressed with such differential operators by the angles. The exact forms of them are given in the following.

(a) In the coordinate system (A),

$$\begin{aligned} R^{01} = R_{01} &= \sin \theta_2 \cos \theta_3 \cosh \theta_4 \frac{\partial}{\partial \theta_1} + \frac{\cos \theta_1 \cos \theta_2 \cos \theta_3 \cosh \theta_4}{\sin \theta_1} \frac{\partial}{\partial \theta_2} \\ &\quad - \frac{\cos \theta_1 \sin \theta_3 \cosh \theta_4}{\sin \theta_1 \sin \theta_2} \frac{\partial}{\partial \theta_3} + \frac{\sin \theta_1 \sin \theta_2 \cos \theta_3 \sinh \theta_4}{\cos \theta_1} \frac{\partial}{\partial \theta_4}, \\ R^{02} = R_{02} &= \sin \theta_2 \sin \theta_3 \cosh \theta_4 \frac{\partial}{\partial \theta_1} + \frac{\cos \theta_1 \cos \theta_2 \sin \theta_3 \cosh \theta_4}{\sin \theta_1} \frac{\partial}{\partial \theta_2} \\ &\quad - \frac{\cos \theta_1 \cos \theta_3 \cosh \theta_4}{\sin \theta_1 \sin \theta_2} \frac{\partial}{\partial \theta_3} + \frac{\sin \theta_1 \sin \theta_2 \sin \theta_3 \sinh \theta_4}{\cos \theta_1} \frac{\partial}{\partial \theta_4}, \\ R^{03} = R_{03} &= \cos \theta_2 \cosh \theta_4 \frac{\partial}{\partial \theta_1} - \frac{\cos \theta_1 \sin \theta_2 \cosh \theta_4}{\sin \theta_1} \frac{\partial}{\partial \theta_2} + \frac{\sin \theta_1 \cos \theta_2 \sinh \theta_4}{\cos \theta_1} \frac{\partial}{\partial \theta_4}, \\ R^{23} = R_{23} &= \sin \theta_3 \frac{\partial}{\partial \theta_2} + \frac{\cos \theta_2 \cos \theta_3}{\sin \theta_2} \frac{\partial}{\partial \theta_3}, \\ R^{31} = R_{31} &= \cos \theta_3 \frac{\partial}{\partial \theta_2} - \frac{\cos \theta_2 \sin \theta_3}{\sin \theta_2} \frac{\partial}{\partial \theta_3}, \\ R^{12} = R_{12} &= \partial / \partial \theta_3, \\ R^{04} &= -R_{04} = \partial / \partial \theta_4, \\ R^{14} = -R_{14} &= \sin \theta_2 \cos \theta_3 \sinh \theta_4 \frac{\partial}{\partial \theta_1} + \frac{\cos \theta_1 \cos \theta_2 \sinh \theta_4}{\sin \theta_1} \frac{\partial}{\partial \theta_2} \\ &\quad - \frac{\cos \theta_1 \sin \theta_3 \sinh \theta_4}{\sin \theta_1 \sin \theta_2} \frac{\partial}{\partial \theta_3} + \frac{\sin \theta_1 \sin \theta_2 \cos \theta_3 \cosh \theta_4}{\cos \theta_1} \frac{\partial}{\partial \theta_4}, \\ R^{24} = -R_{24} &= \sin \theta_2 \sin \theta_3 \sinh \theta_4 \frac{\partial}{\partial \theta_1} + \frac{\cos \theta_1 \cos \theta_2 \sin \theta_3 \sinh \theta_4}{\sin \theta_1} \frac{\partial}{\partial \theta_2} \\ &\quad + \frac{\cos \theta_1 \cos \theta_3 \sinh \theta_4}{\sin \theta_1 \sin \theta_2} \frac{\partial}{\partial \theta_3} + \frac{\sin \theta_1 \sin \theta_2 \sin \theta_3 \cosh \theta_4}{\cos \theta_1} \frac{\partial}{\partial \theta_4}, \\ R^{34} = -R_{34} &= -\cos \theta_2 \sinh \theta_4 \frac{\partial}{\partial \theta_1} + \frac{\cos \theta_1 \sin \theta_2 \sinh \theta_4}{\sin \theta_1} \frac{\partial}{\partial \theta_2} \\ &\quad - \frac{\sin \theta_1 \cos \theta_2 \cosh \theta_4}{\cos \theta_1} \frac{\partial}{\partial \theta_4} \end{aligned} \quad (2 \cdot 22)$$

(b) The case of the system (B) we neglect. It can be obtained from the case (A) by virtue of the formal connection between (A) and (B).

(c) This differs only in the spacial part with (D).

(d) In the coordinate system (D),

$$\begin{aligned}
 R^{01} &= \partial / \partial \phi_3, \\
 R^{02} &= \cos \phi_3 \cos \phi_4 \frac{\partial}{\partial \phi_2} + \frac{\sin \phi_2 \sin \phi_3 \cos \phi_4}{\cos \phi_2} \frac{\partial}{\partial \phi_3} - \frac{\cos \phi_2 \cos \phi_3 \sin \phi_4}{\sin \phi_2} \frac{\partial}{\partial \phi_4}, \\
 R^{03} &= \cos \phi_2 \sin \phi_1 \frac{\partial}{\partial \phi_2} + \frac{\sin \phi_2 \sin \phi_3 \sin \phi_4}{\cos \phi_2} \frac{\partial}{\partial \phi_3} + \frac{\cos \phi_2 \cos \phi_3 \cos \phi_4}{\sin \phi_2} \frac{\partial}{\partial \phi_4}, \\
 R^{23} &= \partial / \partial \phi_4, \\
 R^{13} &= \sin \phi_3 \sin \phi_4 \frac{\partial}{\partial \phi_2} - \frac{\sin \phi_2 \cos \phi_3 \sin \phi_4}{\cos \phi_2} \frac{\partial}{\partial \phi_3} + \frac{\cos \phi_2 \sin \phi_3 \cos \phi_4}{\sin \phi_2} \frac{\partial}{\partial \phi_4}, \\
 R^{12} &= \sin \phi_3 \cos \phi_4 \frac{\partial}{\partial \phi_2} - \frac{\sin \phi_2 \cos \phi_3 \cos \phi_4}{\cos \phi_2} \frac{\partial}{\partial \phi_3} - \frac{\cos \phi_2 \sin \phi_3 \sin \phi_4}{\sin \phi_2} \frac{\partial}{\partial \phi_4}, \\
 R^{04} &= \cos \phi_2 \cos \phi_3 \frac{\partial}{\partial \phi_1} + \frac{\sinh \phi_1 \sin \phi_2 \cos \phi_3}{\cosh \phi_1} \frac{\partial}{\partial \phi_2} + \frac{\sinh \phi_1 \sin \phi_3}{\cosh \phi_1 \cos \phi_2} \frac{\partial}{\partial \phi_3}, \\
 R^{14} &= \cos \phi_2 \sin \phi_3 \frac{\partial}{\partial \phi_1} + \frac{\sinh \phi_1 \sin \phi_2 \sin \phi_3}{\cosh \phi_1} \frac{\partial}{\partial \phi_2} - \frac{\sinh \phi_1 \cos \phi_3}{\cosh \phi_1 \cos \phi_2} \frac{\partial}{\partial \phi_3}, \\
 R^{24} &= \sin \phi_2 \cos \phi_4 \frac{\partial}{\partial \phi_1} - \frac{\sin \phi_1 \cos \phi_2 \cos \phi_4}{\cosh \phi_1} \frac{\partial}{\partial \phi_2} + \frac{\sinh \phi_1 \sin \phi_4}{\cosh \phi_1 \sin \phi_2} \frac{\partial}{\partial \phi_3}, \\
 R^{34} &= \sin \phi_2 \sin \phi_4 \frac{\partial}{\partial \phi_1} - \frac{\sinh \phi_1 \cos \phi_2 \sin \phi_4}{\cosh \phi_1} \frac{\partial}{\partial \phi_2} - \frac{\sinh \phi_1 \cos \phi_4}{\cosh \phi_1 \sin \phi_2} \frac{\partial}{\partial \phi_3}, \quad (2.23)
 \end{aligned}$$

Particularly in this coordinate system the following combinations of  $R^{\lambda\mu}$  have familiar forms to us, if we introduce

$$\phi = 2\phi_1, \quad \phi_+ = \phi_3 + \phi_4, \quad \phi_- = \phi_3 - \phi_4, \quad (2.24)$$

$$R^{01} + R^{23} = 2\partial / \partial \phi_+,$$

$$R^{02} + R^{31} = 2 \left\{ \cos \phi_+ \frac{\partial}{\partial \phi} - \sin \phi_+ \left( \frac{\cos \phi}{\sin \phi} \frac{\partial}{\partial \phi_+} - \frac{1}{\sin \phi} \frac{\partial}{\partial \phi_-} \right) \right\},$$

$$R^{03} + R^{12} = 2 \left\{ \sin \phi_+ \frac{\partial}{\partial \phi} + \cos \phi_+ \left( \frac{\cos \phi}{\sin \phi} \frac{\partial}{\partial \phi_+} - \frac{1}{\sin \phi} \frac{\partial}{\partial \phi_-} \right) \right\}, \quad (2.25)$$

$$(R^{02} + R^{31}) \pm i(R^{03} + R^{12}) = 2e^{\pm i\phi} \left[ \frac{\partial}{\partial \phi} \pm i \left( \frac{\cos \phi}{\sin \phi} \frac{\partial}{\partial \phi_+} - \frac{1}{\sin \phi} \frac{\partial}{\partial \phi_-} \right) \right] \quad (2.25')$$

$$R^{01} - R^{23} = 2 \cdot \partial / \partial \phi_-,$$

$$R^{02} - R^{31} = 2 \left\{ \cos \phi_- \frac{\partial}{\partial \phi} - \sin \phi_- \left( \frac{\cos \phi}{\sin \phi} \frac{\partial}{\partial \phi_-} - \frac{1}{\sin \phi} \frac{\partial}{\partial \phi_+} \right) \right\},$$

$$R^{03} - R^{12} = 2 \left\{ -\sin \phi_- \frac{\partial}{\partial \phi} - \cos \phi_- \left( \frac{\cos \phi}{\sin \phi} \frac{\partial}{\partial \phi_-} - \frac{1}{\sin \phi} \frac{\partial}{\partial \phi_+} \right) \right\}, \quad (2 \cdot 26)$$

$$(R^{02} - R^{31}) \pm i(R^{03} - R^{12}) = 2e^{\mp i\phi} \left[ \frac{\partial}{\partial \phi} \mp i \left( \frac{\cos \phi}{\sin \phi} \frac{\partial}{\partial \phi_-} - \frac{1}{\sin \phi} \frac{\partial}{\partial \phi_+} \right) \right], \quad (2 \cdot 26')$$

The simple systematical forms of these operators allow group theoretical interpretation of wave functions.

(iv) *The pseudo de Sitter space*

Let us consider the four dimensional world described as a surface of another type of hyperboloid

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2 = R^2, \quad (2 \cdot 27)$$

embedded in the five dimensional flat space whose metric is

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2. \quad (2 \cdot 28)$$

Let us call this space as "pseudo de Sitter space". In this space we can introduce the ordinary coordinate system  $x, y, z, t$

$$\begin{aligned} x^0 &= R \sqrt{1 + r^2/R^2} \cos(ct/R), \\ x^1 &= x, \quad x^2 = y, \quad x^3 = z, \\ x^4 &= R \sqrt{1 + r^2/R^2} \sin(ct/R). \end{aligned} \quad (2 \cdot 29)$$

The metric (2.20) takes the following form in this coordinate system

$$ds^2 = -\frac{dr^2}{1 + r^2/R^2} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + (1 + r^2/R^2)c^2 dt^2. \quad (2 \cdot 30)$$

It can be easily shown that this world is also a solution of the Einstein's fundamental law for the gravitational field. The cosmological constant of this world is  $\lambda = -3/R^2$ . The difference between this world and the de Sitter world lies only in the opposite sign of the curvature of the world. The connexion between this world and the Minkowski space is that in a small region about the point  $(R, 0, 0, 0, 0)$  the coordinates  $x^1, x^2, x^3, x^4$  becomes the  $x, y, z, ct$  and the pseudo de Sitter space becomes to the Minkowski space when  $R$  tends to infinity. In this space we can also introduce the spherical coordinate system (E)

$$\begin{aligned} x^0 &= R \cosh \theta_1 \cos \theta_4, \\ x^1 &= R \sinh \theta_1 \sin \theta_2 \cos \theta_3, \\ x^2 &= R \sinh \theta_1 \sin \theta_2 \sin \theta_3, \\ x^3 &= R \sinh \theta_1 \cos \theta_2, \\ x^4 &= R \cosh \theta_1 \sin \theta_4, \end{aligned} \quad (2 \cdot 31)$$

The connexion between these angles and the original space time coordinate is

$$\begin{aligned}
 r &= R \sinh \theta_1, \\
 \theta_2 &= \theta, \quad \theta_3 = \varphi, \\
 \theta_4 &= ct/R,
 \end{aligned}
 \tag{2.32}$$

The forms of  $K^{\lambda\mu}$  in this coordinate system can be easily obtained from their form (2.22) in the system (A) by virtue of the formal connexion between the systems (A) and (E).

The two types of hyperboloid can be expressed in the form

$$(x^0)^2 + \epsilon \{ (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 \} = R^2, \tag{2.33}$$

where  $\epsilon=1$  for the de Sitter hyperboloid and  $\epsilon=-1$  for the pseudo de Sitter hyperboloid.

### § 3. The spinor calculus for the de Sitter group

In order to obtain the general wave equations for fields of elementary particles with arbitrary spin values in the de Sitter space, we must use some rules of the five dimensional spinor analysis for the de Sitter group.

Five dimensional spinors were introduced formerly. The covariant spinors of the first rank in the five dimensional flat space have four components  $(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$  and transform according to

$$S = \begin{pmatrix} \alpha & \beta & \lambda & \mu \\ \gamma & \delta & \nu & \sigma \\ \bar{\sigma} & -\nu & \delta & -\bar{\gamma} \\ -\bar{\mu} & \bar{\lambda} & -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \tag{3.1}$$

where

$$\left. \begin{aligned} \alpha\delta - \beta\gamma - \lambda\sigma + \mu\nu &= 1 \\ -\alpha\nu - \beta\sigma + \gamma\lambda + \delta\mu &= 0 \\ \lambda\bar{\alpha} + \mu\bar{\beta} &= \text{real} \\ \gamma\bar{\nu} + \delta\bar{\sigma} &= \text{real} \end{aligned} \right\}. \tag{3.2}$$

If a system of five matrices  $P^0, P^1, P^2, P^3, P^4$  can generate the complete matrix algebra of the same order and satisfy the commutation relation

$$P^\lambda P^\mu + P^\mu P^\lambda = 2(g^{\lambda\mu} E \text{ (or } -g^{\lambda\mu} E)) \quad (E: \text{unit}), \tag{3.3}$$

there are always a spinor transformation having the relation

$$\begin{aligned}
 SP^\lambda S^{-1} &= a_{\mu}^{*\lambda} P^\mu \quad (a_{\mu}^{*\lambda} = a_{\lambda}^{\mu}), \\
 (\text{or } SP_{\mu} S^{-1} &= a_{\mu}^{\lambda} P_{\lambda}),
 \end{aligned}
 \tag{3.4}$$

for any de Sitter transformation  $(a_{\mu}^{\lambda})$ :

$$x^{1\lambda} = a_{\mu}^{\lambda} x^{\mu}. \tag{3.5}$$



As such a set of matrices we can take the following forms

$$P^0 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad P^1 = \pm \sqrt{\varepsilon} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad P^2 = \pm \sqrt{\varepsilon} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix},$$

$$P^3 = \pm \sqrt{\varepsilon} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad P^4 = \pm \sqrt{\varepsilon} \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \quad (3.6)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$(3.7)$$

Any de Sitter transformation  $(a_\mu^\lambda)$  can be constructed as a product of appropriate two dimensional de Sitter transformations  $T^{\lambda\mu}(\theta_{\lambda\mu})$  concerning the rotational angle  $\theta_{\lambda\mu}$  in the  $x^\lambda x^\mu$  plane, where

$$T^{lm}(\theta) : \begin{cases} x'^l = \cos \theta x^l - \sin \theta x^m, \\ x'^m = \sin \theta x^l + \cos \theta x^m, \\ x'^\lambda = x^\lambda \quad (\lambda \neq l, m). \end{cases} \quad (3.8a)$$

$$T^{l4}(\theta) : \begin{cases} x'^l = \cosh \theta x^l + \sinh \theta x^4, \\ x'^m = \sinh \theta x^l + \cosh \theta x^4, \\ x'^\lambda = x^\lambda \quad (\lambda \neq l, m). \end{cases} \quad (3.8b)$$

If we obtain the spinor transformations  $S^{\lambda\mu}(\theta_{\lambda\mu})$  for these special de Sitter transformations  $T^{\lambda\mu}(\theta_{\lambda\mu})$ , we can construct the spinor transformation  $S$  for any de Sitter transformation  $(a_\mu^\lambda)$ . The  $S^{\lambda\mu}(\theta)$  can be obtained by solving the equation (3.4) for  $(a_\mu^\lambda) T^{\lambda\mu}(\theta)$ .

$$S^{12}(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 & & \\ 0 & e^{i\theta/2} & & \\ & & e^{-i\theta/2} & 0 \\ & & 0 & e^{i\theta/2} \end{pmatrix},$$

$$S^{14}(\theta) = \begin{pmatrix} \cosh \theta/2 & \sinh \theta/2 & & \\ \sinh \theta/2 & \cosh \theta/2 & & \\ & & \cosh \theta/2 & -\sinh \theta/2 \\ & & -\sinh \theta/2 & \cosh \theta/2 \end{pmatrix},$$

$$S^{23}(\theta) = \begin{pmatrix} \cos \theta/2 & -i \sin \theta/2 & & \\ -i \sin \theta/2 & \cos \theta/2 & & \\ & & \cos \theta/2 & -i \sin \theta/2 \\ & & -i \sin \theta/2 & \cos \theta/2 \end{pmatrix},$$

$$S^{24}(\theta) = \begin{pmatrix} \cosh \theta/2 & i \sinh \theta/2 & & \\ -i \sinh \theta/2 & \cosh \theta/2 & & \\ & & \cosh \theta/2 & -i \sinh \theta/2 \\ & & i \sinh \theta/2 & \cosh \theta/2 \end{pmatrix},$$

$$S^{13}(\theta) = \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 & & \\ \sin \theta/2 & \cos \theta/2 & & \\ & & \cos \theta/2 & -\sin \theta/2 \\ & & \sin \theta/2 & \cos \theta/2 \end{pmatrix},$$

$$S^{34}(\theta) = \begin{pmatrix} e^{\theta/2} & 0 & & \\ 0 & e^{-\theta/2} & & \\ & & e^{-\theta/2} & 0 \\ & & 0 & e^{\theta/2} \end{pmatrix},$$

$$S^{01}(\theta) = \begin{pmatrix} \cos \theta/2 & 0 & 0 & -i \sin \theta/2 \\ 0 & \cos \theta/2 & -i \cos \theta/2 & 0 \\ 0 & -i \sin \theta/2 & \cos \theta/2 & 0 \\ -i \sin \theta/2 & 0 & 0 & \cos \theta/2 \end{pmatrix},$$

$$S^{03}(\theta) = \begin{pmatrix} \cos \theta/2 & 0 & -i \sin \theta/2 & 0 \\ 0 & \cos \theta/2 & 0 & i \sin \theta/2 \\ -i \sin \theta/2 & 0 & \cos \theta/2 & 0 \\ 0 & i \sin \theta/2 & 0 & \cos \theta/2 \end{pmatrix},$$

$$S^{02}(\theta) = \begin{pmatrix} \cos \theta/2 & 0 & 0 & \sin \theta/2 \\ 0 & \cos \theta/2 & -\sin \theta/2 & 0 \\ 0 & \sin \theta/2 & \cos \theta/2 & 0 \\ -\sin \theta/2 & 0 & 0 & \cos \theta/2 \end{pmatrix},$$

$$S^{04}(\theta) = \begin{pmatrix} \cosh \theta/2 & 0 & -i \sinh \theta/2 & 0 \\ 0 & \cosh \theta/2 & 0 & -i \sinh \theta/2 \\ i \sinh \theta/2 & 0 & \cosh \theta/2 & 0 \\ 0 & i \sinh \theta/2 & 0 & \cosh \theta/2 \end{pmatrix},$$

For Lorentz transformations among  $x^1, x^2, x^3, x^4$  fixing the zeroth axis  $x'^0 = x^0$ ,  $S$  must satisfy the relation  $SP^0S^{-1} = P^0$ . Solving this equation we have the condition

$$\lambda = \mu = \nu = \sigma = 0, \quad (3.10)$$

therefore in this case our spinor reduces to a pair of the two components-spinors  $(\psi_1, \psi_2)$  and  $(\dot{\varphi}^1, \dot{\varphi}^2)$  introduced by van der Waerden in the following way

$$(\Psi_1, \Psi_2, \Psi_3, \Psi_4) \rightarrow (\psi_1, \psi_2; \dot{\varphi}^1, \dot{\varphi}^2). \quad (3.11)$$

The contravariant spinors  $(\phi^1, \phi^2, \phi^3, \phi^4)$  are defined as those which transform contragradiently to the covariant spinors or according to the transposed matrix  $(S^{-1})'$  of the inverse of  $S$ :

$$(S^{-1})' = \begin{pmatrix} \delta & -\gamma & -\sigma & \nu \\ -\beta & \alpha & \mu & -\lambda \\ -\bar{\lambda} & -\bar{\mu} & \bar{\alpha} & \bar{\beta} \\ -\bar{\nu} & -\bar{\sigma} & \bar{\gamma} & \bar{\delta} \end{pmatrix}. \quad (3.12)$$

The inner product of these two sorts of spinors

$$\phi^r \Psi_r = \phi^1 \Psi_1 + \phi^2 \Psi_2 + \phi^3 \Psi_3 + \phi^4 \Psi_4 \quad (3.13)$$

is an invariant. The rule for taking up and down the spinor indices is

$$(\Psi^1, \Psi^2, \Psi^3, \Psi^4) = (\Psi_2, -\Psi_1, -\Psi_4, \Psi_3), \quad (3.14)$$

or

$$\Psi^r = \epsilon^{rs} \Psi_s, \quad (3.15)$$

where

$$C = (\epsilon^{rs}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.16)$$

$$(S^{-1})' = CSC^{-1}. \quad (3.17)$$

So the next rule holds

$$\phi^r \Psi_r = -\phi_r \Psi^r. \quad (3.18)$$

Spinors of higher ranks are defined as those which transform in the same way as products of spinors of the first rank. The way of reduction of the higher rank spinors in the case where the transformations are confined to Lorentz transformations may be easily obtained by the fundamental formula (3.11) for the basic rank spinors.

We can build a vector as a product of a covariant spinor  $\Psi$  and contravariant spinor  $\phi$ :

$$\begin{aligned} A^\lambda &= 1/4 \cdot \phi P^\lambda \psi \\ &= 1/4 \cdot \sigma^\lambda_{rs} \psi^r \psi^s, \end{aligned} \quad (3 \cdot 19)$$

where  $\sigma^\lambda_{rs}$  is the  $(r, s)$  element of the matrix  $P^\lambda$ . So a vector  $A^\lambda$  can be expressed by a spinor of the second rank  $\psi_{rs}$ :

$$A^\lambda = 1/4 \cdot \sigma^\lambda_{rs} \psi^r \psi^s, \quad (3 \cdot 20)$$

$$\begin{aligned} A^0 &= 1/4 \cdot (-\psi_{12} + \psi_{21} - \psi_{31} + \psi_{13}), \\ A^1 &= \pm \sqrt{\varepsilon}/4 \cdot (\psi_{21} - \psi_{42} - \psi_{13} + \psi_{31}), \\ A^2 &= \pm \sqrt{\varepsilon}i/4 \cdot (\psi_{21} - \psi_{42} + \psi_{13} - \psi_{31}), \\ A^3 &= \pm \sqrt{\varepsilon}/4 \cdot (\psi_{23} - \psi_{32} + \psi_{14} - \psi_{41}), \\ A^4 &= \pm \sqrt{\varepsilon}/4 \cdot (\psi_{23} - \psi_{32} - \psi_{14} + \psi_{41}). \end{aligned} \quad (3 \cdot 21)$$

The distinctive form of these relations show that the vector  $A^\lambda$  can be expressed by an antisymmetric spinor  $\psi^-_{rs} = (\psi_{rs} - \psi_{sr})/2$  as follows.

$$\begin{aligned} A^0 &= 1/2 (-\psi_{12}^- - \psi_{34}^-), \\ A^1 &= \pm \sqrt{\varepsilon}/2 (\psi_{24}^- - \psi_{13}^-), \\ A^2 &= \pm \sqrt{\varepsilon}i/2 (\psi_{24}^- + \psi_{13}^-), \\ A^3 &= \pm \sqrt{\varepsilon}/2 (\psi_{23}^- + \psi_{14}^-), \\ A^4 &= \pm \sqrt{\varepsilon}/2 (\psi_{23}^- - \psi_{14}^-). \end{aligned} \quad (3 \cdot 22)$$

The quantity

$$A = 1/2 (\psi_{12}^- - \psi_{34}^-) \quad (3 \cdot 23)$$

can be shown to be an invariant<sup>1)</sup>. The antisymmetric spinor  $\psi^-_{rs}$  which have the property  $\psi_{12}^- = \psi_{34}^-$  is irreducible<sup>1)</sup> and corresponds to a vector, and for such one

$$A^0 = -\psi_{12}^- = -\psi_{34}^-. \quad (3 \cdot 24)$$

Conversely the irreducible antisymmetric spinor  $\psi^-_{rs}$  can be expressed by a vector  $A^\lambda$ :

$$\begin{aligned} \psi_{12}^- &= -A^0, \\ \psi_{13}^- &= \pm 1/\sqrt{\varepsilon} (-A^1 - iA^2), \\ \psi_{11}^- &= \pm 1/\sqrt{\varepsilon} (A^3 - iA^4), \\ \psi_{23}^- &= \pm 1/\sqrt{\varepsilon} (A^3 + iA^4), \\ \psi_{24}^- &= \pm 1/\sqrt{\varepsilon} (A^1 - iA^2), \\ \psi_{34}^- &= \pm 1/\sqrt{\varepsilon} (-A^0). \end{aligned} \quad (3 \cdot 25)$$

In the same way tensors can be generally expressed by spinors of even rank:

$$A^{\lambda\mu} = (1/4 \cdot \sigma^{\lambda}_{,r}{}^s) (1/4 \cdot \sigma^{\mu}_{,t}{}^u) \dots \Psi^{rst}{}_{\dots} \dots \quad (3.26)$$

But special tensors with certain symmetry characters can be expressed by special tensors with the corresponding symmetry characters. For example, the antisymmetric tensor  $F^{-\lambda\mu}$  can be expressed by a symmetric spinor  $\Psi_{rs}^{+} = (\Psi_{rs} + \Psi_{rs})/2$ :

$$\begin{aligned} F^{-\lambda\mu} &= 1/4 \cdot \phi^{\lambda\lambda} \phi^{\mu\mu} \Psi \\ &= 1/4 \cdot \sigma^{\lambda}_{,r}{}^s \sigma^{\mu}_{,t}{}^u \Psi_{rs} (\Psi_{st} = \phi_{st} \Psi_t), \end{aligned} \quad (3.27)$$

or

$$\begin{aligned} F^{-12} &= \varepsilon i/2 \cdot (-\Psi_{12}^{+} + \Psi_{34}^{+}), \\ F^{-13} &= \varepsilon/4 \cdot (-\Psi_{22}^{+} - \Psi_{11}^{+} + \Psi_{44}^{+} + \Psi_{33}^{+}), \\ F^{-23} &= -\varepsilon i/4 \cdot (\Psi_{22}^{+} - \Psi_{11}^{+} - \Psi_{44}^{+} + \Psi_{33}^{+}), \\ F^{-14} &= \varepsilon/4 \cdot (-\Psi_{22}^{+} + \Psi_{11}^{+} - \Psi_{44}^{+} + \Psi_{33}^{+}), \\ F^{-24} &= -\varepsilon i/4 \cdot (\Psi_{22}^{+} + \Psi_{11}^{+} + \Psi_{44}^{+} + \Psi_{33}^{+}), \\ F^{-34} &= \mp \sqrt{\varepsilon}/2 \cdot (\Psi_{12}^{+} + \Psi_{34}^{+}), \\ F^{-10} &= \pm \sqrt{\varepsilon}/2 \cdot (\Psi_{13}^{+} - \Psi_{24}^{+}), \\ F^{-20} &= \mp \sqrt{\varepsilon} i/2 \cdot (\Psi_{13}^{+} + \Psi_{24}^{+}), \\ F^{-30} &= \mp \sqrt{\varepsilon}/2 \cdot (\Psi_{14}^{+} + \Psi_{23}^{+}), \\ F^{-40} &= \pm \sqrt{\varepsilon}/2 \cdot (\Psi_{14}^{+} - \Psi_{23}^{+}). \end{aligned} \quad (3.28)$$

Conversely the symmetric spinor  $\Psi_{rs}^{+}$  can be expressed by an antisymmetric tensor  $F^{-\lambda\mu}$ :

$$\begin{aligned} \Psi_{11}^{+} &= 1/2\varepsilon \cdot (-F^{-13} + iF^{-24} - iF^{-23} + F^{-14}), \\ \Psi_{12}^{+} &= 1/\varepsilon \cdot (iF^{-12} - F^{-34}), \\ \Psi_{13}^{+} &= \pm 1/\sqrt{\varepsilon} \cdot (F^{-10} + iF^{-20}), \\ \Psi_{14}^{+} &= \pm 1/\sqrt{\varepsilon} \cdot (-F^{-30} + F^{-40}), \\ \Psi_{22}^{+} &= \pm 1/2\varepsilon \cdot (-F^{-13} + iF^{-24} + iF^{-23} - F^{-14}), \\ \Psi_{23}^{+} &= \pm 1/\sqrt{\varepsilon} \cdot (-F^{-30} - F^{-40}), \\ \Psi_{24}^{+} &= \pm 1/\sqrt{\varepsilon} \cdot (-F^{-10} + iF^{-20}), \\ \Psi_{33}^{+} &= 1/2\varepsilon \cdot (F^{-13} + iF^{-24} + F^{-23} + F^{-14}), \\ \Psi_{34}^{+} &= 1/\varepsilon \cdot (-iF^{-12} - F^{-34}), \\ \Psi_{44}^{+} &= 1/2\varepsilon \cdot (F^{-13} + iF^{-24} - iF^{-23} - F^{-14}), \end{aligned} \quad (3.29)$$

Using the above obtained correspondence rules between tensors and spinors, we can write tensor equations as spinor equations and vice versa. If the spinors corresponding to vectors  $A^{\lambda}, B^{\lambda}$  and antisymmetric tensors  $A^{-\lambda\mu}, B^{-\lambda\mu}$  are respectively  $\Psi_{rs}^{-}, \phi_{rs}^{-}, \Psi_{rs}^{+}, \phi_{rs}^{+}$ :



$$\begin{aligned} A^\lambda &\sim \Psi_{rs}^-, & B^\lambda &\sim \Phi_{rs}^-, \\ A^{-\lambda\mu} &\sim \Psi_{rs}^{+}, & B^{-\lambda\mu} &\sim \Phi_{rs}^{+}, \end{aligned} \quad (3.30)$$

so the tensor forms constructed by taking product of them correspond to the spinor forms respectively in the following way.

$$A^\lambda B_\lambda = 1/4 \cdot \Psi^{-rs} \Phi_{rs}^-, \quad (3.31)$$

$$A^{-\lambda\mu} B_{\lambda\mu} = 1/4 \cdot \Psi^{+rs} \Phi_{rs}^+, \quad (3.32)$$

$$A^\lambda B^\mu - A^\mu B^\lambda \sim 1/2 \cdot (\Psi_{rs}^- \Phi_{st}^- + \Psi_{ts}^- \Phi_{sr}^-), \quad (3.33)$$

$$A^\lambda B_{\lambda\mu}^- \sim 1/2 \cdot (\Psi_{rs}^- \Phi_{st}^+ - \Psi_{ts}^- \Phi_{sr}^+), \quad (3.34)$$

$$\sum_{(e,p)} A^\lambda B^{-\mu\nu} \sim 1/2 \cdot (\Psi_{rs}^- \Phi_{st}^+ + \Psi_{ts}^- \Phi_{sr}^+), \quad (3.35)$$

$$A^{-\lambda\mu} B_\mu \sim 1/2 \cdot (\Psi_{rs}^+ \Phi_{st}^- - \Psi_{ts}^+ \Phi_{sr}^-), \quad (3.36)$$

$$\sum_{(e,p)} A^{-\lambda\mu} B_\mu^\nu \sim 1/2 \cdot (\Psi_{rs}^+ \Phi_{st}^- + \Psi_{ts}^+ \Phi_{sr}^-), \quad (3.37)$$

$$\sum_{(e,p)} A^{-\lambda\mu} B^{-\nu\rho} \sim 1/2 \cdot (\Psi_{rs}^+ \Phi_{st}^+ - \Psi_{ts}^+ \Phi_{sr}^+), \quad (3.38)$$

$$A^{-\lambda\nu} B_\nu^\mu - A^{-\mu\nu} B_\nu^\lambda \sim 1/2 \cdot (\Psi_{rs}^+ \Phi_{st}^+ + \Psi_{ts}^+ \Phi_{sr}^+), \quad (3.39)$$

where  $(\lambda, \mu, \nu, \rho, \sigma)$  is an even permutation of  $(0, 1, 2, 3, 4)$  and  $\sum_{(e,p)}$  denotes that summation must be taken with respect to the different terms gotten from taking even permutations of the suffices, for example

$$\sum_{(e,p)} A^\lambda B^{-\mu\nu} = A^\lambda B^{-\mu\nu} + A^\mu B^{-\nu\lambda} + A^\nu B^{-\lambda\mu}, \quad (3.40)$$

$$\begin{aligned} \sum_{(e,p)} A^{-\lambda\mu} B^{-\nu\rho} &= A^{-\lambda\mu} B^{-\nu\rho} + A^{-\lambda\nu} B^{-\rho\mu} + A^{-\lambda\rho} B^{-\mu\nu} \\ &\quad + A^{-\mu\nu} B^{-\lambda\rho} + A^{-\mu\rho} B^{-\nu\lambda} + A^{-\nu\rho} B^{-\lambda\mu}. \end{aligned} \quad (3.41)$$

Now we can construct the rotational spinor operator  $\Omega_{rs}$  corresponding to the antisymmetric tensor operator  $K^{-\lambda\mu} = x^\lambda \partial^\mu - x^\mu \partial^\lambda$  and the differential spinor operator  $\partial_{rs}$  corresponding to the vector operator  $\partial^\lambda$ .

Namely we put, omitting the  $+$ ,  $-$  symbols, as follows.

$$\Omega_{11} = 1/2\varepsilon \cdot (-K^{13} + iK^{24} - iK^{23} + K^{14}),$$

$$\Omega_{12} = 1/2\varepsilon \cdot (iK^{11} - K^{11}),$$

$$\Omega_{13} = \pm 1/\sqrt{\varepsilon} \cdot (R^{10} + iR^{20}),$$

$$\Omega_{14} = \pm 1/\sqrt{\varepsilon} \cdot (-R^{30} + R^{40}),$$

$$\Omega_{22} = 1/2\varepsilon \cdot (-K^{13} + iR^{24} + iR^{23} - K^{14}),$$

$$\Omega_{23} = \pm 1/\sqrt{\varepsilon} \cdot (-R^{30} - R^{40}),$$

$$\Omega_{24} = \pm 1/\sqrt{\varepsilon} \cdot (-K^{11} + iK^{20}),$$

$$\Omega_{33} = 1/2\varepsilon \cdot (K^{13} + iK^{24} + iK^{23} + K^{14}),$$

$$\begin{aligned}
\mathcal{Q}_{34} &= 1/\varepsilon \cdot (-iR^{12} - R^{34}), \\
\mathcal{Q}_{44} &= 1/2\varepsilon \cdot (R^{13} + iR^{24} - iR^{23} - R^{14}), \\
(\mathcal{Q}_{rs} &= \mathcal{Q}_{sr}),
\end{aligned} \tag{3.43}$$

and

$$\begin{aligned}
\partial_{12} &= \partial_0, \\
\partial_{13} &= +1/\sqrt{\varepsilon} \cdot (\partial_1 + i\partial_2), \\
\partial_{14} &= \pm 1/\sqrt{\varepsilon} \cdot (-\partial_3 - \partial_4), \\
\partial_{23} &= \pm 1/\sqrt{\varepsilon} \cdot (-\partial_1 + \partial_4), \\
\partial_{24} &= \pm 1/\sqrt{\varepsilon} \cdot (-\partial_1 + i\partial_2), \\
(\partial_{rs} &= -\partial_{sr}), \\
\partial_{11} &= \partial_{22} = \partial_{33} = \partial_{44} = 0.
\end{aligned} \tag{3.44}$$

Using the relation (3.31) (3.39) we can write tensor operations in the form of spinor operations in the following way.

$$\partial_\lambda A^\lambda = 1/4 \cdot \partial^{rs} \mathcal{P}_{rs}^-, \tag{3.45}$$

$$R^{\lambda\mu} A_{\lambda\mu}^- = 1/4 \cdot \mathcal{Q}^{rs} \mathcal{P}_{rs}^+, \tag{3.46}$$

$$\partial_{\lambda\mu} A^{\lambda\mu} \sim 1/2 \cdot (\partial_{rs} \mathcal{P}_t^{-s} + \partial_{ts} \mathcal{P}_r^{-s}), \tag{2.47}$$

$$\partial_\lambda A^{-\lambda\mu} \sim 1/2 \cdot (\partial_r^s \mathcal{P}_{st}^+ - \partial_t^s \mathcal{P}_{rs}^+), \tag{2.48}$$

$$\sum_{(e,p)} \partial^\lambda A^{-\lambda\mu} \sim 1/2 \cdot (\partial_r^s \mathcal{P}_t^{-s} + \partial_t^s \mathcal{P}_{st}^+), \tag{2.49}$$

$$R^{\lambda\mu} A_\mu \sim 1/2 \cdot (\mathcal{Q}_{rs} \mathcal{P}_t^{-s} - \mathcal{Q}_{ts} \mathcal{P}_r^{-s}), \tag{3.50}$$

$$\sum_{(e,p)} R^{\lambda\mu} A^\nu \sim 1/2 \cdot (\mathcal{Q}_{rs} \mathcal{P}_t^{-s} + \mathcal{Q}_{ts} \mathcal{P}_r^{-s}), \tag{3.51}$$

$$\sum_{(e,p)} R^{\lambda\mu} A^{-\nu\rho} \sim 1/2 \cdot (\mathcal{Q}_r^s \mathcal{P}_{st}^+ - \mathcal{Q}_t^s \mathcal{P}_{sr}^+), \tag{3.52}$$

$$R^{\lambda\nu} A_\nu^- - R^{\mu\nu} A_\nu^- \sim 1/2 \cdot (\mathcal{Q}_r^s \mathcal{P}_{st}^+ + \mathcal{Q}_t^s \mathcal{P}_{sr}^+). \tag{3.53}$$

#### § 4. Wave equations for particles with arbitrary spin values

The conditions for the general wave equation which describes a particle with arbitrary spin values in the de Sitter space may be the next three:

(i) it must be composed of quantities and operators along the de Sitter's hyperboloidal surface,

(ii) it must be relativistic invariant or invariant under the de Sitter transformation,

(iii) it must approach, near the point  $(R, 0, 0, 0, 0)$ , to the ordinary general wave equation for arbitrary spin values in the Minkowski space.

The general wave equation in Minkowski space was proposed by Dirac and investigated in detail by Fierz, but another type of equation was also proposed by Bhabha: these two

sorts of equations coincides with each other in the case of lower spin values 0,  $\frac{1}{2}$ , 1. We reported formerly a general wave equation in the de Sitter space which approaches to the Bhabha's type of equation. But by using the further developed rules for five dimensional spinors summarized in the previous section, we can now obtain the equation which approaches to the Dirac-Fierz's type of equation.

(1) *The equation of Dirac-Fierz's type*

We propose the equation

$$(1/R) \cdot \Omega_r^s \Psi_{s, \dots} = \pm \sqrt{\varepsilon} i \kappa \Psi_{r, \dots} \quad (4.1)$$

as the general wave equation in the de Sitter space which approaches to the ordinary Dirac-Fierz's wave equation in the Minkowski space.

(i) The operator  $\Omega_r^s$  is the spinor operator corresponding to the rotational operator  $R^{\lambda\mu}$  which means the infinitesimal displacement along the de Sitter's hyperboloidal surface. Therefore, if the wave function  $\Psi$  is taken from quantities in the de Sitter space, the proposed equation (4.1) becomes an equation defined entirely along the world.

(ii) The equation (4.1) is, moreover, a sort of spinor equation in the five dimensions, so it is invariant under any five dimensional rotation or under any de Sitter transformation. The induced transformation from it in the de Sitter world build up a wider group of transformation than the Lorentz group in the Minkowski space, namely involving not only the four dimensional rotation  $R^{\lambda\mu}$  ( $\lambda, \mu = 1, 2, 3, 4$ ) but also the four dimensional translation  $R^{\lambda 5}$ . Thus the proposed equation (4.1) is relativistic invariant in a somewhat wider meaning than in the case of Minkowski space.

(iii) In order to see how the equation behaves itself when the space becomes flat, it is sufficient to see its behaviours near the point  $(R, 0, 0, 0, 0)$  because of the symmetrical shape of the hyperboloidal surface. Near that point, operators  $R^{\lambda\mu}$  approach to four dimensional differential operators in the following way.

$$\begin{aligned} (1/R) \cdot R^{\lambda\mu} &\rightarrow 0, \quad (\lambda, \mu = 1, 2, 3, 4) \\ (1/R) \cdot R^{0\lambda} &\rightarrow \partial_\lambda, \quad (\lambda = 1, 2, 3, 4). \end{aligned} \quad (4.2)$$

So the operators  $\Omega_r^s$  approach near that point to the differential operators as follows.

$$\begin{aligned} (1/R) \Omega_{13} &\rightarrow \pm 1/\sqrt{\varepsilon} \cdot (-\partial^1 - \partial^2) = \pm \sqrt{\varepsilon} \partial_1^{\dot{1}}, \\ (1/R) \Omega_{14} &\rightarrow \pm 1/\sqrt{\varepsilon} \cdot (\partial^3 - \partial^4) = \pm \sqrt{\varepsilon} \partial_1^{\dot{2}}, \\ (1/R) \Omega_{23} &\rightarrow \pm 1/\sqrt{\varepsilon} \cdot (\partial^3 + \partial^4) = \pm \sqrt{\varepsilon} \partial_2^{\dot{1}}, \\ (1/R) \Omega_{24} &\rightarrow \pm 1/\sqrt{\varepsilon} \cdot (\partial^1 - i\partial^2) = \pm \sqrt{\varepsilon} \partial_2^{\dot{2}}, \\ \text{the other } \Omega_{rs} &\rightarrow 0. \end{aligned} \quad (4.3)$$

The spinor  $\Psi_{r, \dots}$  approaches then to spinors in four dimensions according to (3.11), namely the indices 3, 4 of  $\Psi_{r, \dots}$  become  $\dot{1}$ ,  $\dot{2}$  and change their positions upwards, so we can write

$$(\Psi_1 \dots, \Psi_{2\dots}, \Psi_{3\dots}, \Psi_{4\dots}; \dots) \rightarrow (\phi_{1\dots}, \phi_{2\dots}, \phi_{1\dots}^{\dot{1}}, \phi_{2\dots}^{\dot{2}}; \dots), \quad (4.4)$$

Thus the equation (5.1) approaches near that point to

$$\begin{aligned} 1/R \cdot (\Omega_1^3 \Psi_{3\dots} + \Omega_1^4 \Psi_{4\dots}) &= \pm \sqrt{\varepsilon} i\kappa \Psi_{1\dots}, \\ 1/R \cdot (\Omega_2^3 \Psi_{3\dots} + \Omega_2^4 \Psi_{4\dots}) &= \pm \sqrt{\varepsilon} i\kappa \Psi_{2\dots}, \\ 1/R \cdot (\Omega_3^1 \Psi_{1\dots} + \Omega_3^2 \Psi_{2\dots}) &= \pm \sqrt{\varepsilon} i\kappa \Psi_{3\dots}, \\ 1/R \cdot (\Omega_4^1 \Psi_{1\dots} + \Omega_4^2 \Psi_{2\dots}) &= \pm \sqrt{\varepsilon} i\kappa \Psi_{4\dots}. \end{aligned} \quad (4.5)$$

This becomes by virtue of (4.3) (4.4)

$$\begin{aligned} \partial_{11} \phi_{1\dots}^{\dot{1}} + \partial_{12} \phi_{2\dots}^{\dot{2}} &= i\kappa \phi_{1\dots}, \\ \partial_{21} \phi_{1\dots}^{\dot{1}} + \partial_{22} \phi_{2\dots}^{\dot{2}} &= i\kappa \phi_{2\dots}, \\ \partial^{11} \phi_{1\dots} + \partial^{21} \phi_{2\dots} &= i\kappa \phi_{1\dots}^{\dot{1}}, \\ \partial^{12} \phi_{1\dots} + \partial^{22} \phi_{2\dots} &= i\kappa \phi_{2\dots}^{\dot{2}}. \end{aligned} \quad (4.6)$$

which is the general wave equation of Dirac-Fierz's type in the Minkowski space:

$$\begin{aligned} \partial^{p\dot{q}} \phi_{p\dots}^{\dot{m}} &= i\kappa \phi_{l\dots}^{\dot{m}}, \\ \partial_{p\dot{q}} \phi_{l\dots}^{\dot{m}} &= i\kappa \phi_{p\dots}^{\dot{m}}. \end{aligned} \quad (4.7)$$

Therefore, the proposed equation (4.1) satisfies the above related three conditions and thus it has been shown to be satisfactory.

The equation (4.1) can be written also in another form. By the definition of  $\Omega_{rs}$  it takes the form

$$1/2R \cdot \sigma^\lambda_{\phantom{\lambda}r} \sigma^\mu_{\phantom{\mu}s} R_{\lambda\mu} \Psi_{s\dots} = \pm \sqrt{\varepsilon} i\kappa \Psi_{r\dots}. \quad (4.8)$$

If we set

$$\Psi = \begin{pmatrix} \Psi_{1\dots} \\ \Psi_{2\dots} \\ \Psi_{3\dots} \\ \Psi_{4\dots} \end{pmatrix} \quad (4.9)$$

the equation (4.10) is the  $r$ -component of the matrix equation

$$\{1/2R \cdot P^\lambda P^\mu R_{\lambda\mu} - (\pm \sqrt{\varepsilon}) i\kappa\} \Psi = 0. \quad (4.10)$$

This is the matrix form of our fundamental equation (4.1).

Multiplying (4.12) by the operator

$$(1/2R) P^\lambda P^\mu R_{\lambda\mu} + (\pm \sqrt{\varepsilon}) i\kappa$$

on the left, we obtain

$$\{ (1/2 R \cdot P^\lambda P^\mu R_{\lambda\mu})^2 + \varepsilon \kappa^2 \} \Psi = 0. \quad (4.11)$$

Now in order to simplify this equation, using the commutation relation (3.3) we write

$$\begin{aligned} 1/2 P^\lambda P^\mu R_{\lambda\mu} &= 1/2 P^\lambda P^\mu (x_\lambda \cdot \partial / \partial x^\mu - x_\mu \partial / \partial x^\lambda) \\ &= 1/2 \cdot \{ P^\lambda P^\mu x_\lambda \cdot \partial / \partial x^\mu + (P^\mu P^\lambda - 2g^{\lambda\mu}) x_\mu \cdot \partial / \partial x^\lambda \} \\ &= (P^\lambda x_\lambda) (P^\mu \cdot \partial / \partial x_\mu) - x^\lambda \cdot \partial / \partial x^\lambda. \end{aligned} \quad (4.12)$$

Then

$$\begin{aligned} (1/2 \cdot P^\lambda P^\mu R_{\lambda\mu})^2 &= (P^\lambda x_\lambda) (P^\mu \partial / \partial x^\mu) (P^\nu x_\nu) (P^\nu \cdot \partial / \partial x^\nu) - (P^\lambda x_\lambda) (P^\mu \cdot \partial / \partial x^\mu) (x^\nu \cdot \partial / \partial x^\nu) \\ &\quad - (x^\lambda \cdot \partial / \partial x^\lambda) (P^\nu x_\nu) (P^\mu \cdot \partial / \partial x^\mu) + (x^\lambda \cdot \partial / \partial x^\lambda) (x^\mu \partial / \partial x^\mu) \\ &= -x^\lambda x_\lambda \cdot \partial^2 / \partial x^\mu \partial x_\mu + 3 P^\lambda x_\lambda P^\mu \cdot \partial / \partial x^\mu + (x^\lambda \cdot \partial / \partial x^\lambda) (x^\mu \partial / \partial x^\mu) \\ &= -x^\lambda x_\lambda \cdot \partial^2 / \partial x^\mu \partial x_\mu + (x^\lambda \cdot \partial / \partial x^\lambda) (x^\mu \cdot \partial / \partial x^\mu) + 3 x^\lambda \cdot \partial / \partial x^\lambda + 3/2 \cdot P^\lambda P^\mu R_{\lambda\mu} \\ &= -1/2 \cdot R^{\lambda\mu} R_{\lambda\mu} + 3/2 \cdot P^\lambda P^\mu R_{\lambda\mu}, \end{aligned}$$

with the help of (4.12) and the relation

$$\frac{1}{2} R^{\lambda\mu} R_{\lambda\mu} = x^\lambda x_\lambda \frac{\partial^2}{\partial x^\mu \partial x_\mu} - \left( x^\lambda \frac{\partial}{\partial x^\lambda} \right) \left( x^\mu \frac{\partial}{\partial x^\mu} \right) - 3 x^\lambda \frac{\partial}{\partial x^\lambda}.$$

So (4.11) becomes

$$\{ 1/2 R^2 \cdot R^{\lambda\mu} R_{\lambda\mu} - 3/2 R^2 \cdot P^\lambda P^\mu R_{\lambda\mu} - \varepsilon \kappa^2 \} \Psi = 0.$$

Using the wave equation (4.10) we get finally

$$\left\{ \frac{1}{2 R^2} R^{\lambda\mu} R_{\lambda\mu} - \varepsilon \kappa^2 - \frac{(\pm \sqrt{\varepsilon}) 3 i \kappa}{R} \right\} \Psi = 0. \quad (4.13)$$

This is the second order wave equation which is satisfied by each of the components of the wave function, namely

$$\left\{ \frac{1}{2 R^2} R^{\lambda\mu} R_{\lambda\mu} - \varepsilon \kappa^2 - \frac{(\pm \sqrt{\varepsilon}) 3 i \kappa}{R} \right\} \Psi_{r, \dots} = 0. \quad (4.14)$$

This equation approaches near the point  $(R, 0, 0, 0, 0)$  to the second order wave equation of Klein-Gordon :

$$\{ \square - \kappa^2 \} \Psi_{r, \dots} = 0 \quad (4.15)$$

for then

$$(1/2 R^2) R^{\lambda\mu} R_{\lambda\mu} \rightarrow \varepsilon \square. \quad (4.16)$$

(2) *The equation of Bhabha's type.*

Now we must notice that another standpoint may be also possible, as mentioned above,



namely we can construct the general wave equation which approaches to the Bhabha's type of equation in the Minkowski space. The wave equation investigated by Bhabha is

$$\{\alpha^l \partial_l - i\kappa\} \psi = 0, \quad l=1, 2, 3, 4, \quad (4.17)$$

where the matrices  $\alpha^l$  is defined as

$$\alpha^l = I^{0l}$$

by means of the representation matrices  $I_{\lambda\mu}$  of the infinitesimal transformations in the  $x^\lambda$ ,  $x_\mu$ , planes in the five dimensional space, in another word  $I^{\lambda\mu}$  are the generators of a representation of the continuous de Sitter group. As well known, the finite dimensional irreducible representations of the de Sitter group can be distinguished by the symbol  $D(p, q)$ , where  $p$  and  $q$  take the values  $0, 1, 2, 3, \dots$  for one valued representations and the values  $1/2, 3/2, 5/2, \dots$  for two valued representations. When the representation  $D(p, q)$  of the de Sitter group is taken, the wave function  $\psi$  of the Bhabha's equation has the same number of components as the dimensions of the representation  $D(p, q)$ , and it is composed of several number of irreducible spinors in the four dimensions which split out from the five dimensional spinor transforming according to  $D(p, q)$  by confining the five dimensional transformations to the four dimensional. For  $D(1/2, 1/2)$  or for the one by the five dimensional spinors the Bhabha's equation (4.17) becomes the electron wave equation of Dirac and  $\alpha^l$  become the Dirac's matrices. For  $D(1.1)$  and  $D(1.0)$  or for the representation by the symmetric or antisymmetric irreducible spinors of the second rank the Bhabha's equation (4.17) becomes the vector and scalar meson wave equation of Kemmer's type and  $\alpha^l$  become the Kemmer's  $\beta^l$  matrices of degree 10 and 5, respectively. Thus within the region for the particles discovered in the nature (spin  $0, 1/2, 1$ ), Bhabha's wave equation coincides with the Dirac-Fierz's wave equation, even if for particles with higher spin values than 1 they do not so in general.

Now we can propose the following equation as the wave equation in the de Sitter space which approaches to the Bhabha's wave equation near the point  $(R, 0, 0, 0, 0)$ :

$$\{1/2 R \cdot \alpha^{\lambda\mu} R_{\lambda\mu} - i\kappa\} \Psi = 0, \quad \alpha^{\lambda\mu} = I^{\lambda\mu}, \quad (4.19)$$

where  $I^{\lambda\mu}$  is the generator of the irreducible representation  $D(p, q)$  of the de Sitter group,  $\Psi$  is a five dimensional spinors transforming according to  $D(p, q)$ . (i) This equation is defined, of course, entirely along the de Sitter's hyperboloidal surface. (ii) It is also relativistic invariant, because of the tensor character of  $I^{\lambda\mu}$  and  $R_{\lambda\mu}$ . (iii) It approaches, near the point  $(R, 0, 0, 0, 0)$ , to the Bhabha's equation (4.17), for then

$$\alpha^{0l} \rightarrow \alpha^l, \quad R_{0l} \rightarrow \partial_l, \quad \Psi \rightarrow \psi. \quad (4.20)$$

In the later section we will consider the fields for lower spin values in detail, and will see that the two types of our proposed wave equations (4.1) and (4.17) coincide with each other for the lower spin values  $(0, 1/2, 1)$  in the same way as in the Minkowski space. The effect of the value  $+1$  or  $-1$  of  $\varepsilon$  is involved in the fact that the fundamental form which the group leaves invariant is

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2,$$

or

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2.$$

### § 5. Fields for particles with lower spin values (0, 1/2, 1)

#### (1) The electron field

The electron field is described with a spinor of the first rank  $\Psi$ , satisfying the wave equation

$$\left\{ \frac{1}{2R} P^\lambda P^\mu R_{\lambda\mu} \mp i \frac{1}{\sqrt{\varepsilon}} \kappa \right\} \Psi = 0, \quad \Psi = \begin{Bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{Bmatrix}. \quad (5.1)$$

We can easily show that this equation approaches to the Dirac's electron wave equation in Minkowski space when the space becomes flat, for then

$$R_{\lambda\mu} \rightarrow R_{0\mu} = -\varepsilon \partial_\mu, \\ P^0 P^\mu = \pm \sqrt{\varepsilon} \gamma^\mu, \quad (5.2)$$

$\gamma^\mu$  being the Dirac's matrices. Thus we have in this case the same result as that discussed by Dirac formerly.<sup>1)</sup>

The adjoint wave function  $\Phi$  is defined as the one satisfying the adjoint wave equation

$$\Phi \left\{ \frac{1}{2R} P^\lambda P^\mu R_{\lambda\mu} \pm i \frac{1}{\sqrt{\varepsilon}} \kappa \right\} = 0, \quad \Phi = (\Phi^1, \Phi^2, \Phi^3, \Phi^4) \quad (5.3)$$

$\Phi^\mu$  being a contravariant spinor of the first rank, the operators operating to the left. The charge current density  $J$  can be introduced by

$$J^\mu = (1/2R) \Phi (P^\lambda P^\mu - P^\mu P^\lambda) \Psi, \quad (5.4)$$

which satisfies the condition

$$x_\mu J^\mu = 0 \quad (5.5)$$

and the continuity condition

$$\partial_\mu J^\mu = 0 \quad (5.6)$$

and approaches to the ordinary charge current density when the space becomes flat.

As already shown by Dirac<sup>1)</sup> a constant of the motion is

$$R^{\lambda\mu} + \frac{1}{2} P^\lambda P^\mu. \quad (5.7)$$

This shows that  $\frac{1}{2} P^\lambda P^\mu$  is to be interpreted as the spin angular momentum, which must be added to the orbital momentum  $L^{\lambda\mu}$  to give a constant of the motion. To see the spin matrices in details let us write the quantity (5.7) as follows

linear momenta :

$$R^{0\nu} + 1/2 \cdot (\mp \sqrt{\varepsilon}) \gamma^\nu \quad (\nu=1, 2, 3, 4), \quad (5.8a)$$

angular momenta :

$$R^{kl} + 1/2 \cdot \varepsilon \sigma^{kl} \quad (k, l=1, 2, 3), \quad (5.8b)$$

Lorentz acceleration :

$$R^{k4} + 1/2 \cdot \varepsilon \sigma^{lk} \quad (5.8c)$$

where

$$\sigma^{kl} = \begin{pmatrix} \sigma^k \sigma^l & 0 \\ 0 & \sigma^k \sigma^l \end{pmatrix}, \quad \sigma^{lk} = \begin{pmatrix} -\sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \quad (5.8d)$$

Thus we have seen that the generalized spin matrices coincide to the ordinary spin matrices for the angular momenta. It involves furthermore another type of spins concerning to the linear momenta and acceleration which did not occur in the theory in the Minkowski space. As easily shown by considering with the coordinate system (A), the spin concerning to the angular momenta causes the same effect on the wave function as the ordinary theory, since  $R^{kl}$  have the same expression in terms of  $\theta_2, \theta_3$ . However the spin concerning with other parts causes new effects on it depending on other arguments also.

The wave equation (5.1) takes the form writing with the components

$$\begin{aligned} & -i\varepsilon(R_{12} + R_{34})\Psi_1 - i\varepsilon(R_{32} + R_{14} + iR_{31} + iR_{24})\Psi_2 \\ & \pm \sqrt{\varepsilon}(R_{03} + iR_{04})\Psi_3 \pm \sqrt{\varepsilon}(R_{01} + iR_{02})\Psi_4 \pm i\frac{1}{\sqrt{\varepsilon}}\kappa'\Psi_1 = 0, \\ & -i\varepsilon(R_{32} + R_{14} - iR_{31} - iR_{24})\Psi_1 + i\varepsilon(R_{12} + R_{34})\Psi_2 \\ & \pm \sqrt{\varepsilon}(R_{01} - iR_{02})\Psi_3 \pm \sqrt{\varepsilon}(-R_{03} + iR_{04})\Psi_4 \pm i\frac{1}{\sqrt{\varepsilon}}\kappa'\Psi_2 = 0, \\ & \pm \sqrt{\varepsilon}(-R_{03} + iR_{04})\Psi_1 \pm \sqrt{\varepsilon}(-R_{01} - iR_{02})\Psi_2 - i\varepsilon(R_{12} - R_{34})\Psi_3 \\ & - i\varepsilon(R_{32} - R_{14} + iR_{31} - iR_{24})\Psi_4 \pm i\frac{1}{\sqrt{\varepsilon}}\kappa'\Psi_3 = 0, \\ & \pm \sqrt{\varepsilon}(-R_{01} + iR_{02})\Psi_1 \pm \sqrt{\varepsilon}(R_{03} - iR_{04})\Psi_2 - i\varepsilon(R_{32} - R_{14} - iR_{31} + iR_{24})\Psi_3 \\ & - i\varepsilon(-R_{12} + R_{34})\Psi_4 \pm i\frac{1}{\sqrt{\varepsilon}}\kappa'\Psi_4 = 0. \end{aligned} \quad (5.9)$$

This equation may be solved by expanding in the terms of the functions obtained as the solutions of the second order wave equation and using the expression of  $R^{\lambda\mu}$  given in (2.22). But we will notice here only the fact we can easily obtain the solution of the equation with mass of opposite sign from the solution of this equation in the de Sitter space. In the de Sitter space it takes the form

$$\begin{aligned}
& (R_{12} + R_{34})\Psi_1 + (R_{32} + R_{14} + iR_{31} + iR_{24})\Psi_2 \\
& \quad \pm (iR_{03} - R_{04})\Psi_3 \pm (iR_{01} - R_{02})\Psi_4 \pm \kappa'\Psi_1 = 0, \\
& (R_{32} + R_{14} - iR_{31} - iR_{24})\Psi_1 + (-R_{12} - R_{34})\Psi_2 \\
& \quad \pm (iR_{01} + R_{02})\Psi_3 \pm (-iR_{03} - R_{04})\Psi_4 \mp \kappa'\Psi_2 = 0, \\
& \pm (-iR_{03} - R_{04})\Psi_1 \pm (-iR_{01} + R_{02})\Psi_2 + (R_{12} - R_{34})\Psi_3 \\
& \quad + (R_{32} - R_{14} + iR_{31} - iR_{24})\Psi_4 \mp \kappa'\Psi_3 = 0, \\
& \pm (-iR_{01} - R_{02})\Psi_1 \pm (iR_{03} + R_{04})\Psi_2 + (R_{32} - R_{14} - iR_{31} + iR_{24})\Psi_3 \\
& \quad + (-R_{12} + R_{34})\Psi_4 \mp \kappa'\Psi_4 = 0.
\end{aligned} \tag{5.10}$$

This equation is invariant under the transformation

$$\kappa' \rightarrow -\kappa',$$

$$\Psi_1 \rightarrow \bar{\Psi}_2, \quad \Psi_2 \rightarrow -\bar{\Psi}_1, \quad \Psi_3 \rightarrow -\bar{\Psi}_4, \quad \Psi_4 \rightarrow \bar{\Psi}_3. \tag{5.11}$$

This shows that the field

$$(\Psi'_1, \Psi'_2, \Psi'_3, \Psi'_4) = (\bar{\Psi}_2, -\bar{\Psi}_1, -\bar{\Psi}_4, \bar{\Psi}_3) \tag{5.12}$$

describes the particle with negative energy.

Lastly we will show that the Bhabha's type wave equation for the electron coincides with (5.1) which obtained as a special case of the Dirac-Fierz's type equation (4.10). The electron wave equation of Bhabha's type concerns with the irreducible representation  $D(1/2, 1/2)$  or the one by the first rank spinor. In this representation we can take

$$I^{\lambda\mu} = 1/4 \cdot (P^\lambda P^\mu - P^\mu P^\lambda) = 1/2 \cdot P^\lambda P^\mu, \tag{5.13}$$

for this form of  $I^{\lambda\mu}$  satisfies the representation condition

$$[I_{ij}, I_{kl}] = -g_{ik}I_{jl} + g_{jk}I_{il} + g_{il}I_{jk} - g_{jl}I_{ik} \tag{5.14}$$

and the spinor condition

$$S_{\lambda\mu}(\theta) I^\lambda_\nu S^{-1}_{\lambda\mu}(\theta) = \sum_\nu \alpha_{\nu\lambda} I^\nu_\nu \tag{5.15}$$

with

$$\begin{aligned}
S_{\lambda\mu}(\theta) &= e^{I_{\lambda\mu}\theta} = e^{(\theta/2)P^\lambda P^\mu}, \\
(\alpha_{\lambda\mu}) &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ or } \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}.
\end{aligned} \tag{5.16}$$

Thus we obtain the equation (5.1) as a special case of (4.19).

## (2) The meson field

(i) The vector meson can be described as a field of an antisymmetric tensor, so it is the field of a symmetric spinor of the second rank. The general wave equation (4.1) takes the form in this case

$$\frac{1}{R} \mathcal{D}_s \Psi_{st}^+ = \pm \frac{i}{\sqrt{\varepsilon}} \kappa \Psi_{rs}^+, \quad (\Psi_{sr}^+ = \Psi_{rs}^+). \quad (5.17)$$

Using (3.53) this equation can be written also in the following tensor form

$$R^{\lambda\mu} F_\nu^\mu - R^{\mu\nu} F_\nu^\lambda = \pm \frac{i}{\sqrt{\varepsilon}} \kappa F^{\lambda\mu} \quad (5.18)$$

where  $F^{\lambda\mu}$  is the antisymmetric tensor corresponding to the symmetric spinor  $\Psi_{rs}^+$ . This tensor splits into an antisymmetric tensor (or a so-called six vector)  $(F_1, F_2, F_3, G_1, G_2, G_3)$  and a vector  $(U_1, U_2, U_3, U)$  in the four dimensions near the point  $(R, 0, 0, 0)$  in the following way

$$\begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} & F_{04} \\ & 0 & F_{12} & F_{13} & F_{14} \\ & & 0 & F_{23} & F_{24} \\ & & & 0 & F_{34} \\ & & & & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -G_3 & G_2 & F_1 & iU_1 \\ & 0 & -G_1 & F_2 & iU_2 \\ & & 0 & F_3 & iU_3 \\ & & & 0 & iU_0 \\ & & & & 0 \end{pmatrix} \quad (5.19)$$

and the equation (5.18) approaches to the Yukawa's equation of the vector meson.

From (5.18)

$$(x_\lambda \partial_\mu - x_\mu \partial_\lambda) F_\nu^\mu - (x_\nu \partial_\mu - x_\mu \partial_\nu) F_\lambda^\mu = \kappa F_{\lambda\nu}$$

or

$$(x_\lambda \partial_\mu F_\nu^\mu - x_\nu \partial_\mu F_\lambda^\mu) + x_\mu (\partial_\nu F_\lambda^\mu - \partial_\lambda F_\nu^\mu) = \kappa F_{\lambda\nu}.$$

Operating  $\partial_\mu$  on the condition

$$x_\lambda F_\nu^\lambda = 0 \quad (5.20)$$

we have

$$x_\lambda \partial_\mu F_\nu^\lambda = -F_{\mu\nu}.$$

So combining this with the above relation

$$x_\lambda \partial_\mu F_\nu^\mu - x_\nu \partial_\mu F_\lambda^\mu - 2F_{\nu\lambda} = \kappa F_{\lambda\mu}.$$

Multiplying with  $x^\lambda$  we have

$$\partial_\mu F_\nu^\mu = 0. \quad (5.21)$$

Putting

$$F^{*\lambda\mu} = x_\nu F_{\rho\sigma} + x_\rho F_{\sigma\nu} + x_\sigma F_{\nu\rho}, \quad (5.22)$$

$((\lambda, \mu, \nu, \rho)$  is an even permutation of  $(0, 1, 2, 3, 4)$ ),

we can show the following relation with the help of (5.18), (5.20), (5.21).

$$\sum_{(\lambda\mu\nu\rho)} R^{\lambda\mu} F^{*\nu\rho} = \sum_{(\lambda\mu\nu\rho)} R^{\lambda\mu} (x_\sigma F_{\lambda\mu} + x_\lambda F_{\mu\sigma} + x_\mu F_{\sigma\lambda}) = 0 \quad (5.23)$$



Thus we can give the following form as the current vector and energy momentum tensor using the same consideration as Husimi's treatment of the electromagnetic field.

$$J^\lambda = \sum R^{\mu\nu} F^{*\rho\sigma},$$

$$T^{\lambda\mu} = F^{*\lambda\epsilon} F_\epsilon^{*\mu} + 1/4 \cdot \delta^{\lambda\mu} F^{*\alpha\beta} F_{\alpha\beta}^*. \quad (5.24)$$

(ii) The scalar meson can be described as a field of a vector, so it is the field of an irreducible antisymmetric spinor of the second rank. The general wave equation (4.1) becomes in this case

$$(1/R) \partial_r^s \Psi_{st}^- = \pm (i/\sqrt{\epsilon}) \kappa \Psi_{rt}^-, \quad (\Psi_{st}^- = -\Psi_{ts}^-). \quad (5.26)$$

Using the relation (3.46) this equation can be written in the following tensor form

$$(1/R) R_\mu^\lambda U^\mu = \pm (i\sqrt{\epsilon}) \kappa U^\lambda \quad (5.27)$$

where  $U^\lambda$  is the vector corresponding to the irreducible antisymmetric spinor  $\Psi_{rs}^-$ . This five dimensional vector  $U^\lambda$  splits into a vector  $U_1, U_2, U_3, U_4$  and a scalar  $U^5$  in the four dimensions near the point  $P$  in the following way.

$$(U_0, U_1, U_2, U_3, U_4) \longrightarrow (iU, U_1, U_2, U_3, U_4) \quad (5.28)$$

and the equation (5.26) approaches to the Yukawa's equation of the scalar meson.

From the equation (5.27)

$$(x_\lambda \partial_\mu - x_\mu \partial_\lambda) U^{\lambda\mu} = \kappa U^{\lambda\lambda}.$$

Differentiating the condition

$$x_\mu U^{\lambda\mu} = 0, \quad (5.29)$$

$$g_{\mu\lambda} U^\mu + x_\mu \partial_\lambda U^\mu = 0.$$

$$x_\mu \partial_\lambda U^\mu = -U_\lambda. \quad (5.30)$$

So we have

$$x_\lambda \partial_\mu U^{\lambda\mu} - U_\lambda = \kappa U_\lambda,$$

$$x^\lambda x_\lambda \partial_\mu U^\mu = 0,$$

$$\partial_\mu U^{\lambda\mu} = 0. \quad (5.31)$$

Thus we can easily show that we are able to take the following form as the current vector and the energy momentum tensor

$$J^\lambda = (1/R) x^\mu \partial_\mu U^\lambda, \quad (5.32)$$

$$T^{\lambda\mu} = U^\lambda U^\mu, \quad (5.33)$$

for by using the relation (5.27), (5.29), (5.30), (5.31)

$$x_\lambda J^\lambda = (1/R) x_\lambda x^\mu \partial_\mu U^\lambda$$

$$= (1/R) x^\mu x_\lambda x_\mu U^\lambda$$

$$= -(1/R)x_\lambda U^\lambda = 0, \quad (5.34)$$

$$\begin{aligned} \partial_\lambda J^\lambda &= 1/R \cdot (g_\lambda^\mu \partial_\mu U^\lambda + x^\mu \partial_\lambda \partial_\mu U^\lambda) \\ &= 1/R \cdot (\partial_\lambda U^\lambda + x^\mu \partial_\mu \partial_\lambda U^\lambda) \\ &= 0, \end{aligned} \quad (5.35)$$

$$x_\lambda T^{\lambda\mu} = x_\lambda U^\lambda U^\mu$$

$$= 0,$$

$$\begin{aligned} \partial_\lambda T^{\lambda\mu} &= (\partial_\lambda U^\lambda) U^\mu + U^\lambda \partial_\lambda U^\mu \\ &= 0. \end{aligned} \quad (5.37)$$

(iii) As we have already reported in letters, the meson wave equations of Bhabha's type coincide with the equation introduced above as a special case of the Dirac-Fierz's type equation. The vector and the scalar meson fields concern with the irreducible representation  $D(1 \cdot 0)$ ,  $D(1 \cdot 1)$  or with the symmetric and antisymmetric irreducible spinor of the second rank respectively. The transformation of these spinors can be given easily for  $S^{\lambda\mu}(\theta)$  corresponding to the plane transformation (see our previous reports about five dimensional spinors<sup>6)</sup>). We can put

$$I^{\lambda\mu} = (\partial S^{\lambda\mu}(\theta) / \partial \theta)_{\theta=0} \quad (5.38)$$

where this differentiation of matrix denotes only the differentiation of each matrix element. Thus we have  $I^{\lambda\mu}$  for  $D(1 \cdot 0)$ ,  $D(1 \cdot 1)$ . In the case of de Sitter space we will give their exact forms.

(vector meson)

$$I^{12} = i \begin{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{pmatrix},$$

$$I^{11} = \frac{1}{2} \left( \begin{array}{ccc|ccc|ccc} 0 & \sqrt{2} & 0 & & & & & & & \\ \sqrt{2} & 0 & \sqrt{2} & & & & & & & \\ 0 & \sqrt{2} & 0 & & & & & & & \\ & & & 0 & -\sqrt{2} & 0 & & & & \\ & & & -\sqrt{2} & 0 & -\sqrt{2} & & & & \\ & & & 0 & -\sqrt{2} & 0 & & & & \\ & & & & & & 0 & -1 & 0 & 0 \\ & & & & & & -1 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & -1 \\ & & & & & & 0 & 0 & -1 & 0 \end{array} \right),$$

$$I^{12} = \frac{1}{2} \left( \begin{array}{ccc|ccc|ccc} 0 & -\sqrt{2} & 0 & & & & & & & \\ \sqrt{2} & 0 & -\sqrt{2} & & & & & & & \\ 0 & \sqrt{2} & 0 & & & & & & & \\ & & & 0 & -\sqrt{2} & 0 & & & & \\ & & & \sqrt{2} & 0 & -\sqrt{2} & & & & \\ & & & 0 & \sqrt{2} & 0 & & & & \\ & & & & & & 0 & -1 & -1 & 0 \\ & & & & & & 1 & 0 & 0 & -1 \\ & & & & & & 1 & 0 & 0 & -1 \\ & & & & & & 0 & 1 & 1 & 0 \end{array} \right),$$

$$I^{14} = \frac{i}{2} \left( \begin{array}{ccc|ccc|ccc} 0 & \sqrt{2} & 0 & & & & & & & \\ -\sqrt{2} & 0 & \sqrt{2} & & & & & & & \\ 0 & -\sqrt{2} & 0 & & & & & & & \\ & & & 0 & -\sqrt{2} & 0 & & & & \\ & & & \sqrt{2} & 0 & -\sqrt{2} & & & & \\ & & & 0 & \sqrt{2} & 0 & & & & \\ & & & & & & 0 & -1 & 0 & 0 \\ & & & & & & 1 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & 0 & -1 \\ & & & & & & 0 & 0 & 1 & 0 \end{array} \right),$$

$$F^{23} = \frac{i}{2} \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F^{34} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F^{01} = \frac{i}{2} \begin{pmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$f^{02} = \frac{-1}{2}$$

$$\begin{pmatrix} 0 & -\sqrt{2} & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -\sqrt{2} & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$f^{03} = \frac{i}{2}$$

$$\begin{pmatrix} -\sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -\sqrt{2} \end{pmatrix},$$

$$\begin{pmatrix} \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$f^{04} = \frac{i}{2}$$

$$\begin{pmatrix} -\sqrt{2} & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -\sqrt{2} \\ -\sqrt{2} & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -\sqrt{2} \end{pmatrix},$$

$$\begin{pmatrix} -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} \end{pmatrix}$$



(unwritten rectangulars are filled with zeros).

(scalar meson)

$$\begin{aligned}
 I^{12} &= i \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline & & & 0 \end{pmatrix}, \quad I^{14} = \frac{1}{2} \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ \hline & & & 0 \end{pmatrix}, \\
 I^{13} &= \frac{1}{2} \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ \hline & & & 0 \end{pmatrix}, \quad I^{24} = \frac{i}{2} \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ \hline & & & 0 \end{pmatrix}, \\
 I^{23} &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline & & & 0 \end{pmatrix}, \quad I^{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline & & & 0 \end{pmatrix}, \\
 I^{01} &= \frac{i}{2} \begin{pmatrix} & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & -1 \\ \hline -1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad I^{02} = \frac{1}{2} \begin{pmatrix} & & & 1 \\ & & & 0 \\ & & & 0 \\ & & & 1 \\ \hline 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \\
 I^{03} &= \frac{i}{2} \begin{pmatrix} & & & 0 \\ & & & -1 \\ & & & -1 \\ & & & 0 \\ \hline 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad I^{04} = \frac{i}{2} \begin{pmatrix} & & & 0 \\ & & & -1 \\ & & & 1 \\ & & & 0 \\ \hline 0 & -1 & 1 & 0 & 0 \end{pmatrix},
 \end{aligned}$$

The wave functions in this case are symmetric spinor

$$\mathcal{V} = (\mathcal{V}_{11}^+, \mathcal{V}_{12}^+, \mathcal{V}_{22}^+; \mathcal{V}_{33}^+, \mathcal{V}_{34}^+, \mathcal{V}_{44}^+; \mathcal{V}_{13}^+, \mathcal{V}_{14}^+, \mathcal{V}_{23}^+, \mathcal{V}_{24}^+) \quad (5 \cdot 40a)$$

and antisymmetric spinor

$$\mathcal{V} = (\mathcal{V}_{13}^-, \mathcal{V}_{14}^-, \mathcal{V}_{23}^-, \mathcal{V}_{24}^-; \mathcal{V}_{12}^- + \mathcal{V}_{34}^-), (\mathcal{V}_{12}^- - \mathcal{V}_{34}^- = 0) \quad (5 \cdot 40b)$$

respectively for vector and scalar case. In this representation the equations approach to the Bhabha's equation when the space becomes flat.

If we want to have wave equations of Kemmer's type we have only to perform the following transformation

$$\begin{aligned} \phi_1 &= i/\sqrt{2} \cdot (\psi_{11} - \psi_{22} + \psi_{33} - \psi_{44}), \\ \phi_2 &= (\psi_{12} + \psi_{34}), \\ \phi_3 &= 1/\sqrt{2} \cdot (\psi_{11} + \psi_{22} + \psi_{33} + \psi_{44}), \\ \phi_4 &= 1/\sqrt{2} \cdot (\psi_{11} - \psi_{22} - \psi_{33} + \psi_{44}), \\ \phi_5 &= \psi_{12} - \psi_{34}, \\ \phi_6 &= i/\sqrt{2} \cdot (\psi_{11} + \psi_{22} - \psi_{33} - \psi_{44}), \\ \phi_7 &= -i(\psi_{13} - \psi_{24}), \\ \phi_8 &= i(\psi_{14} + \psi_{23}), \\ \phi_9 &= -(\psi_{13} + \psi_{24}), \\ \phi_{10} &= -i(\psi_{14} - \psi_{23}), \end{aligned} \quad (5 \cdot 41a)$$

and

$$\begin{aligned} \varphi_1 &= -i/\sqrt{2} \cdot (\psi_{13} - \psi_{24}), \\ \varphi_2 &= i/\sqrt{2} \cdot (\psi_{11} + \psi_{22}), \\ \varphi_3 &= 1/\sqrt{2} \cdot (\psi_{13} + \psi_{24}), \\ \varphi_4 &= i/\sqrt{2} \cdot (\psi_{14} - \psi_{23}), \\ \varphi_5 &= \psi_{12} + \psi_{34} \end{aligned} \quad (5 \cdot 41b)$$

respectively for vector and scalar cases. Taking the imaginary  $x^4$  we have the representation corresponding to Kemmer's. Writing  $I^{\lambda\mu}$  in this representation as  $\beta^{\lambda\mu}$  we have the following form, where  $\beta^{00}$  is the same as the Kemmer's  $\beta^i$  matrices.

$$\beta^{12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & -1 & 0 & 0 \\ & & 0 & 0 & 1 \\ & & 0 & 0 & 0 \\ & & -1 & 0 & 0 \\ & & & & & & & & 0 \end{pmatrix},$$

$$\beta^{14} = \begin{pmatrix} & & & 0 & 0 & 0 \\ & & & 0 & 0 & 1 \\ & & & 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ & & & & & & & & 1 \\ & & & & & & & & 0 \\ & & & & & & & & 0 \\ & & & & & & & & 0 \\ & & & & & & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\beta^{13} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & -1 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 & 0 \\ & & -1 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & -1 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & & & & & 0 \end{pmatrix},$$

$$\beta^{21} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta^{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\beta^{21} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \quad (5 \cdot 42a)$$

and

$$\begin{aligned}
 \beta^{12} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \beta^{14} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
 \beta^{13} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \beta^{24} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
 \beta^{23} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \beta^{34} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
 \end{aligned} \tag{5.42b}$$

respectively for vector and scalar cases. In this case the wave equation approaches to the Kemmer's type of wave equation for mesons in the Minkowski space when the space becomes flat.

In order to obtain equations of Yukawa's type we have only to put

$$\begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} & F_{04} \\ & 0 & F_{12} & F_{13} & F_{14} \\ & & 0 & F_{23} & F_{24} \\ & & & 0 & F_{34} \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\phi_7 & i\phi_8 & i\phi_9 & i\phi_{10} \\ & 0 & -\phi_6 & \phi_5 & \phi_1 \\ & & 0 & \phi_4 & \phi_2 \\ & & & 0 & \phi_3 \\ & & & & 0 \end{pmatrix} \tag{5.43a}$$

and

$$(U_0, U_1, U_2, U_3, U_4) = (i\varphi_5, \varphi_1, \varphi_2, \varphi_3, \varphi_4). \tag{5.43b}$$

The equation in this case are (5.18) and (5.27) which were obtained as special cases of the general equation of Dirac-Fierz's type. By the way it is very interesting that the form of the mesons in the de Sitter space is very comprehensive and splits into many equations when the space becomes flat.

(iv) Recently an interesting discussion was given by S, A, Raje<sup>11</sup>. The proposed equation by him is



$$\left\{ \frac{1}{2R} S^{\lambda\mu} R_{\lambda\mu} - \kappa \right\} \Psi = 0, \quad (5.44a)$$

$$S^{\lambda\mu} = \beta^\lambda \beta^\mu - \beta^\mu \beta^\lambda, \quad (5.44b)$$

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = g^{\mu\nu} \beta^\lambda + g^{\lambda\nu} \beta^\mu. \quad (5.44c)$$

This equation approaches directly to the Kemmer's equation for mesons in Minkowski space when the space becomes flat. But it will not be easy to obtain exact forms of the irreducible set of these  $\beta$  matrices for  $\lambda, \mu, \nu = 0, 1, 2, 3, 4$ . Our opinion is, however, that the matrices  $\beta^{\lambda\mu}$  of (5.42a), (5.42b) may be obtained then.

### (3) The electromagnetic field

The electromagnetic field can also included in our fundamental equation (4.1). The electromagnetic field is described as an antisymmetric tensor field without rest mass, so it is a symmetric spinor  $\Psi_{st}^+$  field with  $\kappa = 0$ . Thus the photon wave equation in the absence of the charge current is

$$(1/2R) \mathcal{Q}_r^s \Psi_{st}^+ = 0. \quad (5.45)$$

But if the charge-current exists, the wave equation may take the form

$$(1/2R) \mathcal{Q}_r^s \Psi_{st}^+ = \chi_{rt}^- \quad (5.46)$$

where  $\chi_{rt}^-$  is the irreducible antisymmetric spinor corresponding to the vector  $J^\lambda$ . Because of the antisymmetric character with regard to the suffices  $r, t$ , the tensor equation corresponding to (5.45) and (5.46) are

$$1/R \cdot \sum_{(e,p)} R^{\lambda\mu} F^{\nu\rho} = 0 \quad (5.47)$$

and

$$1/R \cdot \sum_{(e,p)} R^{\lambda\mu} F^{\nu\rho} = J^*_{\sigma} \quad (5.48)$$

respectively, owing to (3.52). These equations are the electromagnetic field equation proposed by Husimi from an entirely different standpoint, and these equation approach to the Maxwell equations in the tensor form when the space becomes flat. It is important that the results obtained from the two different standpoints of Husimi and us are entirely the same. It is also quite interesting that there exists a close correlation between the equation of the electromagnetic field, namely the left hand sides of them are the antisymmetric and the symmetric part of the same spinor operation respectively; see the relation between the rules (3.52), and (3.53).

## § 6. Solutions of the second order wave equations

### (1) De Sitter space

#### (i) Fundamental solutions

Each component of the wave functions for charge free fields must satisfy the second order wave equation

$$\left\{ \frac{1}{2R^2} R^{\lambda\mu} R_{\lambda\mu} - K \right\} A = 0, \quad (6.1)$$

$$K = \kappa^2 \pm 3i\kappa/R. \quad (6.2)$$

Since all the physical quantities must lie in the four dimensional hyperboloidal surface, we have to seek for the solutions among such quantities on the surface. Thus we use the five dimensional spherical coordinate (A) introduced in § 2.

The differential operator of (6.1) becomes in this coordinate system

$$\begin{aligned} \Omega &= \frac{1}{2} R^{\lambda\mu} R_{\lambda\mu} \\ &= \frac{1}{(\sin \theta_1)^2 \cos \theta_1} \frac{\partial}{\partial \theta_1} \left( (\sin \theta_1)^2 \cos \theta_1 \frac{\partial}{\partial \theta_1} \right) + \frac{1}{(\sin \theta_1)^2 \sin \theta_2} \frac{\partial}{\partial \theta_2} \left( \sin \theta_2 \frac{\partial}{\partial \theta_2} \right) \\ &\quad + \frac{1}{(\sin \theta_1)^2 (\sin \theta_2)^2} \left( \frac{\partial}{\partial \theta_3} \right)^2 - \frac{1}{(\cos \theta_1)^2} \left( \frac{\partial}{\partial \theta_4} \right)^2 \end{aligned} \quad (6.3)$$

and the wave equation becomes

$$\{\Omega - K'\} A = 0, \quad K' = R^2 K = \kappa^2 R^2 \pm 3i\kappa R. \quad (6.4)$$

Separating variables we put

$$A(\theta_1, \theta_2, \theta_3, \theta_4) = A_1(\theta_1) A_2(\theta_2) A_3(\theta_3) A_4(\theta_4). \quad (6.5)$$

Putting this expression into the wave equation (6.4) and assuming the so called separating constants as usual, we obtain the solution in the form

$$A(\theta_1, \theta_2, \theta_3, \theta_4) = A_1(\theta_1) P_l^m(\cos \theta_2) e^{im\theta_3} e^{i\nu\theta_4} \quad (6.6)$$

where  $P_l^m$  is the Legendre associated function and  $A_1$  is a function satisfying the equation

$$\left\{ \frac{1}{(\sin \theta_1)^2 \cos \theta_1} \frac{d}{d\theta_1} \left( (\sin \theta_1)^2 \cos \theta_1 \frac{d}{d\theta_1} \right) - \frac{l(l+1)}{(\sin \theta_1)^2} + \frac{\nu^2}{(\cos \theta_1)^2} - K' \right\} A_1(\theta_1) = 0. \quad (6.7)$$

In order to solve the equation we put

$$\mu = \cos 2\theta_1 \quad (6.8)$$

and then the equation (6.7) becomes

$$\left\{ 2(1-\mu^2) \frac{d^2}{d\mu^2} - (1+5\mu) \frac{d}{d\mu} - \frac{l(l+1)}{1-\mu} + \frac{\nu^2}{1+\mu} - 3 + K' \right\} A_1(\mu) = 0. \quad (6.9)$$

Further putting

$$A_1(\mu) = (1-\mu)^{-3/4} (1+\mu)^{-1/2} u(\mu) \quad (6.10)$$

we have

$$\left[ \frac{d^2}{d\mu^2} + \left\{ \frac{3/4 - l(l+1)}{4(1-\mu)^2} + \frac{1+\nu^2}{4(1+\mu)^2} + \frac{3-K'-l(l+1)+\nu^2}{4(1-\mu^2)} \right\} \right] u(\mu) = 0. \quad (6.11)$$

Now according to the properties of Jacobi polynomials and hypergeometric functions we see that the solution of the equation

$$\left[ \frac{d^2}{d\mu^2} + \left\{ \frac{1-\alpha^2}{4(1-x)^2} + \frac{1-\beta^2}{4(1+\mu)^2} + \frac{\gamma(\gamma+\alpha+\beta+1) + (\alpha+1)(\beta+1)/2}{1-\mu^2} \right\} \right] u(\mu) = 0 \quad (6.12)$$

can be obtained as

$$u(\mu) = (1-\mu)^{(\alpha+1)/2} (1+\mu)^{(\beta+1)/2} P_{\gamma}^{(\alpha, \beta)}(\mu) \quad (6.13)$$

where  $P_{\gamma}^{(\alpha, \beta)}$  is defined with hypergeometric functions  $F(a, b, c; x)$  as

$$P_{\gamma}^{(\alpha, \beta)}(\mu) = F(-\gamma, \gamma + \alpha + \beta + 1; \alpha + 1; (1-\mu)/2) \quad (6.14)$$

and for integral values  $n$  of  $\gamma$  it becomes the Jacobi polynomial  $P_n^{(\alpha, \beta)}(\mu)$ , which construct an orthogonal set in the region  $-1 \leq \mu \leq 1$ . Comparing the two equations (5.11) and (5.12) we see that the solutions of (6.11) can be obtained in the form (5.13) with the arguments satisfying the relation

$$1-\alpha^2 = 3/4 - l(l+1) \quad \text{or} \quad \alpha = \pm(l+1/2), \quad (6.15a)$$

$$1-\beta^2 = 1+\nu^2 \quad \text{or} \quad \beta = \pm i\nu, \quad (6.15b)$$

$$n(n+\alpha+\beta+1) + (\alpha+1)(\beta+1)/2 = \{3-K'-l(l+1)+\nu^2\}/4. \quad (6.15c)$$

These relations can be solved if  $\nu$  is not purely real but  $\nu = \nu_1 + i\nu_2$ . This imaginary part of  $\nu$  causes the timely damping factor  $e^{-\nu_2 \theta_1}$ . This factor does not oscillate but it is useful for solving first rank field equations. The arguments  $\alpha, \beta, \nu$  have discrete values depending on  $l$  and  $n$ . The solution of the equation for  $A_1$  is obtained with these arguments as

$$A_1(\mu) = (1-\mu)^{\alpha/2-1/4} (1+\mu)^{\beta/2} P_n^{(\alpha, \beta)} \quad (6.16)$$

or

$$A_1(\theta_1) = (\sin \theta_1)^{\alpha-1/2} (\cos \theta_1)^{\beta} P_n^{(\alpha, \beta)}(\cos 2\theta_1). \quad (6.17)$$

Thus we have a fundamental system of normal solutions of the wave equation (6.1), which is composed of discrete values.

If we want to have purely oscillating solutions,  $\nu_2=0$  and  $P_n^{(\alpha, \beta)}$  must be substituted by  $P_{\gamma}^{(\alpha, \beta)}$  with  $\gamma$  satisfying

$$\gamma(\gamma+\alpha+\beta+1) + (\alpha+1)(\beta+1)/2 = \{3-K'-l(l+1)+\nu^2\}/4.$$

In this case  $\beta$  is pure imaginary so  $\gamma$  is complex and cannot be an integral number. Since, however the condition that  $F(a, b, c; x)$  converges for  $|x|=1$  is

$$\text{real part of } (c-a-b) > 0 \quad (6.18)$$

and for  $P_{\gamma}^{(\alpha, \beta)}$  the real part of  $(c-a-b) = \beta$ , so  $P_{\gamma}^{(\alpha, \beta)}$  does not converge at the point

$\mu = \pm 1$ . Thus we cannot obtain non singular solutions in this way, even frequencies would take continuous values in this case. In order to obtain solutions converging also at this point, it is convenient to use the coordinate system (D). In this coordinate system

$$\begin{aligned} Q = & -\frac{1}{\cosh^3 \phi_1} \frac{\partial}{\partial \phi_1} \left( \cosh^3 \phi_1 \frac{\partial}{\partial \phi_1} \right) + \frac{1}{\cosh \phi_1 \sin \phi_2 \cos \phi_2} \frac{\partial}{\partial \phi_2} \left( \sin \phi_2 \cos \phi_2 \frac{\partial}{\partial \phi_2} \right) \\ & + \frac{1}{\cosh^2 \phi_1 \cos^2 \phi_2} \left( \frac{\partial}{\partial \phi_1} \right)^2 + \frac{1}{\cosh^2 \phi_1 \sin^2 \phi_2} \left( \frac{\partial}{\partial \phi_2} \right)^2. \end{aligned} \quad (6.19)$$

Separating variables we have the solution of the wave equation in the form

$$A(\phi_1, \phi_2, \phi_3, \phi_4) = A'(\phi_1) A''(\phi_2) e^{im_1 \theta_3} e^{im_2 \theta_4}$$

where  $A'$  and  $A''$  satisfy the equation

$$\left[ \frac{1}{\sin \phi_2 \cos \phi_2} \frac{\partial}{\partial \phi_2} \left( \sin \phi_2 \cos \phi_2 \frac{\partial}{\partial \phi_2} \right) - \frac{m_1^2}{\cos^2 \phi_2} - \frac{m_2^2}{\sin^2 \phi_2} + k \right] A''(\phi_2) = 0, \quad (6.20)$$

$$\left[ -\frac{1}{\cosh^3 \phi_1} \frac{\partial}{\partial \phi_1} \left( \cosh^3 \phi_1 \frac{\partial}{\partial \phi_1} \right) + k + K' \right] A'(\phi_1) = 0, \quad (6.21)$$

$k$  being a separating constant. The equation (6.20) can be solved in the same way as (6.7), and we have

$$A''(\phi_2) = (\sin \phi_2)^{\alpha'} (\cos \phi_2)^{\beta'-1} P_{n'}^{(\alpha', \beta')}(\cos 2\phi_2) \quad (6.22)$$

where  $P_{n'}^{(\alpha', \beta')}$  is the Jacobi polynomial with the arguments satisfying the relations

$$\alpha'^2 = m_1^2 - 1/2 \quad \text{or} \quad \alpha' = \pm \sqrt{m_1^2 - 1/2}, \quad (6.23a)$$

$$\beta'^2 = m_1^2 + 1/2 \quad \text{or} \quad \beta' = \pm \sqrt{m_2^2 + 1/2}, \quad (6.23b)$$

$$(2n' + \alpha' + \beta' + 1)^2 = k.$$

In order to solve the equation (6.21) we put  $\cos 2\phi_1 = \mu$ , the equation becomes

$$\left[ 4(\mu^2 - 1) \frac{d^2}{d\mu^2} + 2(5\mu - 3) \frac{d}{d\mu} + \frac{2k}{\mu + 1} + K' \right] A'(\mu) = 0. \quad (6.24)$$

Putting

$$A'(\mu) = (\mu - 1)^{-1/4} (\mu + 1)^{-1/4} v'(\mu) \quad (6.25)$$

we have

$$\left[ \frac{d^2}{d\mu^2} + \left\{ \frac{1 - 1/4}{4(1 - \mu)^2} + \frac{1 - (k + 1)}{4(1 + \mu)^2} + \frac{-K' - k}{4(1 - \mu^2)} \right\} \right] v'(\mu) = 0 \quad (6.26)$$

The solutions of this equation can be obtained in the form

$$v'(\mu) = (\mu - 1)^{(\alpha+1)/2} (\mu + 1)^{(\beta+1)/2} Q_{\frac{1}{2}}^{(\alpha, \beta)}(\mu) \quad (6.27)$$

where

$$Q_{\mp}^{(\alpha, \beta)}(\mu) = (1 + \mu)^{\mp} F(-\gamma, -\gamma - \beta; -2\gamma - \alpha - \beta; 2/(1 + \mu)). \quad (6 \cdot 28)$$

Thus we have as the solutions of the equation for  $A'(\mu)$

$$A'(\mu) = (\mu - 1)^{\alpha/2 + 1/4} (\mu + 1)^{\beta/2 - 1/2} Q_{\mp}^{(\alpha, \beta)}(\mu). \quad (6 \cdot 29)$$

The arguments  $\alpha, \beta, \gamma$  must satisfy the relations

$$\alpha^2 = 1/4, \quad (6 \cdot 30a)$$

$$\beta^2 = k + 1, \quad (6 \cdot 30b)$$

$$\gamma(\gamma + \alpha + \beta + 1) + (\alpha + 1)(\beta + 2)/2 = -(K' + k)/4 \quad (6 \cdot 30c)$$

so we have

$$\alpha = \pm 1/2, \quad (6 \cdot 31a)$$

$$\beta = \pm \sqrt{k + 1}, \quad (6 \cdot 31b)$$

$$\gamma = \frac{-(\alpha + \beta + 1) \pm \sqrt{1/4 - K'}}{2}. \quad (6 \cdot 31c)$$

The hypergeometric functions appearing in this solutions converge for  $\mu > 1$ , and since for real part of  $(c - a - b) = -\alpha$  they converge for  $\alpha < 0$  at  $\mu = 1$  also. Thus we have obtained a fundamental set of solutions converging all over the de Sitter surface. These solutions have discrete modes of oscillations, for all the arguments take discrete values. Indeed the function  $A'(\phi_1)$  approaches for small values of  $\phi_1$  to a plane wave with discrete frequencies

$$\exp(\pm i \sqrt{(2n' + \alpha' + \beta' + 1)^2 + K'} \cdot \phi_1). \quad (6 \cdot 32)$$

By the way we will give some remarks about the group theoretical characters of these solutions. In this coordinate system  $R^{\lambda\mu}$  have the form of (2.23). Setting as (2.24) the relations (2.25) and (2.26) hold. These relations show that eigen functions of such combinations of  $R^{\lambda\mu}$  can be obtained as follows:

$$e^{\pm i m_+ \phi_+} e^{\mp i m_- \phi_-} (\sin \phi/2)^{\alpha} (\cos \phi/2)^{\beta} P_{n-1}^{(\alpha, \beta)}(\cos \phi) \quad (6 \cdot 33)$$

the arguments  $\alpha, \beta$  being

$$\alpha = m_+ + m_-, \quad \beta = m_- - m_+ \quad \text{for } (2 \cdot 6), \quad (6 \cdot 34a)$$

$$\alpha = m_+ + m_-, \quad \beta = m_+ - m_- \quad \text{for } (2 \cdot 7). \quad (6 \cdot 34b)$$

These facts can be shown easily by using the relation

$$(d/d\mu) P_n^{(\alpha, \beta)}(\mu) = 1/2 \cdot (n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1, \beta+1)}(\mu). \quad (6 \cdot 35)$$

The representation adopted by Thomas is a special case of ours according to the  $\phi, \phi_+, \phi_-$  coordinate. Since the Legendre associated function  $P_l^m(\mu)$  can be expressed through the Jacobi polynomial  $P_n^{(\alpha, \alpha)}(\mu)$  whose arguments are equal  $\alpha = \beta$ , as follows

$$P_l^m(\mu) = \text{const} \cdot (1 - \mu^2)^m P_{n-m}^{(m, m)}(\mu) \quad (6 \cdot 36)$$



the part of (6.35) which depend on  $\phi$  in the case  $\alpha=\beta$  becomes  $P_l^{m+}(\mu)$  or  $P_l^{m-}(\mu)$  for (5.36a) or (6.36b) respectively. Thus we have the following eigenfunctions

$$e^{\pm im+\phi} P_l^{m+}(\cos \phi) \quad \text{for } (2.6), \quad (6.37a)$$

$$e^{\mp im-\phi} P_l^{m-}(\cos \phi) \quad \text{for } (2.7) \quad (6.37b)$$

which correspond to the Thomas representation.

## (ii) Static solutions

Static solutions with spherical symmetry may be interpreted as the static potentials. In order to obtain such solutions it is convenient to use the coordinate system (A). Since the required solutions are functions of  $r$  only, they involve  $\theta_1$  only and are independent of  $\theta_2, \theta_3, \theta_4$ , so the wave equation becomes

$$\left\{ \frac{1}{(\sin \theta_1)^2 \cos \theta_1} \frac{\partial}{\partial \theta_1} \left( (\sin \theta_1)^2 \cos \theta_1 \frac{\partial}{\partial \theta_1} \right) - K \right\} A(\theta_1) = 0. \quad (6.38)$$

The solutions can be easily obtained from the general solution (6.16) by setting  $l=0$ ,  $m=0$ ,  $\nu=0$ , or  $\alpha=\pm 1/2$ ,  $\beta=0$ . Thus we have as the spherically symmetrical static solutions

$$F\left(-\gamma, \gamma + \frac{3}{2}; \frac{3}{2}; r^2/R^2\right), \gamma = (-3 \pm \sqrt{9-4K'})/4. \quad (6.39a)$$

$$1/r^2 \cdot F\left(-\gamma', \gamma' + \frac{2}{1}; \frac{1}{2}; r^2/R^2\right), \gamma' = (-1 \pm \sqrt{9-4K'})/4 \quad (6.39b)$$

where  $r=R \sin \theta_1$  is the spatial distance in the three dimensional sense. This results are also attained by solving directly the transformed equation of (6.40) through the substitution  $\rho=\sin \theta_1$

$$\left[ \frac{1}{\rho^2} \frac{d}{d\rho} \left\{ \rho^2 (1-\rho^2) \frac{d}{d\rho} \right\} - K' \right] A(\rho) = 0. \quad (6.40)$$

## (iii) Plane waves

Solutions corresponding to the plane waves in the Minkowski space do not exist in this curved space. Although purely plane waves do not exist in general, we can superpose such a solution that approaches to any plane wave when the space becomes flat. The solutions discussed in the subsection (i) correspond to the spherical waves in the Minkowski space. We can show that the function  $A_1(\theta_1)$  of the general solutions (6.6) approaches to Bessel functions when  $\theta_1$  becomes small. So such a combination of the solution (6.6) approaches to a plane wave that has the same coefficient as that of the Rayleigh's expansion of a plane wave.

Now it is interesting to consider about the meaning of the fundamental solution (6.6). For  $\theta_1=\theta_1^0$ ,  $\theta_2=\theta_2^0$  the solution takes the form  $A_0 e^{(im\theta_3+\nu\theta_4)}$ . This shows that the wave travels in the same way as a plane wave propagating in the direction of increasing  $\theta_3$  in the region for constant  $\theta_1, \theta_2$ . Since  $A_0$  is a function of  $\cos 2\theta_1$  and  $\cos \theta_2$ , it is symmetrical according to the planes  $\theta_1=0$  and  $\theta_2=0$ . Thus the wave can be interpreted as a

wave travelling along the great circle determined by the  $x_1, x_2$  plane.

In the same way we can obtain also the waves travelling along the great circle determined by the  $x_0, x_1$  plane, which may correspond to the plane waves propagating in the  $x_1$  direction in the Minkowski space. In order to obtain such solutions we take the coordinate system (B). In this coordinate system  $\Omega$  becomes

$$\Omega = \frac{1}{(\sin \omega_1)^2 \cos \omega_1} \frac{\partial}{\partial \omega_1} \left( (\sin \omega_1)^2 \cos \omega_1 \frac{\partial}{\partial \omega_1} \right) + \frac{1}{(\sin \omega_1)^2 \sin \omega_2} \frac{\partial}{\partial \omega_2} \left( \sin \omega_2 \frac{\partial}{\partial \omega_2} \right) + \frac{1}{(\cos \omega_1)^2} \left( \frac{\partial}{\partial \omega_3} \right)^2 - \frac{1}{(\sin \omega_1)^2 (\sin \omega_2)^2} \left( \frac{\partial}{\partial \omega_4} \right)^2. \quad (6.41)$$

This form of the operator  $\Omega$  shows that the solution of the wave equation can be obtained in the form

$$A(\omega_1, \omega_2, \omega_3, \omega_4) = A_1'(\omega_1) A_2^2(\omega_2) e^{i(m\omega_3 + \nu\omega_4)}. \quad (6.42)$$

For this solution the above mentioned interpretation may be possible.

## (2) Pseudo de Sitter space.

In the case of the pseudo de Sitter space the second order wave equation which is satisfied by each component of the wave function is

$$\left\{ \frac{1}{2R^2} R^{\lambda\mu} R_{\lambda\mu} + K \right\} A = 0, \quad (6.43)$$

$$K = \kappa^2 \pm 3\kappa/R, \quad (6.44)$$

or

$$\{ \Omega + K' \} A = 0, \quad (6.45)$$

$$K' = \kappa^2 K^2 + 3\kappa K.$$

We take the coordinate system (E), then

$$\Omega = -\frac{1}{(\sinh \theta_1)^2 \cosh \theta_1} \frac{\partial}{\partial \theta_1} \left( (\sinh \theta_1)^2 \cosh \theta_1 \frac{\partial}{\partial \theta_1} \right) - \frac{1}{(\sinh \theta_1)^2 \sin \theta_2} \frac{\partial}{\partial \theta_2} \left( \sin \theta_2 \frac{\partial}{\partial \theta_2} \right) - \frac{1}{(\sinh \theta_1)^2 (\sin \theta_2)^2} \left( \frac{\partial}{\partial \theta_3} \right)^2 + \frac{1}{(\cosh \theta_1)^2} \left( \frac{\partial}{\partial \theta_4} \right)^2. \quad (6.46)$$

The wave equation can be solved in the form

$$A(\theta_1, \theta_2, \theta_3, \theta_4) = A_1(\theta_1) P_l^m(\cos \theta_2) e^{im\theta_3} e^{i\nu\theta_4} \quad (6.47)$$

where  $A_1(\theta_1)$  satisfies the equation

$$\left\{ -\frac{1}{(\sinh \theta_1)^2 \cosh \theta_1} \frac{\partial}{\partial \theta_1} \left( (\sinh \theta_1)^2 \cosh \theta_1 \frac{\partial}{\partial \theta_1} \right) + \frac{l(l+1)}{(\sinh \theta_1)^2} - \frac{\nu^2}{(\cosh \theta_1)^2} + K' \right\} A_1(\theta_1) = 0. \quad (6.48)$$

Putting

$$\mu = \cosh 2\theta_1,$$

$$A_1(\mu) = (\mu-1)^{-1/4}(\mu+1)^{-1/2}u'(\mu)$$

we have

$$\left[ \frac{d^2}{d\mu^2} + \left\{ \frac{3/4 - l(l+1)}{4(1-\mu)^2} + \frac{1-\nu^2}{4(1+\mu)^2} + \frac{3+K'-l(l+1)-\nu^2}{4(1-\mu^2)} \right\} \right] u'(\mu) = 0. \quad (6.49)$$

The function  $u'(\mu)$  can be obtained as

$$u'(\mu) = (\mu-1)^{(\alpha+1)/2}(\mu+1)^{(3+1)/2} Q_n^{(\alpha, \beta)}(\mu).$$

The function  $Q_n^{(\alpha, \beta)}$  is the Jacobi polynomial of the second kind and the arguments satisfy the relation

$$1 - \alpha^2 = 3/4 - l(l+1) \quad (6.50a)$$

$$1 - \beta^2 = 1 - \nu^2 \quad (6.50b)$$

$$n(n + \alpha + \beta + 1) + (\alpha + 1)(\beta + 1)/2 = \{3 + K' - l(l+1) - \nu^2\}/4 \quad (6.50c)$$

or

$$\alpha = \pm 1/2 \quad (6.51a)$$

$$\beta = \pm \nu \quad (6.51b)$$

$$n = \frac{-(\alpha + \beta + 1) \pm \sqrt{9/4 + K'}}{2}, \text{ or } \beta = -2n - \alpha - 1 \pm \sqrt{9 + 4K'}/4. \quad (6.51c)$$

Thus we have as the solution for  $A_1(\theta_1)$ ,  $A_1(\mu) = (\mu-1)^{\alpha/2-1/4}(\mu+1)^{\beta/2} Q_n^{(\alpha, \beta)}$ , (6.52)

We have obtained the fundamental set of solutions. This is also composed of discrete modes of oscillations. Since in this case  $i\nu$  is pure imaginary, any damping factor does not appear.

We can obtain also the static solutions in the same way as the previous case, putting  $l=0$ ,  $m=0$ ,  $\nu=0$  and substituting  $Q_{\tau}^{(\alpha, \beta)}$  for  $Q_n^{(\alpha, \beta)}$

$$Q^{(\alpha, \beta)}(\mu) = (\mu-1)^{-\tau-\alpha-1}(\mu+1)^{-\beta} F(\gamma + \alpha + 1, \gamma + 1; 2\gamma + \alpha + \beta + 2; 2/(1-\mu)). \quad (6.53)$$

Thus we have as spherical symmetrical static solutions

$$r^{-2\tau-\beta} F(\gamma + 3/2, \gamma + 1; 2\gamma + 5/2; R^2/r^2), \quad \gamma = (-3 \pm \sqrt{9 + 4K'})/4, \quad (6.54)$$

$$r^{-2\tau-\beta} F(\gamma + 1/2, \gamma + 1; 2\gamma + 3/2; R^2/r^2), \quad \gamma = (-1 \pm \sqrt{9 + 4K'})/4. \quad (6.55)$$

Now let us consider circumstances in spatially small regions, putting

$$\rho = \sinh \theta_1 = r/R \quad (6.56)$$

the equation becomes for the waves independent of  $\theta_2, \theta_3, \theta_4$

$$-\frac{1}{\rho^2} \frac{d}{d\rho} \left\{ \rho^2 (1 + \rho^2) \frac{dA}{d\rho} \right\} + K' A = 0 \quad (6.57)$$

Setting  $\rho A = v$ , we have

$$(1 + \rho^2) \frac{d^2 v}{d\rho^2} + 2\rho \frac{dv}{d\rho} - (2 + K')v = 0. \quad (6.58)$$

For small values of  $\rho$  this becomes

$$d^2 v / d\rho^2 = (2 + K')v. \quad (6.59)$$

So we have

$$v = e^{-\sqrt{2+K'} \cdot \rho} \quad (6.60)$$

and thus the static solution must have the form

$$A = e^{-\sqrt{2+K'} \cdot \rho} / \rho, \quad (\rho \ll 1),$$

$$A = e^{-\sqrt{K+2/K^2} \cdot r} / r, \quad (r \ll R). \quad (6.61)$$

This is the form of static potentials for spatially small regions ( $r \ll R$ ) in the pseudo de Sitter space. This shows that the Yukawa potentials in such regions take the form

$$\frac{e^{-\kappa' r}}{r}, \quad \kappa' = \sqrt{\kappa^2 \pm 3\kappa/R + 2/R^2}. \quad (6.62)$$

Moreover in this region we can put  $\sinh \theta_1 = \theta_1$  approximately, and  $\Omega$  can be written in the form

$$\begin{aligned} \Omega = & -\frac{1}{(\theta_1)^2} \frac{\partial}{\partial \theta_1} \left( (\theta_1)^2 \frac{\partial}{\partial \theta_1} \right) \\ & - \frac{1}{(\theta_1)^2} \left\{ \frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \left( \sin \theta_2 \frac{\partial}{\partial \theta_2} \right) + \frac{1}{(\sin \theta_3)^2} \left( \frac{\partial}{\partial \theta_3} \right)^2 \right\} + \left( \frac{\partial}{\partial \theta_4} \right)^2 \\ = & -\partial^2 / \partial \xi_1^2 - \partial^2 / \partial \xi_2^2 - \partial^2 / \partial \xi_3^2 - \partial^2 / \partial \xi_4^2 + \partial^2 / \partial \xi_4^2 \end{aligned} \quad (6.63)$$

where

$$\xi_1 = x_1 / R, \quad \xi_2 = x_2 / R, \quad \xi_3 = x_3 / R, \quad \xi_4 = \sin^{-1}(x_4 / R). \quad (6.64)$$

Thus in this region also plane waves exist approximately, and the quantum theory can be developed in the same way as in the ordinary quantum theory in Minkowski space. Particles behave themselves as if they were particles with other masses in the Minkowski space.

Finally we should express sincere thanks to Professor Husimi who investigated this problem and encouraged us continuously. We must also express hearty thanks to Professor Yamanouchi for his kind interests and valuable discussions.

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## Kinematical Investigations of Meson-Nucleon Reactions

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The angular distributions and the polarizations of recoil nucleons for the meson-nucleon scattering as well as for the photomeson production are studied from the kinematical point of view. Similar to the case of meson-nucleon scattering, as given in our previous papers, there exists a simple symmetry property also in the photomeson production. The angular distribution is invariant under the simultaneous interchange of the reaction amplitudes of electric and magnetic radiations with the same multipole order and the same total angular momentum. This interchange upsets the sense of the polarization of recoil nucleons. For practical purposes the tables of the angular distributions and the polarizations are given for  $J$ 's up to  $3/2$ .

### § 1. Introduction

The purpose of this paper is to give the kinematical relations in the meson-nucleon scattering and the photomeson production in a summarized form, in order to facilitate the analysis of experiments. We think it worth while to do, because the kinematical relations concerned appear in published literatures only fragmentally and also because there exist some important points that have thus far been overlooked. A part of them has been noticed by one of us (S. M.)<sup>1)</sup> concerning the meson-nucleon scattering, that is, the angular distribution is invariant under the simultaneous interchange of every pair of the phase shifts belonging to the same total angular momentum. In other words there is an intrinsic degeneracy in phase shifts apart from the ambiguity of their values due to the computational difficulty in obtaining the set of phase shifts from existing experiments.

Such a symmetry property seems inherent, if there are dichotomic variables concerning angular momenta<sup>†</sup>. In the meson-nucleon system of relative momentum  $q$ , an operator  $(q \cdot \sigma)$ , where  $\sigma$  is the spin operator on nucleon, does not give rise to the change in the total angular momentum  $J$  but to the change of parity  $\Pi$ . The transformation of matrix elements in meson-nucleon scattering by  $(q \cdot \sigma)$  is found to affect nothing as far as the angular distribution is concerned. In the photon-nucleon system of relative momentum  $k$ ,

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† A simple proof of the symmetry property is kindly communicated to us by Professors Dyson and Nambu, to whom we express our thanks in this occasion.



there is another degree of freedom, i.e., the circular polarization of photon that is represented by an operator  $\epsilon$  similar to  $\sigma$ . Now  $(\mathbf{k} \cdot \sigma)(\mathbf{k} \cdot \epsilon)$  leaves  $J$  as well as  $L$  of the whole system invariant, because  $\epsilon$  is transformed as an axial vector and takes two eigen values as  $\sigma$  does. The factor  $(\mathbf{k} \cdot \epsilon)$  of this operator further takes a role to leave the total angular momentum of a photon  $L$ , (orbital + spin), i.e., the order of multipole radiation, invariant. In order that the photon-nucleon system is covariant with the transformation  $(\mathbf{q} \cdot \sigma)$  in the meson-nucleon system, another operation is needed. This can be achieved by the parity change due to the interchange of electric and magnetic radiations belonging to the same order of multipoles. Thus we obtain a theorem: the angular distribution in the photo-meson production is invariant under the simultaneous interchange of every pair of the reaction amplitudes belonging to the same  $J$  as well as the same  $L$ . In other words, if the interchange of the electric and magnetic radiations with the same multipole order in the initial system as well as that of the orbital angular momenta in the final system, both belonging to the same  $J$ , is carried out simultaneously, nothing is altered in the angular distribution of photomesons.

In order to remove the degeneracy of phase shifts in meson-nucleon scattering due to the symmetry property the observation of polarization of recoil nucleons is suggested in our previous report<sup>2)</sup>. The polarizations in the degenerate states are just opposite in sense but of the same magnitude, as these states result from the nucleon spins of opposite directions. Formally speaking, as the spin operator whose component is perpendicular to the reaction plane is anticommutate with the operator  $(\mathbf{q} \cdot \sigma)$ , that expectation value changes its sign as a consequence of the operation under consideration. The same thing holds for the polarization of recoil nucleons in the photomeson production as well. The polarizations of recoil nucleons in the two degenerate states are different only in their signs.

Taking the symmetry properties discussed above into consideration, the thorough expressions are given to the angular distributions and the polarizations for the meson-nucleon scattering as well as for the photo-meson reaction. In § 2 those for the meson-nucleon scattering, which appeared in our previous papers<sup>1,2)</sup>, are shown in tables for the sake of completeness. In § 3 the angular distribution and in § 4 the polarization of recoil nucleons are given for the photomeson reaction.

## § 2. Meson-nucleon scattering

As the differential cross sections and the polarizations of recoil nucleons for meson-nucleon scattering have been given in the previous papers<sup>1,2)</sup>, we summarize here only those results for the completeness of this paper.

$$\begin{aligned}
 (4 \sin^2 \theta / \lambda^2) d\sigma/d\omega = & \sum_{l, l'} (l+1)(l'+1) \{ (R_l^+ R_{l'}^{+*} + R_{l+1}^- R_{l'+1}^{-*}) \\
 & \times [P_l P_{l'} + P_{l+1} P_{l'+1} - \cos \theta (P_l P_{l'+1} + P_{l+1} P_{l'})] \\
 & + (R_l^+ R_{l'+1}^{-*} + R_{l+1}^- R_{l'}^{+*}) [P_l P_{l'+1} + P_{l+1} P_{l'} - \cos \theta (P_l P_{l'} + P_{l+1} P_{l'+1})] \}.
 \end{aligned}
 \tag{2.1}$$

$$\mathbf{P} = (A/B)\mathbf{n},$$

$$(4 \sin \theta / \lambda^2) A = i \sum_{l, l'} (l+1) (l'+1) \{ (R_l^+ R_{l'}^{*} - R_{l+1}^- R_{l'+1}^{-*}) \\ \times (P_l P_{l'+1} - P_{l+1} P_{l'}) + (R_l^+ R_{l'+1}^{-*} - R_{l+1}^- R_{l'}^{*}) (P_l P_{l'} - P_{l+1} P_{l'+1}) \},$$

$$B = d\sigma/d\omega \quad (2.2)$$

where  $P_l$  is the Legendre function.

Table 1. Differential cross sections for meson-nucleon scattering\* (up to  $J=3/2$ )

$(4/\lambda^2) d\sigma/d\omega$	$s_{1/2}(R_0^{+*})$	$p_{1/2}(R_1^{-*})$	$p_{3/2}(R_1^{+*})$	$d_{3/2}(R_2^{-*})$
$s_{1/2}(R_0^+)$	1	$\cos \theta$	$2 \cos \theta$	$3 \cos^2 \theta - 1$
$p_{1/2}(R_1^-)$	$\cos \theta$	1	$3 \cos^2 \theta - 1$	$2 \cos \theta$
$p_{3/2}(R_1^+)$	$2 \cos \theta$	$3 \cos^2 \theta - 1$	$1 + 3 \cos^2 \theta$	$9 \cos^3 \theta - 5 \cos \theta$
$d_{3/2}(R_2^-)$	$3 \cos^2 \theta - 1$	$2 \cos \theta$	$9 \cos^3 \theta - 5 \cos \theta$	$1 + 3 \cos^2 \theta$

Table 2. Polarization of recoil nucleons for meson-nucleon scattering\*\* (up to  $J=3/2$ )

$(4/\lambda^2 \sin \theta) A$	$s_{1/2}(R_0^{+*})$	$p_{1/2}(R_1^{-*})$	$p_{3/2}(R_1^{+*})$	$d_{3/2}(R_2^{-*})$
$s_{1/2}(R_0^+)$	0	$i$	$-i$	$3i \cos \theta$
$p_{1/2}(R_1^-)$	$-i$	0	$-3i \cos \theta$	$i$
$p_{3/2}(R_1^+)$	$i$	$3i \cos \theta$	0	$(9 \cos^2 \theta - 1)i$
$d_{3/2}(R_2^-)$	$-3i \cos \theta$	$-i$	$(1 - 9 \cos^2 \theta)i$	0

### § 3. Differential cross section for photomeson production

The electromagnetic wave propagating along the positive  $z$  direction is expanded in terms of electric and magnetic multipoles<sup>3)</sup>. We denote the total angular momentum of photon by  $L$ . As the total angular momentum,  $J$ , and the parity of the whole system are conserved throughout the photo-meson reaction, we decompose the state with a given total angular momentum,  $J=L+1/2$ , into four different cases, as shown in Table 3.

Table 3.

Classification of radiation	Parity	Orbital angular momentum of meson	Reaction amplitude
$E \ L$	$(-1)^L$	$L+1$	$A_J(e)$
$M \ L+1$			$B_J(m)$
$E \ L+1$	$(-1)^{L+1}$	$L$	$B_J(e)$
$M \ L$			$A_J(m)$

\*  $R_l^+$  and  $R_l^-$  are the  $R$ -matrices with regard to the states of  $J=\pm 1/2$ , corresponding to  $e_{J-1}$  in reference 1), respectively.

\*\* The notations are the same as those in reference 2).

$A_{J\ell}$  and  $B_{J\ell}$  are such reaction amplitudes that are composed of the multipoles of orders  $L$  and  $L+1$  respectively, and the symbols  $(m)$  and  $(\epsilon)$  serve to distinguish between magnetic and electric radiations.

The explicit expressions of  $A_{J\ell}(m)$  and  $A_{J\ell}(\epsilon)$  are easily derived using Clebsch-Gordan coefficients as

$$\begin{aligned} A_{J\ell}^{\alpha+}(m) &= \sqrt{\pi} [\alpha((L+2)/(2L+1))^{1/2} Y_{L,1} \\ &\quad + \beta((L-1)/(2L+1))^{1/2} Y_{L,0}] ((L+2)/(2L+1))^{1/2} a_J(m), \\ A_{J\ell}^{\alpha-}(m) &= \sqrt{\pi} [\alpha(L/(2L+1))^{1/2} Y_{L,-1} \\ &\quad + \beta((L+1)/(2L+1))^{1/2} Y_{L,0}] (L/(2L+1))^{1/2} a_J(m), \\ A_{J\ell}^{\beta-}(m) &= \sqrt{\pi} [\beta((L+2)/(2L+1))^{1/2} Y_{L,-1} \\ &\quad + \alpha((L-1)/(2L+1))^{1/2} Y_{L,-2}] ((L+2)/(2L+1))^{1/2} a_J(m), \\ A_{J\ell}^{\beta+}(m) &= \sqrt{\pi} [\beta(L/(2L+1))^{1/2} Y_{L,1} \\ &\quad + \alpha((L+1)/(2L+1))^{1/2} Y_{L,0}] (L/(2L+1))^{1/2} a_J(m), \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} A_{J\ell}^{\alpha+}(\epsilon) &= \sqrt{\pi} [-\alpha(L/(2L+3))^{1/2} Y_{L+1,1} \\ &\quad + \beta((L+3)/(2L+3))^{1/2} Y_{L+1,2}] ((L+2)/(2L+1))^{1/2} a_J(\epsilon), \\ A_{J\ell}^{\alpha-}(\epsilon) &= \sqrt{\pi} [-\alpha((L+2)/(2L+3))^{1/2} Y_{L+1,-1} \\ &\quad + \beta((L+1)/(2L+3))^{1/2} Y_{L+1,0}] (L/(2L+1))^{1/2} (-a_J(\epsilon)), \\ A_{J\ell}^{\beta-}(\epsilon) &= \sqrt{\pi} [\beta(L/(2L+3))^{1/2} Y_{L+1,-1} \\ &\quad - \alpha((L+3)/(2L+3))^{1/2} Y_{L+1,-2}] ((L+2)/(2L+1))^{1/2} (-a_J(\epsilon)), \\ A_{J\ell}^{\beta+}(\epsilon) &= \sqrt{\pi} [\beta((L+2)/(2L+3))^{1/2} Y_{L+1,1} \\ &\quad - \alpha((L+1)/(2L+3))^{1/2} Y_{L+1,0}] (L/(2L+1))^{1/2} a_J(\epsilon). \end{aligned} \quad (3.2)$$

Here  $\alpha$  and  $\beta$  are spin eigenfunctions of a nucleon corresponding to up and down states respectively and the definition of the spherical harmonics  $Y_{l,m}$  is the same as that of reference 3. They are also used as superscripts to indicate initial spin states. Other superscripts  $+$  and  $-$  are to specify the polarizations of photon. The corresponding equations for  $B_{J\ell}$  can be written down in a similar way, but we omit them in order to save space.

$a_{J\ell}$  and  $b_{J\ell}$  may be decomposed further into three terms, each having the orbital angular momentum of photon,  $l=L-1$ ,  $L$  or  $L+1$ , depending upon the electric or magnetic field strength belonging to the multipole radiation of order  $L$ , that is:

$$\begin{aligned} \text{for } E\ L, \quad & E^{L-1}, \quad H^L, \quad E^{L+1}, \\ \text{for } M\ L, \quad & H^{L-1}, \quad E^L, \quad H^{L+1}. \end{aligned}$$

$E^l(H^l)$  indicates the electric (magnetic) field strength of orbital angular momentum  $l$ . Then the decomposition can readily be carried out as

$$\begin{aligned}
a_J(m) &= \sqrt{(L+1)/2} a_{L-1}(\mathbf{H}) + \sqrt{(2L+1)/2} a_L(\mathbf{E}) - \sqrt{L/2} a_{L+1}(\mathbf{H}), \\
a_J(c) &= \sqrt{(L+1)/2} a_{L-1}(\mathbf{E}) - \sqrt{(2L+1)/2} a_L(\mathbf{H}) - \sqrt{L/2} a_{L+1}(\mathbf{E}), \quad (3.3) \\
b_J(m) &= \sqrt{(L+2)/2} b_L(\mathbf{H}) + \sqrt{(2L+3)/2} b_{L+1}(\mathbf{E}) - \sqrt{(L+1)/2} b_{L+2}(\mathbf{H}), \\
b_J(c) &= -\sqrt{(L+2)/2} b_L(\mathbf{E}) + \sqrt{(2L+3)/2} b_{L+1}(\mathbf{H}) + \sqrt{(L+1)/2} b_{L+2}(\mathbf{E}),
\end{aligned}$$

where  $a_l(\mathbf{E})$ , etc. are the decomposed amplitudes with electric or magnetic field strengths of orbital angular momentum  $l$  and the coefficients standing by them are Clebsch-Gordan ones multiplied by  $\sqrt{2l+1}$ , resulting from the composition of spin and orbital angular momentum of a photon.

Making use of the above relations, we are able to express the differential cross section for this process as follows:

$$\begin{aligned}
\frac{4 \sin^2 \theta}{k^2} \frac{d\sigma}{d\omega} &= \frac{1}{2} \sum_{J,J'} \{ a_J^*(m) a_{J'}(m) + a_J^*(c) a_{J'}(c) \} \{ \xi_1 + LL' \xi_2 \} f(L, L') \\
&+ \{ a_J^*(m) a_{J'}(c) + a_J^*(c) a_{J'}(m) \} \{ -\eta_1 + LL' \eta_2 \} f(L, L') \\
&+ \{ b_J^*(m) b_{J'}(m) + b_J^*(c) b_{J'}(c) \} \{ \xi_1 + (L+2)(L'+2) \xi_2 \} f(L+1, L'+1) \\
&+ \{ b_J^*(m) b_{J'}(c) + b_J^*(c) b_{J'}(m) \} \{ -\eta_1 + (L+2)(L'+2) \eta_2 \} f(L+1, L'+1) \\
&+ \{ a_J^*(m) b_{J'}(c) + a_J^*(c) b_{J'}(m) + b_{J'}^*(c) a_J(m) + b_{J'}^*(m) a_J(c) \} \\
&\quad \{ -\xi_1 + L(L'+2) \xi_2 \} f(L, L'+1) \\
&+ \{ a_J^*(m) b_{J'}(m) + a_J^*(c) b_{J'}(c) + b_{J'}^*(m) a_J(m) + b_{J'}^*(c) a_J(c) \} \\
&\quad \{ \eta_1 + L(L'+2) \eta_2 \} f(L, L'+1) ]. \quad (3.4)
\end{aligned}$$

In this formula we have used the following abbreviations.

$$\begin{aligned}
f(L, L') &= [(2L+1)(2L'+1)L(L+1)L'(L'+1)]^{-1/2}, \\
\xi_1 &= (L+2)(L'+2)P_L^{-1}P_{L'}^{-1} + LL'P_{L+1}^{-1}P_{L'+1}^{-1} \\
&\quad - \{ L(L'+2)P_{L+1}^{-1}P_{L'}^{-1} + L'(L+2)P_{L'+1}^{-1}P_L^{-1} \} \cos \theta, \\
\eta_1 &= L(L'+2)P_{L+1}^{-1}P_{L'}^{-1} + L'(L+2)P_{L'+1}^{-1}P_L^{-1} \\
&\quad - \{ (L+2)(L'+2)P_L^{-1}P_{L'}^{-1} + LL'P_{L+1}^{-1}P_{L'+1}^{-1} \} \cos \theta, \\
\xi_2 &= P_L^{-1}P_{L'}^{-1} + P_{L+1}^{-1}P_{L'+1}^{-1} - (P_L^{-1}P_{L'+1}^{-1} + P_{L'}^{-1}P_{L+1}^{-1}) \cos \theta, \\
\eta_2 &= P_L^{-1}P_{L'+1}^{-1} + P_{L'}^{-1}P_{L+1}^{-1} - (P_L^{-1}P_{L'}^{-1} + P_{L+1}^{-1}P_{L'+1}^{-1}) \cos \theta, \quad (3.5)
\end{aligned}$$

where  $P_l^m$  is the associated Legendre function.

It is worth while to note that  $d\sigma/d\omega$  is symmetric with respect to  $a_J(m)$  and  $a_J(c)$  or  $b_J(m)$  and  $b_J(c)$ . Therefore the angular distribution is invariant under the simultaneous substitution of  $a_J(m) \rightleftharpoons a_J(c)$  and  $b_J(m) \rightleftharpoons b_J(c)$  for all  $J$ 's. This substitution induces the change of the orbital angular momentum of a produced meson, similar to the case of

the scattering of meson by nucleon. This circumstance is due to the fortune that the photon can take only two values of magnetic quantum numbers in spite of its spin to be unity, as discussed in Introduction. We emphasize that the symmetry property stated in reference 1 in the pion-nucleon system is stronger than that in the present case, because another degree of freedom comes in when a meson is replaced by a photon.

The coefficients of  $a_J^*(\nu)$   $b_J(m)$  etc, that represent the angular distribution contributed by respective multipoles, are listed in Table 4 for  $J$  up to  $3/2$ .

Table 4,

$\frac{4}{\lambda^2} \frac{d\sigma}{d\omega}$	$M1(J=1/2)$ $b_{1/2}^*(m)$	$E1(J=1/2)$ $b_{1/2}^*(e)$	$M1(J=3/2)$ $a_{3/2}^*(m)$	$E1(J=3/2)$ $a_{3/2}^*(e)$	$M2(J=3/2)$ $b_{3/2}^*(m)$	$E2(J=3/2)$ $b_{3/2}^*(e)$
$M1(J=1/2)$ $b_{1/2}(m)$	$\frac{1}{3}$	$-\frac{\cos \theta}{3}$	$\frac{1-3\cos^2 \theta}{6}$	$\frac{\cos \theta}{3}$	$\frac{\cos \theta}{\sqrt{5}}$	$\frac{1-3\cos^2 \theta}{2\sqrt{5}}$
$E1(J=1/2)$ $b_{1/2}(e)$	$-\frac{\cos \theta}{3}$	$\frac{1}{3}$	$\frac{\cos \theta}{3}$	$\frac{1-3\cos^2 \theta}{6}$	$\frac{1-3\cos^2 \theta}{2\sqrt{5}}$	$\frac{\cos \theta}{\sqrt{5}}$
$M1(J=3/2)$ $a_{3/2}(m)$	$\frac{1-3\cos^2 \theta}{6}$	$\frac{\cos \theta}{3}$	$\frac{2+3\sin^2 \theta}{6}$	$-\frac{\cos \theta}{3}$	$\frac{\cos \theta(2-3\cos^2 \theta)}{\sqrt{5}}$	$-\frac{(1-3\cos^2 \theta)}{2\sqrt{5}}$
$E1(J=3/2)$ $a_{3/2}(e)$	$\frac{\cos \theta}{3}$	$\frac{1-3\cos^2 \theta}{6}$	$-\frac{\cos \theta}{3}$	$\frac{2+3\sin^2 \theta}{6}$	$\frac{(1-3\cos^2 \theta)}{2\sqrt{5}}$	$\frac{\cos \theta(2-3\cos^2 \theta)}{\sqrt{5}}$
$M2(J=3/2)$ $b_{3/2}(m)$	$\frac{\cos \theta}{\sqrt{5}}$	$\frac{1-3\cos^2 \theta}{2\sqrt{5}}$	$\frac{\cos \theta(2-3\cos^2 \theta)}{\sqrt{5}}$	$-\frac{(1-3\cos^2 \theta)}{2\sqrt{5}}$	$\frac{3(1+\cos^2 \theta)}{10}$	$\frac{3\cos \theta(1-2\cos^2 \theta)}{5}$
$E2(J=3/2)$ $b_{3/2}(e)$	$\frac{1-3\cos^2 \theta}{2\sqrt{5}}$	$\frac{\cos \theta}{\sqrt{5}}$	$-\frac{(1-3\cos^2 \theta)}{2\sqrt{5}}$	$\frac{\cos \theta(2-3\cos^2 \theta)}{\sqrt{5}}$	$\frac{3\cos \theta(1-2\cos^2 \theta)}{5}$	$\frac{3(1+\cos^2 \theta)}{10}$

Our result agrees with that of Feld<sup>1)</sup>, but does not with the general formula given by Simon<sup>5)\*</sup>. It is almost needless to say that the angular distribution remains unchanged if the produced particle is scalar but not pseudoscalar, as is evident from our symmetry property. Moreover this result is valid as well in nuclear reactions such as

$$\gamma + (\text{spin } 1/2 \text{ nucleus}) \rightleftharpoons (\text{nucleon}) + (\text{spin } 0 \text{ nucleus}).$$

#### § 4. Polarization of recoil nucleon

Polarization of a recoil nucleon is examined along the same line as in § 2. For the unpolarized initial state, the polarization in the direction normal to the plane in which the reaction occurs is expressed as<sup>6, 7)</sup>

$$P = (A/B) n, \quad **$$

\* According to his formula (6.8) no interference term of magnetic and electric multipoles appears in the differential cross section for  $\gamma\pi$  process, whereas it does for  $\pi\gamma$  process. So long as the incident gamma ray is unpolarized, we can hardly understand his procedure on the phase of incident photon. It is also surprising to us that his formulas violate the principle of detailed balance. On this point we have enjoyed many helpful discussions with Messrs. Morita, Sugie and Yoshida.

\*\* Note that we take the sense of  $n$  opposite to Lepore's<sup>7)</sup>.



$$\begin{aligned}
- (4 \sin \theta / \lambda^2) A = & (1/2) i \sum_{J, J'} [\{a_J^*(m) a_{J'}(m) - a_J^*(e) a_{J'}(e)\} \{\zeta_1 + LL' \zeta_2\} f(L, L') \\
& + \{a_J^*(e) a_{J'}(m) - a_J^*(m) a_{J'}(e)\} \{\chi_1 - LL' \chi_2\} f(L, L') \\
& + \{b_J^*(e) b_{J'}(e) - b_J^*(m) b_{J'}(m)\} \{\zeta_1 + (L+2)(L'+2) \zeta_2\} f(L+1, L'+1) \\
& + \{b_J^*(m) b_{J'}(e) - b_J^*(e) b_{J'}(m)\} \{\chi_1 - (L+2)(L'+2) \chi_2\} f(L+1, L'+1) \\
& + \{a_J^*(e) b_{J'}(m) - a_J^*(m) b_{J'}(e) + b_J^*(e) a_{J'}(m) - b_J^*(m) a_{J'}(e)\} \\
& \quad \{\zeta_1 - L(L'+2) \zeta_2\} f(L, L'+1) \\
& + \{a_J^*(m) b_{J'}(m) - a_J^*(e) b_{J'}(e) + b_J^*(e) a_{J'}(e) - b_J^*(m) a_{J'}(m)\} \\
& \quad \{\chi_1 + L(L'+2) \chi_2\} f(L, L'+1)],
\end{aligned}
\tag{4.1}$$

$$B = d\sigma/d\omega,$$

where

$$\begin{aligned}
\zeta_1 &= (L+2) L' P_L^1 P_{L'+1}^1 - L(L'+2) P_{L+1}^1 P_{L'}^1, \\
\chi_1 &= (L+2)(L'+2) P_L^1 P_{L'}^1 - LL' P_{L+1}^1 P_{L'+1}^1, \\
\zeta_2 &= P_L^1 P_{L'+1}^1 - P_{L+1}^1 P_{L'}^1, \\
\chi_2 &= P_L^1 P_{L'}^1 - P_{L+1}^1 P_{L'+1}^1.
\end{aligned}
\tag{4.2}$$

The expressions of the coefficients appearing in (4.1) are shown for  $J$  up to  $3/2$  in Table 5.

Table 5.

$\frac{8}{\lambda^2 \sin \theta} A$	$M1(J=1/2)$ $b_{1/2}^*(m)$	$E1(J=1/2)$ $b_{1/2}^*(e)$	$M1(J=3/2)$ $a_{3/2}^*(m)$	$E1(J=3/2)$ $a_{3/2}^*(e)$	$M2(J=3/2)$ $b_{3/2}^*(m)$	$E2(J=3/2)$ $b_{3/2}^*(e)$
$M1(J=1/2)$ $b_{1/2}(m)$	0	$\frac{2}{3}i$	$i \cos \theta$	$\frac{1}{3}i$	$\frac{1}{\sqrt{5}}i$	$\frac{3 \cos \theta}{\sqrt{5}}i$
$E1(J=1/2)$ $b_{1/2}(e)$	$-\frac{2}{3}i$	0	$-\frac{1}{3}i$	$-i \cos \theta$	$-\frac{3 \cos \theta}{\sqrt{5}}i$	$-\frac{1}{\sqrt{5}}i$
$M1(J=3/2)$ $a_{3/2}(m)$	$-i \cos \theta$	$\frac{1}{3}i$	0	$-\frac{4}{3}i$	$\frac{2(1-3\cos^2\theta)}{\sqrt{5}}i$	0
$E1(J=3/2)$ $a_{3/2}(e)$	$-\frac{1}{3}i$	$i \cos \theta$	$\frac{4}{3}i$	0	0	$-\frac{2(1-3\cos^2\theta)}{\sqrt{5}}i$
$M2(J=3/2)$ $b_{3/2}(m)$	$-\frac{1}{\sqrt{5}}i$	$\frac{3 \cos \theta}{\sqrt{5}}i$	$-\frac{2(1-3\cos^2\theta)}{\sqrt{5}}$	0	0	$\frac{12 \cos^2 \theta}{5}i$
$E2(J=3/2)$ $b_{3/2}(e)$	$-\frac{3 \cos \theta}{\sqrt{5}}i$	$\frac{i}{\sqrt{5}}$	0	$\frac{2(1-3\cos^2\theta)}{\sqrt{5}}i$	$-\frac{12 \cos^2 \theta}{5}i$	0

One can readily see from (3.1) that the interchange,  $a_J(e) \leftrightarrow a_J(m)$  and  $b_J(e) \leftrightarrow b_J(m)$ , upsets the sign of the polarization. Thus the method to discriminate between the contributions from  $ML$  and  $EL$  amplitudes is to examine the polarization.

The relation between the reaction amplitudes of photomeson processes and the phase shifts of meson-nucleon scattering as well as its application to the analysis of experiments will appear in a forthcoming paper.

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## Mass Spectrum of Elementary Particles, II\*

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The considerations of mass spectrum of elementary particles are presented. First, the eigenvalue problem of the mass of a Dirac particle is discussed; the mass eigenvalues of the Dirac particle which interacts with itself through scalar photon fields are estimated by a method similar to that developed in the case of a deuteron for a Yukawa well. Secondly, the ambiguities which appear in the calculation of the nucleon self-energy caused by scalar mesons with vector coupling are discussed. It is shown that, in spite of the ambiguities, we may assume a fermion ( $\sim 2350m_e$ ) and a scalar cohesive meson ( $\sim 1100m_e$ ) to get convergence of the nucleon self-energy due to  $\pi$  mesons. A brief survey of the model of heavy particles which seems to be qualitatively consistent with the recent experimental results, is also given.

### Introduction

In the first part of this paper<sup>1)</sup> we have investigated the mass problem of Dirac particles for a Coulomb type self-potential which has been derived from a complicated expression of self-energy on the basis of some assumptions. The essential point of the approach in *A* was its prediction of a simple mass spectrum. Although it is rather gratifying to see that a considerable amount of information about mass spectrum can be drawn for a simple form of the potential, it is clear that the Coulomb well is not a good approximation for the physical situation. The purpose of the present paper is to ascertain if it is also possible to get a reasonable mass spectrum for a Yukawa type self-potential which is supposed to be a better approximation than the Coulomb well. This does not lead to any difficulty in practice. We recall that the techniques of determining the energy levels of the deuteron for a Yukawa well without tensor forces have been described in detail by Hulthén and Laurikainen<sup>2)</sup>. It is an easy matter to generalize their method for the case of the mass levels. We shall discuss a general procedure to determine mass values in terms of a rest-mass, coupling constants and quantum numbers. No thorough consideration with respect to the experimental material is attempted.

Another aim is to investigate some ambiguities contained in the calculation of the nucleon self-energy; in previous papers<sup>3)</sup>, we have suggested a simple model of mixing fields in order to cancel the divergences of the self-energy and at the same time to get a

\*) This paper is a continuation of the paper "Mass Spectrum of Elementary Particles, I" Prog. Theor. Phys. 11 (1954), 125. It will be quoted as *A*.

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mass relation of particles involved. In that case, we have assumed a cohesive meson to be of a scalar type with vector coupling. However, as is well known, there are some discrepancies between formal and actual calculations in terms of derivative couplings. In this note we shall remark the following point. In spite of these ambiguities, we may conclude that, if we restrict ourselves to the terms of second order in the coupling constants, we still have to assume the existences of the Dirac particle and scalar meson, whose masses are supposed to be  $m_1 \sim 2350m_e$  and  $\mu \sim 1100m_e$  respectively, to get convergence for the nucleon self-energy caused by  $\pi$  mesons which are supposed to be of a pseudoscalar type with pseudovector coupling.

## Part I

### The mass eigenvalues of Dirac particles

In the previous paper<sup>1)</sup> a method to formulate the eigenvalue problem for the masses of elementary particles was given for a simple form of self-potential. Since the ideas stated there are slightly different from those of the ordinary quantum mechanics in which time  $t$  is regarded as a special variable, we briefly explain the fundamental assumptions involved in our treatment. First of all we must mention a possible generalization of the present formalism, which is suggested by the theory of relativity. It seems that the difficulties of the present quantum field theory are not due to an inadequate mathematical handling of the problem, but due to the incomplete treatment of it in physical aspects. That is to say, from our point of view the troubles may directly be correlated with the fact that the synthesis of the relativity and quantum theory has not yet been achieved in a satisfactory way. In fact, in the quantum field theory it has been assumed that a space-like surface which means infinitesimally small instant of *the present time* has a significance with respect to the time dependence of state vectors. However, if we consider a point in a Minkowski space, we must take into account the following circumstance; a light cone which belongs to the point exists at every point in space-time. In other words, the locus of points which are simultaneous to the point considered is not a surface but a region which lies outside the light cone. In order to be consistent with the consequence of the relativity, it seems to be necessary to introduce a formulation of the quantum field theory that preserves this relativistic feature at least in the particle aspect of the theory. This can be done by employing the formulations of the theory which contains a *proper time parameter* instead of the time  $t$ , and which we have developed so far. The notion of space like region gives rise to new features that are entirely lacking in the usual treatment. As a result of this postulate, we are in a position to consider an important question: What kind of action is in the space-like domain? It cannot be the action through medium in the ordinary sense. It must be a kind of actions at a distance which one does not know at all, while in the present field theory the action is supposed to be transmitted by a field. However, the notion of action at a distance cannot be overlooked unnoticed by the quantum field theory; if one considers Feynman's  $D_F(x)$  function, one easily sees that it can still have a value outside the light cone whose vertex is at the origin. This is also the case

for the self-energy which arises from the product of some Feynman functions. Thus we may suppose that as a result of *quantum fluctuations* a new kind of action at a distance does appear in the space-like region, which is called as the self-energy of elementary particles. Of the mathematical expression of the self-action at a distance, it is enough to illustrate it by the *correspondence principle*. We recall the representation of the electron self-energy caused by electromagnetic field, which was given by Weisskopf<sup>(4)</sup>. The one-electron theory (i.e. all negative energy states are left empty) gives us

$$\delta m_{one} = e^2/\alpha, \quad (\alpha \rightarrow 0), \quad (1)$$

where  $\alpha$  is a length denoting the dimension of the electron. On the contrary the hole theory leads to

$$\delta m_{hole} = (3\alpha/2\pi)m \log 1/m\alpha, \quad (\alpha \rightarrow 0). \quad (2)$$

This suggests that we may employ the latter as a self-potential in the hole theory, which just corresponds to the Coulomb type self-action (1) in the classical and one-electron theories. Relativistically, according to Schwinger<sup>(5)</sup> the expression (2) is written as

$$\delta m = (3\alpha/4\pi)m \log 1/\gamma ms, \quad (s \rightarrow 0), \quad (3)$$

where  $s$  is regarded as a four dimensional distance in a space-like domain. With these considerations, we have set up an eigenvalue problem of mass in  $A$ : a generalized Hamiltonian or mass operator which involves the self-energy of elementary particles as a self-potential which is a function of  $s$  is given, so that we still have the Hamiltonian formalism in which the invariant parameter plays an important role instead of the time  $t$ . Then, in order to get the mass eigenvalues, a generalized wave equation is set up for the Hamiltonian and is solved outside the light cone under a boundary condition which is similar to that of the ordinary quantum mechanics. Generally speaking, the characteristic feature of our method is its analogy to the relativistic quantum mechanics. We thus easily understand how the correspondence principle provides a guide in the mass problem of elementary particles.

Now we shall proceed to investigate the mass eigenvalues of the Dirac particle interacting with scalar photons. We have considered this case in order to simplify the problem and to see the qualitative feature of mass spectrum. Therefore the following discussion is only a sketch, but it will be found to be sufficient for further application of the present method. We start with the ordinary wave equation:

$$(\gamma_\mu \partial/\partial x_\mu + m)\psi(x) = 0, \quad (4)$$

where  $m$  is assumed to be the observed mass of the particle. When we use the expression (4), we understand that the symbol  $x_\mu$ 's represent the variables describing the motion of the particle in the usual sense; although in this case theory appears to be relativistically invariant, the notion of the space-like domain is completely discarded. However, if we take into account the effects due to its self-fields, we have to assume a new wave equation in the following way. All variables  $x_\mu$ 's may be considered as the functions of the invariant parameter, and the extended Hamiltonian formalism is assumed, in which the



mass of the particle is a constant of motion, and the concept of space-like domain is reinstated. This modification is physically reasonable since the inclusion of the self-energy means to some extent a matter in a *proper space* or an *internal space* of the particle, as Yukawa has suggested<sup>(6)</sup>. Now the actual mass  $m$  in (4) is replaced by

$$m_0 + \delta m(m), \quad (5)$$

where  $m_0$  is the rest-mass. The term  $\delta m(m)$  shows the mass correction caused by self-interactions, and is a function of the actual mass  $m$  (not of the rest-mass  $m_0$ ). Thus the fundamental wave equation is assumed to be

$$(\gamma_\mu \partial/\partial y_\mu + m_0 + \delta m(m))\psi(x_\mu(\sigma), x_\mu(o)) = 0, \quad (6)$$

where  $y_\mu = x_\mu(\sigma) - x_\mu(o)$ , and  $x_\mu(\sigma)$  denote the variables outside the light cone whose vertex is at the point  $x_\mu(0)$ . The special form of the equation (6), in which the initial values  $x_\mu(0)$  are supposed to be zero is given in A. Here we remark again that the equations (4) and (6) represent the dualistic character of our theory. Namely, it is clear that the equation (4) describes in general real transitions and scattering processes of the Dirac particle in the four dimensional space-time. However, it is inconvenient to discuss the system including the effect of the self-energy by means of the equation (4). It is rather reasonable to represent such a system by the equation (6) which is valid in the proper space-time to which five coordinates are assigned, i.e. *four space-like variables* and *one time-like parameter*.

A detailed derivation of the Dirac equation in terms of polar coordinates in the proper space-time has been given in A. It will suffice to write it here explicitly: the radial functions  $\phi_1(R)$ , and  $\phi_2(R)$ , ( $R^2 = (x_\mu(\sigma) - x_\mu(o))^2$ ), satisfy the equations,

$$d\phi_1/dR - Q/R\phi_1 = -(V + \mu_0)\phi_2, \quad (7)$$

and

$$d\phi_2/dR + Q/R\phi_2 = -(V + \mu_0)\phi_1, \quad (8)$$

where

$$|Q| = 3/2, 5/2, 7/2, \dots, \\ \mu_0 = m_0 - g_1^2 m, \quad g_1^2 = 7g^2/64\pi^2, \quad g_0^2 = 3g^2/32\pi^2, \quad (9)$$

and

$$V = -g_0^2 m (2/mR) e^{-mR/2}. \quad (10)$$

It should be noted that the eigenvalue to be determined is the actual mass  $m$ , and the attractive potential form (10) is chosen by the following reason. To simplify the calculation and to be consistent with the boundary condition  $\psi(R) \rightarrow 0$  ( $R \rightarrow \infty$ ), we have used the asymptotic form for large  $R$  of the divergent part of self-energy,

$$\delta m \sim -2g_0^2 m e^{-mR/2} \{1/mR - 2/m^2 R^2 + 8/m^3 R^3 - \dots\}, \quad (11)$$

which is shown in the expression (47) in A. Because of the higher singularities in (11) a further approximation is introduced, which leads to the expression (10). In other

words, we will take up only the first term of the series (11). The boundary conditions for  $\phi_i(R)$  are that they become zero both for  $R \rightarrow 0$  and for  $R \rightarrow \infty$ .

Introducing the transformations

$$\phi_1 = e^{-\mu_0 R} (\chi_1 - \chi_2) \quad (12)$$

and

$$\phi_2 = e^{-\mu_0 R} (\chi_1 + \chi_2), \quad (13)$$

and eliminating the component  $\chi_2$ , in view of (7) and (8) we have

$$\frac{d^2 \chi_1}{dR^2} + \left( \frac{1}{R} - 2\mu_0 \right) \frac{d\chi_1}{dR} + \left( -2\mu_0 V - \frac{Q^2}{R^2} - V^2 + \frac{dV}{dR} + \frac{V}{R} \right) \chi_1 = 0. \quad (14)$$

To facilitate the solution by the polynomial method, it is convenient to replace the equation (14) by

$$\frac{d^2 w}{dR^2} - 2\mu_0 \frac{dw}{dR} + \left\{ -2\mu_0 V - V^2 + \frac{dV}{dR} + \frac{V}{R} + \frac{\mu_0}{R} - \frac{(Q^2 - 1/4)}{R^2} \right\} w = 0 \quad (15)$$

through the substitution

$$\chi_1 = R^{-1/2} w. \quad (16)$$

Furthermore, we rewrite (15) in a dimensionless form by introducing the dimensionless independent variable  $x = mR/2$ . We have

$$\frac{d^2 w}{dx^2} - a \frac{dw}{dx} + \left\{ b \frac{e^{-x}}{x} + c \frac{1}{x} - g_2 \frac{e^{-2x}}{x^2} - \frac{(Q-1/2)(Q+1/2)}{x^2} \right\} w = 0, \quad (17)$$

where

$$\begin{aligned} a &= 4\mu_0/m, & b &= 8\mu_0 g_0^2/m + 2g_0^2, \\ c &= 2\mu_0/m, & g_2 &= 4g_0^4. \end{aligned} \quad (18)$$

The object of our calculation is to determine the  $m$  which is involved in the factors  $a$ ,  $b$  and  $c$  as a function of the rest-mass  $m_0$ , coupling constant  $g_0^2$  and quantum numbers. If the differential equation (17) has an eigenfunction  $w$  for a value  $m$ , then we may find a relation between  $a$ ,  $b$  and  $c$ . It is often convenient to write  $b$  as a function of  $a$  and  $c$ . For large enough values of  $m_0$ , the relation can be expressed as a power series as follows:

$$b = \rho/\tau + \sum_{\mu=0}^{\infty} B_{\mu} \tau^{\mu}, \quad (19)$$

where

$$\tau = 1/(\alpha + \hbar), \quad (20)$$

and the factors  $\rho$ ,  $\hbar$  and  $B_{\mu}$ 's will be given later. The existence of the series such as (19) was shown by Hulthén and Laurikainen<sup>2)</sup> for the case of the deuteron problem in 3-dimensional space. Since the series has infinitely many terms, we cannot solve it

to get  $m$  in terms of other parameters. However, for practical purposes the first few terms of the power series (19) will clearly suffice. In this way the eigenvalues of mass will be estimated. Here we will follow the method developed by Laurikainen<sup>7)</sup>.

We now find a solution for  $w$  in the form

$$w = \sum_{\nu=-1}^{\infty} \alpha_{\nu} x^{\rho+\nu}, \quad \rho \geq 0, \quad \alpha_0 \neq 0. \quad (21)$$

Substitution of (21) into the equation (17) gives

$$\{(\rho+\nu+1)(\rho+\nu) - (Q-1/2)(Q+1/2)\} \alpha_{\nu+1} + \{c - a(\rho+\nu)\} \alpha_{\nu} \\ + b \sum_{\sigma=0}^{\nu} (-1)^{\sigma} / \sigma! \alpha_{\nu-\sigma} - g_2 \sum_{\sigma=0}^{\nu+1} (-2)^{\sigma} / \sigma! \alpha_{\nu+1-\sigma} = 0. \quad (22)$$

When  $\nu = -1$ , we get

$$\rho^2 - \rho - Q^2 + 1/4 - g_2 = 0 \quad (23)$$

or

$$\rho = 1/2 \pm (Q^2 + g_2)^{1/2}. \quad (24)$$

The boundary condition for  $w$  requires that we must choose the positive sign. In order to see the general nature of the asymptotic solution of the equation (17), we assume the conditions:

- (1) For large  $x$  the terms containing the function  $1/x^2$  in (17) can be neglected.
- (2) The function  $w$  is assumed to be a slowly varying one for large  $x$ .

Hence we can neglect  $d^2 w / dx^2$  in comparison with  $adw/dx$ . Then the equation (17) becomes

$$d\bar{w}/dx = (be^{-x}/ax + c/ax)\bar{w}. \quad (25)$$

We now find a solution for (25) in the form

$$\bar{w} = \sum_{\nu=0}^{\infty} \beta_{\nu} x^{\rho+\nu}, \quad \beta_0 \neq 0. \quad (26)$$

Substitution of (26) into (25) gives us the relation for  $\beta_{\nu}$ ;

$$\beta_{\nu}(\rho+\nu-c/a) = (b/a) \sum_{\sigma=0}^{\nu} (-2)^{\sigma} / \sigma! \beta_{\nu-\sigma}. \quad (27)$$

When  $\nu=0$ , one obtains

$$\rho = (b+c)/a \quad (28)$$

and

$$\bar{w} = x^{(b+c)/a} \sum_{\nu=0}^{\infty} \beta_{\nu} x^{\nu}. \quad (29)$$

Since the asymptotic form of the solution (21) is of the form (29), we find

$$1/2 + (Q^2 + 4g_2^2)^{1/2} = (b+c)/a, \quad (30)$$

in view of (24) and (28). Hence, from (27) we get

$$\beta_{\nu} = (1/\nu) (Q^2 + 4g_2^2)^{1/2} \sum_{\sigma=1}^{\nu} (-1)^{\sigma} / \sigma! \beta_{\nu-\sigma} = (1/\nu) (\rho - 1/2) \sum_{\sigma=1}^{\nu} (-1)^{\sigma} / \sigma! \beta_{\nu-\sigma}. \quad (31)$$

We can use the relation (30) to get an approximate expression for the mass of the particle :

$$m = \{2m_0 g_0^{-2} (Q^2 + 4g_0^4)^{1/2} - 4m_0\} \{1 - 4g_1^2 + 2g_1^2 g_0^{-2} (Q^2 + 4g_0^4)^{1/2}\}^{-1}. \quad (32)$$

This is a generalized formula of the mass spectrum derived in  $A$  on the basis of the Coulomb well, *viz.*

$$m = m_0 g_0^{-2} (n + |Q|). \quad (33)$$

We shall now look for a somewhat more detailed expression. For that purpose we assume the series (19) which includes  $c$  as a term:  $B_0 = -c$ . The expansion parameter  $\tau$  contains the unknown parameter  $h$  which will be determined later. We also expand  $\alpha_\nu$  as power series in  $\tau$  :

$$\alpha_\nu = \sum_{\lambda=0}^{\infty} \alpha_{\nu, \lambda} \tau^\lambda. \quad (34)$$

The expansions (19) and (34) are substituted into the recurrision relation (22) to give

$$\begin{aligned} & \{(\rho + \nu + 1)(\rho + \nu) - (Q - 1/2)(Q + 1/2)\} \alpha_{\nu+1, \lambda} - (\rho + \nu) \alpha_{\nu, \lambda+1} + \{c + h(\rho + \nu)\} \alpha_{\nu, \lambda} \\ & + \sum_{\sigma=0}^{\nu} \sum_{\mu=-1}^{\lambda} (-1)^\sigma / \sigma! \alpha_{\nu-\sigma, \lambda-\mu} B_\mu - 4g_0^4 \sum_{\sigma=0}^{\nu+1} (-2)^\sigma / \sigma! \alpha_{\nu+1-\sigma, \lambda} = 0. \end{aligned} \quad (35)$$

We assume that the arbitrary constants  $\alpha_{0, \lambda}$  are given by

$$\alpha_{0,0} = 1, \quad \alpha_{0, \lambda} = 0, \quad (\lambda \neq 0), \quad (36)$$

and that the following relation is valid :

$$\alpha_{\nu, 0} = \beta_\nu. \quad (37)$$

Then, with the help of (31) one finds

$$\begin{aligned} \alpha_{0,0} &= 1, \\ \alpha_{1,0} &= -(\rho - 1/2), \\ \alpha_{2,0} &= \rho(\rho/2 - 1/4), \\ &\dots\dots\dots \end{aligned} \quad (38)$$

If we take  $\nu=0$  and  $\lambda=0$  in (35), we see that

$$h = \{\rho(\rho+1) - Q^2 + 1/4 - 4g_0^4\} (Q^2 + 4g_0^4)^{1/2} \rho^{-1} - 8g_0^4 \rho^{-1} = 2\rho - 1 - 8g_0^4 \rho^{-1}. \quad (39)$$

If we set  $\nu=0$  in (35), with the aid of (39) we find that

$$B_\lambda = \{Q^2 - 1/4 + 4g_0^4 - \rho(\rho+1)\} \alpha_{1, \lambda} = -2\rho \alpha_{1, \lambda}. \quad (40)$$

Therefore, each of these expansion coefficients  $B_\lambda$  is defined by  $\alpha_{1, \lambda}$ . In this way we can obtain the asymptotic expansion for  $b$ . In order to get the term to first order in  $\tau$ , we have to know the expression for  $\alpha_{1,1}$ . Setting  $\nu=1$  and  $\lambda=0$  in (35), one obtains

$$\begin{aligned} \alpha_{1,1} &= \rho(\rho/2 - 1/4) \{(\rho+2)(\rho+1) - Q^2 + 1/4 - 4g_0^4\} \\ & - (\rho - 1/2) \{h(\rho+1) + 2g_2^2\} + c - 8g_0^4. \end{aligned} \quad (41)$$

With these formulas,  $B_1$  can be written as

$$B_1 = -2\rho(\rho - 1/2 - 4g_0^4 \rho^{-1} + c). \quad (42)$$

Now we find from (19)

$$b = \rho/\tau - c - 2\rho(\rho - 1/2 - 4g_0^4 \rho^{-1} + c)\tau + \sum_{\mu=2}^{\infty} B_{\mu} \tau^{\mu}. \quad (43)$$

There is no sense in calculating (43) exactly since it involves only an approximate expression for the self-potential. To the extent that the higher terms ( $\mu \geq 2$ ) in (43) can be neglected, we get a formula as follows:

$$b = 2\rho^2 + (a-1)\rho - 8g_0^4 - c - \{2\rho^3 + (2c-1)\rho^2 - 8g_0^4\rho\} \{2\rho^2 + (a-1)\rho - 8g_0^4\}^{-1}. \quad (44)$$

In terms of following notations,

$$a = 4y, \quad b = 8g_0^2 y + 2g_0^2, \quad c = 2y, \quad (45)$$

and

$$y = \mu_0/m = m_0/m - g_1^{-2}, \quad (46)$$

the relation (44) becomes

$$(32g_0^2 + 8 - 16\rho)y^2 + 2\{4(1-L)(g_0^2 - \rho) + L + 6\rho\}y + L\{2g_0^2 + \rho(1-L)\} = 0, \quad (47)$$

where

$$\rho = 1/2 + (Q^2 + 4g_0^4)^{1/2} \quad (48)$$

and

$$L = 2\rho - 8g_0^4 \rho^{-1} - 1. \quad (49)$$

It should be noted that the mass  $m$  is contained only in the roots of the second order equation (47). One might think here that a qualification is necessary to use (44) instead of the series (43). We observe that (44) is not a poor approximation to (43) provided  $a$  is large enough so that the higher terms may be neglected. We feel however that it seems to be useless at this time to determine the mass  $m$  by any more detailed relation than (44) since we have replaced the complicated self-potential by a Yukawa well which is not an adequate expression for it at small values of  $x$ . Although the present treatment would not give us any accurate information about the mass spectrum, it should be remarked that with the aid of (47) the order of the magnitude of the mass  $m$  would be estimated. As an illustration of this point, we have solved (47). The equation (47) for the mass  $m$  involves three parameters: the coupling constant  $g_0^2$ , quantum number  $Q$ , and rest-mass  $m_0$ . Calculations are made for several values of  $g_0^2$  and  $Q$ , and for a fixed  $m_0$ . The numerical values of  $m$  are shown in Table I. It is noted there that the  $Q$  dependence of the mass seems to be qualitatively consistent with the supposition that the states of the heavy particles may be regarded as the *excited states* of a stable particle. However, it is not reasonable that there appear some negative masses in the result. It need hardly be stressed that, since in the limit of  $g_0^2 \rightarrow 0$  we should have the relation  $m \rightarrow m_0$ , they arise from the crude approximations both for the self-potential and for the



series (43). Finally, we must take notice of the fact that, although the example mentioned above is the most simple one, our method provides a way to see what properties of mass spectrum can be inferred for such a simple case. It is gratifying to see that a qualitative understanding of the existence of higher mass states, which has been hard to interpret,

 Table I. Mass values  $m$  in units of  $m_0$ 

$g_0^2$	1.5	1.2	1.0	0.8	0.6	0.5
$Q=3/2$	0.81 0.96	1.29 1.60	1.72 5.92	3.20 -13.99	28.57 -3.72	-9.80 -2.89
$Q=5/2$	1.23 -1.38	2.54 -1.17	9.43 -1.09	-5.56 -1.03	— —	— —
$Q=7/2$	3.06 -0.66	-8.62 -0.65	— —	— —	— —	— —

can be obtained on the basis of such a simple assumption. Here no consideration has been done for the actual system of the elementary particles to account for the observed mass spectrum if it is to be predicted from the theory developed here, since for the present the experimental data are not sufficient to draw reliable informations about the type of couplings and other properties of the elementary particles. Therefore we shall be content to remark that it will in general be possible to extend the present method to any cases concerning the problem of the mass spectrum. To such a problem we shall come back in a future paper.

## Part II

### Problems of scalar mesons with vector coupling

In the previous papers<sup>3)</sup> we have developed a method of avoiding the divergence difficulties of the quantum field theory by means of the method of mixing various fields. In this section we shall investigate the ambiguities which appeared in the computation of the nucleon self-energy.

Before entering into detailed calculations we wish to mention some points connected with the so-called mixed theory. First, it has been clear by recent experiments<sup>8)</sup> that there seem to be many new particles in nature. As a result of this discovery, the problem of the mixed theory appears in a new light. That is to say, it would seem that we may now analyze the self-energy problem in terms of the mixed theory based on a physical ground. Secondly, the analyses given in the preceding sections and in the paper A show that it is easy to understand why the method of compensation would be effective for the mass problem. We now know that the object of the method is to cancel higher singularities of the self-potentials by compensation or at least to make the self-energy negative in order that a mass spectrum can be obtained. This assumption seems to be natural. At any rate, the recognition of the fundamental connection between the self-energy and mass

spectrum might clarify the difficulty of the quantum field theory.

Now we shall proceed to investigate the ambiguities. As an example of compensations, we have considered so far the following set of interactions:

$$i(g/\kappa)\bar{\psi}_N\gamma_5\gamma_\nu\psi_N\partial\phi_\pi/\partial x_\nu + h.c., \quad (50)$$

$$i(G/\mu)\bar{\psi}_\Lambda\gamma_\nu\psi_N\partial\phi_0/\partial x_\nu + h.c., \quad (51)$$

and

$$i(I/\kappa)\bar{\psi}_\Lambda\gamma_5\gamma_\nu\psi_N\partial\phi_\pi/\partial x_\nu + h.c., \quad (52)$$

where  $\psi_N$ ,  $\psi_\Lambda$ ,  $\phi_\pi$  and  $\phi_0$  are the wave functions of the nucleon,  $\Lambda$  particle,  $\pi$  meson and  $\theta$  meson respectively. The ambiguity concerning the self-energy manifests itself in a special case of the interaction (51). Namely, if we assume that the mass of the  $\Lambda$  particle is equal to that of the nucleon (i.e.  $M_\Lambda = M$ ;  $M$ : nucleon mass), then (51) becomes the usual vector coupling of a scalar meson, which does not contribute to the nucleon self-energy. This is proved by a formal calculation or by the well known equivalence theorem. However, this is not always the case for the actual calculations<sup>11</sup>. The paradoxical feature of the theory lies even in the case when one considers the interaction (51). itself (i.e.  $M_\Lambda \neq M$ ). Usually, the ambiguities arise in the calculations by means of the momentum representation, the results then are sensitive to the method used.

We will consider in the following two different methods of computation of the nucleon self-energy in terms of (50) and (51), and will discuss the results of compensation. As will be seen later, even in the case of actual calculation it is possible to make the self-energy zero for the special case of (51) in accordance with the formal proof. We use the same notation as that employed in B.

Confining our attention now to the essential parts of the matrix elements of the nucleon self-energy to the second order in the coupling constants, we find from (51) and (50)

$$I_s = \int (dk) \frac{(\gamma k) [i\gamma(k-p) + M_\Lambda] (\gamma k)}{[(p-k)^2 + M_\Lambda^2](k^2 + \mu^2)}, \quad (\gamma k) = (\gamma_\mu k_\mu), \quad (53)$$

and

$$I_{ps} = \int (dk) \frac{(\gamma k) [i\gamma(k-p) - M] (\gamma k)}{[(p-k)^2 + M^2](k^2 + \kappa^2)}, \quad (54)$$

where  $M_\Lambda$ ,  $\kappa$  and  $\mu$  denote the masses of the  $\Lambda$  particle,  $\pi$  meson and  $\theta$  meson. Since we are interested in the interaction (51), we want to show the detailed calculation of (53). The numerator of (53) can be modified as follow\*:

$$\begin{aligned} I_s &= (\gamma k) [i\gamma(k-p) + M_\Lambda] (\gamma k) \\ &= (\gamma k) [i\gamma(k-p) + M] (\gamma k) + (M_\Lambda - M) (\gamma k) (\gamma k). \end{aligned} \quad (55)$$

With the help of the relations

\*) This possibility was first pointed out by Prof. N. Kroll. The author wishes to thank him for his discussion on this point.

$$i(\gamma p) + M = 0, \quad (56)$$

and

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}, \quad (57)$$

the first term on the right-hand side of (55) becomes

$$(\gamma k) [i\gamma(k-p) + M] (\gamma k) = i(\gamma k) [k^2 - (\gamma p)(\gamma k)] + (\gamma k)(\gamma k)M = i(\gamma k) [k^2 - 2(k \cdot p)].$$

Thus we find

$$I_s = i(\gamma k) [k^2 - 2(k \cdot p) + M_1^2 - M^2] - i(\gamma k)(M_1^2 - M^2) + (\gamma k)(\gamma k)(M_1 - M). \quad (58)$$

The integration (53) then can be written as

$$I_s = I_s^1 + I_s^2 + I_s^3, \quad (59)$$

where

$$I_s^1 = \int (dk) \frac{i(\gamma k) [k^2 - 2(k \cdot p) + M_1^2 - M^2]}{[(p-k)^2 + M_1^2 - M^2 + M^2](k^2 + \mu^2)}, \quad (60)$$

$$I_s^2 = \int (dk) \frac{-i(\gamma k)(M_1^2 - M^2)}{[(p-k)^2 + M_1^2](k^2 + \mu^2)}, \quad (61)$$

and

$$I_s^3 = \int (dk) \frac{(\gamma k)(\gamma k)(M_1 - M)}{[(p-k)^2 + M_1^2](k^2 + \mu^2)}. \quad (62)$$

With the aid of the relations

$$\begin{aligned} [k^2 - 2(k \cdot p) + M_1^2 - M^2 + (p^2 + M^2)]^{-1} &= [k^2 - 2(k \cdot p) + M_1^2 - M^2]^{-1} \\ &- (p^2 + M^2) \int_0^1 [k^2 - 2(k \cdot p) + M_1^2 - M^2 + (P^2 + M^2)t]^{-2} dt, \end{aligned} \quad (63)$$

and

$$P^2 + M^2 = 0, \quad (64)$$

$I_s^1$  can be transformed as follows:

$$I_s^1 = \int (dk) \frac{i(\gamma k) [k^2 - 2(k \cdot p) + M_1^2 - M^2]}{[k^2 - 2(k \cdot p) + M_1^2 - M^2](k^2 + \mu^2)} = i \int (dk) \frac{(\gamma k)}{(k^2 + \mu^2)} = 0 \quad (65)$$

from the symmetry in the  $k$ -space.

The integrations of the remaining terms  $I_s^2$  and  $I_s^3$  are carried out by the method described in B.

Making use of the identity

$$a^{-1}b^{-1} = \int_0^1 [au + b(1-u)]^{-2} du, \quad (66)$$

and substituting  $k_\mu$  by  $k_\mu + up_\mu$  we obtain

$$I_s^2 = - (M_1^2 - M^2) \int_0^1 du \int (dk) (1+D) A^{-2} [i(\gamma k) + iu(\gamma p)]$$

$$= -(M_1^2 - M^2) \int \int_0^1 du (dk) \bar{I}_s^2, \quad (67)$$

where

$$A = k^2 + A^2 = k^2 + M^2 u^2 + (M_1^2 - M^2 - \mu^2) u + \mu^2 + (p^2 + M^2) u (1 - u), \quad (68)$$

and

$$D = -u p_\alpha \partial / \partial k_\alpha. \quad (69)$$

After the differentiation with respect to  $k_\alpha$  we find

$$\bar{I}_s^2 = i A^{-2} (\gamma k) + 4i A^{-3} (p \cdot k) [u (\gamma k) + u^2 (\gamma p)]. \quad (70)$$

If we neglect the terms which are linear in  $k_\alpha$  and put

$$(p \cdot k) (\gamma k) = (\gamma p) k^2 / 4, \quad (71)$$

we have

$$I_s^2 = -(M_1^2 - M^2) \int \int_0^1 du (dk) i u (\gamma p) k^2 = M (M_1^2 - M^2) \int \int_0^1 du (dk) [u, 1^{-2} - u, 1^2 A^{-3}]. \quad (72)$$

Performing the integrations, one gets for the divergent part

$$I_s^2 = i\pi^2 M (M_1^2 - M^2) \log (K + K_0) / M_1 \Big|_{K \rightarrow \infty}, \quad K_0 = (K^2 + M_1^2)^{1/2}. \quad (73)$$

We can now do the integrals of  $I_s^3$  in the same way.

$$I_s^3 = (M_1 - M) \int \int_0^1 du (dk) (1 + D) \cdot 1^{-2} [\gamma (k + u p)] (\gamma (k + u p)) = (M_1 - M) \int \int_0^1 du (dk) I_s^3, \quad (74)$$

where

$$D = -u p_\alpha \partial / \partial k_\alpha + (u^2 / 2!) P_\alpha P_\beta \partial^2 / \partial k_\alpha \partial k_\beta. \quad (75)$$

After some calculations, we find the result in the form

$$\bar{I}_s^3 = A^{-1} + A^{-2} (-A^2 + u^2 p^2) + A^{-3} (2u^2 p^2 A^2 + u^4 p^4 - 6u^2 p^2 A^2) + A^{-4} (3u^2 p^2 A^4 - 3u^4 p^4 A^2). \quad (76)$$

Thus the divergent parts of  $I_s^3$  is

$$\begin{aligned} I_s^3 &= (M_1 - M) \int \int_0^1 du (dk) [A^{-1} + A^{-2} \{-2M^2 u^2 + (-M_1^2 + M^2 + \mu^2) u^2 - \mu^2\}] \\ &= 2i\pi^2 (M_1 - M) M^2 (K K_0 / M^2 - \beta^2 \log (K + K_0) / M_1) \\ &\quad - i\pi^2 (M_1 - M) (M_1^2 + M^2 / 3 + \mu^2) \log (K + K_0) / M_1, \end{aligned} \quad (77)$$

where

$$\beta = M_1 / M. \quad (78)$$

Collecting terms, we see then that the infinite self-energy of the nucleon which is caused by the interaction (51) is equal to

$$\begin{aligned} \delta M_{s(n)} &= M \left( \frac{\mu^2}{4\pi} \right) \left( \frac{M}{\mu} \right)^2 \left[ \left( \frac{1}{2\pi} \right) (\beta - 1) (K K_0 / M^2) + \left( \frac{1}{4\pi} \right) (1 - \beta) \right. \\ &\quad \times \{ 3\beta^2 - \beta - 2/3 + (\mu / M)^2 \} \log (K + K_0) / M_1]. \end{aligned} \quad (79)$$

We have thus shown that there is no contribution to the self-energy if and only if the mass of the  $M$  particle agrees with that of the nucleon. It is also seen from the fact that the terms  $I_s^2$  and  $I_s^3$  given by (61) and (62) respectively vanish in the limit of  $M_1 \rightarrow M$ .

Next, let us consider the case of the interaction (60): pseudoscalar  $\pi$  meson with pseudovector coupling. In order to cancel the divergences by compensation it should be noted that the computation must be carried out by the method analogous to that applied to the interaction (61).

Consider now the transformations such as (55)–(58) in the numerator of  $I_{ps}$  given by (54):

$$I_{ps} = \int (dk) \frac{(\gamma k) [i\gamma(k-p) - M](\gamma k)}{[(p-k)^2 + M^2](k^2 + \kappa^2)}. \quad (80)$$

In this case, we see that

$$\begin{aligned} I_{ps} &= (\gamma k) [i\gamma(k-p) - M](\gamma k) \\ &= (\gamma k) [i\gamma(k-p) + M](\gamma k) - 2M(\gamma k)(\gamma k) \\ &= i(\gamma k) [k^2 - 2(k \cdot p)] - 2M(\gamma k)(\gamma k). \end{aligned} \quad (81)$$

Thus

$$I_{ps} = I_{ps}^1 + I_{ps}^2,$$

where

$$I_{ps}^1 = \int (dk) \frac{i(\gamma k) [k^2 - 2(k \cdot p)]}{[(p-k)^2 + M^2](k^2 + \kappa^2)} = i \int (dk) \frac{(\gamma k)}{(k^2 + \kappa^2)} = 0, \quad (82)$$

and

$$I_{ps}^2 = -2M \int (dk) \frac{(\gamma k)(\gamma k)}{[(p-k)^2 + M^2](k^2 + \kappa^2)} = -2M \int (dk) \bar{I}_s^3 \Big|_{M_1 \rightarrow M, \mu \rightarrow \kappa}. \quad (83)$$

Therefore, according to (78), we can readily get the result in the form

$$I_{ps} = i\pi^2 \left[ -4M^3 (KK_0/M^2 - \log(K+K_0)/M) + 2M((4/3)M^2 + \kappa^2) \log(K+K_0)/M \right]. \quad (84)$$

Alternatively, the infinite self-energy of the nucleon due to the interaction (50) is written as

$$\begin{aligned} \delta M_{ps(pv)} &= M \left( \frac{g^2}{4\pi} \right) \left( \frac{M}{\kappa} \right)^2 \left[ -\left( \frac{1}{\pi} \right) (KK_0/M^2) + (1/4\pi) \{20/3 + 2(\kappa/M)^2\} \right. \\ &\quad \left. \times \log(K+K_0)/M \right]. \end{aligned} \quad (85)$$

Comparing (79) with (85), we finally obtain the conditions for compensation as follows:

$$(G/\mu)^2 = (2/\beta - 1) (g/\kappa)^2, \quad (86)$$

and



$$3\beta^2 - \beta - 4 + (\mu^2 - \kappa^2) / M^2 = 0. \quad (87)$$

In Table II we give the mass values for the  $A$  particle and  $\theta$  meson which are derived from the relation (87) when we assume that the masses of the nucleon and  $\pi$  meson are  $1836 m_e$  and  $276 m_e$  respectively. Although the result obtained is in general not reliable

Table II. Masses of  $A$  particles and  $\theta$  mesons in units of  $m_e$ . ( $M=1836m_e$ ,  $\kappa=276 m_e$ ).

$A$ mass	2280	2300	2360	2380	2400	2420
$\theta$ mass	1463	1380	1092	969	832	608

because of the approximation adopted, it can be seen from the Table II that the masses are of the order of magnitude of those which have been observed by many people in the recent experiments.<sup>8)</sup> In order to compare the results obtained above with that given in  $C$ , the evaluation analogous to that in  $B$  will be here stated for the case of the interaction (51). We begin with the expression

$$I_s = \int (dk) \frac{(\gamma k) [i\gamma(k-p) + M_1](\gamma k)}{[(p-k)^2 + M_1^2](k^2 + \mu^2)}. \quad (88)$$

The first point to which we turn our attention is the treatment of the numerator of (88). We shall perform the integrations without making use of the transformations such as (55) and (58). According to the formula (66) and the change of variables,  $k_\mu \rightarrow k_\mu + u p_\mu$ , we have

$$I_s = \int_0^1 du \int (dk) (1+D) A^{-2} [(\gamma(k+up)) \{i\gamma(k+(u-1)p) + M_1\} (\gamma(k+up))], \quad (89)$$

where

$$D = -up_\alpha \frac{\partial}{\partial k_\alpha} + \frac{u^2}{2!} p_\alpha p_\beta \frac{\partial^2}{\partial k_\alpha \partial k_\beta} - \frac{u^3}{3!} p_\alpha p_\beta p_\gamma \frac{\partial^3}{\partial k_\alpha \partial k_\beta \partial k_\gamma}. \quad (90)$$

and  $A$  is given by (68).

Differentiation with respect to  $k_\mu$  gives

$$I_s = I_s^1 + I_s^2 + I_s^3 + I_s^4, \quad (91)$$

$$I_s^1 = \int_0^1 du \int (dk) A^{-2} [(-2\beta u + 2\beta + 3u^2 + u - 1) (Mk^2/2) + (-7\beta u^4 - \beta u^3 + 6u^5 - 4u^4 - u^3) (M^3/3)], \quad (92)$$

$$I_s^2 = \int_0^1 du \int (dk) A^{-3} [(-uMk^4 + (-2\beta u^4 + 2\beta u^2 + \beta u^2 + 3u^5 + u^4 + 3u^3 - u^2) M^3 k^2 + 2(3\beta u^5 + \beta u^4 - 3u^6 + u^5 + u^4) M^5)], \quad (93)$$

$$I_s^3 = \int_0^1 du \int (dk) A^{-4} [3(\beta u^3 - \beta u^2 - 2u^4 - u^3) M^3 k^4 + 3(-3\beta u^5 - 3\beta u^4 + 3u^6 + 2u^5 - 3u^4) M^5 k^2], \quad (94)$$

and

$$I_s^4 = \int_0^1 du \int (dk) A^{-5} [4u^3 M^3 k^6 + 4(2\beta u^4 - 3u^5 + 2u^4) M^5 k^4]. \quad (95)$$

$I'_s$  can be modified with the help of the relations :

$$k^2 \rightarrow (k^2 + A^2) - A^2, \quad k^4 \rightarrow (k^2 + A^2)^2 - 2A^2(k^2 + A^2) + A^4, \dots \quad (96)$$

After some evaluations, if we neglect finite terms, we have

$$\begin{aligned} I_s = \int_0^1 du (dk) [ & (M/2A) (-2\beta u + \beta + 3u^2 - u - 1) + (M^3/A^2) \{5u^5 \\ & - (26\beta + 41)(u^4/6) + (-3\beta^2 + 8\beta + 17 + 3\mu^2/M^2)(u^3/3) \\ & + (2\beta^3 + 5\beta^2 - 2\beta - 11 - 2\beta\mu^2/M^2 - 11\mu^2/M^2)(u^2/4) \\ & + (2\beta + 3)(\mu^2 u/2M^2) - (\beta - 1/2)(\mu^2/M^2)\} ]. \end{aligned} \quad (97)$$

By integrating over  $u$  and  $k$ , there follows directly

$$\begin{aligned} \partial M_{s(v)} = M(G^2/4\pi) (M/\mu)^2 [ & (1/8\pi) (2\beta - 1) (KK_0/M^2) + (1/120\pi) \\ & \times \{-20\beta^3 + 25\beta^2 - 22\beta - 2 + (35\mu^2/M^2) - (40\beta\mu^2/M^2)\} \log(K + K_0)/M_1], \end{aligned} \quad (98)$$

when expressed in the form of self-energy. One sees at once that (98) diverges even in the limit of  $M_1 \rightarrow M$ . In the same way, as was shown in  $B$  we can get the nucleon self-energy due to (50) as

$$\begin{aligned} \partial M_{ps(pv)} = M(G^2/4\pi) (M/\kappa)^2 [ & -(3/8\pi) (KK_0/M^2) + (1/120\pi) \\ & \times \{(65 + (75\kappa^2/M^2))\} \log(K + K_0)/M]. \end{aligned} \quad (99)$$

The conditions of compensation resulting from (98) and (99) are therefore

$$(G/\mu)^2 = 3/(2\beta - 1) (g/\kappa)^2, \quad (100)$$

and

$$(\mu/M)^2 = [(2\beta - 1) \{(74\kappa^2/M^2) + 65\} - (60\beta^3 - 75\beta^2 + 66\beta + 6)] / (120\beta - 105), \quad (101)$$

which are the results given in  $C$ . The mass values which are derived from (101) are again shown in Table III.

Table III. Masses of  $A$  particles and  $\theta$  mesons in units of  $m_e$ . ( $M=1836 m_e$ ,  $\kappa=276 m_e$ ).

	2100	2200	2250	2300	2350	2400
$A$ mass	2100	2200	2250	2300	2350	2400
$\theta$ mass	1152	1042	988	923	857	787

Now it remains to study in more detail the inconsistency involved. However, at the present stage we have no convention to remove such ambiguities which seem to be due to the use of the different methods for carrying out the divergent integrals. In spite of the incompleteness of the present method, the presumption is strong that the approach mentioned above may be nearly but not quite correct on the way toward a deeper understanding of the nature of elementary particles. In fact, as is easily seen from the Tables II and III, the masses of the  $A$  particle and  $\theta$  meson seem to be not at variance with those which have been observed in the recent experiments, when either method of calculation is used. Therefore we should like to put forward here a hypothesis that

the mass values thus estimated may be looked upon as indices to consider the model of elementary particles on the basis of the mixed theory. Of course, a further uncertainty arises from the neglect of higher order terms in our calculation, so that our model may not be free from a serious change. Nevertheless, we shall emphasize that our point of view is most practical and comprehensive to solve the problems involving heavy unstable particles, as was discussed more fully in the paper C'. In the following shall we be concerned chiefly with the processes involving neutral  $\Lambda$  particles and  $\theta$  mesons. From our point of view it seems that the electromagnetic field does not play an important role in studying the problem of mass quantization. The reason is that, since it gives us a positive self-energy, we may take into account it as a perturbation for the mass levels. Accordingly, the neutral particles are the central object of our investigation.

### (1) Interactions :

The interactions which we have assumed in the paper C from the field theoretical considerations are as follows :

$$\left. \begin{array}{ll} (NN\pi), & PS(pv), \\ (\Lambda N\theta), & S(v), \\ (\Lambda\Lambda\pi), & \text{and } PS(pv), \end{array} \right\} g^2/4\pi\hbar c \sim 10^{-1}, \quad (102)$$

$$(NN\theta) \quad \text{and} \quad (\Lambda\Lambda\theta), \quad S(s), \quad f^2/\pi\hbar c \sim 10^{-9} - 10^{-12}.$$

### (2) Masses :

The order of magnitude of mass is estimated in terms of the condition for the compensation of divergences.

#### (i) Self-energy of the nucleon ;

$$(NN\pi) \text{ and } (\Lambda N\theta) \text{ furnish } M_1 \sim 2200m_e, \mu \sim 1000m_e.$$

#### (ii) Self-energy of the $\Lambda$ particle ;

$$(\Lambda N\theta) \text{ and } (\Lambda\Lambda\pi) \text{ give } M_1 \sim 2300m_e, \mu \sim 1000m_e.$$

We have assumed that  $M=1836m_e$  and  $\kappa=276m_e$ .

### (3) Production of particles :

The  $\Lambda$  and  $\theta$  are produced in pairs.

#### (i) $(NN\pi)$ and $(\Lambda\Lambda\theta)$ predict

$$N+N \rightarrow N+\Lambda+\theta, \text{ and } N+N \rightarrow \Lambda+\Lambda.$$

#### (ii) $(NN\pi)$ and $(\Lambda\Lambda\theta)$ also predict

$$\pi+N \rightarrow \Lambda+\theta.$$

It must be noted that this is in accordance with the production scheme which has been observed in the experiments by the Brookhaven cosmotron.

### (4) Decay of Particles :

$$(i) \quad (\Lambda\Lambda\theta), \quad (NN\pi) \text{ and } (NN\theta),$$

or  $(\Lambda\Lambda\theta)$ ,  $(\Lambda\Lambda\pi)$  and  $(\Lambda N\theta)$ .

Either combination gives rise to

$$\Lambda \rightarrow N + \pi.$$

(ii)  $(NN\theta)$  and  $(NN\pi)$ ,

or  $(\Lambda\Lambda\theta)$  and  $(\Lambda\Lambda\pi)$ .

Either combination gives rise to

$$\theta \rightarrow \pi + \pi.$$

In this way the consequences which are derived by making use of the set of interactions (102), which is employed to understand at least processes concerning the  $\Lambda$  particle and  $\theta$  meson, appear to be qualitatively not at variance with experimental results. While many details remain to be considered, it should be remarked that the field theoretical point of view may furnish a new light on the deep-lying relations between elementary particles.

### Outlook

From the foregoing works it appears that we may use the generalized Hamiltonian formalism and the method of mixed fields as profitable tools in order to investigate the mass problem of elementary particles. The advantage of the new Hamiltonian formalism is that in a system interacting with itself it permits a treatment of the problem of self-interaction in a relativistically invariant way. It is satisfactory from the physical standpoint: it has been formulated in such a way that it preserves one of the most important features of quantum mechanics. Recently several authors have put forward the idea similar to that involved in our treatment. Proca<sup>9)</sup> has suggested a possible extension of the point mechanics of a Dirac particle, in which a proper time plays an important part. The necessity of introducing the proper time or *absolute time* in the quantum electrodynamics has been stressed also by Dirac<sup>10)</sup>.

At any rate, it seems that there is reason to believe that the new point of view will prove fruitful for the present quantum field theory.

I would like to thank Professor H. Yukawa for his kind interest he has taken in this work.

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## Vapour Pressures of $\text{He}^3\text{-He}^4$ Mixtures

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Recently, Heer and Daunt<sup>1)</sup> have developed smoothed potential ideal (Bose or Fermi) gas models for the study of mixtures of Bose and Fermi liquids, that is,  $\text{He}^4$  and  $\text{He}^3$ , respectively. They have assumed that the liquids are incompressible and that they form statistically independent systems in the mixture. Thus they have written for the total volume  $V$ , and the total potential  $\chi$ , of the mixture, the expressions:

$$V = N_3 V_3^0 + N_4 V_4^0, \quad (1)$$

$$\chi = N_3 \chi_3^0 + N_4 \chi_4^0. \quad (2)$$

In order to obtain expressions for the partial vapour pressures over the liquids Heer and Daunt have assumed that the Gibbs energy

$$G = G_3 + G_4 = F_3 + F_4 + (p_3 + p_4) (N_3 V_3^0 + N_4 V_4^0)$$

can, for sufficiently small pressures, be replaced by the Helmholtz free energy  $F$ , i. e., that

$$G \simeq F = F_3 + F_4. \quad (3)$$

Using this assumption Heer and Daunt have carried out detailed calculations for the total vapour pressure  $(p_3 + p_4)$  over 20 per cent mixture of  $\text{He}^3$  in  $\text{He}^4$  in the case when the former is regarded as essentially Boltzmannian. It is found that there is good agreement between their theoretical predictions and the observations of Weinstock et al<sup>2)</sup> for 20.3 per cent mixture.

However, it is important to note that Heer and Daunt's "model solutions" are actually mixtures of ideal (Bose or Fermi) gases (except for the constant potential wells) and therefore their assumption (3) must be regarded with a due amount of reserve. Recently, we<sup>3)</sup> calculated values of  $(p_3 + p_4)$  from expressions derived directly from the Gibbs energies and found that the results were everywhere considerably higher than those reported by Heer and Daunt. Hence the good agreement obtained by them is seemingly more apparent than real.

Here we propose to consider a slightly different model for the mixture. We assume the mixture to be a single system in a smoothed potential well  $-\chi_m$ , where  $\chi_m$  is given by

$$\chi_m = X_3 \chi_3^0 + X_4 \chi_4^0 + X_3 X_4 \chi_{34}, \quad (4)$$



where  $X_3$  and  $X_4$  are the molar fractions and, as before,  $-\chi_3^0$  and  $-\chi_4^0$  are the potential wells for the pure liquids. Thus we have included in the potential for the mixture a contribution due to the interaction between the Bose and Fermi ( $\text{He}^4$  and  $\text{He}^3$ ) particles. Next, we assume that despite this interaction the liquids form statistically independent systems in the mixture. This may be expected since the form of the interaction term is, clearly, analogous to the random mixing term in the theory of regular solutions<sup>(1)</sup>. However this analogy may not be regarded as a proof of eq. (4).

Now using eq. (4) we obtain, from the Gibbs energies, the following expressions for the partial vapour pressures (in terms of the saturated pressures).

(a) *Below the  $\lambda$ -temperature:*

$$\frac{p_4}{p_4^0} = \exp \left[ -\frac{V_4^0}{V_3^0} X_3^1 - X_3^2 \frac{\chi_{34}}{kT} \right], \quad (5)$$

$$\frac{p_3}{p_3^0} = X_3^1 \exp \left[ X_4^1 - X_4^2 \frac{\chi_{34}}{kT} \right]. \quad (6)$$

(b) *Above the  $\lambda$ -temperature:*

$$\begin{aligned} \frac{p_4}{p_4^0} = X_4^1 \exp \left[ \left( 1 - \frac{V_4^0}{V_3^0} \right) X_3^1 - 0.24 \left( 1 + X_4' - \frac{1}{X_4'} \right) \left( \frac{T_\lambda}{T} \right)^{3/2} - \right. \\ \left. - 0.033 \left( 1 + 2X_3' - \frac{1}{(X_4')^2} \right) \left( \frac{T_\lambda}{T} \right)^3 - \dots - X_3^2 \frac{\chi_{34}}{kT} \right], \quad (7) \end{aligned}$$

$$\begin{aligned} \frac{p_3}{p_3^0} = X_3' \exp \left[ \left( 1 - \frac{V_3^0}{V_4^0} \right) X_4' + X_4' \left\{ 0.924 \left( \frac{T_\lambda}{T} \right)^{3/2} + 0.066 \left( \frac{T_\lambda}{T} \right)^3 + \dots \right\} - \right. \\ \left. - X_4^2 \frac{\chi_{34}}{kT} \right]. \quad (8) \end{aligned}$$

It may be noted that we have, following Heer and Daunt, considered the Fermi liquid to be essentially Boltzmannian. Using the observed values<sup>(5)</sup> of the saturated vapour pressures we have computed the total pressure  $P$  ( $p_3 + p_4$ ) over 20 per cent mixture from eqs. (5), (6), (7) and (8). The molar volumes, as usual were taken to be 27.6 c.c. and 37.6 c.c. for  $\text{He}^4$  and  $\text{He}^3$ , respectively. The results are plotted in Fig. 1 as a function of temperature (full curve). The value of  $\chi_{34}$ ,  $1.5R$ , used for calculating the pressure is obtained by fitting up the theoretical expression with the observed pressure (276m.m.) at  $3^\circ\text{K}$ . We have also included the curve obtained from Heer and Daunt's model when assumption (3) is disregarded (dotted curve). (See reference 3). The measurements of Weinstock et al. are indicated by circles.

It is seen that the theoretical predictions agree fairly closely with the observed pressures for temperatures above the  $\lambda$ -temperature ( $\sim 1.63^\circ\text{K}$ ). Below the  $\lambda$ -temperature the theoretical results are too low. This may be due to the fact that we have neglected the temperature variation in  $\chi_{34}$ . In practice, we should expect  $\chi_{34}$  to decrease (numerically) as the absolute zero is approached. Hence, it may be expected that when the temperature

variation is included our results would come closer to the experiment. We hope to publish the details elsewhere.

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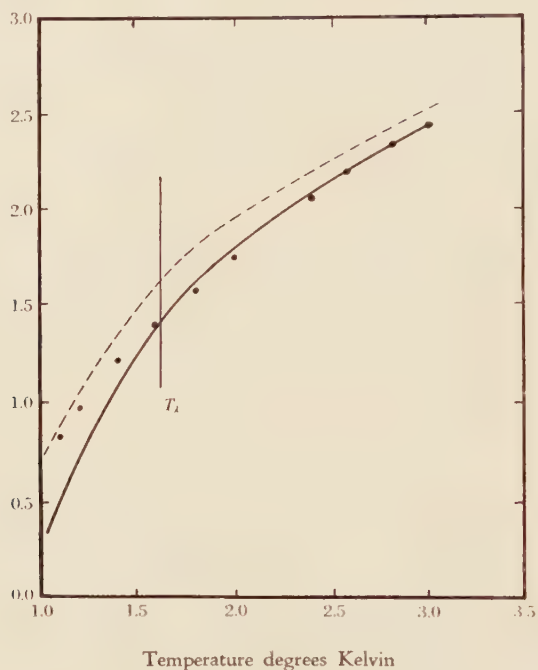


Fig. 1. Plots of the total vapour pressure over 20 per cent mixture of  $\text{He}^3$  in  $\text{He}^4$ . The full curve is computed from the expressions derived in this note and the dotted curve is computed from Heer and Daunt's model (but not assuming  $G=F$ ) (3). The circles indicate the measurements of Weinstock et al. (Reference 2).

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## On the Time Reversal in the Quantized Field Theory

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The possibility of time reversal transformation is investigated from the general viewpoint of the quantized field theory. The time reversal transformation is so defined as to preserve the canonical formalism of the theory. It is found in § 2 that there are two kinds of such transformations, which are shown to correspond to the so-called Pauli's and Wigner's transformations, respectively. In § 3 the transformation properties of physical quantities under these transformations are examined and then the physical meanings of time reversal are clarified. In § 4 some considerations are made as to whether the invariance requirement under the time reversal really imposes restrictions on the form of interaction Hamiltonian and the otherwise undetermined phase factors of the coupling constants of the Fermi and Yang-Tiomno interactions. Finally the connection between the generalized time reversal transformation and the concept of the family of elementary particles is studied. In the appendix the transformation is further applied to the Pais' theory of baryon-meson-photon system.

### § 1. Introduction

Present theory of the elementary particles is considered to serve as a mean for phenomenological approach to the general rule which governs the world of the elementary particles. Since we have no definite information on the structure of the elementary particles, it seems necessary to study how many quantities will fix the structure of an elementary particle apart from their structure. In the present theory these quantities are obtained in reference to a point particle in connection with the various invariance properties of the theory. From the relativistic invariance of the theory, the ten quantities, namely, the energy-momentum, angular momentum and center of mass (or Lorentz transformation) are found to be of definite meaning.<sup>1)</sup> Further, in this connection, the mass and spin also come out as the intrinsic quantities of the elementary particles. Empirical information on these ten quantities will be contained in the mass-spectrum and selection rules for the mutual transition of the elementary particles. The electric charge of the field appears subject to the gauge invariance of the theory. On the other hand, the requirement for the invariance of the theory under the space inversion and time reversal sometimes gives rise to restrictions on the interaction form of fields. However, in the quantized field theory, the definition of the time reversal ceases to be unique.

It is the main purpose of this paper to find how many types of transformations are possible subject to the conditions of invariance under the time reversal and to what extent the types of interactions are restricted in respective cases. In § 2 it is shown that there

exist two stand-points concerning the time reversal, namely, Pauli's<sup>2)</sup> and Wigner's<sup>3)</sup> transformations under the Hamiltonian and Schwinger's variational formalisms.<sup>4)</sup> In § 3 we investigate the physical meaning of these two transformations by giving their intuitive images. Finally in § 4 the restriction on the Boson-Fermion and Fermi interactions are studied from Wigner's transformation. We also consider briefly the connection between such transformations and the family of the elementary particles. In the appendix the transformation is further applied to the Pais' theory of  $\omega$ -space.

## § 2. The general consideration of time reversal

In this section we shall discuss the general formulation of the theory of time reversal. The following treatment can also be applied to the other transformations, i.e., space reflection, and total reflection, etc. The fundamental equations of the canonical theory are given by

$$-\partial_\mu Q_\alpha(x) = [Q_\alpha(x), T_\mu^0 Q_\alpha], \quad (2.1)$$

$$i \cdot \partial \Psi[\sigma] / \partial \sigma(x) = H(x) \Psi[\sigma], \quad (2.2)$$

$$[Q_\alpha(x), Q_\beta(x')] = i R_{\alpha\beta}(\partial) A(x-x') \quad (2.3)$$

in the interaction representation, where  $T_\mu^0 |Q_\alpha|$  is the interaction free part of the energy-momentum four vector and  $H(x)$  is the interaction Hamiltonian.  $A_{\alpha\beta}(\partial)$  is a differential operator which is defined by the following equation;

$$A_{\alpha\beta}(\partial) R_{\beta\gamma}(\partial) = (\square - \kappa^2) \delta_{\alpha\gamma}, \quad (2.4)$$

where  $A_{\alpha\beta}(\partial)$  is a differential operator (including the sign), which arises in the field equation derived by the variation of the Lagrangian, that is,

$$A_{\alpha\beta}(\partial) Q_\beta(x) = 0. \quad (2.1)'$$

The type of transformation of the time reversal is restricted by the requirement for the invariance of equations (2.1) and (2.3), while the interactions are restricted by the invariance of the Schrödinger equation (2.2).

It can be proved that the field equations (2.1)' are invariant under time reversal  $x_k \rightarrow \underline{x}_k = x_k$ ,  $x_4 \rightarrow \underline{x}_4 = -x_4$ , and

$$\begin{aligned} \psi_{\mu_1 \mu_2 \dots \mu_n}(x) &\rightarrow \psi'_{\mu_1 \mu_2 \dots \mu_n}(\underline{x}) = S \psi_{\mu_1 \mu_2 \dots \mu_n}(x), \\ S &= (-1)^m \rho \gamma_5 \gamma_4, \quad \rho = \pm 1, \pm i \end{aligned} \quad (2.5)$$

for the half integer spin field,

or

$$\begin{aligned} U_{\mu_1 \mu_2 \dots \mu_n}(x) &\rightarrow U'_{\mu_1 \mu_2 \dots \mu_n}(\underline{x}) = S U_{\mu_1 \mu_2 \dots \mu_n}(x), \\ S &= (-1)^m \rho', \quad \rho' = \pm 1 \end{aligned} \quad (2.5)'$$

for the integer spin field,

where  $m$  denotes the number of the 4th suffices in  $\mu_1, \mu_2, \dots, \mu_n$ ,  $\psi_{\mu_1\mu_2\dots\mu_n}$  describes a half integer spin field in the mixed representation, where the spinor suffix of  $\psi$  is abbreviated. In this case it is easily shown that the free part of the Lagrangian is transformed as follows,

$$\int_{-\infty}^{\infty} d^4x L^0(x) = \begin{cases} \text{odd scalar for the half integer spin field,} \\ \text{even scalar for the integer spin field.} \end{cases} \quad (2.6)^*$$

Therefore,

$$T_{\mu}^0[Q_{\alpha}] = \begin{cases} \text{even vector for h.i.s. field,} \\ \text{odd vector for i.s. field,} \end{cases} \quad (2.7)^{**}$$

due to the odd vector character of  $\int d\sigma_{\mu}$ .\*\*\*

For the complex field, the field quantities consist of  $Q_{\alpha}$  and its complex conjugate quantity  $Q_{\alpha}^+$ . Therefore, the whole of the field quantities are given by  $(Q_{\alpha}, Q_{\beta}, \dots, Q_{\alpha}^+, Q_{\beta}^+, \dots)$ . The above transformation  $S$  transforms the equation of  $Q_{\alpha}$  not to that of  $Q_{\alpha}^+$ , but to that of  $Q_{\alpha}$  itself. However, the invariance of the theory only requires that the system of equations of  $Q_{\alpha}$  and  $Q_{\alpha}^+$  are invariant as a whole.

In order to discuss generally the possibility of such a transformation, we shall consider the following most extensive transformation;

$$\begin{aligned} \Psi[\sigma] &\rightarrow \underline{\Psi}[\sigma] = \Psi^*[\sigma] \mathbf{R} + \mathbf{R}' \Psi[\sigma] \\ Q_{\alpha}(x) &\rightarrow \underline{Q}_{\alpha}(x) = U_{\alpha\beta} Q_{\beta}^{\tau}(x), \end{aligned} \quad (2.8)^{****}$$

where  $\mathbf{R}$  and  $\mathbf{R}'$  are the unitary operators to be operated on the state vector, and  $U$  is a unitary operator whose suffices  $\alpha, \beta$  are extended over the whole species of the field quantities  $(Q_{\alpha}, Q_{\beta}, \dots, Q_{\alpha}^+, Q_{\beta}^+, \dots)$  and the tensor suffices of the respective fields. In order that the phase of the state vector is not observable, either  $\mathbf{R}$  or  $\mathbf{R}'$  must vanish.

We shall consider first the case  $\mathbf{R}'=0$ .  $\mathbf{R}$  is clearly commutable with  $S$ ,  $U$ ,  $\gamma_{\mu}$ ,  $\tau$  etc., and also with the coordinates  $x_{\mu}$  and the differential operators  $\partial_{\mu}$ . According to

\* We shall call a scalar  $A$  and a vector  $A_{\mu}$  even or odd when they transform under a time reversal as follows;

$$\begin{aligned} A &\rightarrow A' = A & (\text{even}) \text{ or } & -A \text{ (odd)} \\ A_k &\rightarrow A'_k = A_k & (\text{even}) \text{ or } & -A_k \text{ (odd)} \quad (k=1, 2, 3) \\ A_4 &\rightarrow A'_4 = -A_4 & (\text{even}) \text{ or } & A_4 \text{ (odd).} \end{aligned}$$

\*\* This difference in the transformation property of  $T_{\mu}^{*0}$  leads to difficulties when we make the transformations (2.5) and (2.5)'. For instance, in the case of i.s. field the both sides of (2.1) shows the different transformation properties. On the other hand, this is not the case for h.i.s. field, but (2.3) can not retain, in this case, the original form after transformation. This is the reason why we have to make such a tremendous transformation as defined below.

\*\*\* As is well-known, the infinitesimal element of the hypersurface (three dimensional) in the four dimensional space is the volume of the infinitesimal parallelepiped, and the latter is represented by the anti-symmetric tensor  $dS_{\lambda\mu\nu}$ .  $d\sigma_{\mu}$  is a dual four vector to it, and is given by  $d\sigma_{\mu} = 1/6 \epsilon_{\mu\nu\lambda\kappa} dS_{\nu\lambda\kappa}$ . Therefore,  $d\sigma_{\mu}$  is an odd vector.

\*\*\*\* The matrix with superscript  $T$  indicates the transposed matrix hereafter.



(2.8), the matrix element of any function  $F(Q)$  is rewritten as follows:

$$(\Psi[\sigma_2], F(Q) \Psi[\sigma_1]) = (\Psi[\sigma_1], (\mathbf{R}^{-1} F(U^{-1} \underline{Q}^T(x)) \mathbf{R})^T \Psi[\sigma_2]). \quad (2.9)$$

If we call the worlds before and after the transformation the reciprocal worlds each other, the  $\sigma_1 \rightarrow \sigma_2$  transition corresponds to the  $\sigma_2 \rightarrow \sigma_1$  transition in the reciprocal world.

Now, we shall assume\* that the field quantities are transformed linearly,

$$\mathbf{R}^{-1} \underline{Q}^T(x) \mathbf{R} = \underline{Q}^T(x) \quad (2.10)$$

$$\text{or} \quad U^{-1} \mathbf{R}^{-1} \underline{Q}^T(x) \mathbf{R} = U^{-1} \underline{Q}^T(x), \quad (2.11)$$

where the suffices of the operator  $R$ , like as the operator  $U$ , are generally extended over whole species of field quantities  $(Q_\alpha, Q_1, \dots, Q_\alpha^\dagger, Q_1^\dagger, \dots)$  and the tensor suffices of the respective fields.

The equations (2.9) and (2.10) show that the operator  $U$  appears always in a form  $U^{-1}R$ . We can, therefore, set especially  $U \equiv S$ , because either  $U$  or  $R$  may be chosen arbitrarily.

Rewriting the canonical equation (2.10), we have the following equation

$$-\partial_\mu Q_\alpha'^T(x) = \sigma [Q_\alpha'^T(x), (T_\mu^0[RQ'(x)])^T], \quad (2.12)$$

$$\sigma = \begin{cases} -1 & \text{for h.i.s. field} \\ +1 & \text{for i.s. field,} \end{cases} \quad (2.13)$$

where we have used the relation (2.7).

In order that canonical equation (2.1) is invariant, it is necessary and sufficient that the equation (2.12) is identical with the equation

$$-\partial_\mu Q_\alpha'^T(x) = [Q_\alpha'^T(x), T_\mu^0[Q'^T(x)]], \quad (2.14)$$

because  $Q_\alpha'^T(x) (= Q_\alpha(x))$  is the field quantity in the new coordinate system. On the other hand,  $T_\mu^0[Q]$  can generally be written in the following form,

$$T_\mu^0[Q] = \int Q_\alpha^\dagger(x) \mathcal{Q}_{\alpha\beta}^\mu(\partial) Q_\beta(x) d^3x, \quad (2.15)$$

where  $\mathcal{Q}_{\alpha\beta}^\mu(\partial)$  is an appropriate differential operator.

Therefore, the equations (2.12) and (2.14) are identical with each other provided that

$$Q'(x) (R^{-1} \mathcal{Q}^\mu(\partial) R)^T Q'^\dagger(x) = \sigma Q'^\dagger(x) \mathcal{Q}^\mu(\partial) Q'(x). \quad (2.16)$$

We can separate solutions of (2.16) into the following two types.

[ $T_1$ ]: Assuming the commutation relations of the positive and negative types for the half integer spin field, respectively, we can determine the matrix  $R$  by the equation

\* This requirement means physically that transition from a state of the occupation number  $m$  to that of occupation number  $n$  corresponds to the transition from the state of  $n$  to that of  $m$  in the reciprocal world.

$$R^{-1}\underline{\mathcal{Q}}^\mu(\underline{\partial})R=\underline{\mathcal{Q}}^\mu(\underline{\partial}). \quad (2.17)$$

Apart from the constant factor,  $R=1$  is a solution of (2.17).<sup>\*</sup> However, it is not unique solution. For instance, it is easily shown that  $R=\gamma_5$  also satisfies the equation (2.17) for the field with vanishing mass and spin one half.<sup>\*\*</sup>

$[T_2]$ : Now we shall replace the matrix  $R$  by  $C$  defined by

$$RQ'(x)=Q'^{\dagger}(x)C^{-1}. \quad (2.18)$$

Then the condition (2.16) is replaced by

$$C^{-1}\underline{\mathcal{Q}}^{\mu T}(\underline{\partial})C=\sigma\underline{\mathcal{Q}}^\mu(\underline{\partial}). \quad (2.19)$$

We can obtain this equation without assuming the commutation relation. Since, as is well-known,  $\underline{\mathcal{Q}}^{\mu T}(\underline{\partial})=\underline{\mathcal{Q}}^\mu(\underline{\partial})$  holds for the integer spin field,  $C=1$  is a unique solution of (2.19), that is,

$$RQ'(x)=Q'^{\dagger}(x). \quad (2.18)'$$

We can express this transformation concretely as follows,

$$\begin{aligned} U_{\mu_1\mu_2\cdots\mu_n} &\rightarrow U_{\mu_1\mu_2\cdots\mu_n}^{\dagger}, \\ U_{\mu_1\mu_2\cdots\mu_n}^{\dagger} &\rightarrow U_{\mu_1\mu_2\cdots\mu_n}. \end{aligned} \quad (2.20)$$

This transformation coincides with charge conjugation of the Bose field.

For the half integer spin field, it is easily shown that the charge conjugation matrix is a solution of (2.17). For example, in the case of the field with spin one half,  $\underline{\mathcal{Q}}^4$  is given by  $\underline{\mathcal{Q}}^4=\gamma_4\{\gamma_k\partial_k+\kappa\}$ ,<sup>\*\*\*</sup> we have

$$C^{-1}\gamma_4^T C=-\gamma_4, \quad C^{-1}\gamma_k^T C=\gamma_k \quad (2.21)$$

according to (2.19).

It is easily shown that the matrix  $C$  which is determined by (2.21) satisfies the equation (2.19) also for  $k=1, 2, 3$ . The equation (2.21) is a well-known relation for the matrix of the charge conjugation. That there exist no other possibilities for the time reversal than  $[T_1]$  and  $[T_2]$ , can be understood from the fact that the field quantities before and after the time reversal can not give such a correspondence to each other that

$$Q \sim aQ' + bQ'^{\dagger}, \quad (a \neq 0, b \neq 0).$$

The cases  $a=0$  and  $b=0$  correspond to  $[T_1]$  and  $[T_2]$ , respectively.

\* In general, we can choose  $R=\rho$  ( $|\rho|=1$ ) for the complex field and  $R=\pm 1$  for the real field. However, these factors can be absorbed into the constant factors  $\rho$  and  $\rho'$  in the equations (2.5) and (2.5)'. See § 3.

\*\* It can be shown that  $R=1$  is a unique solution for the field with nonvanishing mass, when we do not introduce other transformations than the Lorentz transformations.

\*\*\*  $\mu=(1, 2, 3, 4)$ ,  $k=(1, 2, 3)$

For the differential operator  $A(\partial)$  in the equation of motion derived from the free Lagrangian, we can prove the following relations:

$$R^{-1}A(\partial)R = A(\partial), \quad \text{for } [T_1], \quad (2.22)$$

$$C^{-1}A^T(\partial)C = \sigma A(\partial), \quad \text{for } [T_2], \quad (2.23)$$

$$S^{-1}A(\partial)S = \sigma A(\partial). \quad (2.24)$$

It can be shown that the free Lagrangians in the representation appropriate to  $\Psi$  and  $\bar{\Psi}$ , respectively are related to each other as even scalar, irrespective of their spin, under the time reversal defined above, according to (2.6), (2.9), (2.11) and (2.22) – (2.24).

Using (2.4), (2.22) and (2.23), we have

$$R^{-1}R(\partial)R = R(\partial), \quad \text{for } [T_1], \quad (2.25)$$

$$C^{-1}R^T(\partial)C = \sigma R(\partial), \quad \text{for } [T_2], \quad (2.26)$$

$$S^{-1}R(\partial)S = \sigma Q(\partial). \quad (2.27)$$

Considering these relations and (2.9), we can prove the invariance of commutation relation. In this proof, the definite type of the commutation relation (i.e. + or – types for fields of half integer or integer spin, respectively) is used in the case  $[T_1]$ . Distinguishing  $R$  and  $R'$  for  $[T_1]$  and  $[T_2]$  by suffices 1 and 2, respectively, we shall define  $R'$  by

$$R'^{-1}(R_1^{-1}Q'(x)R_1)R' = R_2^{-1}Q'(x)R_2,$$

in order to investigate the relation between  $R_1$  and  $R_2$ , that is,  $[T_1]$  and  $[T_2]$ . Setting  $R_1 = 1$ , we obtain

$$R'^{-1}Q'(x)R' = R_2Q'(x) = Q'(x)C^{-1}.$$

That is,  $R'$  is just a unitary operator of the charge conjugation. Therefore  $R'$  is equivalent to the product of the transformation  $[T_1]$  and the charge conjugation.\*

$[T_1]$  is the so-called Pauli's transformation,<sup>2)</sup> and also corresponds to the transformation used by J. Schwinger.<sup>4)</sup>  $[T_2]$  is the so-called Wigner's transformation.<sup>10)</sup>

It is clear that the above discussion holds even for the case where the matrix  $R$  involves the unitary matrix of other transformations than the Lorentz transformation. For instance, in the theory of the nucleon and meson system,  $R$  may involve the unitary operator in the  $\tau$ -space. In this manner, there may exist various types of the time reversal, which we shall discuss later in detail.

The various transformations restrict the type of the interaction in their different ways. In fact, the restriction on the interaction Hamiltonian is derived from the requirement of the invariance of equation (2.2). From the equation (2.2), we have

\* It can be also shown that this  $R'$  is equivalent to unitary operator for the charge conjugation given by Wolfenstein and Ravenhall. L. Wolfenstein and D. G. Ravenhall, Phys. Rev. **88** (1952), 279.

$$i\partial\Psi[\sigma]/\partial\sigma(\underline{x}) = (\mathbf{R}^{-1}H(\underline{x})\mathbf{R})^T\Psi[\sigma]. \quad (2.28)$$

Therefore, the condition that the Schrödinger equation (2.2) is invariant under the time reversal is given by

$$H[Q'_\alpha(\underline{x})] = (\mathbf{R}^{-1}H[Q_\alpha(\underline{x})]\mathbf{R})^T. \quad (2.29)$$

The transformation corresponding to  $\mathbf{R}=0$ ,  $\mathbf{R}' \neq 0$  in (2.8) is excluded by the following physical reason. In this case, starting from the equations (2.9) and (2.10) without their transpose suffices, we can derive the condition for the invariance of the canonical equations (2.1)—(2.3) through the same method above used. As a result, we find that the free energy operators in the representation corresponding to  $\Psi[\sigma]$  and  $\bar{\Psi}[\sigma]$ , respectively, have opposite sign. Therefore, in order that the eigenvalue of the energy remains positive and definite, it is necessary to invert the sign of the commutation relation after the transformation. However, calculation shows that the commutation relation remains invariant. Thus the inversion of the sign of the energy in the reciprocal world becomes unavoidable. Therefore we shall exclude the transformation  $\mathbf{R}=0$  and  $\mathbf{R}' \neq 0$  for physical reasons. We shall refer to this point again at the end of this section.

Now, we shall repeat briefly the above discussion using the Schwinger's variational formalism.<sup>4)</sup> In the following discussion we use the Heisenberg representation.

The fundamental equation in the Schwinger's formalism is

$$\delta(\zeta'_1\sigma_1|\zeta''_2\sigma_2) = i(\zeta'_1\sigma_1|\delta\int_{\sigma_2}^{\sigma_1} dx^4 L(Q(\underline{x}))|\zeta''_2\sigma_2). \quad (2.30)$$

Transforming the state vector by (2.8)  $\mathbf{R} \neq 0$  and  $\mathbf{R}'=0$ , and setting  $\underline{\zeta} = (\mathbf{R}^{-1}\underline{\zeta}\mathbf{R})^T$ , we can replace the equation (2.30) by

$$\delta(\zeta''_2\sigma_2|\zeta'_1\sigma_1) = i(\zeta''_2\sigma_2|\delta\{[\mathbf{R}^{-1}L(S^{-1}Q'(\underline{x}))\mathbf{R}]^T dx^4|\zeta'_1\sigma_1\}). \quad (2.31)$$

Therefore, in order that the theory is invariant under the time reversal, it is necessary and sufficient that the next equation is satisfied,

$$[\mathbf{R}^{-1}L(S^{-1}Q'(\underline{x}))\mathbf{R}]^T = L(\dot{Q}'^T(\underline{x})), \quad (2.32)$$

if we regard  $Q'^T(\underline{x})$  as the field quantity in the new coordinate system. Since the transformation  $\mathbf{R}$  should transform the free and the interaction Lagrangian to the free and the interaction part in the new coordinate system, respectively, we must first determine the operator  $\mathbf{R}$  in such a way that the free part of the Lagrangian is invariant and then determine the interaction form of Lagrangian to be invariant under this given transformation  $\mathbf{R}$ . As an example, we consider the free field with spin one half in detail. Since we can take  $\psi$  and  $\bar{\psi}$  as the independent variables, the matrix  $\mathbf{R}$  which satisfies the above condition is determined by considering either of the following two equations:

$$\mathbf{R}_1^{-1}\psi'(\underline{x})\mathbf{R}_1 = R_1\psi'(\underline{x}), \quad (2.33)$$

$$\mathbf{R}_2^{-1}\psi'(\underline{x})\mathbf{R}_2 = R_2\psi'^{\dagger}(\underline{x}). \quad (2.34)$$



After the simple algebraic calculation, we obtain

$$R_1=1, \quad (2.35)$$

$$R_2=(C^{-1})^T, \quad (2.36)$$

and therefore these two cases correspond to the previous  $[T_1]$  and  $[T_2]$ .

Finally, we shall consider the case  $R=0$  and  $R' \neq 0$ . As previously related, this transformation gives rise to the difficulty concerning to the transformation of the energy. This difficulty arises from the fact that  $R'$  is no longer commutable with the differential operator  $\partial_\mu$ . However, it is shown that if we take as the field quantities at the reciprocal world points\* instead of those at the same world points as in the previous case, then  $R'$  becomes commutable with  $\partial_\mu$  and we can construct the consistent theory, and that the invariant system under this transformation satisfies the condition of the reversibility introduced by S. Watanabe.<sup>(6)</sup>

In this case, the transformation of the state vector  $\Psi[\sigma]$ , the field operator  $Q_\alpha(x)$  and the observable  $\zeta[\sigma]$  is given by

$$\Psi[\sigma] \rightarrow \underline{\Psi}[\underline{\sigma}] = R' \Psi[\sigma], \quad (2.37)$$

$$Q_\alpha(x) \rightarrow \underline{Q}_\alpha(x) = R' Q_\alpha(x) R'^{-1}, \quad (2.38)$$

$$\zeta[\sigma] \rightarrow \underline{\zeta}[\underline{\sigma}] = R' \zeta[\sigma] R'^{-1}, \quad (2.39)$$

where  $\underline{\sigma}$  is a reciprocal surface with  $\sigma$  as shown by Fig. 1.

Using these transformations the fundamental equation (2.30) can be rewritten in the variables in the new coordinate system as follows; after replacing the integral variable in the right hand side by the new coordinate variable, we have the result,

$$\delta(\zeta'_1 \sigma_1 | \zeta''_2 \sigma_2) = i \left( \zeta'_2 \sigma_1 | \partial \int_{\sigma_2}^{\sigma_1} L(Q(x), \partial_\mu Q(x)) dx^\mu | \zeta''_2 \sigma_2 \right). \quad (2.30)'$$

Comparing the equations (2.30) and (2.30)', we find that the fundamental equation is formally invariant under the transformations (2.37)–(2.39) apart from the four dimensional region of integration. However, since the fundamental equation of the Schwinger's formalism consists of the equation (2.30) for the whole of the arbitrary set  $(\sigma_1, \sigma_2)$ , the fundamental equation is completely invariant as a whole.

To complete the discussion of the invariance of this type, it is further necessary to require that the equations of motion etc., derived from the new Lagrangian

$$\underline{L} = L(\underline{Q}(x), \partial_\mu \underline{Q}(x)) \quad (2.40)$$

\* We shall call the world points  $P_1$  and  $P_2$  the reciprocal world points of each other, when the coordinate of  $P_1$  in the old coordinate system is numerically equal to that of  $P_2$  in the new coordinate system and vice versa.

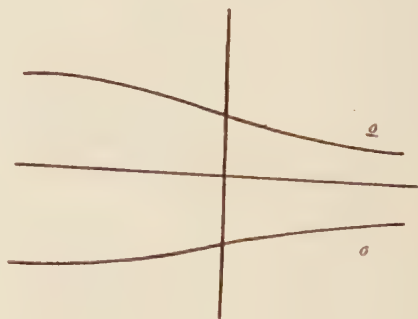


Fig. 1



are compatible with those derived from  $L$ . This is satisfied if the following relation holds ;

$$\underline{L} = I^T(Q(x), \partial_\mu Q(x)). \quad (2.41)$$

According to (2.40) and (2.41), we find that  $Q(x)$  is generally given by

$$\begin{aligned} \underline{Q}_\alpha(x) &= \mathbf{R}' Q_\alpha(x) \mathbf{R}'^{-1} = R'_{\alpha\beta} Q'_\beta(x) \\ &= (R' S^{-1})_{\alpha\beta} Q'^T_\beta(x), \end{aligned} \quad (2.42)$$

where the operator  $R'$  is of the same kind as  $R$  used in (2.10). By (2.42), the above condition (2.41) for the free Lagrangian part

$$\begin{aligned} L^0 &= Q^\dagger(x) \Lambda(\partial) Q(x) \text{ is rewritten by} \\ \{Q'^{T\dagger}(x) (R' S^{-1})^\dagger \Lambda(\partial) (R' S^{-1}) Q'^T(x)\}^T \\ &= Q^\dagger(x) \Lambda(\partial) Q(x). \end{aligned} \quad (2.43)$$

The determination of the matrix  $R'$  is done, separating the following two cases.

$[T_1]$ : Using the commutation relation of the positive and negative types for the half integer and integer spin field, respectively, we can rewrite (2.43) as

$$\begin{aligned} Q'^{T\dagger}(x) (R' S^{-1})^\dagger \Lambda(\partial) (R' S^{-1}) Q'^T(x) \\ = Q'^{T\dagger}(x) \Lambda(\partial) Q'^T(x). \end{aligned}$$

The solution is given by

$$R' = S. \quad (2.44)$$

From (2.42), we have

$$\mathbf{R}' Q(x) \mathbf{R}'^{-1} = S Q^T(x). \quad (2.45)$$

$[T_2]'$ : Now we introduce the matrix  $C'$  defined by

$$R' S^{-1} Q'^T(x) = Q'^{T\dagger}(x) C'^{-1}. \quad (2.46)$$

Using (2.46), we transform (2.43) into

$$\begin{aligned} Q'^{T\dagger}(x) C'^{-1} \Lambda(\partial) C' Q'^T(x) \\ = \sigma Q'^{T\dagger}(x) \Lambda(\partial) Q'^T(x). \end{aligned} \quad (2.47)$$

Therefore, the matrix  $C'$  which satisfies the following relation

$$C'^{-1} \Lambda(\partial) C' = \sigma \Lambda(\partial) \quad (2.48)$$

becomes a solution of (2.47). From (2.23) and (2.48), we find that  $C'$  is equivalent to  $C$  apart from a constant factor. By use of (2.42), we have

$$\mathbf{R}' Q(x) \mathbf{R}'^{-1} = Q'^{T\dagger}(x) S^{-1} C'^{-1}. \quad (2.49)$$

It will be noticed that the transformation equations (2.45) and (2.49) are completely identical with that given by S. Watanabe.<sup>6)</sup> Now we shall explain their connection in more detail. We assume that the system is invariant under the time reversal (2.37) — (2.39). If  $\Psi(t)$  is a solution of the Schrödinger equation

$$i\partial\Psi(t)/\partial t = H(t)\Psi(t), \quad (2.50)$$

$\Psi'(t)$  which is defined by

$$\Psi'(t) = \Psi^*(-t) \mathbf{R}'^{-1}$$

is also a solution of (2.50), provided that  $H(t)$  satisfies the following relation

$$\mathbf{R}' H(t) \mathbf{R}'^{-1} = H^T(-t). \quad (2.51)$$

However, (2.51) is just a result of (2.39), that is, the condition for the invariance under the time reversal. Thus, we can conclude that the system which is invariant under the time reversal (2.37) — (2.39) has a property of the reversibility.

In conclusion, we shall discuss briefly the space inversion. As mentioned above, that there exist two stand points  $[T_1]$  and  $[T_2]$  for the time reversal is due to the fact that the theory without interaction is invariant under the charge conjugation. Thus, for the space inversion there arise two stand-points which are connected with each other by the charge conjugation. We shall call these  $[L_1]$  and  $[L_2]$ . For instance, in the case of spin one half, the transformation operator is given by  $\rho\gamma_1$  and  $\rho\gamma_1 C$  in  $[L_1]$  and  $[L_2]$ , respectively. In this case, the stand point  $[L_2]$  imposes the restriction on the commutation relation as  $[T_1]$  does in the time reversal.

The fact that only one of the two stand point in the space or time inversion imposes the restriction on the commutation relation of the field quantities shows that it is the charge conjugation what essentially imposes the restriction on the commutation relation.\*

### § 3. The transformation property of the physical quantities

Since the explicit form and the physical meaning of the transformation operator  $\mathbf{R}'$  regarding the time reversal (2.37) — (2.39) has been investigated by S. Watanabe in detail, we shall study exclusively the case (2.8)  $\mathbf{R} \neq 0$  and  $\mathbf{R}' = 0$ . By studying the nonvanishing matrix element of  $\mathbf{R}$ , we can obtain the correspondence between the reciprocal worlds. The explicit form of  $\mathbf{R}_1$  and  $\mathbf{R}_2$  can be determined by expanding (2.33) and (2.34) in the Fourier series and comparing the both side.

As mentioned in the previous section,  $\mathbf{R}_1$  and  $\mathbf{R}_2$  have an arbitrary factor produced by the unitary transformation which changes the arbitrary phase factor in the right hand side of the equations (2.17), (2.19), (2.39) and (2.36). This arbitrary factor comes from a different origin from that of the well-known  $\rho$  and  $\rho'$  which appear in (2.5) and

\* Therefore, if the theory was invariant under the charge conjugation, two stand points  $[T_1]$  and  $[T_2]$  are equivalent each other. However, we can make interactions which are not invariant under the charge conjugation but invariant under one of two transformations  $[T_1]$  and  $[T_2]$ .

(2.5)'. The unitary operators which change the phase factor of the field operator by an arbitrary phase  $\varphi$  are given by

$$\begin{aligned} & \exp [i\varphi \cdot i \int (\partial^* / \partial x_\mu U - \partial U / \partial x_\mu U^*) d\sigma_\mu], \\ & \exp [i\varphi \cdot i \int \bar{\psi} \gamma_\mu \psi d\sigma_\mu] \end{aligned} \quad (3.1)$$

for spin zero and one half, respectively. These operators have generally the form of  $\exp [i\varphi \cdot \text{charge}]$ , that is, the gauge transformation of the first kind.\* Especially for the real field, the phase is restricted to  $\pm 1$  only. Thus, for this case of  $-1$ , we need only to multiply the previous result for  $\mathbf{R}$  by the factor

$$(-1)^{\text{number operator}}. \quad (3.2)$$

Therefore, if we assign to the respective fields an intrinsic "charge" a new kind of the charge conservation may be obtained. We shall discuss this problem in the next section.

We shall now consider transformation properties of the relevant physical quantities  $F[Q(x)]$ . The physical quantity in the reciprocal world corresponding to  $F[Q(x)]$  is  $F[Q^T(x)]$ . Between both quantities there exists in general the following relation,

$$F[Q^T(x)] = \pm (\mathbf{R}^{-1} F[Q(x)] \mathbf{R})^T. \quad (3.3)$$

If  $F[Q(x)]$  has an eigenvalue  $f$ , that is,

$$F[Q(x)] \Psi[\sigma] = f \Psi[\sigma]. \quad (3.4)$$

then  $F[Q^T(x)]$  has the eigenvalue  $\pm f$  with the corresponding sign to that of (3.3) and its eigenvector  $\underline{\Psi}[\sigma]$  which is connected with  $\Psi[\sigma]$  by (2.8), that is,

$$F[Q^T(x)] \underline{\Psi}[\sigma] = \pm f \underline{\Psi}[\sigma]. \quad (3.3)'$$

We shall call the physical quantities a plus and minus quantity or even and odd tensor (in taking into consideration the tensor character) corresponding to the sign which appears in (3.3) and (3.4)'.

For instance, the energy-momentum tensor  $T_{\mu\nu}$  is an even tensor since the Lagrangian  $L$  is an even scalar in this sense. On the other hand, the energy momentum four vector is an odd vector, because  $d\sigma_\mu$  is an odd vector. Therefore, in the reciprocal world, the sign of the momentum is inverted and that of the energy leaves invariant. For the angular momentum  $J_{\mu\nu} = \int d\sigma_\lambda M_{\lambda\mu\nu}$ , the same argument holds, that is, the direction of the spin is inverted in the reciprocal world. The transformation properties of the various physical quantities are shown in Table I and II. In these Tables, it should be noted that if the two spinors in the bilinear form belong to the different fields, there arise differences of their phase after the time reversal and further that each term changes into the corresponding Hermite conjugate quantity after the transformation  $[T_2]$ .

\* This transformation is also possible for the neutral field, provided that it is represented by the complex field variables.

From Table II, we find that there exists the following correspondence between the reciprocal worlds. Let us consider the state  $\mathcal{U}[\sigma]$ , where there exists one electron with the momentum  $\mathbf{k}$  and spin  $\sigma$ . Then, in the corresponding state in the reciprocal world, there exists one positron with the momentum  $-\mathbf{k}$ , spin  $-\sigma$  if we take the stand point  $[T_1]$ , while there exists one electron in the same state, if we take the stand point  $[T_2]$ .

The distinction of the plus or minus quantity plays an important role for the consideration of the detailed

Table I

spin 1/2 field	$[T_1]$	$[T_2]$
$\bar{\psi}\psi$	+	+
$\bar{\psi}\gamma_\mu\psi$	$\frac{k}{4}$	$\pm$
$\bar{\psi}\gamma_\mu\gamma_\nu\psi$	$\frac{ik}{4i}$	$\pm$
$\bar{\psi}\gamma_5\psi$	-	-
$\bar{\psi}\gamma_5\gamma_\mu\psi$	$\frac{k}{4}$	$\mp$
$\bar{\psi}\gamma_5\gamma_\mu\gamma_\nu\psi$	$\frac{ik}{4i}$	$\mp$

Table II

	$[T_1]$	$[T_2]$
charge (total, density)	-	+
current density	+	-
momentum (tot., dens.)	-	-
spin orbital angular momentum (tot., dens.)	-	-
energy (tot., dens.)	+	+
vector potential (electromagnetic)	+	-
scalar potential (electromagnetic)	-	+

balance. In the theory of the detailed balance the transition probability of the process  $A \rightarrow B$  does not coincide with that of  $B \rightarrow A$ , but of  $\underline{B} \rightarrow \underline{A}$ , where  $\underline{A}$  and  $\underline{B}$  are the corresponding states in the reciprocal world to the states  $A$  and  $B$ , respectively.\*

§ 4. The restriction of the interaction and the family of the elementary particles

In this section we shall study to what extent the forms of the intereaction are restricted by the requirement of the invariance under the time reversal  $[T_1]$  and  $[T_2]$ .

From the time reversal  $[T_1]$ , we can not obtain any new restriction. We find that the requirement of the invariance under the time reversal  $[T_1]$  coincides with that for the limited Lorentz transformation. On the other hand, the time reversal  $[T_2]$  leads to new conditions. For instance, let us consider the interaction of the neutral field and a spinor field. Since the transformation  $[T_2]$  is done by the unitary operator  $R$ , the coupling constant remains invariant. From Table I, it is easily shown in  $[T_2]$  (not in  $[T_1]$ !) that the coexistence of the following interactions is forbidden; that is, "scalar coupling and vector coupling" for the neutral scalar field, "pseudovector coupling and pseudotensor coupling" for the neutral pseudovector field.

\* From the invariance requirement under time reversal transformation the principle of detailed balance can only be derived if we further apply the hypothesis of molecular chaos (or symmetry of space-time) in connection with the odd physical quantities. In this respect, therefore,  $[T_2]$  is more suitable than  $[T_1]$ . As to this point see further F. Coester, Phys. Rev. 84 (1951), 1259 and S. Watanabe (unpublished result).



This results have been ever pointed out by G. Lüders.<sup>7)</sup>

However, this restriction is redundant, when the source density has been bilinear form of spinors which belong to the different fields. Because, the coupling constant may be complex for this case and the above requirement can be satisfied by choosing its phase appropriately. In fact, let us consider the following interaction

$$f \bar{\psi}^{\alpha} \Omega \psi^{\beta} \phi \pm f^{*} \bar{\psi}^{\beta} \Omega \psi^{\alpha} \phi^{*}, \quad (4.1)$$

where  $\pm$  signs of the second term depend on  $\Omega$ , that is,  $+$  for  $\Omega=S$ ;  $-$  for  $\Omega=V, T, P_s, P_v$ . We express the phase factors as  $\rho_{\alpha}, \rho_{\beta}, \rho$  which arise from  $\psi^{\alpha}, \psi^{\beta}, \phi$  under the time reversal  $[T_2]$ . Then, the invariance condition under  $[T_2]$  is given by

$$\pm f \rho_{\alpha}^{*} \rho_{\beta} \rho = f^{*}, \quad (4.2)$$

where  $\pm$  signs are the product of the sign of the second term in (4.1) and that due to  $[T_2]$  given by Table I in § 3, that is;  $+$  for  $\Omega=P_s, P_v$ ;  $-$  for  $\Omega=V, T, S$ . Now (4.2) determines only the relation between  $f$  and  $\rho$ .

Secondly we shall consider the direct interaction of the Fermi particles. The invariance condition is given by the following equation for the even interaction (the conventional Fermi-interaction),

$$g \rho_N \rho_F^{*} \rho_e^{*} \rho_v = g^{*}. \quad (4.3)$$

Since the interaction Hamiltonian in this case is bilinear with respect to  $\bar{\psi} \Omega \psi$ ,  $\pm$  sign does not appear in (4.3) in contradistinction to (4.2).<sup>8)</sup> Thus, for the odd interaction (Yang-Tiomno interaction)<sup>9)</sup>, the appearance of the sign factor dependent on the interaction type may be expected. However, we have only the same condition (4.3) even in this case, because the two sign factors which arise from the Hermitian conjugate term and from the time reversal  $[T_2]$  cancel each other. Therefore, from the requirement of the time reversal  $[T_2]$ , the phase of  $g$  is restricted to the straight lines shown in Fig. 2, corresponding to the  $\rho$ -values. Thus, unfortunately, it is impossible to obtain the different relative phases for the five interactions, although this seems to be interesting for the theory of the beta-decays.

Finally we shall consider the charge independent interaction of the nucleon  $\psi$  and meson field  $\phi_x$ . This interaction is invariant under the following transformation;

$$\phi \rightarrow \phi' = R(\alpha, \mathbf{n}) \phi, \quad (4.4)$$

$$\phi \rightarrow \phi' = A(\alpha, \mathbf{n}) \phi, \quad (4.5)$$

$$R(\alpha, \mathbf{n}) \tau_k R^{-1}(\alpha, \mathbf{n}) = a_{ik} \tau_i, \quad (4.6)$$

where  $R(\alpha, \mathbf{n})$  expresses the rotation around a unit vector by an angle  $\alpha$  for the spinor in the  $\tau$ -space, and  $A(\alpha, \mathbf{n}) = (a_{ik})$  is the corresponding matrix of the coordinate

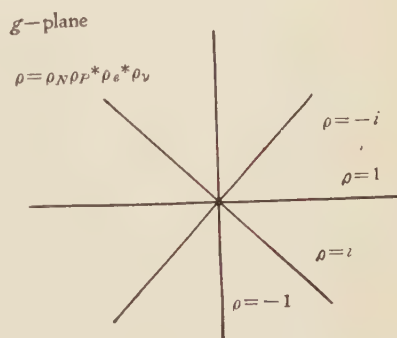


Fig. 2



transformation in this space.

From (4.6), we have

$$\begin{aligned} R(\alpha, \mathbf{n}) &= \exp[-i\alpha/2 \cdot \boldsymbol{\tau} \cdot \mathbf{n}] \\ &= \cos \alpha/2 + i\boldsymbol{\tau} \cdot \mathbf{n} \sin \alpha/2. \end{aligned} \quad (4.7)$$

This charge independent meson-nucleon interaction is invariant under  $[Z_2]$ , because the fields  $\phi_\alpha$  are asymmetrically transformed as  $\phi_{1,3} \rightarrow \rho\phi_{1,3}$ ,  $\phi_2 \rightarrow -\rho\phi_2$  while the transformation of the nucleon source is also  $\tau_{1,3} \rightarrow \tau_{1,3}$ ,  $\tau_2 \rightarrow -\tau_2^*$ . Furthermore, it is invariant, even if we replace the matrix  $R$  given by (2.35) or (2.36) by the following  $R_\psi$  and  $R_\phi$ , that is;

$$\left. \begin{aligned} R_\psi &= R \cdot R(\alpha, \mathbf{n}) \\ R_\phi &= R \cdot A(\alpha, \mathbf{n}) \end{aligned} \right\}. \quad (4.8)$$

This is due to the invariance of the theory under the rotation of the  $\tau$ -space. However, in this case, it is necessary to require that the commutation relations of  $\psi_L$  and  $\psi_N$  itself and between them are of the same type, in order that the quantized field theory is invariant under the continuous transformation  $R(\alpha, \mathbf{n})$ .

Now we shall generalize this discussion for the case where the various fields coexist. Now we shall consider a case in which the matrices  $\mathcal{Q}$  in § 2 have suffices corresponding to the various types of field in addition to tensor suffices concerning to the Lorentz transformation. Then,  $\mathcal{Q}^\mu$  can be written as follows,

$$\mathcal{Q}^\mu = \sum_\alpha P^\alpha \mathcal{Q}_\alpha^\mu, \quad (4.9)$$

where  $P^\alpha$  is the projection operator to the  $\alpha$ -field and  $\mathcal{Q}_\alpha^\mu$  is the usual  $\mathcal{Q}^\mu$  for the  $\alpha$ -field and the summation is extended over the special group of the fields. We shall call such a group of the fields a family. Even in this case, the condition for the invariance under the time reversal is given by (2.16). However,  $\sigma \mathcal{Q}_\alpha^\mu$  must be here understood as  $\sum \sigma^\alpha P^\alpha \mathcal{Q}_\alpha^\mu$ , where  $\sigma^\alpha = \pm 1$  for the Bose or Fermi particle, respectively.

The corresponding equations to (2.17) and (2.19) are written as follows,

$$R^{-1} \left\{ \sum_\alpha P^\alpha \mathcal{Q}_\alpha^\mu(\partial) \right\} R = \sum_\alpha P^\alpha \mathcal{Q}_\alpha^\mu(\partial), \quad (2.17)'$$

$$C^{-1} \{ \mathcal{Q}_\alpha^{\mu T}(\partial) \} C = \sum_\alpha \sigma^\alpha P^\alpha \mathcal{Q}_\alpha^\mu(\partial). \quad (2.19)'$$

It should be noted here that the condition (2.17)' and (2.19)' are weaker than the condition that (2.17) and (2.19) hold for the respective  $\alpha$ 's. Therefore, there arises the possibility of the existence of the various types of the time reversal. For instance,  $R_1$  and  $R_2$  may involve the unitary operator in the family space which is commutable with the whole set of  $P^\alpha$ . Further, if  $\mathcal{Q}_\alpha^\mu$  is common for all  $\alpha$ , that is, the case where

<sup>1</sup> This circumstance has been often considered in the phenomenological treatment of the theory. See, for example, G. J. Kynch, Phys. Rev. **81** (1951), 1060 and L. L. Foldy, Phys. Rev. **92** (1953), 178.

all  $\alpha$ -fields which belong to the same family have the identical spin and mass,  $R_1$  and  $R_2$  may involve the arbitrary operator in the family space, since  $\sum_{\alpha} P^{\alpha} = 1$ . In fact, if we regard the proton and neutron system as such a family and neglect their mass difference,  $R_1$  and  $R_2$  may in general involve a unitary operator of the form  $\sum_k \alpha_k \tau_k$ , so long as we restrict our discussion only to the invariance of the interaction free part. This is the result which we have shown in the above consideration of the charge independent theory. If we take into account the mass difference of the proton and neutron, it may still involve the unitary operator of the form  $\alpha + b\tau_3$ , considering  $P^P, N = (1 \pm \tau_3)/2$ . Some examples of the time reversal transformation of a family theory are given in Appendix.

Constructing the product of these transformation and the transformation of the conventional time reversal  $[T_1]$  or  $[T_2]$  in a similar way as (4.7), we can obtain the various kind of the time reversal. Of course, these various transformation may be restricted by the actually existing interactions. It is also important to notice that the commutation relations between the fields which belong to the same family may be restricted to have the same type, as we have shown previously.\*

In conclusion, the authors should like to express their sincere thanks to Prof. S. Sakata for his continual interest and discussions. They are also indebted to Mr. M. Kawaguchi for his valuable advices and discussions for this work.

## Appendix

As an example of the generalized time reversal transformation for a family of elementary particles, we shall consider the Pais' theory of baryon-meson system<sup>9</sup>. In his theory the free Lagrangian of the system is given by

$$L^0(x) = \int \bar{\psi}(x, \omega) \left[ \gamma_{\mu} \partial_{\mu} + \kappa - \frac{1}{\lambda} (\boldsymbol{\tau} \cdot \mathbf{K}) \right] \psi(x, \omega) d\omega. \quad (\text{A} \cdot 1)$$

At first we shall examine the condition of invariance of the Lagrangian under the time reversal transformation, being given by (2.32) :

$$[\mathbf{R}^{-1} L^0(Q(x)) \mathbf{R}]^T = L(Q'^T(x)). \quad (\text{A} \cdot 2)$$

Now, by putting

$$\mathbf{R}_1^{-1} \psi'(x, \omega) \mathbf{R}_1 = R_1 \psi'(x, \omega), \quad (\text{A} \cdot 3)$$

$$\mathbf{R}_2^{-1} \psi'(x, \omega) \mathbf{R}_2 = \psi'^{\dagger}(x, \omega) (C_L C_{\omega})^{-1} \quad (\text{A} \cdot 4)$$

corresponding respectively to time reversal transformations  $[T_1]$  and  $[T_2]$  as defined in the text, where  $C_L$  is the conventionally used charge conjugation operator for the spinor field and  $C_{\omega}$  is a corresponding operator acting on the isotopic spin space, we study the condition (A.2). It is easily shown that under the time reversal (A.3)  $R_1 = 1$  is a solution

\* The connection between the commutation relation and the family of elementary particles has been formerly examined by S. Oneda and H. Umezawa.<sup>10)</sup>

of (A.2) and so this transformation is carried through straightforwardly. Hence, in the following we shall pay special attention only to the time reversal (A.4). In this case, the left hand side of (A.2) can be rewritten in the form;

$$\int \bar{\psi}^{iT}(\underline{x}, \omega) \left[ \gamma_\mu \partial_\mu + \kappa - \frac{1}{\lambda} C_\omega^{-1} (\boldsymbol{\tau} \cdot \mathbf{K})^T C_\omega \right] \psi^{iT}(\underline{x}, \omega) d\omega, \quad (\text{A.5})$$

where we have used the commutability of  $C_\omega$  with  $\boldsymbol{\tau}$  and  $\mathbf{K}$ . Comparing (A.5) with the right hand side of (A.2) we obtain the condition for  $C_\omega$  as follows;

$$C_\omega^{-1} (\boldsymbol{\tau} \cdot \mathbf{K})^T C_\omega = \boldsymbol{\tau} \cdot \mathbf{K}. \quad (\text{A.6})$$

Using the properties of  $\boldsymbol{\tau}$  and  $\mathbf{K}$ :

$$\begin{aligned} \tau_i^T &= \tau_i, & (i=1, 3), \\ &= -\tau_i, & (i=2), \end{aligned} \quad (\text{A.7})$$

$$K_i^T = -K_i, \quad (i=1, 2, 3), \quad (\text{A.8})$$

we find that the condition (A.6) is satisfied by the following alternative solutions:

$$(\text{A}) \quad \left. \begin{aligned} C_\omega^{-1} K_i C_\omega &= K_i, & (i=1, 2, 3), \\ C_\omega^{-1} \tau_i C_\omega &= -\tau_i, & (i=1, 3), \\ &= \tau_i, & (i=2) \end{aligned} \right\} \quad (\text{A.9})$$

or

$$(\text{B}) \quad \left. \begin{aligned} C_\omega^{-1} \tau_i C_\omega &= \tau_i, & (i=1, 2, 3), \\ C_\omega^{-1} K_i C_\omega &= -K_i, & (i=1, 3), \\ &= K_i, & (i=2). \end{aligned} \right\} \quad (\text{A.10})$$

Next, we shall consider the invariance of Schrödinger equation (in the interaction representation). From this requirement of invariance, the transformation property of meson field is determined. In the Pais' theory the interaction Hamiltonian between baryon and meson is given by

$$H^{in}(x) = ig \int \bar{\psi}(x, \omega) \gamma_5 \boldsymbol{\tau}_i \psi(x, \omega) \phi_i(x, \omega) d\omega. \quad (\text{A.11})$$

According to (2.29), the condition of invariance of Schrödinger equation is given by the equation

$$H^{in}(Q_\alpha^{iT}(\underline{x})) = [\mathbf{R}^{-1} H^{in}(Q_\alpha(x)) \mathbf{R}]^T. \quad (\text{A.12})$$

The right hand side of (A.12) can be rewritten in the form

$$- \int \bar{\psi}^{iT}(\underline{x}, \omega) \gamma_5 C_\omega^{-1} \tau_i^T C_\omega \psi^{iT}(\underline{x}, \omega) [\mathbf{R}^{-1} \phi_i'(\underline{x}, \omega) \mathbf{R}]^T d\omega, \quad (\text{A.13})$$

where we have assumed, for convenience,  $\phi_i'(\underline{x}, \omega) = \phi_i(\underline{x}, \omega)$ , which corresponds to the case  $\rho' = 1$  in (2.5)'.

Thus, using (A.9) and (A.10), we see that (A.12) is satisfied if the meson field is transformed as follows ;

$$\mathbf{R}^{-1}\phi'_i(\underline{x}, \omega) \mathbf{R} = \phi'_i(\underline{x}, \omega) \quad (i=1, 2, 3),$$

or

$$\mathbf{R}^{-1}\phi_i(\underline{x}, \omega) \mathbf{R} = \phi_i(\underline{x}, \omega) \quad (i=1, 2, 3) \quad \text{for the case (A),} \quad (\text{A.14})$$

and

$$\begin{aligned} \mathbf{R}^{-1}\phi'_i(\underline{x}, \omega) \mathbf{R} &= -\phi'_i(\underline{x}, \omega) \quad (i=1, 3) \\ &= \phi'_i(\underline{x}, \omega) \quad (i=2) \end{aligned}$$

or

$$\begin{aligned} \mathbf{R}^{-1}\phi_i(\underline{x}, \omega) \mathbf{R} &= -\phi_i(\underline{x}, \omega) \quad (i=1, 3) \\ &= \phi_i(\underline{x}, \omega) \quad (i=2) \quad \text{for the case (A).} \end{aligned} \quad (\text{A.15})$$

Of course, if we assume  $\phi'_i(\underline{x}, \omega) = -\phi_i(\underline{x}, \omega)$ , which corresponds  $\rho' = -1$  in (2.5), all the terms on the right hand side of (A.14) and (A.15) take the opposite signs.

Finally, we shall study the physical meaning of transformations (A) and (B). For this purpose, we consider the transformation property of the electric charge, which in Pais' theory, is given by

$$q(x) = ie \int \bar{\psi}(x, \omega) \gamma_4 \left( \frac{1 + \tau_3}{2} + K_3 \right) \psi(x, \omega) d\omega. \quad (\text{A.16})$$

The transformation property of a physical quantity is obtained from the sign of the right hand side of (3.3) :

$$F[Q'^T(x)] = \pm (\mathbf{R}^{-1}(F[Q(x)]\mathbf{R})^T. \quad (\text{A.17})$$

Using the relation

$$[\mathbf{R}^{-1}q(x)\mathbf{R}]^T = ie \int \bar{\psi}'^T(\underline{x}, \omega) \gamma_4 C_\omega^{-1} \left( \frac{1 + \tau_3}{2} - K_3 \right) C_\omega \psi'^T(\underline{x}, \omega) d\omega, \quad (\text{A.18})$$

and considering (A.17), we can easily find that the electric charge is an even quantity under the transformation (B), while it shows no definite property of this kind under the transformation (A), because of the additional term  $1/2$  in the definition of electric charge (A.16).

The case (A), which has not been ordinarily considered in the framework of the isotopic spin formalism (§ 4), appears only in the theory of  $\omega$ -space. However this transformation breaks the invariance of such an interaction as the baryon-meson system with electromagnetic field, because the transformation of electric charge shows no definite property from the above discussion. On the other hand, the case (B) still holds for this interaction as usual.

However, the third component of the total isotopic spin  $I_3 = \int \bar{\psi} \gamma_4 ((\tau_3/2) + K_3) \psi d\omega$  is an odd quantity under transformation (A). Thus, under the transformation (A) there

exists the following correspondence  $P \longleftrightarrow N$ ,  $A^+ \longleftrightarrow A^0$ ,  $A^{++} \longleftrightarrow A^-$ , if we assign the isotopic spin states  ${}^2P_{3/2}$  for  $(A^{++}, A^+, A^0, A^-)$ ,  ${}^2S_{1/2}$  for  $(P, N)$  and  ${}^3S_1$  for  $(\pi^+, \pi^0, \pi^-)$ , respectively. Therefore, under the assumption that the theory is invariant under the transformation (A), we lead to the result that, for instance, the transition amplitudes for the following processes

$$A^{++} \rightarrow A^+ + \pi^+ \quad (\text{A} \cdot 19)$$

and

$$A^0 \rightarrow P + \pi^- \quad (\text{A} \cdot 20)$$

are respectively equal to those of the processes

$$A^0 + \pi^- \rightarrow A^- \quad (\text{A} \cdot 19)'$$

and

$$N + \pi^+ \rightarrow A^+, \quad (\text{A} \cdot 20)'$$

for the reason discussed in § 3 (This result is only trivial when we confine ourselves to the lowest order terms in the perturbation approximation).

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## Letters to the Editor

### On the Direct Interaction in Nuclear Reactions

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Since the  $(d, p)$  reaction has been interpreted successfully in terms of a stripping process<sup>1)</sup>, it is noted to be important that the interaction takes considerable part in nuclear reaction outside a nuclear surface at moderate energies, say, several to several tens Mev. This idea is applied by Austern, Butler and McManus<sup>2)</sup> also to  $(n, p)$  reactions. The theory is based on such an assumption that an impinging particle is scattered (including rearrangement) by a nucleon near the nuclear surface through nucleon-nucleon forces and the scattering amplitude is evaluated by means of the overlapping integral for these nucleons only outside a nucleus. The possibility that the impinging particle gets in the nucleus is attributed to the formation of a compound nucleus, which can be separated from the direct interaction on account of the angular distribution of scattered particles.

This way of approach is critically investigated by various authors,<sup>3)-12)</sup> particularly for stripping and pick-up processes. A question which remains yet nuclear is why only the reaction taking place outside a nucleus is chiefly responsible to the process under consideration and to what extent the contribution from the inner part of a nucleus should be taken into account. To clarify these problems we investigated the scattering of a nucleon by one of nucleons in a nucleus. The wave functions of incident, outgoing and initial as well as final bound nucleons were obtained as single particle waves in a square well potential. The scattering amplitude was given by the overlapping integral for these four waves and the interaction potential between two nucleons concerned. Then we were able to see the contribution to the overlapping from each place outside as well as inside a nucleus.

To fix numerical values our method was illustrated

by an example  $^{28}\text{Si}(n, p)^{28}\text{Al}$ . The  $(n, p)$  reaction was assumed to occur through the collision of an impinging neutron and a bound proton in  $3d$  orbit, leaving the neutron in  $2s$  bound orbit. This corresponds to the transition to the ground state of  $^{28}\text{Al}$  through the absorption of energy, 4 Mev. The depths of the square well potentials for  $^{28}\text{Si}$  and  $^{28}\text{Al}$  were obtained from the binding energies of the last nucleons as 38 Mev in both cases, assuming the nuclear radius to be  $4.25 \times 10^{-13}$  cm. ( $1.4 \times 10^{-13} \text{ A}^{1/3}$  cm). In this illustration we neglected the Coulomb interaction between the proton and a nucleus. We further assumed the range of interaction between two nucleons to be zero and neglected the exchange interaction. The last assumptions saved the labour of computations. We further took account only of a  $s$  wave for the impinging neutron and accordingly of a  $d$  wave for the outgoing proton.

Table 1. Ratio of the probabilities of interactions taking place in the inner to those in the outer parts of nuclei.

Energy of incident neutron (Mev)	5	10	14	20	30	50
Ratio	22.9	12.7	4.75	5.16	2.59	13.2

In order to see how the overlapping depends upon energy, we take six energies, as shown in Table 1. The overlapping as a function of radial positions is shown for six incident energies in Fig. 1. The procedure adopted by Austern et al.<sup>2)</sup> is, therefore, not justified, as far as our example is concerned.

It is noticeable, however, that a maximum of the overlapping appears outside and close to the nuclear surface, *e. g.*, at  $4.7 \times 10^{-13}$  cm from the origin at the incident energy of 14 Mev. This seems to indicate the effective nuclear radius for nuclear reactions, which is appreciably larger than the radius for nucleon densities, in spite of that we have neglected the range of interactions. It is also interesting to note the energy dependence of the effective radius. Thus our calculation suggests that the effective nuclear radius depends in a regular manner upon the incident energy as well as the binding energies of the last few nucleons. This

problem will be discussed more in detail separately.

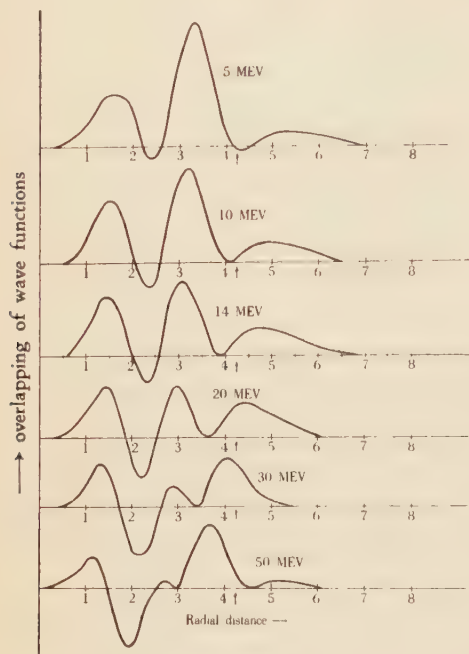


Fig. 1.

Fig. 1. Overlapping of four nucleon waves for six incident energies vs radial distance (in  $10^{-13}$  cm). Arrows in the figures indicate the position of the nuclear surface. Incident energies are written in respective figures.

Integrating the overlapping over the inner and outer parts of the nucleus separately, the contributions from the respective parts were compared. The ratio of the squares of the overlapping integrals for the inner to the outer parts is shown in the last row of Table 1. It is seen that the ratio depends strongly on energy and the contribution from the inner part is larger than that from the outer part. The energy dependence is found to be due to the detailed shape of waves concerned.

The result obtained may account for the reason why the Butler approximation often fails to give the correct absolute value of cross sections. Before drawing conclusions, however, it should be kept in mind that our calculation is merely a numerical example and is subject to the following oversimplification. Firstly, the actual reaction takes place through an interaction with finite range and the potential contains an exchange part. Secondly, the

higher angular momenta of incident nucleons, up to  $l=5$ , give an appreciable contribution. Thirdly, the absorption of nucleons in nuclei makes the contribution from the inner part smaller than the above estimate. These three points will be taken into consideration in a forthcoming paper. Fourthly, there is a considerable possibility that the final nucleon is left in excited states. Fifthly, the Coulomb force pulls the bound proton inward, while it pushes away the scattered proton. Finally, the actual shape of the nuclear potential is not of the square well, but the edge of the well is rounded.

Detailed discussions on these points, together with our general method of treating this problem, will be published shortly.

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## Potential in Quantum Field Theory

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The recent success of renormalization theory is essentially due to its covariant form. This fact makes it difficult to use the concept of "potential" within

the frame-work of the quantum field theory. For, as the "potential" is in general a non-covariant concept, divergent part of the potential can not be removed in a usual way. Therefore it seems to be desirable to express a potential at least in such a form as the part which should be renormalized being covariant. There are some attempts<sup>1)</sup> to do so by starting from Bethe-Salpeter equation; however, their four dimensional wave function makes the general view to be obscure. In order to avoid this difficulty we shall turn our attention to integral equation of  $R$ -matrix. If there is a phenomenological potential  $\psi$ , the integral equation for  $R$ -matrix is written as follows:

$$R = \psi + \psi \frac{1}{E - H_0 + i\epsilon} R \quad (1)$$

On the other hand, using the relation<sup>2)</sup> of  $S$ -matrix and Green-functions we can get the following equation for two fermion problem

$$\begin{aligned} K(\vec{p}_1, \vec{p}_2, q_2) &\equiv (\vec{p}_1 \vec{p}_2 | R | q_1 q_2) = 2J(\vec{p}_1, \vec{p}_2, q_2) \\ &- i(2\pi)^{-7} \int \bar{U}_+^{r_1}(\vec{p}_1) \bar{U}_+^{r_2}(\vec{p}_2) S_F^{-1}(\vec{p}_1) S_F^{-1}(\vec{p}_2) \\ &\times G(\vec{p}_1) G(\vec{p}_2) I(\vec{p}_1, \vec{p}_2, l) S_F(\vec{p}_1 + \vec{p}_2 - l) \\ &\times S_F(l) F(\vec{p}_1 + \vec{p}_2 - l, l, q_2) U_+^{s_1}(\vec{q}_2) U_+^{s_2}(\vec{q}_2) dl. \end{aligned} \quad (2)$$

Here

$$\begin{aligned} K(\vec{p}_1, \vec{p}_2, q_2) &= -i(2\pi)^{-3} \delta(\vec{p}_1 + \vec{p}_2 - \vec{q}_1 - \vec{q}_2) \\ &\times \bar{U}_+^{r_1}(\vec{p}_1) \bar{U}_+^{r_2}(\vec{p}_2) F(\vec{p}_1, \vec{p}_2, q_2) U_+^{s_1}(\vec{q}_1) \bar{U}_+^{s_2}(\vec{q}_2) \\ J(\vec{p}_1, \vec{p}_2, q_2) &= i(2\pi)^{-3} \delta(\vec{p}_1 + \vec{p}_2 - \vec{q}_1 - \vec{q}_2) \\ &\times \bar{U}_+^{r_1}(\vec{p}_1) \bar{U}_+^{r_2}(\vec{p}_2) S_F^{-1}(\vec{p}_1) S_F^{-1}(\vec{p}_2) G(\vec{p}_1) \\ &\times G(\vec{p}_2) I(\vec{p}_1, \vec{p}_2, q_2) U_+^{s_1}(\vec{q}_1) U_+^{s_2}(\vec{q}_2), \end{aligned}$$

$-F/4$  and  $I$  are momentum representations of the coefficient with the factor  $N(\bar{\psi} \psi \bar{\psi} \psi)$  in  $S$ -matrix and Schwinger's interaction kernel for two fermion problem respectively. After modification of the part  $S_F(\vec{p}_1 + \vec{p}_2 - l) S_F(l)$  in (2), (2) turns out to be the following set of integral equations

$$\begin{aligned} K(\vec{p}_1, \vec{p}_2, q_2) &= 2L(\vec{p}_1, \vec{p}_2, q_2) \\ &+ \int L(\vec{p}_1, \vec{p}_2, \vec{l}) \sqrt{\vec{l}^2 + M^2} \\ &\times \frac{1}{\vec{p}_1^0 + \vec{p}_2^0 - \sqrt{\vec{l}^2 + M^2} - \sqrt{(\vec{p}_1 + \vec{p}_2 - \vec{l})^2 + M^2} + i\epsilon} \end{aligned}$$

$$\begin{aligned} &\times K(\vec{l}, \vec{p}_1^0 + \vec{p}_2^0 - \sqrt{\vec{l}^2 + M^2}, \vec{l}) \sqrt{\vec{l}^2 + M^2}, q_2) \\ &\times d\vec{l} d\vec{l}', \end{aligned} \quad (3)$$

$$\begin{aligned} L(\vec{p}_1, \vec{p}_2, q_2) &= J(\vec{p}_1, \vec{p}_2, q_2) + \int J(\vec{p}_1, \vec{p}_2, l_1) \\ &\times \frac{1}{\vec{p}_1^0 + \vec{p}_2^0 - \sqrt{\vec{l}^2 + M^2} - \sqrt{(\vec{p}_1 + \vec{p}_2 - \vec{l})^2 + M^2} + i\epsilon} \\ &\times \frac{i\gamma^0}{\pi} \frac{1}{l^0 - \sqrt{\vec{l}^2 + M^2}} \\ &L(\vec{l}, \vec{p}_1^0 + \vec{p}_2^0 - l^0, l, q_2) d^4l d\vec{l}' \end{aligned} \quad (4)$$

where contributions of the positron from the above part  $S_F, S_F$  to (3) and (4) have been neglected. When we take into account all the contributions, the integral equation corresponding to (4) becomes somewhat complicated. Here it must be remarked that the fact that the factor 2 appears in the first term of (3) whereas it does not in (1) is due to the symmetrization character of the state vector. Comparing (1) and (3) and remembering the situation mentioned just above, we arrive at the following conclusions: (i) the potential of two fermion problem can be given by  $L$  being defined by integral equations (4), and (ii) since the divergent parts have covariant forms this potential  $L$  is just one we expected, the detailed explanation will be published in this journal in the near future.

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## An Empirical "Charge Dependence" Relation to the Rest-masses of Elementary Particles

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Previously<sup>1)</sup> we have shown the precise computation table of our mass spectrum of the form  $\mu_n = (\hbar c/l^2) \omega^n$ , using the value of  $\hbar c/l^2 = 137.036$ . Al-

though a more recent value clarified by Bearden et al.<sup>(2)</sup> is 137.0325, the former value is still numerically sufficient to be referred to, so far as the present short note is concerned.

Now we would like to suggest an empirical formula regarding, so to speak, the "charge dependence" relation to the rest-masses of elementary particles; i.e.,

$$\left. \begin{aligned} \mu_n &= m_{ch} + (-1)^{2T} |\delta m|, \\ \delta m &= 2/\pi \cdot (m_{ch} - m_0), \end{aligned} \right\} \quad (1)$$

where  $\mu_n$  is the calculated value above mentioned (see Table 1 in reference 1) and  $m_{ch}$ ,  $m_0$  represent the actual masses of arbitrary charged and neutral particles, respectively, which are in relationship of the counter-part to each other, and besides,  $T$  is assumed to denote isotopic spin of these particles. Namely, the formula 1 is given in expectation of satisfying the following postulates: (1) The deviation of the calculated  $\mu_n$  from the actual  $m_{ch}$  is proportional to the mass difference  $(m_{ch} - m_0)$  which is generally attributed to electromagnetic effects. (2) The sign of the deviation  $(\mu_n - m_{ch})$  depends on the isotopic spin parity. (3) Now that isotopic spin is introduced here, the formula 1 should never violate the conservation of the  $z$  component  $T_z$  of isotopic spin  $T$  in order that the conservation of charge may be held firmly.

Hereupon it is required by the formula 1 that the sign of the deviation  $(\mu_n - m_{ch})$  is positive or negative respectively when  $2T$  is even or odd, and that, self-evident as it is, the product of  $(\mu_n - m_0)(m_{ch} - m_0)$  is always positive irrespective of isotopic spin  $T$ . Therefore the possible inequality conditions to be considered are as follows:

- 1)  $\mu_n > m_{ch} > m_0$  or  $m_0 > \mu_n > m_{ch}$ ,  
when  $2T$  is even.
- 2)  $m_0 > m_{ch} > \mu_n$  or  $m_{ch} > \mu_n > m_0$ ,  
when  $2T$  is odd.

However, at first we will confine our attention only to the case in which the above deviation and mass difference simultaneously take the same sign, namely  $(\mu_n - m_{ch})(m_{ch} - m_0) > 0$ . Such a case is that which is expressed by the following equation of plus sign apparently applicable to the cases of the  $\pi$  meson and the nucleon; that is,

$$\mu_n = m_{ch} \pm \delta m. \quad (2)$$

Then, in what follows, whether the suggested formulae are actually compatible with observations

will be examined;

#### Case A: Nucleons.

The calculated value,  $\mu_{1,1/2} = 1834.55 m_e$ .

$$\delta m = -2/\pi \text{ (the } n-p \text{ mass difference)}$$

$$= (-2/\pi) \times 2.53 = -1.61 m_e.$$

Accordingly, using eq. (2) of plus sign

$$m_{ch} = m_p = 1836.16 m_e.$$

#### Case B: $\pi$ Mesons.

The calculated value,  $\mu_{1,0} = 280.32 m_e$ .

$$\delta m = 2/\pi \cdot (\text{the } \pi^\pm - \pi^0 \text{ mass difference})$$

$$= (2/\pi) \times 10.6^{(3)} = 6.7 m_e.$$

Accordingly, using eq. (2) of plus sign

$$m_{ch} = m_{\pi^\pm} = 273.6 m_e.$$

The value  $273.6 m_e$  is in good agreement with the value  $(273.5 \pm 1.2) m_e$  obtained at Radiation Laboratory of Berkeley<sup>(1)</sup> for the positively charged  $\pi$  meson in absolute mass measurements. However, with regard to the  $\pi^- - \pi^0$  mass difference, if a more recent value  $(8.8 \pm 0.6) m_e^{(5)}$  is taken instead of  $(10.6 \pm 2.0) m_e^{(3)}$  deservedly we have an appreciably larger mass value of the charged  $\pi$  meson.

Thus, taking into account of the preceding discussion concerning the formula 1, the fact that the predicted nucleon mass is only about  $1.6 m_e$  smaller than the actual mass of the proton is interpreted to be due to the circumstances that the magnitude of the  $n-p$  mass difference is only about  $+2.5 m_e$  and, additionally, nucleons take the half-integer isotopic spin. In the case of the  $\pi$  meson also, the reason why the calculated mass is comparatively larger than the observed mass is easily elucidated quite in like manner.

In the next case the neutral  $\tau$ -meson mass will be given, supposing that the neutral counterpart of the charged  $\tau$  meson really exists. However, not to speak of the case above taken, here we must take into consideration of the case  $(\mu_n - m_{ch})(m_{ch} - m_0) < 0$ , too, which is expressed by Eq. 2 of minus sign.

#### Case C: $\tau$ Mesons.

The calculated value,  $\mu_{1,0} = 980.77 m_e$ .

$$m_{ch} = m_{\tau^\pm} = 965.5 m_e^{(6)}$$

$$(\mu_{1,0} > m_{ch} \text{ } \therefore 2T = \text{even.})$$

Therefore, if eq. (2) of plus sign is used,

$$m_0 = m_{\tau^0} = 941.5 m_e.$$

And if eq. (2) of minus sign is used,

$$m_0 = m_{\tau^0} = 989.5 m_e.$$



Table 1. The summarized results concerning the suggested "charge dependence" relation to the rest-masses of elementary particles.

Fundamental formula	$\mu_n = m_{e\hbar} + (-1)^{2T}  \delta m ,$ $\delta m = 2/\pi \cdot (m_{e\hbar} - m_0).$			
Isotopic spin parity	even		odd	
Possible inequality conditions	$\mu_n > m_{e\hbar} > m_0$	$m_0 > \mu_n > m_{e\hbar}$	$m_0 > m_{e\hbar} > \mu_n$	$m_{e\hbar} > \mu_n > m_0$
Equivalent formulae	$\mu_n = m_{e\hbar} + \delta m$	$\mu_n = m_{e\hbar} - \delta m$	$\mu_n = m_{e\hbar} + \delta m$	$\mu_n = m_{e\hbar} - \delta m$
Examples :	$\pi$ Mesons : $\mu_{4,0} = 280.3$ $m_{e\hbar} = 273.6$ $m_0 = 263.0$	$\tau$ Mesons : $m_0 = 989.5$ $\mu_{11,0} = 980.8$ $m_{e\hbar} = 965.5$	Nucleons : $m_n = 1838.6$ $m_p = 1836.1$ $\mu_{14 \ 1/2} = 1834.5$	$V_1$ Particles : $m_{e\hbar} = 2208$ $\mu_{15 \ 1/2} = 2194$ $m_0 = 2184$
Mass values are shown in the electron mass unit.	$\tau$ Mesons : $\mu_{11,0} = 980.8$ $m_{e\hbar} = 965.5$ $m_0 = 941.5$			

Seeing that the Lorentz-invariance does not allow us to regard the  $V_4^0$  particle as the neutral variety of the  $\tau$  meson,<sup>1, 7)</sup> it would seem to be within the range of possibility that the  $V_4^0$  of mass  $\sim 970 m_e$ <sup>8)</sup> would take the intermediate mass value between the values obtained above.

Since it is probable that so far the charged  $V_1$  particle which has a  $Q$  value of the order of  $Q(V_1^0)$  may be identified only with the positively charged  $V_1$  particle suggested by the Pasadena group,<sup>9)</sup> the hypothesis of the  $V_1$  particles of two charge states, namely the  $V_1^+$  and  $V_1^0$ , may presumedly be reasonable. Therefore, according to the aforesaid inequality condition 2, the calculated mass of  $V_1$  particles will take an intermediate value between  $m_{V_1^+}$  and  $m_{V_1^0}$ , provided the calculated mass is larger than  $m_{V_1^0}$ . In that case, the charged  $V_1$ -particle mass may be given by eq. (2) of minus sign; viz., Case D:  $V_1$  Particles.

The calculated value,  $\mu_{15 \ 1/2} = 2193.99 m_e$ .

$m_0 = m_{V_1^0} = 2184 m_e$ ,<sup>10)</sup>  $\therefore \mu_{15 \ 1/2} > m_{V_1^0}$ .

And  $2T=1=\text{odd}$ .

Therefore, using eq. (2) of minus sign

$$m_{e\hbar} = m_{V_1^+} = 2208 m_e.$$

Although the cosmotron group at Brookhaven<sup>11)</sup> have recently reported the  $V_1$ -particle decay of a high  $Q$

value  $\sim 130$  Mev, it would not seem reasonable to us to attribute the large difference  $Q(V_1^-)$  and  $Q(V_1^0)$  to electro-magnetic effects. Hence it is felt that this is not right place to discuss any relationships between the  $V_1^-$  and  $V_1^0$  particles, granting that some such a situation should actually exist between them.<sup>12)</sup>

Now we will discuss the problem of the conservation of the  $z$  component  $T_z$ , which is insured by the conservation of charge. As for the only particles already discussed such as nucleons,  $\pi$ ,  $\tau$  mesons, and  $V_1$  particles, it appears that, perhaps on account of the self-consistent assignment of isotopic spin to these particles, the formula 1 would not cause the violation of the conservation of the  $z$  component  $T_z$  at the time of their interacting with each other; e.g., in the decay  $\tau \rightarrow 3\pi$  or in the decay  $V_1^0 \rightarrow \rho + \pi^-$ . Furthermore, it is of interest to note that in the decay  $\pi \rightarrow \mu + \nu$  also, the formula 1 would seem to succeed in satisfying the requirement of the  $T_z$  conservation. That is to say, according to the formula 1, the fact that the predicted  $\mu$ -meson mass  $\mu_{2 \ 1/2} = 214 m_e$  is about  $7 m_e$  larger than the actual  $\mu^\pm$ -meson mass<sup>4)</sup> means that the assignment of isotopic spin to  $\mu$  mesons would be integral and the neutral  $\mu$  meson of mass 218 or  $196 m_e$  should, though undiscovered at all yet, exist and that consequently the  $z$  component  $T_z'$  might be conserved in the decay  $\pi \rightarrow \mu + \nu$ , because the neutrino may be



regarded to have the integer isotopic spin in any case. Such being the case, to be brief, it may be said that we have a bright prospect on this problem.

Inferring from what has been described hitherto, though the present measurements are not accurate enough to be safely relied upon, it would be still likely that the formula 1 might in principle satisfy the aforesaid postulates, as expected first, and that these postulates themselves would be founded on real facts well. Here Table 1 shows the summarized results concerning the suggested empirical relations.

In conclusion, the authors wish to express their hearty thanks to Dr. P. H. Fang for his valuable suggestions. Their sincere thanks also go to Professor Y. Nambu for his kind advices and criticism.

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$-Q(V_1^0)$ . See A. Pais, *Prog. Theor. Phys.* **10** (1953), 457.

## Excitation of Metal Electrons by An Energetic Primary Particle

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In the theory of secondary electron emission of metals it is primarily important to investigate the excited states of metal electrons (M. E.) caused by a charged primary particle (P. P.). For this purpose we have worked out the problem in a scheme of the second quantisation employing the "Plasma Theory" developed by D. Bohm and D. Pines.<sup>1)</sup> The essential parts of the theory are as follows.

According to the plasma theory of Bohm and Pines, the M. E. are described by an assembly of harmonic oscillators (Boson) and free electrons interacting each other by short range forces (Fermion) and the interaction energies,  $H_I$ , between Boson and Fermion; that is:

$$H_{\text{osc}} = -\frac{1}{2} \sum_{k < k_c} (\hat{p}_k \hat{p}_{-k} + \omega_p^2 q_k q_{-k}),$$

(Boson) [B.P. (16)]

$$H_{\text{Fermi}} = \sum_i \hat{f}_i^2 / 2m = \sum_i \hbar^2 \mathbf{x}_i^2 / 2m,$$

(Fermion) [B.P. (16-1)]

$$H_I = (4\pi)^{1/2} \frac{e}{m} \sum_i \mathbf{E}_k \left( \mathbf{p}_i - \frac{\hbar \mathbf{k}}{2} \right) q_k e^{i\mathbf{k} \cdot \mathbf{x}_i},$$

[B.P. (17)]

and the wave function,  $\psi$ , is subject to a subsidiary condition (S. C.),

$$Q_k \psi = 0 \quad (k < k_c). \quad [\text{B.P. (12)}]$$

In the scheme of second quantisation, the Boson, [B.P. (16)], and the Fermion, [B.P. (16-1)], can be represented, respectively, by their wave number vectors  $\mathbf{k}$ 's and  $\mathbf{x}$ 's, and the interaction energy,  $H_I$ , is transformed into

$$(H_I) = \frac{e}{m} \left( \frac{2\pi\hbar}{\omega_p} \right)^{1/2} \left\{ \begin{array}{l} \hat{b}^*(n\mathbf{k}) \hat{b}(n\mathbf{k} + \mathbf{k}) \sqrt{n+1} \\ \hat{b}^*(n\mathbf{k}) \hat{b}(n\mathbf{k} - \mathbf{k}) \sqrt{n} \end{array} \right\}$$

$$\times a^*(\mathbf{x}) a(\mathbf{x}-\mathbf{k}) \left( \frac{\hbar \mathbf{k}}{2} \right) \delta_{\mathbf{k}} \\ \left( \begin{matrix} \mathbf{x}=\mathbf{x}_1, \mathbf{x}_2, \dots \\ \mathbf{k}=\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_c \end{matrix} \right) \quad (1)$$

where  $a^*(\mathbf{x})$  is the destruction operator of  $\mathbf{x}$ -electron,  $a(\mathbf{x})$  the creation operator, and  $b^*(\mathbf{k})$  and  $b(\mathbf{k})$  are those for  $\mathbf{k}$ -Boson. Similarly, the S. C., [B.P. (12)], can be transformed into

$$\left. \begin{aligned} b^*(n\mathbf{k}) b(n\mathbf{k}+\mathbf{k}) \} &= \left( \frac{8\pi e^2}{k^2 \hbar \omega_p} \right)^{1/2} a^*(\mathbf{x}) a(\mathbf{x}+\mathbf{k}), \\ -b^*(n\mathbf{k}) b(n\mathbf{k}-\mathbf{k}) \} & \\ (\mathbf{x}=\mathbf{x}_1, \mathbf{x}_2, \dots) & \\ (k < k_c). & \end{aligned} \right\} \quad (2)$$

A P. P., having charge  $Ze$ , mass  $M$  and momentum  $\hbar \mathbf{K}$  in a position  $\mathbf{x}_0$ , may cause some perturbation on the M.E. with perturbation energies

$$V_I = \frac{Ze}{M} (4\pi)^{1/2} \sum_{\mathbf{k}} q_{\mathbf{k}} \delta_{\mathbf{k}} \left( \frac{\hbar \mathbf{K}}{2} - \frac{\hbar \mathbf{k}}{2} \right) e^{i\mathbf{k} \cdot \mathbf{x}_0}, \quad (3)$$

and

$$V_s = 2\pi Ze^2 \sum_{\substack{\mathbf{k} > k_c \\ i \neq 0}} \frac{1}{k^2} e^{i\mathbf{k} \cdot (\mathbf{x}_0 - \mathbf{x}_i)}, \quad (4)$$

together with a S.C. These can be transformed into

$$(V_I) = Ze/M \cdot (2\pi \hbar / \omega_p)^{1/2} \left\{ \begin{aligned} &b^*(n\mathbf{k}) b(n\mathbf{k}+\mathbf{k}) \sqrt{n+1} \\ &b^*(n\mathbf{k}) b(n\mathbf{k}-\mathbf{k}) \sqrt{n} \end{aligned} \right\} \\ \times a^*(\mathbf{K}) a(\mathbf{K}-\mathbf{k}) \left( \frac{\hbar \mathbf{K}}{2} - \frac{\hbar \mathbf{k}}{2} \right) \delta_{\mathbf{k}}, \\ (\mathbf{k}=\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_c), \quad (5)$$

and

$$(V_s) = 2\pi Ze^2 (1/k^2) a^*(\mathbf{K}) a^*(\mathbf{x}_i) a(\mathbf{K}-\mathbf{k}) a(\mathbf{x}_i+\mathbf{k}), \\ (k < k_c, \mathbf{x}_i=\mathbf{x}_1, \mathbf{x}_2, \dots). \quad (6)$$

In the non-perturbed state the Boson and the Fermion may be in their lowest energy states, that is, all  $\mathbf{k}$ 's are zero and all  $\mathbf{x}$ 's lie in the Fermi-sphere. It is convenient to represent this state by a vector,  $\varphi_0^0(\mathbf{K}_0; 0, 0, \dots; 0, 0, \dots)$ , in the configuration space. A vector,  $\varphi_j^i(\mathbf{K}'; \mathbf{x}_1', \mathbf{x}_2', \dots, \mathbf{x}_i'; \mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_j)$ , represents a state in which M.E.  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_i$  are excited to  $\mathbf{x}_1', \mathbf{x}_2', \dots, \mathbf{x}_i'$  and the Boson are in the excited states  $\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_j$  and the P.P. in  $\mathbf{K}'$ -state. Total wave function,  $\Phi$ , in the configuration space is denoted, therefore, by a matrix

$$\Phi = \begin{pmatrix} \varphi_0^0 & \varphi_1^0 & \varphi_2^0 & \dots & \varphi_n^0 \\ \varphi_0^1 & \varphi_1^1 & \varphi_2^1 & \dots & \varphi_n^1 \\ \dots & \dots & \dots & \dots & \dots \\ \varphi_0^m & \varphi_1^m & \varphi_2^m & \dots & \varphi_n^m \end{pmatrix}. \quad (7)$$

The wave function,  $\Phi$ , satisfies the Schrödinger equation

$$[H^{(0)} + (V) + (H) - W_0] \Phi = 0, \quad (8)$$

where

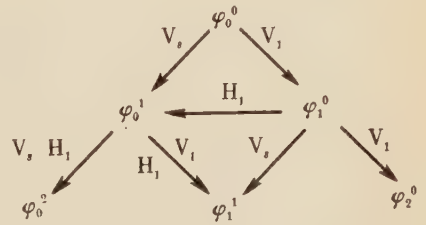
$$\begin{cases} H^{(0)} = \sum_{\mathbf{x}} \hbar^2 \mathbf{x}^2 / 2m + \hbar^2 \mathbf{K}_0^2 / 2M, \\ (V) = (V_I) + (V_s), \\ (H) = (H_I) + (H_s), \end{cases} \quad (9)$$

[( $H_s$ ) is a short range interaction energy having similar form as (6)].

The operators ( $V$ ) and ( $H$ ) having following characters:

- (1) ( $V_I$ ) increases the Boson by one quanta.
- (2) ( $V_s$ ) increases the excited Fermion by one.
- (3) ( $H_I$ ) changes one excited Boson into one excited Fermion and vice versa, remaining  $\mathbf{K}$  invariant. Especially, if we have a highly excited Fermion, the recombination of excited electron and a hole in the Fermi-sphere may be neglected and ( $H_I$ ) acts to increase the Boson by one quanta remaining the number of excited Fermion invariant.
- (4) ( $H_s$ ) increases the excited Fermion by one, remaining  $\mathbf{K}$  invariant.

Considering these properties of operators ( $V$ ) and ( $H$ ), the eq. (8) in each subspace may be readily written down if higher terms in ( $H$ ) and ( $V$ ) than second order are neglected. In fact, we have a following diagram of the states to deal with.



The diagram showing, for instance, a state  $\varphi_0^1$  is produced by ( $V_s$ ) directly from the initial state  $\varphi_0^0$ , and by ( $V_I$ ) and ( $H_I$ ) through a state  $\varphi_1^0$  etc. All the states omitted from this diagram are higher order states in perturbations. Thus we have

$$\begin{aligned}
& (W_0 - H_0') \varphi_0'(\mathbf{x}_1', \mathbf{K}') = (\mathbf{x}_1, \mathbf{K}_0 | V_s | \mathbf{K}', \mathbf{x}_1') \varphi_0^0 \\
& \quad + \sum_{\mathbf{k}_1} (\mathbf{x}_1, \mathbf{k}_1 | I_I | 0, \mathbf{x}_1') \varphi_1^0, \\
& (W_0 - H_1^0) \varphi_1^0(\mathbf{k}_1, \mathbf{K}'') = (0, \mathbf{K}_0 | V_s | \mathbf{K}'', \mathbf{k}_1) \varphi_0^0, \\
& (W_0 - H_2^0) \varphi_2^0(\mathbf{x}_1', \mathbf{x}_2', \mathbf{K}''') \\
& \quad = \sum_{\mathbf{K}'} (\mathbf{x}_1', \mathbf{x}_2, \mathbf{K}' | V_s | \mathbf{K}''', \mathbf{x}_2', \mathbf{x}_1') \varphi_0^1 \\
& \quad + \sum_{\mathbf{x}_1''} (\mathbf{x}_1'', \mathbf{x}_2 | I_s | \mathbf{x}_2', \mathbf{x}_1') \varphi_0^1, \\
& (W_0 - H_1^1) \varphi_1^1(\mathbf{x}_1', \mathbf{k}_1, \mathbf{K}^{(4)}) \\
& \quad = \sum_{\mathbf{K}'} (\mathbf{x}_1', 0, \mathbf{K}' | V_I | \mathbf{K}^{(4)}, \mathbf{k}_1, \mathbf{x}_1') \varphi_0^1 \\
& \quad + \sum_{\mathbf{x}_1''} (\mathbf{x}_1'', 0 | I_I | \mathbf{k}_1, \mathbf{x}_1') \varphi_0^1 \\
& \quad + \sum_{\mathbf{K}'} (\mathbf{x}_1, \mathbf{k}_1, \mathbf{K}' | V_s | \mathbf{K}^{(4)}, \mathbf{k}_1, \mathbf{x}_1') \varphi_1^0, \\
& (W_0 - H_2^0) \varphi_2^0(\mathbf{k}_1, \mathbf{k}_2, \mathbf{K}^{(5)}) \\
& \quad = \sum_{\mathbf{K}''} (0, \mathbf{k}_1, \mathbf{K}'' | V_I | \mathbf{K}^{(5)}, \mathbf{k}_1, \mathbf{k}_2) \varphi_1^0
\end{aligned} \tag{10}$$

where

$$\begin{cases}
H_0^1 = H^{(0)} - (\hbar^2/2m)(\mathbf{x}_1^2 - \mathbf{x}_1'^2) \\
\quad - (\hbar^2/2M)(\mathbf{K}_0^2 - \mathbf{K}'^2), \\
H_1^0 = H^{(0)} + \hbar k_s(k) - (\hbar^2/2M)(\mathbf{K}_0^2 - \mathbf{K}'^2), \\
\quad \dots\dots\dots
\end{cases} \tag{11}$$

The eqs. (10) can be solved successively and we have, for instance,

$$\begin{aligned}
\varphi_1^0(\mathbf{k}_1, \mathbf{K}'') &= (0, \mathbf{K}_0 | V_I | \mathbf{K}'', \mathbf{k}_1) \\
&\quad \times [1/(W_0 - H_1^0) - i\pi\delta(W_0 - H_1^0)], \tag{12}
\end{aligned}$$

$$\begin{aligned}
(W_0 - H_1^1) \varphi_0'(\mathbf{x}_1', \mathbf{K}') &= (\mathbf{x}_1, \mathbf{K}_0 | V_s | \mathbf{K}', \mathbf{x}_1') \varphi_0^0 \\
&+ \sum_{\mathbf{k}_1'} \frac{(0, \mathbf{K}_0 | V_I | \mathbf{K}', \mathbf{k}_1) (\mathbf{x}_1, \mathbf{k}_1 | I_I | 0, \mathbf{x}_1')}{W_0 - H_1^0} \varphi_0^0 \\
&+ \sum_{\mathbf{K}_0'} \delta(\mathbf{K} - \mathbf{K}_0') (0, \mathbf{K}_0 | V_I | \mathbf{K}', \mathbf{k}_1) \\
&\times (\mathbf{x}_1, \mathbf{k}_1 | I_I | 0, \mathbf{x}_1') \\
&\times [1/(W_0 - H_1^0) - i\pi\delta(W_0 - H_1^0)], \tag{13}
\end{aligned}$$

in which  $\mathbf{K}_0'$  is a value of  $\mathbf{K}'$  satisfying  $H_1^0 = H_0^1$ . But these solutions, (12) and (13), are not consistent with the S.C., (2). We can, however, eliminate the Bosen by the S.C., (2), and we have

$$\begin{aligned}
\varphi_1^0(\bar{\mathbf{x}}_1', \mathbf{K}'') &= \left( \frac{8\pi Z e^2}{k^2 \hbar \omega_j} \right)^{1/2} (0, \mathbf{K}_0 | V_I | \mathbf{K}'', \mathbf{k}_1) \\
&\times \left[ \frac{1}{W_0 - H_1^0} - i\pi\delta(W_0 - H_1^0) \right] \varphi_0^0, \\
\text{etc. } (\bar{\mathbf{x}}_1' &= \mathbf{x}_1 + \mathbf{k}_1, \mathbf{x}_1 = \mathbf{x}_1, \mathbf{x}_2, \dots, k < k_c). \tag{14}
\end{aligned}$$

In this way the excited states of M.E. can be calculated easily. The details of the theory will be published soon in elsewhere.

- 1) D. Bohm and D. Pines, Phys. Rev. **92** (1953), 609.

# Relativistic Wave Equations under the Inhomogeneous Lorentz Group

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Relativistic wave equations, which are invariant under the inhomogeneous Lorentz group, are established by referring to the six dimensional pseudo-rotation group. It is also investigated what sorts of new degrees of freedom arise comparing with the case of the ordinary relativistic wave equations under the homogeneous Lorentz group, particularly from the view point of describing the new unstable particles. The Maxwell field is unaltered and the Dirac field has eight components: this coincides with the case of Pais'  $\omega$ -formalism.

## § 1. Introduction

According to the recently discovered phenomena concerning the new unstable particles, it seems that the ordinary relativistic wave equations under the homogeneous Lorentz group are not satisfactory enough to describe all the particles fully. Some endeavours have been made to generalize the wave equations. Pais' formalism<sup>1)</sup> using the  $\omega$ -space is the first trial. But we cannot explain what the  $\omega$ -space means. So it is only a semi-phenomenological theory in this stage of development. It is desired to obtain a generalized formalism based upon a physically meaningful group, which may be necessarily wider than the homogeneous Lorentz group as usual.

The group to be taken into account as the first step may be the inhomogeneous Lorentz group. This group is physically meaningful. The theory of the special relativity is invariant under coordinate translations also, so it is invariant under the inhomogeneous Lorentz group. However the ordinary formalism of the quantum theory is established only on the basis of the homogeneous Lorentz group, and the inhomogeneous Lorentz group is not taken into account fully. In other words the ordinary general wave equations exhaust the finite dimensional representations of the homogeneous Lorentz group, but they correspond, about the inhomogeneous Lorentz group, to the completely restricted representation where all the translations are represented with the identity. It is desired, since fairly before, to obtain general relativistic wave equations under the inhomogeneous Lorentz group. But the representations of it is in general of infinite dimensions. General cases are investigated by Wigner<sup>2)</sup>. But it seems to be too general for our present purpose. It is required to obtain merely slightly generalized equations. Similar circumstances also occurred in the case of the homogeneous Lorentz group. Also in that case irreducible representations are in general of infinite dimensions. We choosed from them the finite dimensional representations by referring to the four dimensional rotation group. Similar treatments are possible also



for the inhomogeneous Lorentz group by referring to the six dimensional pseudo-rotation group. This is an improvement of the method used by Dirac<sup>3)</sup> in order to introduce the wave equations in the conformal space.

The inhomogeneous Lorentz group can be represented by a subgroup of the six dimensional pseudo-rotation group. This subgroup is a linear transformation group of a subspace of the six dimensional pseudo-Euclidean space. The invariant wave equations under this subgroup in this subspace can be introduced without difficulties. The formalism translated from this into the Minkowski space is the one which we require. The Maxwell equation is invariant even under the conformal group, which involves the inhomogeneous Lorentz group and the dilatation. The fact that the electromagnetic field is not necessary to be amended is quite satisfactory for our purpose. Because, for the photon field any other degree of freedom, such as  $\omega$ -freedom, is not necessary according to the Pais' investigation. Moreover the Dirac field has eight components in our case and the baryon field has also eight components in the Pais' formalism. These facts hold automatically in our case without any special assumptions and our purpose is to yield the satisfactory freedom without any physical assumption such as the  $\omega$ -space.

It must be noticed that our treatment is based upon non-completely-reducible representations. In ordinary theory wave functions are concerned with several number of irreducible representations, but in our case they are concerned with a non-completely-reducible representation, by virtue of which finite dimensional faithful representations of the inhomogeneous Lorentz group became possible. In this line of investigations much more endeavours should be made.

## § 2. General formulation

The inhomogeneous Lorentz transformation

$$x'_\lambda = \sum a_\lambda{}^\mu x_\mu + a_\lambda \quad (2.1)$$

can be characterized by the matrix  $A = (a_\lambda{}^\mu)$  of the coefficients of the homogeneous transformation and the vector  $\mathbf{a} = (a_1, a_2, a_3, a_4)$  of the translation, so it can be denoted by the symbol  $L(A, \mathbf{a})$ . The law of its multiplication is

$$L(A, \mathbf{a}) \cdot L(B, \mathbf{b}) = L(AB, \mathbf{a} + A\mathbf{b}). \quad (2.2)$$

The identity transformation  $x'_\lambda = x_\lambda$  is denoted by  $L(E, 0)$ . The inverse transformation of  $L(A, \mathbf{a})$  is  $L(A^{-1}, -A^{-1}\mathbf{a})$ . Thus these transformations form a group, namely the inhomogeneous Lorentz group.

Let us consider the following transformation among the six real variables  $y_l$  ( $l=1, 2, \dots, 6$ ) which was taken up by Dirac<sup>3)</sup> for the first time;

$$\begin{aligned} y'_\lambda &= \sum a_\lambda{}^\mu y_\mu + a_\lambda (y_5 + y_6), \\ y'_5 + y'_6 &= y_5 + y_6, \\ y'_5 - y'_6 &= y_5 - y_6 - 2 \sum a^\lambda{}_\lambda a_\lambda{}^\mu y_\mu - \sum a^\lambda{}_\lambda a_\lambda (y_5 + y_6), \\ &(\lambda, \mu = 1, 2, 3, 4). \end{aligned} \quad (2.3)$$



The coefficients  $a_\lambda^{\mu}$  and  $a_\lambda$  are the same as those of (2.1). This transformation is determined with these coefficients only and can be denoted by the symbol  $T(A, \mathbf{a})$ . As can be easily shown, these transformations satisfy the same multiplication law as (2.2)

$$T(A, \mathbf{a}) \cdot T(B, \mathbf{b}) = T(AB, \mathbf{a} + \mathbf{b}). \quad (2.4)$$

Thus these transformations form a representation of the inhomogeneous Lorentz group. The explicit relation of them can be established by

$$x_\lambda = y_\lambda / (y_5 + y_6), \quad (2.5)$$

and indeed these  $x_\lambda$  transform according to  $L(A, \mathbf{a})$  when  $y_i$  transform according to  $T(A, \mathbf{a})$ .

The transformation  $T$  leaves moreover the quadratic form

$$y_1^2 + y_2^2 + y_3^2 - y_4^2 + y_5^2 - y_6^2 \quad (2.6)$$

invariant, and inversely we can easily show that the homogeneous transformation which leaves (2.6) and  $y_5 + y_6$  invariant takes the form of (2.3). Thus the transformation  $T$  forms a subgroup which leaves  $y_5 + y_6$  invariant among the pseudo-rotation group, which leaves (2.6) invariant, of the pseudo Euclidean space whose metric is

$$dy_1^2 + dy_2^2 + dy_3^2 - dy_4^2 + dy_5^2 - dy_6^2.$$

All the representations of this group are also representations of the inhomogeneous Lorentz group. The former will be obtained from the irreducible representations of the six dimensional pseudo-rotation group. By such a method we can introduce tensors and spinors for the inhomogeneous Lorentz group, even if this means a special restriction of the representation. Our general method is to establish firstly invariant equations under the above stated six dimensional subgroup and then to translate it into those of the Minkowski space.

The meaningful displacements for the group of  $T$  can be defined by the two conditions

$$\partial(y^i y_i) = 0, \quad \partial(y_5 + y_6) = 0, \quad (2.7)$$

and in this case as can be easily shown<sup>3)</sup>

$$\begin{aligned} \partial A &= \sum \partial A / \partial y^i \cdot \partial y^i \\ &= \sum (1 / (y_5 + y_6)) \{ (y_5 \partial A / \partial y^\lambda - y_\lambda \partial A / \partial y^5) \\ &\quad + (y_6 \partial A / \partial y^\lambda - y_\lambda \partial A / \partial y^6) \} dy^\lambda. \end{aligned} \quad (2.8)$$

Thus the meaningful differentiations are only

$$(1 / (y_5 + y_6)) (R_{5\lambda} + R_{6\lambda}), \quad (2.9)$$

where  $R_{lm}$  is the pseudo-rotation operator of the six dimensional space:

$$R_{lm} = y_l \partial / \partial y^m - y_m \partial / \partial y^l.$$

Now let us consider the equation

$$\{ (I_5^\lambda + I_6^\lambda) (R_{5\lambda} + R_{6\lambda}) + \kappa \} \Psi(y) = 0 \quad (2.10)$$

where  $f^{lm}$  is the representation matrix concerning the infinitesimal pseudo-rotation in the  $J'_l$ ,  $J'_m$  plane. The differential operator in this equation is meaningful for our subgroup. This equation is also invariant, because

$$\sum_{\lambda=1}^4 (I_5^\lambda + I_6^\lambda) (R_{5\lambda} + R_{6\lambda}) = \sum_{l=1}^6 (I_5^l + I_6^l) (R_{5l} + R_{6l}) \quad (2.11)$$

and this transforms in the same as  $\sum \{ (y_5 + y_6) y_i^2 \} \{ (y_5 + y_6) y_i \}$ ,  $(y_5 + y_6)$  and  $y^i y_i$  being invariant. Thus the equation (2.10) is satisfactory for the above stated subgroup.

For given  $(\alpha_\lambda^\mu)$  and  $\alpha_\lambda$  the operator  $R_{5\lambda} + R_{6\lambda}$  transforms, as can be easily seen from (4.3b), such as

$$R_{5\lambda}' + R_{6\lambda}' = \alpha_\lambda^\mu (R_{5\mu} + R_{6\mu}) \quad (2.12)$$

that is in the same way as  $\partial_\lambda = \partial / \partial x^\lambda$ . So if we write as  $\Psi(x)$  the quantity which transforms in the same way as  $\Psi(y)$  for given  $(\alpha_\lambda^\mu)$  and  $(\alpha_\lambda)$ , the equation

$$\{ (I_5^\lambda + I_6^\lambda) \partial_\lambda + \kappa \} \Psi(x) = 0 \quad (2.13)$$

is an wave equation invariant under the inhomogeneous Lorentz group. This is our general wave equation.

Now we must give some attention about the representation theoretical features of our method. The transformation  $T$  is not an irreducible representation, but it is a non-completely-reducible representation. In the ordinary theory of group representation the "Irreduzible Bestandteile" of representations are investigated in detail. For example the theory of the group character is powerful in the standpoint where representations whose Irreduzible Bestandteile coincide are not to be distinguished. Also in the ordinary field theory the wave functions are concerned only with irreducible representations, but they are composed of several irreducible parts. For example the vector meson field is described with an antisymmetric tensor and a vector. According to the Bhabha's investigation<sup>1)</sup> these two irreducible parts are two fractions from one irreducible quantity with respect to a five dimensional pseudo-rotation group, namely according to our previous calculation<sup>5)</sup> the vector meson field is concerned with an irreducible representation by the symmetric spinor of the second rank or the antisymmetric tensor in the five dimensions and this five dimensional irreducible quantity separates into the above stated two irreducible parts with respect to the four dimensional Lorentz group. In our present formulation wave functions are concerned with one non-completely-reducible representation. This is not undesirable at all. In this case the components of the wave functions are more closely related than the case of two irreducible parts. This means a new attempt, which should be investigated more seriously in our opinion.

### § 3. Spinor in six dimensions

The introduction of vectors and tensors is self evident, so we study that of spinors. First we introduce spinors for the group  $J'_i J'_l = i\pi\epsilon_{il}$ , and then confine it by the condition  $y_i + y_i = i\pi\epsilon_{il}$ . In this section we obtain the six dimensional spinors by generalizing the method which we used to have the five dimensional spinors formerly<sup>5)</sup>.

According to Brauer and Weyl<sup>(6)</sup>, if we have six matrices  $P^1, P^2, \dots, P^6$  which can generate the complete matrix algebra and satisfy the commutation relation

$$P^i P^j + P^j P^i = 2\sigma^{ij} E, \quad (3.1)$$

$$g_{11} = g_{22} = g_{33} = -g_{44} = g_{55} = -g_{66} = 1, \quad g_{ij} = 0 \quad (i \neq j),$$

$$E: \text{ unit matrix,}$$

there exists to any pseudo-rotation in six dimensions  $(a_l^m)$  the spinor transformation  $S$  with the relation

$$SP^l S^{-1} = \sum a_m^l P^m. \quad (3.2)$$

A representation of  $P^i$  is also given by them and its dimension is  $2^k$ , where  $k = n/2$  or  $(n-1)/2$  for even or odd dimensions  $n$  respectively. For  $n=6$ ,  $k=3$  and the dimension of  $P^i$  is eight. Let us choose another representation than theirs, that is the convenient one for the ordinary four dimensional spinors of van der Waerden and our five dimensional spinors

$$P^1 = \begin{pmatrix} 0 & P^{11} \\ P^{11} & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0 & P^{12} \\ P^{12} & 0 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 0 & P^{13} \\ P^{13} & 0 \end{pmatrix}, \quad P^4 = \begin{pmatrix} 0 & P^{14} \\ P^{14} & 0 \end{pmatrix},$$

$$P^5 = \begin{pmatrix} 0 & P^{15} \\ P^{15} & 0 \end{pmatrix}, \quad P^6 = \begin{pmatrix} E' & 0 \\ 0 & -E' \end{pmatrix} \quad (3.3)$$

where

$$P^{11} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad P^{12} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad P^{13} = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad P^{14} = \begin{pmatrix} 0 & E'' \\ -E'' & 0 \end{pmatrix}, \quad E' = \begin{pmatrix} E'' & 0 \\ 0 & E'' \end{pmatrix}$$

and  $\sigma_1, \sigma_2, \sigma_3$  are the Pauli's spin matrices and  $E''$  is the unit matrix of two dimensions.

First we calculate the  $S_{ij}(\theta)$  corresponding to the  $(a_l^m)$  of the  $ij$ -plane pseudo-rotation by the angle  $\theta$ ,  $T_{ij}(\theta)$  say, by solving the corresponding equation of (3.1). As can be easily shown from  $SP^6 S^{-1} = P^6$ ,  $S_{ij}$   $i, j = 1, 2, 3, 4, 5$  is of the form

$$S_{ij}(\theta) = \begin{pmatrix} S'_{ij}(\theta) & 0 \\ 0 & S'_{ij}(\theta) \end{pmatrix} \quad (3.4)$$

where  $S'_{ij}$  is the five dimensional spinor transformation, for example

$$S'_{5j}(\theta) = \begin{pmatrix} E'' \cos \theta/2 & i\sigma_j \sin \theta/2 \\ i\sigma_j \sin \theta/2 & E'' \cos \theta/2 \end{pmatrix} \quad (j=1, 2, 3), \quad (3.5a)$$

$$S'_{54}(\theta) = \begin{pmatrix} E'' \cosh \theta/2 & iE'' \sinh \theta/2 \\ -iE'' \sinh \theta/2 & E'' \cosh \theta/2 \end{pmatrix}. \quad (3.5b)$$

For the  $S_{6j}(\theta)$  we have

$$S_{6l}(\theta) = \begin{pmatrix} E' \cosh \theta/2 & iP^l \sinh \theta/2 \\ -iP^l \sinh \theta/2 & E' \cosh \theta/2 \end{pmatrix}, \quad (l=1, 2, 3, 5) \quad (3.6a)$$

$$S_{64}(\theta) = \begin{pmatrix} E' \cos \theta/2 & iP^4 \sin \theta/2 \\ -iP^4 \sin \theta/2 & E' \cos \theta/2 \end{pmatrix}. \quad (3.6b)$$

The general spinor transformations are those which are obtained by multiplying the matrices of the types (3.4) (3.6) in all means. This transformation is in general of eight dimensions.

The quantity which transforms according to  $S$  is the covariant spinor of the first rank. It has eight components. If the group is restricted to the five dimensional one by fixing the sixth axis,  $S$  reduces to

$$S \rightarrow \begin{pmatrix} S' & 0 \\ 0 & S' \end{pmatrix} \quad (3.7)$$

by  $SP^6S^{-1}=P^6$ , where  $S'$  is the transformation matrix of the five dimensional spinors. Thus in this case the spinor  $\Psi$  separates into two spinors of five dimensions  $\Psi'$ ,  $\Phi'$ :

$$(\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6, \Psi_7, \Psi_8) \rightarrow (\Psi'_1, \Psi'_2, \Psi'_3, \Psi'_4; \Phi'_1, \Phi'_2, \Phi'_3, \Phi'_4). \quad (3.8)$$

For the transformations where  $P^6$  is not invariant the two parts  $\Psi$  and  $\Phi$  transform mixed together.

The contravariant spinor can be obtained by the transformation  $C\Psi$  with the matrix  $C$  satisfying

$$C^T P^l C^{-1} = P^l \quad (l=1, 2, \dots, 6) \quad (3.9)$$

where  ${}^T P^l$  is the transposed matrix of  $P^l$ . Such matrix  $C$  can be obtained as

$$C = \begin{pmatrix} C' & 0 \\ 0 & C' \end{pmatrix}, \quad C' = \begin{pmatrix} 0 & 1 & \vdots & 0 & 0 \\ -1 & 1 & \vdots & 0 & 0 \\ \hline 0 & 0 & \vdots & 0 & -1 \\ 0 & 0 & \vdots & 1 & 0 \end{pmatrix}. \quad (3.10)$$

Thus we have the rule for taking up or down spinor suffices:

$$(\Psi^1, \Psi^2, \Psi^3, \Psi^4, \Psi^5, \Psi^6, \Psi^7, \Psi^8) = (\Psi_2, -\Psi_1, -\Psi_4, \Psi_3, \Psi_6, -\Psi_5, -\Psi_8, \Psi_7). \quad (3.11)$$

The contravariant spinors transform according to  $CS$  and the inner product of the type  $\sum \Psi^i \Psi_i$  ( $i=1, 2, \dots, 8$ ) becomes a scalar. Thus by defining the spinors of higher ranks in the same way as the tensor analysis, we can obtain a useful spinor analysis of six dimensions. The detailed developements of the 6 dimensional spinor analysis may be given elsewhere if need.

#### § 4. Tensors and spinors for the inhomogeneous Lorentz group

##### (i) Tensors

The transformation  $T(C, \alpha)$  can be expressed in the matrix form as

$$T = \begin{pmatrix} & & & & a_1 & 0 \\ & & & & a_2 & 0 \\ & & & & a_3 & 0 \\ & & & & a_4 & 0 \\ (a_\lambda^\mu) & & & & & \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -2a_1' - 2a_2' - 2a_3' - 2a_4' & -a^\lambda a_\lambda & 1 \end{pmatrix}, \quad (4.1a)$$

or

$$T = \begin{pmatrix} & & & & a_1 & a_1 \\ & & & & a_2 & a_2 \\ & & & & a_3 & a_3 \\ & & & & a_4 & a_4 \\ (a_\lambda^\mu) & & & & & \\ -a'^1 - a'^2 - a'^3 - a'^4 & 1 - 1/2a^\lambda a_\lambda & -1/2a^\lambda a_\lambda \\ a'^1 & a'^2 & a'^3 & a'^4 & 1/2a^\lambda a_\lambda & 1 + 1/2a^\lambda a_\lambda \end{pmatrix}, \quad (4.1b)$$

$$(a'^\lambda = a^\mu a_\mu^\lambda),$$

respectively in the representation of the coordinate system  $(y_1, y_2, y_3, y_4, y_5, y_6, y'_1, y'_2, y'_3, y'_4, y'_5, y'_6)$  or  $(x_1, x_2, x_3, x_4, x_5, x_6)$ . From (4.1a) we can see that  $(A, \alpha)$  is not irreducible but noncompletely-reducible. As stated in § 2,  $T$  forms an one-valued representation of the inhomogeneous Lorentz group. Here we will call as a vector the quantity depending on  $x_\lambda$  and transforming according to  $T(A, \alpha)$  for given inhomogeneous Lorentz transformation  $(A, \alpha)$ .

The covariant vector is defined by

$$A^i = g^{im} A_m. \quad (4.2)$$

The tensors of higher ranks can be defined in the usual way and thus tensor analysis is introduced. However, in this case we must notice that scalars can be made in the two ways, namely as  $A_5 + A_6$  and  $A^i B_i$ .

Now let us consider the transformation of antisymmetric tensor  $F_{lm}$ . The transformation coefficient from  $F_{\mu\nu}$  to  $F'_{lm}$  is  $a_l^\mu a_m^\nu - a_l^\nu a_m^\mu$ , where  $a_i^j$  is the vector transformation coefficient, Latin indices running from 1 to 6. Thus we have

$$F'_{\lambda\mu} = a_{\lambda\mu}^{\rho\sigma} F_{\rho\sigma} + a_{\lambda\mu}^{-\nu} (F_{\nu 5} + F_{\nu 6}), \quad (4.3a)$$

$$F'_{\lambda 5} + F'_{\lambda 6} = a_\lambda^\mu (F_{\mu 5} + F_{\mu 6}), \quad (4.3b)$$

$$F'_{\lambda 5} - F'_{\lambda 6} = a_\lambda^\mu (F_{\mu 5} - F_{\mu 6}) + a_\lambda^{-\mu\nu} F_{\mu\nu} + a_\lambda F_{56}, \quad (4.3c)$$

$$F'_{56} = a'^\lambda (F_{\lambda 5} - F_{\lambda 6}) + F_{56}, \quad (4.3d)$$

where

$$\begin{aligned} a_{\lambda\mu}^{-\rho\sigma} &= a_\lambda^\rho a_\mu^\sigma - a_\lambda^\sigma a_\mu^\rho, \\ a_{\lambda\mu}^{-\nu} &= a_\lambda^\nu a_\mu - a_\lambda a_\mu^\nu, \end{aligned} \quad (4.4)$$



$$\alpha'^{\mu\nu} = \alpha_\lambda^\mu \alpha'^\nu - \alpha'^\mu \alpha_\lambda^\nu.$$

From these results we can see that non-completely-reducible sets are  $(F_{\lambda 5} + F_{\lambda 6})$ ,  $(F_{\lambda u}$ ,  $F_{v5} + F_{v6})$  and (all  $F_{lm}$ ).

Here we must notice that ordinary four vectors and tensors are also possible. They coorespond to particular representations where all translations are represented by the unity always. For example, displacements and differential operators are ordinary four vectors.

## (ii) Spinors

The spinors for the group  $T$  can be obtained easily by solving the corresponding equation (3.2). It is evident that we have the ordinary four dimensional spinor transformation for the homogeous transformation  $(A, 0)$ . Since in this case

$$\mathcal{V}'_\lambda = \alpha_\lambda^\mu \mathcal{V}_\mu, \quad \mathcal{V}'_5 = \mathcal{V}_5, \quad \mathcal{V}'_6 = \mathcal{V}_6,$$

we have

$$S_{\lambda\mu}(\theta) = \begin{pmatrix} S'_{\lambda\mu}(\theta) & 0 \\ 0 & S'_{\lambda\mu}(\theta) \end{pmatrix} \quad (S_{54} = S_{64} = E) \quad (4.6)$$

and the spinor separates into two irreducible ordinary spinors with four components.

For a pure translation for example in the  $x_1$  direction the transformation  $T(E, \alpha_1)$  can be decomposed in the following way

$$T(E, \alpha_1) = T_{15}(\theta_1) \cdot T_{56}(\theta_2) \cdot T_{16}(\theta_3), \quad (4.7)$$

where

$$\tan \theta_1 = \frac{\alpha_1}{1 - \alpha_1^2/2}, \quad \tanh \theta_2 = \frac{\alpha_1^2/2}{\sqrt{1 + \alpha_1^2/4}}, \quad \tanh \theta_3 = \frac{\alpha_1}{1 + \alpha_1^2/2}. \quad (4.8)$$

Thus the matrix

$$S(E, \alpha_1) = S_{15}(\theta_1) \cdot S_{56}(\theta_2) \cdot S_{16}(\theta_3) \quad (4.9)$$

represents the translation by  $\alpha_1$  in the  $x_1$  direction  $(\alpha_1, 0, 0, 0)$ . Representations of the translations in other directions can be obtained in the same way. The general spinor transformations are those which are obtained by multiplying the matrices of the types (4.6) and (4.9) in all means. Thus we have the spinor transformations for the inhomogeneous Lorentz group. Treatments of higher rank spinors are possible in the usual way.

## § 5. The Maxwell field

We introduce a vector potential  $A_l$ ,  $l=1, \dots, 6$ , which transforms according to  $T(A, \alpha)$ , in order to describe an electromagnetic field. We proceed in the previously stated general method, namely to establish invariant equations under the previously stated subgroup of the six dimensional pseudo rotation group and then to translate it into those of the Minkowski space. The natural generalization of the second order wave equation may be

$$(R_5^l + R_6^l)(R_{5l} + R_{6l})A_m(y) = 0. \quad (5.1)$$

In this equation the case for  $l=5, 6$  vanish automatically, and we can write it as

$$(R_5^\lambda + R_6^\lambda) (R_{5\lambda} + R_{6\lambda}) A_m(y) = 0. \quad (5.2)$$

The corresponding equation in the Minkowski space is

$$\partial^\lambda \partial_\lambda \cdot A_m(x) = 0. \quad (5.3)$$

The Lorentz condition cannot be generalized in the form

$$(R_5^l + R_6^l) A_l(y) = 0, \quad (5.4)$$

because this involves the case  $l=5, 6$  explicitly and has no meaning unless the condition

$$R_{56}(A_5 + A_6) = 0 \quad (5.5)$$

holds. The meaningful generalization may be

$$(I_5^l + I_6^l) (R_{5l} + R_{6l}) (A) = 0, \quad (5.6)$$

where  $(A)$  is the six column vector of  $A_l$  and  $I^{rs}$  is the representation matrix for the vector transformation  $D_{1,0,0}$

$$(I^{rs})_{lm} = 1/2 \cdot (g_l^r g_m^s - g_m^r g_l^s). \quad (5.7)$$

The equation (5.6) can be written in the following way with components

$$(R_5^\lambda + R_6^\lambda) A_\lambda(y) = 0, \quad (5.8a)$$

$$(R_5^\lambda + R_6^\lambda) (A_5(y) + A_6(y)) = 0. \quad (5.8b)$$

The corresponding equation in the Minkowski space is

$$\partial^\lambda A_\lambda(x) = 0, \quad (5.9a)$$

$$\partial^\lambda (A_5(x) + A_6(x)) = 0. \quad (5.9b)$$

The field strength  $F_{lm}$  may be introduced by

$$F_{lm} = (R_{5l} + R_{6l}) A_m - (R_{5m} + R_{6m}) A_l. \quad (5.10)$$

With this definition and the equation (5.1), (5.8) we can easily show first that the cases of  $l=5, 6$  in the left of the following expression vanish automatically and so

$$(R_5^l + R_6^l) F_{lm}(y) = (R_5^\lambda + R_6^\lambda) F_{\lambda m}(y) \quad (5.11)$$

and secondly that it satisfies automatically the equation

$$(R_5^l + R_6^l) F_{lm}(y) = 0, \quad (5.12)$$

or

$$(R_5^\lambda + R_6^\lambda) F_{\lambda m}(y) = 0. \quad (5.13)$$

However, we must notice that the formal definition (5.10) is not meaningful for every  $l, m$ , for  $R_l + R_{6l}$  has a meaning only for  $l=1, \dots, 4$ . Thus we must restrict to the class  $(F_{\lambda\mu}, F_{55} + F_{66})$ . But  $F_{55} + F_{66}$  vanishes on account of (5.8). Thus the meaningful non vanishing field is not to be amended. In the Minkowski space evidently we have

$$F_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda, \quad (5 \cdot 14)$$

$$\partial^\lambda F_{\lambda\mu}(x) = 0. \quad (5 \cdot 15)$$

The gauge transformation may be given formally by

$$A_l \rightarrow A_l + (R_{5l} + R_{6l})S, \quad (5 \cdot 16)$$

with arbitrary  $S$  satisfying

$$(R_5^l + R_6^l)(R_{5l} + R_{6l})S = 0. \quad (5 \cdot 17)$$

After this transformation all the equations for  $A_l$  (6,2), (5,9), (5,9) are also satisfied automatically as before. The significant parts of the transformation may be

$$A_\lambda \rightarrow A_\lambda + (R_{5\lambda} + R_{6\lambda})S(y),$$

$$A_5 + A_6 \rightarrow A_5 + A_6,$$

with

$$(R_5^\lambda + R_6^\lambda)(R_{5\lambda} + R_{6\lambda})S(y) = 0.$$

The gauge transformation in the Minkowski space becomes

$$A_\lambda(x) \rightarrow A_\lambda(x) + \partial_\lambda S(x), \quad (5 \cdot 18a)$$

$$A_5(x) + A_6(x) \rightarrow A_5(x) + A_6(x), \quad (5 \cdot 18b)$$

with

$$\partial^\lambda \partial_\lambda S(x) = 0. \quad (5 \cdot 18c)$$

This is equivalent to the one in the ordinary theory. Thus we have seen that any intrinsic alteration does not occur for the Maxwell field. Further developments are possible in the same way as usual. For example the equation

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

and the current

$$j_\mu = \partial^\lambda F_{\lambda\mu}$$

satisfying the conservation law

$$\partial^\mu j_\mu = 0$$

are unaltered. This is, however, selfevident. As were verified by several investigators,<sup>(3,7)</sup> the Maxwell equation is invariant under the conformal group which involves the inhomogeneous Lorentz group.

## § 6. The Dirac field

The Dirac field is the one which is described by a spinor of the first rank. The representation matrix  $I^{ij}$  of the infinitesimal pseudo-rotation concerning to the spinor representation can be obtained from the matrix of the spinor transformation  $S^{ij}(\theta)$  as

$$I^{ij} = (\partial S^{ij}(\theta) / \partial \theta)_{\theta=0}. \quad (6 \cdot 1)$$

In this method we have

$$I^{5\mu} = i/2 \begin{pmatrix} P'^5 & P'^\mu & 0 \\ 0 & P'^5 & P'^\mu \end{pmatrix}, \quad I^{6\mu} = i/2 \begin{pmatrix} 0 & P'^\mu \\ -P'^\mu & 0 \end{pmatrix} \quad (6.2)$$

for our representation. According to the general method introduced in § 2 we will determine the special form of (2,13) for the spinor representation. Thus we have

$$\{\alpha^\lambda \partial_\lambda + i\kappa\} \Psi = 0, \quad (6.3)$$

where

$$\alpha^\lambda = 1/2 \begin{pmatrix} P'^5 & P'^\mu & -P'^\mu \\ P'^\mu & P'^5 & P'^\mu \end{pmatrix}. \quad (6.4)$$

Using the two parts  $\Psi'$  and  $\Phi'$ , having four components, of  $\Psi = (\Psi', \Phi')$  the proposed equation (6,3) can be written in the following way

$$\{P'^\mu \partial_\mu + i\kappa P'^5\} \Psi' = i\kappa \Phi', \quad (6.5a)$$

$$\{P'^\mu \partial_\mu + i\kappa P'^5\} \Phi' = -i\kappa \Psi'. \quad (6.5b)$$

This is the generalized wave equation invariant under the inhomogeneous Lorentz group. This equation involves one more degree of freedom than the usual one by virtue of the two parts of the wave function. Indeed it is similar to the Pais' equation for baryons in the point that their wave functions have eight components.

As can be easily shown, each component of the wave function satisfies the second rank wave equation

$$\begin{aligned} \{\square - \kappa^2\} \Psi' &= 0, \\ \{\square - \kappa^2\} \Phi' &= 0. \end{aligned} \quad (6.6)$$

Interactions with the electromagnetic field  $A_\mu$  can be introduced as usual by the substitution

$$\partial_\mu \rightarrow \partial_\mu - ie A_\mu \quad (6.7)$$

without any difficulty. The spin magnetic moment which arises due to this substitution is the same as the ordinary one.

The charge current density can be introduced in the usual way. The adjoint wave function  $\Psi^*$  is defined as the one which satisfies the adjoint wave equation

$$\Psi^* \{\alpha^\lambda \partial_\lambda - i\kappa\} = 0, \quad (6.7)$$

where  $\partial_\lambda$  operates from the right. The charge current vector is defined as

$$\begin{aligned} j^\lambda &= \Psi^* \alpha^\lambda \Psi \\ &= 1/2 (\Psi'^* \gamma^\lambda \Psi' + \Phi'^* \gamma^\lambda \Phi' - \Psi'^* \gamma^5 \gamma^\lambda \Phi' + \Phi'^* \gamma^5 \gamma^\lambda \Psi'), \end{aligned} \quad (6.9)$$

which satisfies the conservation law

$$\partial_\lambda j^\lambda = 0. \quad (6.10)$$

The quantization can be performed in the usual way, namely

$$|\Psi^*(x), \Psi(x')|_+ = (\alpha^\lambda \partial_\lambda - i\kappa) J(x-x') \quad (6 \cdot 11)$$

where  $J$  is the ordinary Jordan and Pauli's delta. Although  $J(x)$  is not invariant but  $J(x-x')$  is invariant under the inhomogeneous group, momenta being ordinary four vectors.

Further detailed developments may be evident. The difference from the ordinary case is that  $\alpha^\lambda$  differs from Dirac's  $\gamma^\lambda$  and the wave function has twice more components than the usual one, transforming according to a wider group.

Wave equations for particles with higher spins can be obtained by considering representations by means of spinors of higher ranks and establishing the equation (2·13). Another method of generalization to higher spins is the following spinor analytical one. We write the equation (6·6) in the spinor form

$$\partial_r^s \Psi_s = i\kappa \Psi_r \quad (6 \cdot 12)$$

where  $i\partial_r^s$  is the  $(\ )_r^s$  element of the matrix  $(\alpha^\lambda \partial_\lambda)$ . The fields for higher spins may be introduced in the form

$$\partial_r^s \Psi_{st\dots} = i\kappa \Psi_{rt\dots}, \quad (6 \cdot 13)$$

even if the non-completely-reducible condition might cause complications about the higher rank spinors.

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## Theory of the Electronic Polaron and Ionization of a Trapped Electron by an Exciton

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Introducing the concept of the electronic polaron, which is an analogue of the usual lattice polaron, we have considered its motion under the influence of a trapping potential, by using a simplified model for the electronically polarizable crystal such that the excited states of crystal electrons consist of a single exciton band. By a suitable transformation we obtain an expression for the Hamiltonian which consists of the following three parts: the first part describes the motion of an electron under the potential in a dielectric medium in the classical way, the second part represents the energy of the electronic polarization field the quanta of which are nothing but free excitons, and the last part is the interaction energy between these two parts which causes the production or annihilation of excitons in the neighbourhood of the potential by exchanging energy with the electron. Treating the last part as a perturbation, we have calculated the probability for the ionization of a trapped electron by an exciton. After correcting the oversimplification of the model, we have applied the obtained result to actual crystals, leading to a conclusion that in alkali-halides the excitons annihilate rather by ionizing  $F$ -electrons than by spontaneous emission if the density of the  $F$ -centers is larger than  $10^{16}$  per c.c., in consistence with the experimental results of Apker and Taft on the external photoelectric effect.

### § 1. Introduction and basic idea

The interaction of a conduction electron with lattice vibrations in an ionic crystal has been investigated very actively of late years,<sup>(1)-(3)</sup> because of its theoretical and practical importances, as well as its mathematical interest and difficulty. When the velocity of the electron is so slow that no phonon can be emitted, we can visualize the *stationary state* of the system by considering the electron, clothed with the polarization or virtual phonons around itself, to move freely in the crystal with a self-energy and a corrected effective mass both of which are the main subjects of the so-called *polaron* problem (we are considering only the case of the absolute zero of temperature). On the other hand, when the velocity of the electron exceeds some critical value characteristic of the crystal, the stationary state description is no longer appropriate; we should treat the interaction to be responsible for the *transition* of the electron between different states, though in actual crystals such as alkali-halides the interaction is so strong that the above statement has only an approximate meaning.

Now, as is well known, the general polarization consists of two parts: "displacement polarization" or lattice polarization as state above on the one hand, and "electronic polarization" on the other hand, the latter being more general as it plays a rôle also in

non-polar crystals where the former does not. A conduction electron in an insulator or a semiconductor interacts also with the electronic polarization of the crystal, thus leading to the concept of the "electronic polaron"<sup>4)</sup> analogous to the usual or "lattice polaron" stated above. For the ordinary velocities of the electron, the electronic polarization induced around it is generally considered to follow the motion of the electron almost perfectly, thus it is the usual method of description<sup>5)</sup> to consider the electron as a classical point charge imbedded in a medium with a dielectric constant  $\kappa_0$  which is to be identified with the optical dielectric constant of the crystal. As we have shown previously,<sup>6)</sup> this description corresponds to the quantum mechanical adiabatic approximation between the conduction and the crystal electrons. In the electronic polaron theory the point charge has to be replaced by somewhat spread cloud of charge due to uncertainty principle as in the case of a lattice polaron.<sup>1)</sup> Now, in our case too, there is a critical velocity of the electron beyond which the electronic polarization cannot follow the electron, or the stationary state description is inappropriate. If there is any analogy or parallelism between the electronic and lattice polarizations, the electron with such high velocities can emit *quanta of electronic polarization*, and we have intimate connection between the electronic polaron state and the processes such as emission and absorption of these quanta by the electron, both of which are to be treated under a unified formulation in the same way as the scattering of an electron by lattice vibrations is considered to be nothing but another aspect of the lattice polaron problem.

What is meant by the "quanta of electronic polarization" is easy to understand if we consider in analogy with the case of an isolated atom. In insulating and semiconducting crystals, we should take them as the excitation of filled band electrons to the conduction band or the exciton levels if any. It is qualitatively inferred from perturbation theory that the narrower the energy gap between the filled and conduction band is, the more easily the crystal can be polarized. In fact, there is substantially an antiparallelism between high frequency dielectric constant  $\kappa_0$  and the gap energy  $\varepsilon$  for many crystals of different types, as is seen from Tab. 1. Roughly speaking the gap energy  $\varepsilon$  in this case plays the same rôle as the phonon energy  $\hbar\omega$  does in case of the lattice polaron, and the relaxation time of the electronic polarization is given by

$$\tau \sim \hbar/\varepsilon.$$

It is interesting to note that the interaction of the conduction electron with the electronic polarization in an insulating crystal is analogous to the interaction of a free electron with vacuum polarization<sup>7)</sup> which is caused by the virtual creation of pairs of an electron and a positron. The most essential difference lies in the circumstance that a conduction electron in a crystal, if its velocity is sufficiently large, can emit pairs of another conduction electron and a positive hole, while it is impossible for a free electron in vacuum to create pairs of an electron and a positron except in the neighbourhood of

Tab. 1. Gap energies  $\varepsilon$  and optical dielectric constants  $\kappa_0$  of various crystals

	$\varepsilon(\text{eV})$	$\kappa_0$
NaCl	9.6	2.25
MgO	6	2.95
diamond	5.5	5.85
AgCl	5	4.01
BaO	4.8	4
Si	1.1	12.5
Ge	0.8	18.5

a strong field such as is produced by a nucleus, as is evident from the Lorentz invariance.

Owing to the complexity of the band structures and the lack of the accurate knowledge on the band wave functions in actual crystals, it seems very difficult to treat quantitatively the interaction stated above. In this work we take a model as simple as possible without spoiling the most essential aspect of the problem.

First of all we assume only one exciton band for the excited states of the crystal electrons, instead of the series of exciton bands and an ionization continuum. It is true that this model is inappropriate to the discrimination between the production of excitons and the ionization of filled band electrons to the conduction band\*, but it is convenient for our present aim, which consists in discussing the electronic polaron problem in connection with the annihilation and production of an exciton by an electron. With this model we shall derive, in § 2, the Hamiltonian for the system composed of crystal electrons and an additive electron, which has a form characteristic of the Hamiltonian for the system of a particle and a boson field.

One of the subjects which are interesting in case of the electronic polaron problem is to see whether and how the classical picture of a conduction electron as a point charge in a dielectric medium is derived from general formulation. For this purpose it is convenient to consider a fixed charge distribution due to any kind of imperfection which interacts with the electron in the field of the electronic polarization. In § 3 we set up the Hamiltonian for this system by making use of the result obtained in § 2, and then express it in the new variables by a suitable transformation so as to derive the classical picture stated above. The new expression thus obtained of the Hamiltonian has diagonal elements which correspond to the two independent systems, one for the electronic polaron moving under the influence of the potential due to the imperfection with a shielding factor  $1/\kappa_0$ , the other for the free excitons or the electronic polarization waves, the origin of whose co-ordinates are somewhat modulated near the imperfection. The expression has also off-diagonal elements corresponding to the interaction between the two systems in the neighbourhood of the imperfection.

Treating the last term as a perturbation, we have calculated, in § 4, the transition probability for the process in which a free exciton annihilates near the imperfection by giving its energy to the trapped electron (strictly speaking, the electronic polaron) and ionizing it.

The oversimplification of the model on which we have formulated the above method can be corrected by connecting the interaction constant with the oscillator strength of the exciton transition so as to be applicable to real crystals. This is done in § 5, the result of which tells us, for instance, that in alkali-halides an exciton annihilates by ionizing  $F$ -center electrons instead of emitting radiation spontaneously if the density of the  $F$ -centers is larger than  $10^{16}/\text{c.c.}$  This is in accordance with the experimental results by Apker and Taft<sup>10 11)</sup> on the one hand, and explains to some extent, as Seitz<sup>12)</sup> has been

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\* For these two processes, Kubo and Takano<sup>8)</sup> calculated the transition probabilities, using Wannier's wave function<sup>9)</sup> for the exciton states

expecting, the absence of emission by an exciton from any experiment carried out so far, on the other hand.

## § 2. The electronic polarization as a boson field

Let us consider an insulating crystal  $C$ , and take into account the motion of *valence* electrons only which are responsible for the most part of the electronic polarization, the *inner* electrons and the nuclei being considered as the sources of the potential for the former. Denoting the coordinates of the valence electrons by  $\mathbf{r}_1, \mathbf{r}_2, \dots$  and  $\mathbf{r}_N$ , we can set up the Hamiltonian  $H_C$  for these  $N$ -electrons. In addition to this system we consider an *extra* electron  $e$  to be introduced into the conduction band of the crystal. The Coulomb interaction of  $e$  with the valence electrons is denoted by

$$U(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \sum_{i=1}^N e^2 / |\mathbf{r} - \mathbf{r}_i|, \quad (2.1)$$

and the interaction of  $e$  with the inner electrons and the nuclei of the crystal, together with the kinetic energy of  $e$  itself, is written as  $H_v$ . Thus the total Hamiltonian takes the form:

$$H = H_0(\mathbf{r}) + H_v(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) + U(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N). \quad (2.2)$$

Now, the system  $C$  of  $N$ -electrons, in the one body approximation, has excited states in which some of the electrons are excited to the conduction band, but in the next approximation, in which the interaction of the excited electrons and their counterparts—the positive holes—is taken into account, we have to consider a series of the exciton bands below the ionization continuum stated above. In ionic crystals these exciton bands play a very important rôle in some phenomena. Let us assume, for simplicity, that the excited states of the  $N$ -electrons as a whole can be represented by a *single exciton band*, each exciton having an energy  $\varepsilon$  irrespective of its wave number  $\mathbf{w}$ . Then the eigenstates of the system  $C$  can be specified by the number  $n_w$  of excitons for each  $\mathbf{w}$ , and we can write

$$\begin{aligned} H_C \Psi(\dots, n_w, \dots, n_{w'}, \dots | \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \\ = (E_0 + \varepsilon \sum_w n_w) \Psi(\dots, n_w, \dots, n_{w'}, \dots | \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N). \end{aligned} \quad (2.3)$$

The above consideration, of course, is valid only when the total number of excitons  $\sum_w n_w$  is much smaller than  $N$ .

The next procedure is to find the matrix elements of  $U$  in the  $\{n_w\}$  representation. For this purpose we take a tightly bound atomic orbital approximation<sup>(13)</sup> for the wave function  $\Psi$ . Denoting the ground and excited states of an isolated atom by  $\phi^g(\mathbf{r})$  and  $\phi^e(\mathbf{r})$ , respectively, we construct a normalized Slater determinant



$$\Phi(m_1, m_2, \dots, m_n | \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi^g(\mathbf{r}_1 - \mathbf{R}_1) & \phi^g(\mathbf{r}_2 - \mathbf{R}_1) & \dots & \dots & \dots \\ \phi^g(\mathbf{r}_1 - \mathbf{R}_2) & \phi^g(\mathbf{r}_2 - \mathbf{R}_2) & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \phi^e(\mathbf{r}_1 - \mathbf{m}_1) & \phi^e(\mathbf{r}_2 - \mathbf{m}_1) & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \phi^e(\mathbf{r}_1 - \mathbf{m}_2) & \phi^e(\mathbf{r}_2 - \mathbf{m}_2) & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \phi^g(\mathbf{r}_1 - \mathbf{R}_N) & \phi^g(\mathbf{r}_2 - \mathbf{R}_N) & \dots & \dots & \dots \end{vmatrix} \quad (2.4)$$

for the state in which the  $m_1$ -th,  $m_2$ -th, ... and  $m_n$ -th atoms of the crystal are excited. We have written  $\mathbf{m}_1, \mathbf{m}_2, \dots$  instead of  $\mathbf{R}_{m_1}, \mathbf{R}_{m_2}, \dots$  for the sake of simplicity. The overlaps between all pairs of atomic orbitals are neglected. The eigenstates of  $H_0$  are such that each of the excited positions (for example,  $m_1$ ) propagates from atom to atom with a definite wave number, and these excitation waves can be taken as independent of each other if the total number  $n$  of excitons is much smaller than  $N$  as has been assumed above; because the effect of the coincidence of  $m_i$  and  $m_j$  for  $i \neq j$  (a collision between two excitons) can be neglected for such cases. Thus the normalized eigenfunctions  $\Psi$  of  $H_0$  can be written down as follows:

$$\begin{aligned} \Psi(\dots, n_w, \dots, n_{wt}, \dots | \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \\ = \{N^n \prod_w (n_w!)\}^{-1/2} \sum_{m_1} \sum_{m_2} \dots \sum_{m_n} \exp \left[ \sum_w i\mathbf{w} \cdot (\mathbf{m}_{w1} + \mathbf{m}_{w2} + \dots + \mathbf{m}_{wn, n_w}) \right] \\ \times \Phi(m_1, m_2, \dots, m_n | \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)^* \\ \sum_w n_w = n. \end{aligned} \quad (2.5)$$

In this expression the sets of  $\{\mathbf{m}_{w1}, \mathbf{m}_{w2}, \dots, \mathbf{m}_{wn, n_w}\}$  for all  $\mathbf{w}$ 's coincide with  $\{m_1, m_2, \dots, m_n\}$ , as a whole; we have only to allow the latter elements to each wave number,  $n_w$ , elements assigned to the wave number  $\mathbf{w}$ .

Now that the explicit form of  $\Psi$  is known, we can calculate the matrix elements of the Coulomb interaction  $U$ . Using (2.1) and (2.5), we have

$$\begin{aligned} (\dots, n_w, \dots, n_{wt}, \dots | U | \dots, n'_{wt}, \dots, n'_w, \dots) \\ = \{N^n \prod_w (n_w!)\}^{-1/2} \{N^{n'} \prod_w (n'_w!)\}^{-1/2} \end{aligned}$$

\* In the previous report (letters to the editor) the author has restricted the values of  $n_w$  to 0 and 1 only, leading to the Hamiltonian in which the creation and annihilation operators are those for the Fermions. Although it makes no difference so far as the results obtained in this work are concerned whether we use the Fermion or Boson operators, there is no reason in principle for the restriction of  $n_w$  values. The author is indebted to Professor H. Fröhlich who has kindly suggested to him that the exciton should be treated as a Boson.



$$\begin{aligned} & \times \sum_{m_1} \cdots \sum_{m_n} \sum_{m'_1} \cdots \sum_{m'_n} \exp \left[ \sum_{\mu} i \mathbf{r} \cdot (m'_{w,1} + \cdots + m'_{w,n'} - m_{w,1} - \cdots - m_{w,n_n}) \right] \\ & \times \sum_{i=1}^N \int \frac{e^2}{|\mathbf{r} - \mathbf{r}_i|} \phi^*(m_1, \cdots, m_n | \mathbf{r}_1, \cdots, \mathbf{r}_N) \phi(m'_1, \cdots, m'_{n'} | \mathbf{r}_1, \cdots, \mathbf{r}_N) \prod_{i=1}^N d\mathbf{r}_i. \end{aligned} \quad (2.6)$$

The last integral is different from zero in the following three cases:

- (i)  $n = n'$  and the set  $(m_1, m_2, \cdots, m_n)$  as a whole coincides with  $(m'_1, m'_2, \cdots, m'_{n'})$ .
- (ii)  $n + 1 = n'$  and the  $(m_1, m_2, \cdots, m_n)$  as a whole coincides with  $(m'_1, m'_2, \cdots, m'_{j-1}, m'_{j+1}, \cdots, m'_{n'})$ , where  $j$  may take any one of  $(1, 2, \cdots, n')$ .
- (iii) The reverse case of (ii).

In case (i), we have the electrostatic potential at the position  $\mathbf{r}$  due to the  $N$ -electrons, those which belong to the  $m_1$ -th,  $m_2$ -th,  $\cdots$  or  $m_n$ -th atom being excited. Now the charge distribution

$$|\phi^e(\mathbf{r})|^2 - |\phi^o(\mathbf{r})|^2$$

has neither a total charge nor a dipole moment, because both wave functions are of atomic type and have definite parities. If we take account of the point and dipole potentials only, and neglect the potentials of the higher multipoles we can equate the integral in (2.6) to that integral in which the ground state wave function  $\phi_0$  of  $H_0$  (no excited atoms) is substituted. Then the summation over  $(m_1, m_2, \cdots, m_n)$  leads to a non-zero value only when the set  $(m'_{w,1}, \cdots, m'_{w,n'})$  as a whole coincides with the set  $(m_{w,1}, \cdots, m_{w,n_w})$  for each  $\mu$ . In the case (i), therefore, we have non-zero matrix elements only when  $n_w = n'_{w'}$  for all  $\mu$ 's, moreover their values are all equal:

$$\begin{aligned} & (\cdots, n_w, \cdots, n_{w'}, \cdots | U | \cdots, n_w, \cdots, n_{w'}, \cdots) \\ & = (0, 0, \cdots | U | 0, 0, \cdots) = U_0(\mathbf{r}). \end{aligned} \quad (2.7)$$

In case (ii) the integral in (2.6) can be calculated, in the approximation stated above, as

$$\begin{aligned} & \sum_{i=1}^N \int \frac{e^2}{|\mathbf{r} - \mathbf{r}_i|} \phi^*(m_1, \cdots, m_n | \mathbf{r}_1, \cdots, \mathbf{r}_N) \phi(m_1, \cdots, m_n, m'_j | \mathbf{r}_1, \cdots, \mathbf{r}_N) \prod_{i=1}^N d\mathbf{r}_i \\ & = \frac{N(N-1)!}{N!} \varphi_{m'_j}(\mathbf{r}), \end{aligned} \quad (2.8)$$

where  $\varphi_{m'_j}(\mathbf{r})$  is the potential due to the dipole moment  $\mu$  located at the  $m'_j$ -th atom,  $\mu$  being given by

$$\mu = \int \phi^{o*}(\mathbf{r}) \mathbf{r} \phi^e(\mathbf{r}) d\mathbf{r}, \quad (2.9)$$

that is, the dipole-moment for the atomic transition. The summation over  $m_1, m_2, \cdots$  and  $m_n$  ( $m'_j$  being fixed) leads to a non-zero value only when  $n'_w = n_w$  for all  $\mu$ 's except the wave number to which  $m'_j$  belongs. For the latter wave number we have necessarily

$n_w' = n_w + 1$ . Paying attention to the fact that  $j$  can take any of  $(1, 2, \dots, n_w + 1)$ , we see that

$$\begin{aligned} & (\dots, n_w, \dots, n_w, \dots | L^\dagger | \dots, n_w + 1, \dots, n_w, \dots) \\ &= \frac{\epsilon}{\sqrt{N}} \sqrt{n_w + 1} \sum_m \exp(i\mathbf{w} \cdot \mathbf{m}) \varphi_m(\mathbf{r}). \end{aligned} \quad (2.10)$$

Let us now assume more explicit forms for the atomic wave functions  $\phi^o$  and  $\phi^e$ . In order that the exciton absorption is permitted optically,  $\phi^o$  has to be of different parity from  $\phi^e$ . If  $\phi^o$  is of the  $s$ -type, we have to consider triply degenerate  $p$ -orbitals for  $\phi^e$ , which necessarily lead to three types of excitation waves. As Heller and Marcus<sup>14)</sup> have shown, the approximate eigenstates resulting from this degeneracy are *one longitudinal* and *two transverse* waves for each  $\mathbf{w}$ , where the "longitudinal" means that the  $p$ -state which propagates from atom to atom with wave number  $\mathbf{w}$  should itself be directed toward  $\mathbf{w}$ , the "transverse" corresponding to the  $p$ -states directed perpendicular to  $\mathbf{w}$ . Then the summation in (2.10), when it is replaced by the integration, vanishes for the transverse waves; that is, in this approximation the transverse exciton waves do not interact with the extra electron. For the longitudinal waves (2.10) can be calculated as

$$(-e) / \sqrt{L^3} \cdot \gamma \cdot i / w \cdot \sqrt{n_w + 1} \exp(i\mathbf{w} \cdot \mathbf{r}),$$

where

$$\gamma = 4\pi e \mu / \sqrt{v_0}. \quad (2.11)$$

and  $L^3$ ,  $v_0$  and  $\mu$  mean the volume of the crystal, the volume of a unit cell and the absolute value of  $\mu$  given by (2.9), respectively.

The matrix elements in case (iii) can be calculated as the conjugate complexes of those in case (ii), and finally, introducing the annihilation and production operator  $b_w$  and  $b_w^*$  for a boson;

$$b_w = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \dots\dots\dots \\ 0 & 0 & \sqrt{2} & 0 & \dots\dots\dots \\ 0 & 0 & 0 & \sqrt{3} & \dots\dots\dots \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \end{bmatrix}, \quad etc. \quad (2.12)$$

we can express  $U$  in the  $\{n_w\}$  representation:

$$U(\mathbf{r}; \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \equiv U_0(\mathbf{r}) + (-e)\varphi(\mathbf{r}), \quad (2.13)$$

$$\varphi(\mathbf{r}) = \gamma / \sqrt{L^3} \cdot \sum_w i / w \{ b_w \exp(i\mathbf{w} \cdot \mathbf{r}) - b_w^* \exp(-i\mathbf{w} \cdot \mathbf{r}) \}. \quad (2.14)$$

$U_0(\mathbf{r})$  is the potential energy for the extra electron  $e$  when the crystal electrons are in the ground state as is seen from the definition (2.7), thus it will be quite evident that  $\varphi(\mathbf{r})$  is to be interpreted as the electrostatic potential due to the electronic polarization which is caused by the deviation of  $\psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$  from  $\phi_0(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$ , if we call

to mind the above approximation in which we have taken into account only up to the dipole potential.

The above expression is not altered, when the situation is reversed such that the ground state of the atom consists of triply degenerate  $p$ -orbitals all being filled with electrons and the excited state is of  $s$ -type. In either case, after dropping from  $H_0$  the energy of transverse exciton waves which do not interact with  $e$ , we can write the Hamiltonian for the system as follows:

$$H = \{H_0 + U_0(\mathbf{r})\} + \{E_0 + \varepsilon \sum_w b_w^* b_w\} + (-e)\varphi(\mathbf{r}). \quad (2.15)$$

The first is the Hamiltonian for the extra electron  $e$  under the periodic potential of the crystal when the crystal electrons are fixed to the ground state, so that the eigenstates constitute the conduction band of the crystal in which the electronic polarizability of the crystal is not yet taken into account. If we apply the Wannier-Slater<sup>17)</sup> formulation for the perturbed periodic potential problem, under the assumption that  $\varphi(\mathbf{r})$  is slowly varying perturbation (which is justified for the long wave-length components of  $\varphi(\mathbf{r})$  in (2.14)), we can simplify (2.15) to the form

$$H = \mathbf{p}^2/2m + \varepsilon \sum_w b_w^* b_w + (-e)\varphi(\mathbf{r}) \quad (2.16)$$

after dropping the unimportant terms, where  $m$  means the effective mass of the conduction band, that is:

$$\begin{aligned} \{H_0 + U_0(\mathbf{r})\} \psi_k(\mathbf{r}) &= E(\hbar\mathbf{k}) \psi_k(\mathbf{r}), \\ E(\mathbf{p}) &= E(0) + \mathbf{p}^2/2m + (\mathbf{p}^4). \end{aligned} \quad (2.17)$$

The Hamiltonian (2.16), combined with (2.14), means that the electronically polarizable crystal behaves as if it were a Boson field, in regards not only to its proper energy structure but also to the interaction with an extra electron. Moreover the interaction energy is of the same form as that of the interaction between an electron and the longitudinal modes of optical lattice vibrations in ionic crystals, which was to be expected because both types of polarization are equivalent from the stand-point of macroscopic electrostatics. According to our simplified model, the interaction constant  $\gamma$  is to be connected with the dielectric constant  $\kappa_0$  at high frequencies by the relation

$$\gamma = 2\pi e^2(1 - 1/\kappa_0), \quad (2.18)$$

the proof of which is given in the Appendix I.

### § 3. The motion of an electronic Polaron under the influence of a trapping potential

The Hamiltonian (2.16) and (2.14) obtained in the preceding section show that the low-lying energy levels of the system constitute the "electronic polaron" band in which the electron moves freely in the crystal accompanied by the electronic polarization around itself. While the mathematical structure of the Hamiltonian is the same as that

for the usual (lattice) polaron, the phonons in the latter case are to be replaced by the excitons in the former. We are interested, however, mainly in the behavior of the electronic polaron in the neighbourhood of the trap, so that we proceed directly to a general case. That is, we consider a system composed of three parts: the electronically polarizable crystal  $C$ , an extra electron  $e$ , and an imperfection  $Q$  which may be an ion vacancy or an impurity ion. If we assume that  $Q$  is represented by an extra charge distribution  $\rho(\mathbf{r})$ , its interaction  $V(\mathbf{r})$  with  $e$  is given by a Poisson equation

$$\Delta V(\mathbf{r}) = 4\pi e \rho(\mathbf{r}) \quad (e > 0), \quad (3.1)$$

and the interaction  $H_p$  of  $Q$  with crystal electronic polarization field is given by

$$H_p = \int \rho(\mathbf{r}) \varphi(\mathbf{r}) d\mathbf{r} = -\frac{\gamma}{\sqrt{L^3}} \sum_w \frac{i}{2\omega} \{ \rho_w b_w - \rho_w^* b_w^* \}, \quad (3.2)$$

where we have used the equation (2.14) and the definition of  $\rho_w$ :

$$\rho_w = \int \rho(\mathbf{r}) \exp(i\mathbf{w} \cdot \mathbf{r}) d\mathbf{r}. \quad (3.3)$$

Combining the above with the result obtained in the last section, the energy of the total system can be written down as

$$H = H_e + H_C + H_I + H_p + V(\mathbf{r}), \quad (3.4)$$

where we have used the abbreviations  $H_e$ ,  $H_C$  and  $H_I$  for the first, second and third terms in (2.16).

The third and fourth parts of the expression (3.4) contain linear terms in the polarization co-ordinates  $b_w$  and  $b_w^*$ , but the most convenient description appealing to our intuition is such that in the first approximation the electron clothed with electronic polarization moves in the field due to the fixed charge  $\rho(\mathbf{r})$ , the latter also polarizing the surrounding medium. In order to realize this picture from (3.4), it is a natural method to seek an appropriate unitary transformation so that the interaction terms such as  $H_I$  and  $H_p$  disappear. Such a transformation, for the first time, was used by Bohm and Pines<sup>(6)</sup> for the interaction of electrons and lattice vibrations in metals, and was applied by Morita and Horie<sup>(7)</sup> to the lattice polaron problem in ionic crystals.

In our case the transformation should be as follows. By a suitable unitary transformation

$$\begin{aligned} \mathbf{p} &= \exp(-iS/\hbar) \mathbf{P} \exp(iS/\hbar), \\ \mathbf{r} &= \exp(-iS/\hbar) \mathbf{R} \exp(iS/\hbar), \\ b_w &= \exp(-iS/\hbar) B_w \exp(iS/\hbar), \end{aligned} \quad (3.5)$$

we turn from old variables  $\mathbf{p}$ ,  $\mathbf{r}$  and  $b_w$ 's to new ones  $\mathbf{P}$ ,  $\mathbf{R}$  and  $B_w$ 's, in terms of which we now try to express the total Hamiltonian  $H$ ;

$$H = \exp(-iS/\hbar) \Lambda \exp(iS/\hbar)$$

$$\begin{aligned}
&= \exp(-iS/\hbar) \{A_e + A_c + A_I + A_p + V(\mathbf{R})\} \exp(iS/\hbar) \\
&= A_e + A_c \\
&\quad - i/\hbar \cdot [S, A_e + A_c] + A_I + A_p \\
&\quad - 1/2\hbar^2 \cdot [S, [S, A_e + A_c]] - i/\hbar \cdot [S, A_I + A_p] + \cdots \\
&\quad + V(\mathbf{R}) - i/\hbar \cdot [S, V(\mathbf{R})] + \cdots,
\end{aligned} \tag{3.6}$$

where  $A$  with any suffix means the transform of the Hamiltonian  $H$  with the same suffix, and is obtained by formally putting new variables in place of the corresponding old ones in the expression  $H$ . The operator  $S$  is chosen in such a way that the second line in the right-hand side of (3.6) vanishes:

$$-i/\hbar \cdot [S, A_e + A_c] + A_I + A_p = 0. \tag{3.7}$$

Then the expression (3.6) reduces to

$$H = A_e + A_c + V(\mathbf{R}) - i/2\hbar \cdot [S, A_I + A_p] - i/\hbar \cdot [S, V(\mathbf{R})] + \cdots. \tag{3.8}$$

Now, the equation (3.7) is satisfied by

$$\begin{aligned}
S &= S_I + S_p \\
&= -\frac{e\gamma\hbar}{\sqrt{L^3}} \sum_w \frac{1}{\omega} \left[ \frac{B_w}{\varepsilon + \frac{1}{2m}(-2\hbar\mathbf{w} \cdot \mathbf{P} + \hbar^2\omega^2)} \exp(i\mathbf{w} \cdot \mathbf{R}) \right. \\
&\quad \left. + \exp(-i\mathbf{w} \cdot \mathbf{R}) \frac{B_w^*}{\varepsilon + \frac{1}{2m}(-2\hbar\mathbf{w} \cdot \mathbf{P} + \hbar^2\omega^2)} \right] \\
&\quad + \frac{\gamma\hbar}{\varepsilon\sqrt{L^3}} \sum_w \frac{1}{\omega} (\rho_w B_w + \rho_w^* B_w^*).
\end{aligned} \tag{3.9}$$

Physical meanings of the two parts are as follows:  $S_I$  is an operator by which the electron  $e$  becomes clothed with electronic polarization, while  $S_p$  is that which corresponds to the displacement of the origin of the polarization co-ordinates  $b_n$ , the new origin being the equilibrium position of static polarization around the charge distribution  $\rho(r)$ .

In calculating the commutators appearing in (3.8), we can take advantage of the following situation. In the lattice polaron problem there are finite average number of thermal phonons for each mode of vibration so far as the temperature is not absolutely zero, while in our case the energy of each exciton is so much larger than the thermal energy  $kT$  that the existence of thermal exciton is out of the question. Even if we treat those processes in which excitons are produced or destroyed, as we do in §4, the total number of excitons in the initial or final state is limited to a very few number, while the volume  $L^3$  of the crystal is practically infinite. This situation can be expressed mathematically by the relations



$$\begin{aligned}\langle B_n^* B_n \rangle_n &= 0 (1/L^3), \\ \langle B_m B_m^* \rangle_n &= \langle 1 + B_n^* B_n \rangle_m = 1 + 0 (1/L^3),\end{aligned}\quad (3.10)$$

in which  $\langle \rangle_w$  means the average over all wave numbers  $w$ .

If we take the continuum approximation, that is, make the limit  $w$ , of wave numbers tend to infinity, explicit calculations of the commutators become easier, and we get the following results:

$$\begin{aligned}-\frac{i}{2\hbar} [S_I, A_I] &= -\frac{e^2 \gamma^2}{L^3} \sum_w \frac{1}{w^2} \frac{1}{\varepsilon + \frac{1}{2m} (-2\hbar w \cdot \mathbf{P} + \hbar^2 w^2)} \\ &= -\alpha \varepsilon \sin^{-1}(P/u) / (P/u) \\ &= -\alpha \varepsilon - \alpha/6 \cdot P^2/2m + 0(P^4),\end{aligned}\quad (3.11)$$

where  $u$  and  $\alpha$  are defined by

$$\hbar^2 w^2/2m = \varepsilon, \quad \alpha = 1/2 \cdot (1 - 1/\kappa_0) e^2 u / \varepsilon, \quad (3.12)$$

and

$$\begin{aligned}-\frac{i}{2\hbar} [S_p, A_I] &= \frac{e \gamma^2}{2\varepsilon L^3} \sum_w \frac{1}{w^2} \{ \rho_w^* \exp(iw \cdot \mathbf{R}) + \rho_w \exp(-iw \cdot \mathbf{R}) \} \\ &= -\frac{1}{2} \left( 1 - \frac{1}{\kappa_0} \right) V(\mathbf{R}),\end{aligned}\quad (3.13)$$

$$\begin{aligned}-\frac{i}{2\hbar} [S_p, A_p] &= \frac{e \gamma^2}{2L^3} \sum_w \frac{1}{w^2} \left\{ \frac{\rho_w^*}{\varepsilon + \frac{1}{2m} (-2\hbar w \cdot \mathbf{P} + \hbar^2 w^2)} \exp(iw \cdot \mathbf{R}) + \text{h.c.} \right\} \\ &= -1/2 \cdot (1 - 1/\kappa_0) V(\mathbf{R}) + \dots,\end{aligned}\quad (3.14)$$

where the omitted terms contain the products of the second or higher derivatives of  $V(\mathbf{R})$  with the second or higher powers of  $\mathbf{P}$ , and are not of primary importance as far as the potential  $V(\mathbf{R})$  is rather slowly varying. It is interesting to note that each of the cross commutators (3.13) and (3.14) contributes a half to the shielding effect

$$-(1 - 1/\kappa_0) V(\mathbf{R})$$

of the potential  $V(\mathbf{R})$  by electronic polarization. The next commutator

$$\begin{aligned}-\frac{i}{2\hbar} [S_p, A_p] &= -\frac{\gamma^2}{\varepsilon} \frac{1}{L^3} \sum_w \frac{1}{w^2} \rho_w^* \rho_w \\ &= -\frac{1}{2} \left( 1 - \frac{1}{\kappa_0} \right) \int \rho(\mathbf{r}) \frac{V(\mathbf{r})}{(-e)} d\mathbf{r} = -\varepsilon_p\end{aligned}\quad (3.15)$$

is nothing but the electrostatic energy due to the electronic polarization produced by the field of charge distribution  $\rho(\mathbf{r})$ . Finally the commutator with  $V(\mathbf{R})$  is calculated as

$$-i/\hbar \cdot [S, V(\mathbf{R})] = -i/\hbar \cdot [S_p, V(\mathbf{R})]$$

$$= \frac{e\hbar^2}{m\sqrt{L^3}} \sum_w \frac{1}{\left(\varepsilon + \frac{\hbar^2 w^2}{2m}\right)^2} \frac{\mathbf{w}}{w} \cdot \text{grad } V(\mathbf{R}) \{B_w \exp(i\mathbf{w} \cdot \mathbf{R}) + B_w^* \exp(-i\mathbf{w} \cdot \mathbf{R})\} \\ + \dots, \quad (3.16)$$

where we have expanded the operator  $S_I$  in power series in  $\mathbf{P}$  and omitted unimportant terms.

The transformation (3.9) which we have chosen is the most appropriate one especially from the physical point of view, for it gives a Hamiltonian which corresponds, in the zeroth approximation, to the classical description of an electron in a dielectric medium, as is seen below. Inserting (3.11), (3.13), (3.14), (3.15) and (3.16) into (3.8), we have the Hamiltonian expressed in new co-ordinates:

$$H = -\varepsilon_p - \varepsilon_s \\ + \mathbf{P}^2/2m_c + 1/\kappa_0 \cdot V(\mathbf{R}) \\ + \varepsilon \sum B_w^* B_w \\ + \frac{e\hbar^2}{m\sqrt{L^3}} \sum_w \frac{1}{\left(\varepsilon + \frac{\hbar^2 w^2}{2m}\right)^2} \frac{\mathbf{w}}{w} \cdot \text{grad } V(\mathbf{R}) \{B_w \exp(i\mathbf{w} \cdot \mathbf{R}) \\ + B_w^* \exp(-i\mathbf{w} \cdot \mathbf{R})\}, \quad (3.17)$$

where

$$-\varepsilon_s = -\alpha\varepsilon \quad (3.18)$$

means the self-energy of the electron  $e$  due to the electronic polarization around itself, and  $m_c$  is the effective mass of the conduction electron in which the interaction with the electronic polarization is already taken into account, and is given by

$$\frac{1}{m_c} = \frac{1}{m} \left(1 - \frac{\alpha}{6}\right). \quad (3.19)$$

Thus we see that the bottom of the conduction band is lowered by  $\alpha\varepsilon$  from that which is obtained if we do not take account of the interaction, while the effective mass of the band is given by  $m_c$  instead of  $m$ . As the interaction constant  $\alpha$  is  $0.5 \sim 1$  for real crystals, the self-energy turns out to be pretty large, but because of the overestimation caused by the continuum approximation, the above value should be reduced to about half of it. On the other hand, the correction  $\alpha/6$  of the effective mass is of less importance.

The equation (3.17) is interpreted as follows. As a result of clothing the fixed charge  $\rho(\mathbf{r})$  and the moving electron  $e$  with electronic polarization, there appear in the first line two kinds of self-energy, and these dressed units interact with each other, so that the potential  $V(\mathbf{R})$  is shielded by a factor  $1/\kappa_0$ , leading to the classical description as is seen in the second line. Of course,  $\mathbf{P}$  and  $\mathbf{R}$  should be interpreted as the momentum and the position of the electronic polaron rather than as those of the electron itself. The third line corresponds to the free excitons, which could exist independently of the motion of the electronic polaron, but for the last term. The last term which is

interpreted as the Hamiltonian for the interaction corresponds to the production and annihilation of excitons in the neighbourhood of the imperfection (1) by exchanging energy with the electronic polaron.

If we consider a trapped state of the electronic polaron as a stationary state problem, the interaction term causes the production of virtual excitons in addition to those which always accompany a free electronic polaron. As a result the energy of trapping is deeper than the value which would be obtained from the Hamiltonian

$$\mathbf{P}^2/2m_e + 1/\kappa_0 \cdot V(\mathbf{R}).$$

This is analogous to the Lamb-shift problem<sup>17</sup> in the hydrogen atom, if we replace the electronic polarization field in a crystal by the radiation field in vacuum. But in our case the Hamiltonian (3.17) is only an approximate expression, so that this problem belongs to the interest of rather academic nature. Far more interesting and important application of (3.17) is the calculation of the transition probability for the processes in which free excitons are produced or destroyed in the neighbourhood of the imperfection. We shall now turn to one of such problems.

#### § 4. Ionization of a trapped electron by an exciton

Recently Apker and Taft<sup>(10)(11)</sup> carried out a series of very interesting and excellent experiments on the external photoelectric effects in alkali-halides. They observed an enhanced effect of photoelectric yield in the wave-length region of irradiation which corresponds to the first exciton absorption band of the crystal. They concluded that this effect is due to the ionization of *F*-center electrons by excitons.

On the other hand, it has been generally considered to be somewhat curious that no one has observed the emission due to exciton annihilation. From this reason Seitz<sup>(12)</sup> infers that the excitons annihilate by giving its energy to some kind of imperfection under the ordinary conditions of purity.

Under these circumstances it is highly necessary and interesting to calculate the transition probability for the ionization process of a trapped electron by an exciton.

If we assume that the trapping potential  $V(\mathbf{R})$  is caused by a point charge  $ze$  located at the origin, the Hamiltonian (3.17) is written as follows:

$$H = H_0 + H', \quad (4.1)$$

$$H_0 = \mathbf{P}^2/2m - 1/\kappa_0 \cdot ze^2/R + \varepsilon \sum B_w^* B_w, \quad (4.2)$$

$$H' = \frac{eV\hbar^2}{m\sqrt{L^3}} \sum_w \frac{1}{\left(\varepsilon + \frac{\hbar^2 w^2}{2m}\right)^2} \frac{\mathbf{w}}{w} \cdot \text{grad} \left( -\frac{ze^2}{R} \right) \{ B_w \exp(i\mathbf{w} \cdot \mathbf{R}) + B_w^* \exp(-i\mathbf{w} \cdot \mathbf{R}) \}, \quad (4.3)$$

where we have replaced  $m_e$  in (3.17) by  $m$ , because the difference between them is rather small. We consider the following two eigenstates of (4.2) between which transition occurs owing to the perturbation (4.3).

(i) *Initial state*excitons :  $N_w = B_w^* B_w = 1$ ,  $N_{w'} = 0$  for all other  $w'$ .

electron : 1s-state in a Coulomb field,

$$-\varepsilon_i = -m_0/m \cdot \varepsilon'^2 \varepsilon_H, \quad (4.4)$$

$$\psi_i = \left( \frac{\varepsilon'}{a_H} \right)^{3/2} \frac{1}{\sqrt{\pi}} \exp\left( -\frac{\varepsilon'}{a_H} R \right), \quad (4.5)$$

where  $m_0$  is the electron mass,  $a_H$  the Bohr radius and  $\varepsilon_H = 13.5$  ev the energy of the hydrogen 1s-state.  $\varepsilon'$  is defined by

$$\varepsilon' = \varepsilon / \kappa_0 \cdot m / m_0. \quad (4.6)$$

The energy of the initial state is given by

$$E_0^i = \varepsilon - \varepsilon_i = \varepsilon(1 - J), \quad J = \varepsilon_i / \varepsilon. \quad (4.7)$$

(ii) *Final state*excitons :  $N_w = 0$  for all  $w$ 's.electron : ionized state in a Coulomb field with asymptotic wave number  $k$ ,<sup>18)</sup>

$$\psi_k(\mathbf{R}) = \frac{1}{\sqrt{L^3}} \exp(i\mathbf{k} \cdot \mathbf{R}) I'(1 - i\rho) \exp\left( \frac{1}{2} \rho \pi \right) F(i\rho, 1, i k R (1 - \cos \theta')) \quad (4.8)$$

$$= \frac{1}{\sqrt{L^3}} \exp\left( \frac{1}{2} \rho \pi \right) \sum_{l=0}^{\infty} \frac{\Gamma(l+1-i\rho)}{(2l)!} (2ikR)^l \exp(ikR) \\ \times F(l+1-i\rho, 2l+2, -2ikR) P_l(\cos \theta'), \quad (4.8')$$

where  $F$  means a confluent hypergeometric function, and the parameter  $\rho$  is defined by

$$\rho = \varepsilon' / a_H k. \quad (4.9)$$

Energy conservation requires that

$$E_0^f = \hbar^2 k^2 / 2m, \quad E_0^i = \varepsilon(1 - J), \quad (4.10)$$

from which we have, by (4.9), the relation

$$\rho^2 = J / (1 - J). \quad (4.11)$$

Let us now calculate the transition probability. We use the co-ordinates illustrated in Fig. 1: the incident direction of the exciton ( $w$ ) is taken as  $z$ -axis, and  $x$ -axis is chosen so that the wave vector  $k$  of the ionized electron lies in the  $xz$ -plane. The polar co-ordinates referred to these axes are denoted by  $(R, \theta, \phi)$ , and the angles  $\angle k\hat{z}$  and  $\angle k\hat{R}$  are written  $\theta_0$  and  $\theta'$ , respectively.

According to the time dependent perturbation theory, the transition probability for

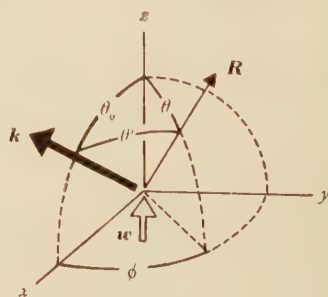


Fig. 1.

the electron to be emitted in the direction  $\theta_0$  within a solid angle  $d\Omega_0$  is given by

$$\begin{aligned} W(\theta_0) d\Omega_0 &= \frac{2\pi}{\hbar} |H'_{if}|^2 \frac{1}{4\pi} \rho(E) d\Omega_0 \\ &= \frac{mkL^3}{(2\pi)^2 \hbar^3} |H'_{if}|^2 d\Omega_0, \end{aligned} \quad (4.12)$$

where  $\rho(E)$  is the state density of a free electronic polaron with energy  $E$ . The matrix element  $H'_{if}$  can be written as

$$H'_{if} = \frac{ze^3 \gamma \hbar^2}{m \sqrt{L^3}} \frac{1}{\left(\varepsilon + \frac{\hbar^2 w^2}{2m}\right)^2} \int \phi_i^* \frac{\cos \theta}{R^2} \exp(-i\mathbf{w} \cdot \mathbf{R}) \phi_k d\mathbf{R} \quad (4.13)$$

by using (2.12) and (4.3). Putting (4.5) and (4.8') into the integral, then making use of addition formulae for Legendre polynomials to express  $P_l(\cos \theta')$  in terms of  $\theta$ ,  $\theta_0$  and  $\phi$ , finally integrating term by term, one finds the following double series for the matrix element :

$$\begin{aligned} H'_{if} &= \frac{1}{L^3} \frac{2i\pi e^3 \gamma \hbar^2 z \varepsilon^{l^{3/2}}}{m a_H^{3/2}} \frac{\exp(\frac{1}{2} p \pi)}{k \left(\varepsilon + \frac{\hbar^2 w^2}{2m}\right)^2} \\ &\times \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma(l+1-i p)}{(2l)!} P_l(\cos \theta_0) \frac{1}{2^{2m}} \frac{(-1)^m (l+2m)(2l+2m-1)!}{m! \Gamma(l+m+\frac{3}{2})} \\ &\times \frac{\lambda^{l+2m-1}}{(p-i)^{2l+2m}} F\left(l+1-i p, 2l+2m, 2l+2, \frac{-2i}{p-i}\right) \end{aligned} \quad (4.13')$$

where  $\lambda$  is defined by

$$\lambda = w/k, \quad (4.14)$$

and  $F$  is the hypergeometric function of customary use.

The total probability for the process is obtained by integrating (4.12) over all directions after inserting (4.13') into (4.12) :

$$\begin{aligned} W_T(p, \lambda) &= \int W(\theta_0) d\Omega_0 \\ &= \frac{32\pi^2}{L^3} \frac{\hbar^2 c^2}{m \varepsilon} \kappa_0^2 \left(1 - \frac{1}{\kappa_0}\right) \frac{p^5 (1+p^2)^2 \exp(p\pi)}{(1+p^2+\lambda^2)^4} S(p, \lambda), \end{aligned} \quad (4.15)$$

where  $S(p, \lambda)$  is given by

$$\begin{aligned} S(p, \lambda) &= \sum_{l=0}^{\infty} \frac{|\Gamma(l+1-i p)|^2}{\{(2l)!\}^2 (2l+1)} \frac{\lambda^{2l-2}}{(1+p^2)^{2l}} \\ &\times \left| \sum_{m=0}^{\infty} \frac{(l+2m)(2l+2m-1)!}{m! \Gamma(l+m+\frac{3}{2})} \left\{ \frac{i}{p-i} \frac{\lambda}{2} \right\}^{2m} \right. \\ &\times \left. F\left(l+1-i p, 2l+2m, 2l+2, \frac{-2i}{p-i}\right) \right|^2. \end{aligned} \quad (4.16)$$



We have carried out the calculation of this series to the fourth power in  $\lambda$ , by making use of the equation for the  $I$ -function:

$$|I'(1-ip)|^2 \exp(p\pi) = \frac{2p\pi}{1 - \exp(-2p\pi)} \quad (4.17)$$

and the explicit forms of the hypergeometric functions  $I$  which are given in the Appendix II. If we use (4.14), (4.9) and (3.12) to rewrite the result in terms of  $p$  and  $w$ , the total probability  $W_T$  can be expressed as follows:

$$W_T(p, w) = \frac{1}{I^3} \frac{\hbar^2 c^2}{m \varepsilon} \kappa_0^2 \left(1 - \frac{1}{\kappa_0}\right) G(p, w), \quad (4.18)$$

where

$$G(p, w) = g_0(p) / \{1 + (w/u)^2\}^4 \cdot \{1 + g_1(p) (w/u)^2 + g_2(p) (w/u)^4 + \dots\}, \quad (4.19)$$

and

$$g_0(p) = \frac{256\pi^2}{3} \frac{p^6 \exp(-4p \cot^{-1} p)}{\{1 - \exp(-2p\pi)\} (p^2 + 1)^3} \quad (4.20)$$

$$\begin{aligned} g_1(p) = & \frac{1}{5(p^2 + 1)(p^2 + 4)} \{ (172p^4 + 147p^2 + 23) \\ & - 2(p^2 + 1)(23p^2 + 11) \exp(2p \cot^{-1} p) \\ & + 3(p^2 + 1)^2 \exp(4p \cot^{-1} p) \}, \end{aligned} \quad (4.21)$$

$$\begin{aligned} g_2(p) = & \frac{1}{35(p^2 + 1)(p^2 + 4)(p^2 + 9)} \{ (29517p^6 + 49646p^4 + 22385p^2 + 3312) \\ & - 8(p^2 + 1)(997p^4 + 1324p^2 + 279) \exp(2p \cot^{-1} p) \\ & + 540(p^2 + 1)^3 \exp(4p \cot^{-1} p) \}. \end{aligned} \quad (4.22)$$

For all values of  $p$  between 0 and  $\infty$  (corresponding to  $0 \leq \varepsilon_t \leq \varepsilon$ ),  $g_1(p)$  and  $g_2(p)$  are comparatively small and  $w$ -dependence of  $G(p, w)$  is mainly determined by the factor

$$1 / \{1 + (w/u)^2\}^4,$$

so that we can conclude that excitons with small wave-number  $w$  are the most effective in ionizing a trapped electron. For the same reason,  $p$ -dependence of  $G(p, w)$  is mainly determined by  $g_0(p)$  so far as  $w$  is small compared with  $u$ . The curve (a) of Fig. 2 shows  $g_0(p)$  as a function of  $\Delta = \varepsilon_t / \varepsilon$  through the relation (4.11). Thus the ionization probability tends to a finite value as the trapping energy becomes nearer to the exciton energy, in spite of the fact that

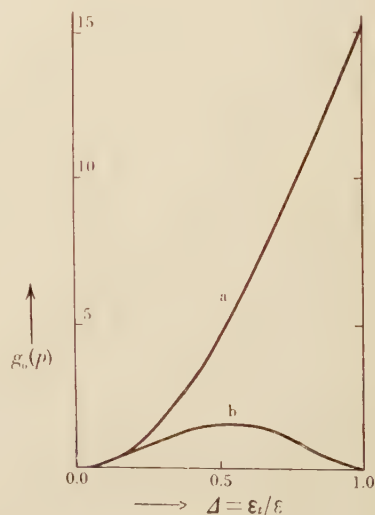


Fig. 2.

the state density of final states of the ionized electron tends to zero in this limit. This is due the property of the wave function (4.8), the amplitude of which at the origin tends to infinity as  $k$  approaches zero. In fact, when we use plane wave approximation instead of the strict wave function (4.8), we obtain the curve (b), which coincides with the curve (a) for small values of  $\Delta$  but deviates from it for larger values, and tends to zero as  $\Delta$  approaches unity.

## § 5. Application to real crystals

Before we apply the result obtained in the preceding section to real crystals, we must correct them because we have been assuming a very simple model according to which the excited states of crystal electrons consist of only one exciton band. In real crystals such as alkali-halides, however, the optical absorption measurements in the fundamental region show that the excited states consist of a few number of exciton bands together with an ionization continuum. This means that the relation (2.18) between  $\gamma$  and  $\kappa_0$  has to be replaced by a generalized one (which is nothing but a dispersion formula for zero frequency) :

$$\sum_i \gamma_i^2 / \varepsilon_i = 2\pi (1 - 1/\kappa_0) \quad (5.1)$$

in which  $\gamma_i$  and  $\varepsilon_i$  are the values for each exciton bands and an ionization continuum. Thus the contribution  $\gamma_i^2$  of a single (for example, the first) exciton band is much smaller than the value  $\gamma^2$  given by (2.18). Since the perturbation term  $H'$  in (4.3) is proportional to  $\gamma$ , we must correct the result (4.18) for the transition probability by multiplying a factor  $\gamma_i^2 / \gamma^2$ . Now each  $\gamma_i$  is connected with the oscillator strength  $f_i$  of the corresponding exciton absorption through the equation (2.11) and the relation between the dipole moment  $\mu_i$  and the oscillator strength :

$$f_i = 2m_0 \varepsilon / 3\hbar^2 \cdot \mu_i^2, \quad (5.2)$$

leading to the correction factor :

$$\frac{\gamma_i^2}{\gamma^2} = \frac{12\pi e^2 \hbar^2}{m_0 \varepsilon^2 \tau_0 (1 - 1/\kappa_0)} f_i. \quad (5.3)$$

In the following we shall drop the suffix  $i$  of  $f$ , confining ourselves to the *first* exciton band.

Assuming the density of trapped electrons to be  $n_t$ /c.c., the probability for a longitudinal exciton with wave number  $w$  to annihilate by ionizing any of the trapped electrons is now given by

$$W_I = \frac{12\pi \hbar^3 e^4 \kappa_0^2}{m_0^2 \varepsilon^3 \tau_0} n_t G(p, w) f \quad (5.4)$$

where we have replaced  $m$  by  $m_0$ . In case of alkali-halides with lattice constant  $a$ ,  $\tau_0$  is equal to  $2a^3$ , therefore (5.4) is rewritten as

$$W_I = 1.0 \times 10^{-6} \times \kappa_0^2 (\varepsilon_H / \varepsilon)^3 (a_H / a)^3 n_t G(p, w) f \text{ sec}^{-1}, \quad (5.4')$$

Without the knowledge on the  $f$ -value, we can compare (5.4') with the probability  $W_E^{(20)}$  for the annihilation of an exciton through spontaneous emission which is also proportional to  $f$  :\*

$$W_E = 8 \times 10^9 (\varepsilon/\varepsilon_H)^2 f \text{ sec}^{-1}. \quad (5.5)$$

Of these two ways of destruction of excitons, the ionization occurs only for longitudinal waves, while the radiation process is possible only for transverse waves. If we neglect the energy difference between the two kinds of waves, the ratio of their probable numbers is  $1/3 : 2/3$ . The factor  $2/3$  is already taken into account in (5.5). In order that the ionization process is more probable than the emission, the following condition must be satisfied :

$$W_I/3 > W_E. \quad (5.6)$$

As an example for numerical estimation we take the case of KI crystal containing  $F^+$ -centers, with which Apker and Taft<sup>(10)</sup> carried out one of their experiments. Inserting the values

$$\varepsilon = 5.6 \text{ eV},$$

$$\varepsilon_i = 2 \text{ eV (presumed from the } F^+\text{-absorption energy } 1.7 \text{ eV)}$$

$$a = 3.53 \text{ \AA}, \quad \kappa_0 = 2.7$$

into (5.4') and (5.5), and assuming that  $w \doteq 0$ , we have

$$W_I = 0.8 \times 10^{-6} n_i f \text{ sec}^{-1}, \quad (5.7)$$

$$W_E = 1.4 \times 10^9 f \text{ sec}^{-1},$$

the condition (5.6) therefore means that

$$n_i > 0.5 \times 10^{16} / \text{c.c.} . \quad (5.8)$$

Thus we can safely conclude that for all alkali-halide crystals, excitons annihilate rather by ionizing  $F^+$ -electrons than by spontaneous emission, if the density of  $F^+$ -centers is larger than  $10^{16} / \text{c.c.}$  .

Apker and Taft<sup>(10)</sup> presume that about  $10^{10} / \text{c.c.}$   $F^+$ -centers are formed in their experiment on KI; in case of such a large density, the ionization of  $F^+$ -centers is predominant according to the above discussion, thus our calculation supports their interpretation of exciton-enhanced photoemission as being caused by the  $F^+$ -electron ionization.

More generally we can conclude that excitons annihilate mainly by ionizing electrons trapped in various impurity levels under the usual condition of purity. (Deeper traps are more efficient in destroying excitons as is seen from the curve (a) of Fig. 3.) This is

\* The optical properties of the system which is described by means of exciton waves are approximately the same as those of a system consisting of free atoms.<sup>(10)</sup> This rule can be applied to the case of the spontaneous emission of an exciton, the probability being given by that of a free atom.<sup>(12)</sup>

qualitatively in agreement with Seitz's<sup>12</sup> speculation on the mechanism of exciton destruction, according to which the impurity content of the order of  $10^{-7}$  is sufficient to suppress spontaneous emission of excitons. It would be very interesting if the relation between the density of impurities and the photoconductivity in the first exciton absorption region were investigated experimentally, though it would be somewhat difficult owing to the formation<sup>10</sup> of color-centers during irradiation.

In the above discussion we have tacitly assumed that the exciton band has a positive effective mass so that only the excitons with wave number  $w \doteq 0$  prevail in thermal equilibrium. If the effective mass is negative, such excitons will decrease through scattering by lattice vibrations more rapidly than they annihilate in any way, so that the emission process (which is possible only for  $w \doteq 0$ ) will be less probable and the condition (5.8) has to be replaced by even more lenient one.

In order to estimate  $W_1$  absolutely instead of in comparison with  $W_1^*$ , it is necessary to know the value of  $f$ . Dexter<sup>21</sup> recently carried out theoretical calculation and obtained the value  $f=0.07$  for NaCl. On the other hand it is difficult to estimate the  $f$ -value from experimental data; perhaps it would be safe to assume that  $f$  is of the order of 0.05 for the first exciton bands in all alkali-halides, on the grounds of (a) absorption data<sup>22, 10, 11</sup>, (b) relation to the  $\beta$ -bands<sup>23</sup> and (c) Mayer's analysis<sup>24</sup> of dispersion data.\* Assuming this value for the first exciton band of KI, and  $n_i=10^{19}$ /c.c. for the density of  $F$ -centers, the largest value experimentally obtainable, we have, by (5.7),

$$W_1/3 \sim 10^{11} \text{ sec}^{-1}. \quad (5.7')$$

This is still much smaller than the frequency of scattering of excitons by lattice vibrations, which is estimated to be of the order of

$$W_v = 10^{12} \sim 10^{13} \text{ sec}^{-1}. \quad (5.9)$$

As was stated in § 2, only the longitudinal excitons are effective in ionizing trapped electrons, while the excitons which are produced by irradiation as in Apker and Taft's experiments are of transverse type. By comparison of (5.9) and (5.7'), however, we see that the equilibrium number of longitudinal waves are produced by scattering in sufficiently short time; thus our estimation remains correct just the same.

Another point which should be discussed is the energy difference between the longitudinal and the transverse waves, which has been neglected in the above estimation. Heller and Marcus<sup>14</sup> recently noticed that even without overlapping of atomic orbitals an exciton band has a finite breadth due to the dipole-dipole interaction of atoms which are excited one after another, and that the longitudinal waves have higher energies than the transverse ones. But in real crystals most of the contribution to the total oscillator strength for the excitation of crystal electrons comes from the transitions to higher states than the

\* In estimating the  $f$ -value from the data (a) and (b), we have used Smakula's formula<sup>25</sup> which seems to the author to be approximately valid also for the first exciton band, because the most part of the characteristic absorption lies in the shorter wave-length region.

first exciton band, so that the dipole-dipole interaction stated above is shielded by a factor  $1/\kappa_0$ , thus leading to a rather small value of the separation energy. Moreover the influence of overlapping would complicate the situation. Therefore their results does not seem to affect our estimation so seriously.

In conclusion the author wishes to express his sincere thanks to Professor T. Muto, Professor T. Inui and Professor F. Seitz for their valuable suggestions and discussions. His thanks are also to Mr. Y. Uemura and Mr. H. Miyazawa for their continual encouragements and stimulations to this problem. This work is indebted to the Scientific Research Expenditure of the Ministry of Education.

### Appendix I. Proof of the equation (2.18)

Equation (2.18) can be proved in the same way as Fröhlich, Pelzer and Zienau<sup>1)</sup> did in the lattice polaron problem. That is, we consider a classical point charge  $e_1$  fixed at  $\mathbf{r}=0$ . Then the energy of the system composed of  $e_1$  and the crystal  $C$  is written, by (2.14) and (2.16), as

$$H_1 = \varepsilon \sum_w \bar{b}_w^* b_w + \frac{e_1 \gamma}{\sqrt{L^3}} \sum_w \frac{i}{\tau w} (b_w - \bar{b}_w^*).$$

The minimum of this expression is realized for the values of  $b_w$ :

$$b_w = \frac{e_1 \gamma}{\varepsilon \sqrt{L^3}} \frac{i}{\tau w}.$$

This electronic polarization causes the potential

$$\varphi_1(\mathbf{r}) = \frac{\gamma}{\sqrt{L^3}} \sum_w \frac{i}{\tau w} \{ \bar{b}_w \exp(i\mathbf{w} \cdot \mathbf{r}) - \bar{b}_w^* \exp(-i\mathbf{w} \cdot \mathbf{r}) \} = -\frac{\gamma^2}{2\pi\varepsilon} \frac{e_1}{r}$$

according to (2.14). In order that this be equal to

$$-(1 - 1/\kappa_0) \cdot e_1 / r$$

as is required by electrostatics, the relation (2.18) has to be satisfied.

### Appendix II. Explicit forms of the hypergeometric functions used in the calculation

The hypergeometric functions appearing in (4.16) are such that the two parameters  $\beta = 2l + 2m$  and  $\gamma = 2l + 2$  are integers. We can, therefore, express them in terms of elementary functions as follows.

Starting out from the relation

$$F(\alpha, \beta, \beta, z) = (1-z)^{-\alpha}$$

which corresponds, in our case, to  $m=1$ , we can utilize the step-up recurrence formula<sup>26)</sup> for the hypergeometric functions



$$zF'(\alpha, \beta, \gamma; z) + \beta F(\alpha, \beta, \gamma; z) = \beta F(\alpha, \beta + 1, \gamma; z)$$

repeatedly in order to obtain the expressions for the cases  $m > 1$ . Thus we have, for example,

$$\begin{aligned} F(l+1-i\hbar, 2l+2, 2l+2, \frac{-2i}{\hbar-i}) &= \left(\frac{\hbar-i}{\hbar+i}\right)^{l+1} \exp(-2\hbar \cot^{-1} \hbar) \\ F(l+1-i\hbar, 2l+4, 2l+2, \frac{-2i}{\hbar-i}) \\ &= \frac{(2l-1)\hbar^2-1}{2l+3} \frac{(\hbar-i)^{l+1}}{(\hbar+i)^{l+3}} \exp(-2\hbar \cot^{-1} \hbar). \end{aligned}$$

When  $m=0$ , it is more convenient to make use of the expressions in infinite series: we can write, for example,

$$\begin{aligned} F(\alpha, 2, 4; z) &= \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1) \cdots (\alpha+n-1) \cdot 2 \cdot 3}{n!(n+2)(n+3)} z^n \\ &= \frac{6}{(\alpha-1)(\alpha-2)(\alpha-3)} \frac{d}{dz} \left\{ \frac{(1-z)^{-(\alpha-3)}}{z^2} - \frac{1}{z^2} - \frac{\alpha-3}{z} \right\}. \end{aligned}$$

The results up to the case  $l=3$  are as follows:

$$\begin{aligned} F\left(2-i\hbar, 2, 4; \frac{-2i}{\hbar-i}\right) &= 3 \frac{\hbar-i}{\hbar+i} \exp(-2\hbar \cot^{-1} \hbar), \\ F\left(3-i\hbar, 4, 6; \frac{-2i}{\hbar-i}\right) &= \frac{5}{2} \left(\frac{\hbar-i}{\hbar+i}\right)^2 \frac{1}{\hbar^2+4} \cdot \{(23\hbar^2+11) \exp(-2\hbar \cot^{-1} \hbar) \\ &\quad - 3(\hbar^2+1)\}, \\ F\left(4-i\hbar, 6, 8; \frac{-2i}{\hbar-i}\right) &= \frac{63}{2} \frac{(\hbar-i)^4}{(\hbar+i)^2} \frac{1}{(\hbar^2+4)(\hbar^2+9)} \\ &\quad \times \{(37\hbar^2+13) \exp(-2\hbar \cot^{-1} \hbar) - 5(\hbar^2+1)\}. \end{aligned}$$

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**Note added in proof :** The estimation (5-8) for the critical density of F-centers is in qualitative agreement with that of the previous work by Dexter and Heller<sup>27)</sup> who considered the F-electron ionization by an exciton to take place through two stages, whereas our formulation permits direct process. Though both are based on somewhat different points of view, they are in accordance as regards the most essential point, that is, the transition dipole field of the exciton is responsible for the process. The author wishes to thank Dr. Dexter for informing his interesting idea about this problem.<sup>28)</sup>

## On the Conductivity of Non-polar Crystals in the Strong Electric Field, I

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Recently Ryder and Shockley have found experimentally that the electrical conduction in a pure germanium crystal shows the marked deviation from Ohm's law in the strong electric field. In order to clarify the mentioned phenomena along the usual theory of electrical conduction, we have calculated the distribution function of conducting electrons in non-polar semiconductor by solving the well-known Bloch's integral equation to the second order approximation, taking account of the interaction of electrons with both the acoustical and optical modes of lattice vibrations. As a result we have found that our theory may well interpret the general behaviour of this phenomenon qualitatively. Furthermore, we have discussed the effect of impurity ions upon the critical field strength.

### § 1. Introduction

In the theory of the electrical conductivity the so-called Bloch's integral equation is well-known to be the fundamental one, which unfortunately has not been solved rigorously so far. For the case of weak electric field, however, we are allowed approximately to use the first order solution of the Bloch's integral-equation, which correctly lead to the Ohm's law of the electrical conductivity. On the other hand, for the case of very strong electric field, the appreciable non-ohmic current has been observed experimentally to appear even in insulating crystals, which finally breaks down rather abruptly at certain critical strength of the applied field. The phenomenon of the so-called electrical breakdown mentioned above has been investigated by many theoretical physicists. In order to determine the breakdown strength by the energy balance condition, they have calculated only the rate at which a conduction electron would lose its energy to the lattice, while the distribution function of the conduction electrons in the strong electric field has never been taken into consideration in detail so far. Owing to the lack of the knowledge of the mentioned distribution function, however, the phenomenon of non-ohmic current or the pre-breakdown current in the very strong field, closely connected with the electrical breakdown, could not be accounted for satisfactorily.

Recently Ryder and Shockley<sup>1)</sup> have found experimentally that the electrical conduction in a pure germanium crystal shows the marked deviation from the Ohm's law in the strong electric field. Furthermore Shockley<sup>2)</sup> has discussed theoretically this very interesting phenomenon by a rather intuitive method and explained the essence of the phenomenon elegantly. Independent of him, we<sup>3)</sup> also have worked theoretically the same phenomenon by the rather orthodox method of solving to the second order approximation the Bloch's

integral equation in non-polar semiconductor, allowing for the action of the strong electric field. We also assume that the conduction electrons in germanium crystals obey the Maxwell-Boltzmann statistics approximately. Our calculations have shown to give the same results as Shockley's in many respects and we are sure that his rather intuitive standpoint has been actually confirmed by our calculation. Since Shockley's second paper shows that the effect of the optical mode of vibrations seems to be quite appreciable for the electron-scattering for some crystals, we shall here calculate the electrical conductivity of the non-polar crystals in the strong electrostatic field by taking account the interactions of electrons with both the acoustical and optical modes of vibrations.

## § 2. The effect of the acoustical mode

In the experiment of Ryder and Shockley the electric field is not yet strong enough to excite the appreciable number of electrons of the full band into the conduction band. In this case the conduction electrons are mainly scattered by the interaction with the lattice vibrations (both acoustical and optical) and by colliding with the impurity ions. As the interaction between electrons and acoustical modes of vibrations may have the most important effect in non-polar crystals, we at first take into account only this modes of vibrations. Let  $f(\vec{k})$  be the distribution function of the conduction electrons having the wave number vector  $\vec{k}$ , then the stationary condition in electronic current is given, as usual, by

$$[\partial f / \partial t]_{\text{field}} + [\partial f / \partial t]_{\text{collision}} = 0, \quad (1)$$

where the first term means the rate of change of  $f(\vec{k})$  caused by drift in the external field and the second term by the above-mentioned collisions. We find after some process of computation along the conventional way,<sup>4)</sup>

$$\left[ \frac{\partial f}{\partial t} \right]_{\text{coll}} = \frac{V_0 C^2}{8\pi^3 M u_0} \frac{\pi}{2\lambda k} \int_0^{2\pi} d\phi \int_{q_{\min}}^{q_{\max}} \frac{q^2 dq}{e^{x_q} - 1} \\ \times [ \{ f(\vec{k} + \vec{q}) e^{x_q} - f(\vec{k}) \} + f(\vec{k} - \vec{q}) + f(\vec{k}) e^{x_q} ] \quad (2)$$

where

- $V_0$ : crystal volume,
- $C$ : an energy parameter describing the coupling between electron and lattice,
- $M$ : reduced mass of two ions,
- $u_0$ : sound velocity,
- $k_0$ : Boltzmann constant,
- $q$ : wave number of lattice vibration,
- $\lambda = \hbar^2 / 2m^*$ ,
- $m^*$ : effective mass of conduction electron,
- $\omega_q$ : circular frequency of the longitudinal mode associated with  $q$ , which is assumed to be  $\omega_q = u_0 q$ ,
- $x_q = \hbar \omega_q / k_0 T$ ,

$T$ : temperature,  
 $K$ : wave number of electron,  
 $E = \hbar K^2/m^*$ : energy of conduction electron.

Here we expand the distribution function  $f(k)$  in the series of Legendre polynomials

$$f(\vec{k}) = f_0(E) + k_x g(E) + \dots \quad (3)$$

Inserting eq. (3) into eq. (2), it becomes:

$$\begin{aligned} \left[ \frac{\partial f}{\partial t} \right]_{coll} = & \frac{V_0 C^2}{8\pi M u_0 \lambda K} \left[ \int_{q_{min}}^{q_{max}} \frac{q^2 dq}{e^{x_q} - 1} [\{f_0(E + \hbar \omega_q) e^{x_q} - f_0(E)\} \right. \\ & + \{f_0(E - \hbar \omega_q) - f_0(E) e^{x_q}\}] \\ & + \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{q_{min}}^{q_{max}} \frac{q^2 dq}{e^{x_q} - 1} [(k_x + q_x) g(E + \hbar \omega_q) e^{x_q} - k_x g(E) \\ & \left. + (k_x + q_x) g(E - \hbar \omega_q) - k_x g(E) e^{x_q}] \right], \quad (4) \end{aligned}$$

Now we expand the functions  $f_0(E + \hbar \omega)$ ,  $f_0(E - \hbar \omega)$  or  $\exp(x)$  with respect to  $\hbar \omega$  or  $x$ , neglecting the higher order terms than the second order one.

$$\begin{aligned} e^x &= 1 + x + 1/2 \cdot x^2, \\ f_0(E + \hbar \omega) &= f_0(E) + f_0'(E) \hbar \omega + 1/2 \cdot f_0''(E) (\hbar \omega)^2, \\ f_0(E - \hbar \omega) &= f_0(E) - f_0'(E) \hbar \omega + 1/2 \cdot f_0''(E) (\hbar \omega)^2, \\ g(E + \hbar \omega) &= g(E - \hbar \omega) = g(E). \end{aligned}$$

This approximate procedure may be allowed except for the very low temperature. Further we must be cautious in determining the upper limit of integration  $q_{max}$ . On integrating  $\{f_0(E + \hbar \omega) e^x - f_0(E)\}$  and  $\{f_0(E - \hbar \omega) - f_0(E) e^x\}$  with respect to  $q$ , the corresponding upper limit  $q_{max}$  is to be taken as

$$q_{max} = 2K \pm 2m^* u_0 / \hbar$$

respectively owing to the usual considerations. On the other hand we shall be allowed to take the lower limit as 0, unless  $K$  is very small ( $K < m u_0 / \hbar$ ). The behaviour of such a slow electron, however, is unimportant in our case except for the very low temperature. Then, after some calculations, eq. (4) becomes:

$$\begin{aligned} \left[ \frac{\partial f}{\partial t} \right]_{coll} = & \frac{V_0 C^2}{8\pi M u_0} \frac{1}{\lambda K} \left[ \int_0^{2K + 2m^* u_0 / \hbar} q^2 dq \{f_0(E) + k_0 T f_0'(E)\} \right. \\ & - \int_0^{2K - 2m^* u_0 / \hbar} q^2 dq \{f_0(E) + k_0 T f_0'(E)\} + \int_0^{2K} q^2 dq (\hbar u_0 q) \{f_0'(E) + k_0 T f_0''(E)\} \\ & \left. - \frac{k_0 T}{\hbar u_0 k^2} k_x g(E) \int_0^{2K} q^3 dq \right]. \quad (5) \end{aligned}$$



After carrying out the integration the final expression for  $(\partial f / \partial t)_{coll}$  has been obtained as follows :

$$\left[ \frac{\partial f}{\partial t} \right]_{coll} = \frac{V_0 C^2}{8\pi M} \frac{1}{\lambda K} \left[ \frac{16m^{*2}}{\hbar} E k_0 T \left\{ E f_0''(E) + \left( \frac{E}{k_0 T} + 2 \right) f_0'(E) \right. \right. \\ \left. \left. + 2/k_0 T \cdot f_0(E) \right\} - (k_0 T / \hbar u_0^2) (8mE / \hbar^2) k_x g(E) \right]. \quad (6)$$

On the other hand,  $(\partial f / \partial t)_{field}$  is written as usual,

$$[\partial f / \partial t]_{field} = eF / \hbar (g(E) + 2/3 \cdot E dg/dE + \hbar^2 / m^* \cdot k_x df_0/dE + \dots) \quad (7)$$

where  $-e$  is the electronic charge. Inserting (7) and (6) into (1) we obtain the following set of differential equations for  $f_0(E)$  and  $g(E)$  :

$$E f_0'' + \left( \frac{E}{k_0 T} + 2 \right) f_0' + \frac{2}{k_0 T} f_0 = - \frac{\pi M \hbar^4 \sqrt{2m^*}}{4V_0 C^2 m^{*3} k_0 T} \frac{1}{\sqrt{E}} \frac{eF}{\hbar} \left( g + \frac{2}{3} E \frac{dg}{dE} \right), \quad (8)$$

$$g(E) = \frac{\pi M u_0^2 \hbar^6}{V_0 C^2 m^{*2} \sqrt{2m^*}} \frac{1}{\sqrt{E}} \frac{eF}{\hbar} \frac{df_0}{dE}. \quad (9)$$

Let us here introduce the mean free path  $l$  in the weak field

$$l = \pi M u_0^2 \hbar^4 / V_0 C^2 m^{*2} k_0 T, \quad (10)$$

then the above equations are written as

$$E f_0'' + (E/k_0 T + 2) f_0' + 2/k_0 T \cdot f_0 = - \frac{l}{2\sqrt{2m^* u_0^2}} \frac{eF}{\hbar} \frac{1}{\sqrt{E}} \left( g + \frac{2}{3} E \frac{dg}{dE} \right), \quad (11)$$

$$g = \hbar l / \sqrt{2m^*} \cdot eF / \sqrt{E} \cdot df_0/dE. \quad (12)$$

Eliminating  $g(E)$  from the above equations we obtain the final differential equation for  $f_0(E)$

$$(E + p k_0 T) f_0'' + (2 + E k_0 T / p k_0 T / E) f_0' + 2/k_0 T \cdot f_0 = 0, \quad (13)$$

where

$$p = (eFl)^2 / 6m^* u_0^2 k_0 T. \quad (14)$$

In the case of weak field we may neglect the second order quantities with respect to  $l^2$ , so eq. (13) reduces to

$$E f_0'' + (2 + E/k_0 T) f_0' + 2f_0/k_0 T = 0. \quad (15)$$

whose solution is readily found as  $f_0(E) = A \exp(-E/k_0 T)$ . According to the usual procedure we also find that the conductivity  $\sigma_0$  is given by :

$$\sigma_0 = 4nc^2 l / 3 \sqrt{2\pi m k_0 T}. \quad (16)$$

We see that in this case our equations lead to the Boltzmann distribution and the Ohmic conductivity.

On the contrary, if we assume  $p k_0 T \gg E$  and  $p \gg 1$  (the case of very strong field),

we obtain the following equation,

$$pk_0 T f_0'' + (E/k_0 T + pk_0 T/E) f_0' + 2f_0/k_0 T f_0 = 0; \quad (17)$$

a solution of which is

$$f_0 = N \exp [-E^2/2p(k_0 T)^2]. \quad (18)$$

By the normalization condition

$$\int f_0(E) dk_x dk_y dk_z = 1$$

we find

$$N = 2\hbar^3/\pi (2m^*k_0 T)^{3/2} (2p)^{3/4} \Gamma(3/4). \quad (19)$$

Now the current density is, as usual, given by

$$j/n = \int (-e) v_x k_x g(E) dk_x dk_y dk_z. \quad (20)$$

Using eqs. (12), (18) and (19) the current density in strong field may be written as follows :

$$j/n = e \sqrt{2\pi}/3^{3/4} \Gamma(3/4) \cdot (eFlu_0)^{1/2}/(m^*k_0 T)^{1/4}. \quad (21)$$

Thus the observed field dependence of the electric current has been found to be correctly given, and the temperature dependence of it is also in qualitative accord with the observation. The ratio of  $\sigma$  in strong field and  $\sigma_0$  in weak field is easily computed as :

$$\frac{\sigma}{\sigma_0} = \frac{\pi}{2 \cdot 2^{1/4} \Gamma(3/4)} p^{-1/4} \sim 1.08 p^{-1/4}, \quad (p \gg 1)$$

which value is tabulated in Table I for several values of  $p$ .

From the distribution function obtained above the number density and the energy density of the electron in strong field are easily seen to reach their maximum at the energy values of

$$E_m = (eFl/2u_0) (k_0 T/3m)^{1/2} \quad (22)$$

and

$$E_{max} = (eFl/2u_0) (k_0 T/m)^{1/2} \quad (23)$$

respectively, together with the average energy of

$$E = 0.854 E_{max}.$$

The energy value determined from the energy balance condition, presupposed usually in the breakdown theory,

$$[dE/dt]_r + [dE/dt]_c = 0 \quad (25)$$

has been found to coincide with our  $E_{max}$  exactly, which fact seems to justify the validity of the mentioned condition for computing the current density in the strong field. Shockley has already derived our eq. (22) on the basis of the above mentioned energy balance

Table I.

$p$	$\sigma/\sigma_0$
6	0.69
10	0.61
50	0.41
100	0.36
300	0.26
500	0.23
1000	0.19

condition. He, however, has not derived the distribution function in strong field, but supposed for the distribution function to be expressed as  $f_0(E) = A \exp(-E/k_0 T_F)$ , where  $T_F$  is the apparent temperature, proportional to the electric field. Such high energy electrons are called by him "Hot Electrons".

Now in order to get the estimate of the critical field strength, where the deviation from the Ohm's law begins to appear, we have to find the exact solution of eq. (13). Fortunately eq. (13) has an analytical solution of the following form.

$$f_0(E) = N_0 (E/k_0 T + p)^p \exp(-E/k_0 T). \quad (26)$$

When  $p$  becomes much larger than  $E/k_0 T$ , eq. (26) is easily shown to reduce to eq. (18). From (26) we have found the more accurate values of  $E_m$  and  $E_{max}$  compared with those of eqs. (22) and (23):

$$E_m = 1/4 \cdot k_0 T + \sqrt{(1/4 k_0 T)^2 + (cFl/2u_0)^2} \cdot k_0 T/3m, \quad (27)$$

and

$$E_{max} = 3/4 \cdot k_0 T + \sqrt{(3/4 k_0 T)^2 + (cFl/2u_0)^2} \cdot k_0 T/m. \quad (28)$$

For the weak field strength, the above expressions are reduced approximately to the usual ones, while for the strong field they are seen to coincide with eqs. (22) and (23).

Table II.

$p$	$\sigma/\sigma_0$
1	0.80
2	0.73
4	0.63
6	0.60

Using the distribution function (26) we have found the conductivity  $\sigma$  for several values of  $p$ , as tabulated in Table II. Now, in order to discuss the detailed behaviors in the real crystal, the numerical relation between the  $p$ -value and the field-one is essentially required, but unfortunately the reliable informations about it can hardly be obtained

at present, since the values of mobility and sound velocity involved in (14) are rather at variance with each other according to the different measurements. We, therefore, have adopted tentatively 3600 (*Ge*), 900 (*Diamond*) and 1000 (*Si*)  $\text{cm}^2/\text{volt}/\text{sec.}$  for the mobilities at room temperature, while for sound velocity,  $0.5 \cdot 10^6$  (*Ge*),  $1.75 \cdot 10^6$  (*Diamond*) and  $0.9 \cdot 10^6$  (*Si*)  $\text{cm}/\text{sec.}$  respectively, which values lead to 200 (*Ge*), 3000 (*Diamond*) and 1000 (*Si*) volts/cm for the field values of  $p=1$  at room temperature respectively. Using the mentioned field value we have computed actually the absolute value of the electrical conductivity for *Ge* in the strong field region, whose comparison with the experiment reveals that the observed deviations from Ohm's law do not occur at field as low as predicted by our computation, namely, the Ohm's law still holds even at several times higher field than that expected by our theory. The reason for such discrepancy seems to be rather complicated one. In this connection it should be pointed out that the value of the interaction parameter  $C$  in the strong field has not worked out theoretically, but the observed value of the mobility in weak field has been used in our estimates. The conventional procedure mentioned above may be considered to introduce much ambiguity in our theory. In order to explain the above discrepancy Shockley has suggested that the energy surface

in the Brillouin zone of  $Ge$  crystals is not of a spherical one, which is presupposed in our theory, but instead a complex surface of two or three sheets. Although his suggestion is very interesting, it seems to be difficult at present to derive a quantitative conclusion from his ideas, since the detailed knowledge of the band structure of  $Ge$  has not been obtained.

### § 3. The effect of the optical mode

The treatment presented above is based entirely upon interaction with the longitudinal acoustical mode of the lattice vibrations of the crystal. The diamond structure lattice, however, is expected to have also the optical mode. The effect of the optical mode for the electron scattering has been thoroughly discussed by Seitz<sup>5)</sup> but the observed variation of the mobility with temperature did not reveal this effect. Recently Shockley showed that the observation of Ryder was well interpreted by taking into account the effects of the optical modes. In view of the above situation, therefore, we shall here also discuss the influence of optical modes upon the electron scattering. We assume for simplicity that the frequency of the optical modes has a constant value  $\omega_0$  and the temperature is so low that the conduction electron is able to emit a phonon but can not absorb a phonon at all. Then, according to Seitz, we may easily write down the expression of  $(\partial f / \partial t)_{coll}$  as follows :

$$[\partial f / \partial t]_{coll} = [eq. (6)] \quad \text{for } E \leq \hbar\omega_0$$

and

$$\begin{aligned} \left[ \frac{\partial f}{\partial t} \right]_{coll} = & [eq. (6)] + \frac{V_0(\rho D)^2}{8\pi M\lambda K\omega_0} \frac{\exp(\hbar\omega_0/k_0T)}{\exp(\hbar\omega_0/k_0T) - 1} \left[ \int_0^{q_{max}} q \, dq f_0(E + \hbar\omega_0) \right. \\ & \left. - \int_0^{q''_{max}} q \, dq f_0(E) - k_x g(E) \int_0^{q_{max}} q \, dq \left( \frac{q^2}{2K^2} - \frac{\hbar\omega_0}{E} \right) \right], \text{ for } E \geq \hbar\omega_0, \end{aligned} \quad (29)$$

where  $D$  is the interaction parameter for the optical mode, analogous to  $C$  in the acoustical one and  $\rho$  the first non-vanishing reciprocal vector of the lattice. When the electron energy  $E$  is nearly equal to  $\hbar\omega_0$ , we are not allowed to expand the function  $f_0(E \pm \hbar\omega_0)$ , in which case Bloch's integral equation is not easily reduced to the usual differential equation. On the other hand, if  $E$  is much larger than  $\hbar\omega_0$ , it may be expanded as follows :

$$\begin{aligned} \left[ \frac{\partial f}{\partial t} \right]_{coll} = & [eq. (6)] + \frac{V_0(\rho D)^2}{8\pi M\lambda K\omega_0} \left\{ (f_0(E) + f_0' \hbar\omega_0 + 1/2 f_0''(E) (\hbar\omega_0)^2) \right. \\ & \times \int_0^{2K + K\hbar\omega_0/E} q \, dq - f_0(E) \int_0^{2K - K\hbar\omega_0/E} q \, dq - k_x g(E) \int_0^{2K} q \, dq \left( \frac{q^2}{2K^2} - \frac{\hbar\omega_0}{2E} \right) \Big\} \\ = & \frac{4V_0 C^2 m^3 k_0 T}{\pi M \hbar^4 \sqrt{2m}} \sqrt{E} \left[ E f_0'' + \left( \frac{E}{k_0 T} + 2 \right) f_0' + \frac{2}{k_0 T} f_0 - \frac{1}{m u_0^2} k_x g(E) \right] \\ & + \frac{V_0(\rho D)^2 m^2}{2\pi M \hbar^2 \sqrt{2m}} \frac{1}{\sqrt{E}} \left[ (E f_0'' + f_0') \hbar\omega_0 + 2(E f_0' + f_0) - \frac{2E}{\hbar\omega_0} \left( 1 - \frac{\hbar\omega_0}{2E} \right) k_x g(E) \right]. \end{aligned} \quad (30)$$

Here we shall denote the relative ratio of the interaction constants of two modes by  $R$ :

$$R = D^2 / 2C^2 \cdot 2mu_0^2 / k_0 T \cdot (\hbar^2 \rho^2 / 2m) / \hbar \omega_0 \approx D^2 / C^2. \quad (31)$$

Then eq. (30) becomes

$$\left[ \frac{\partial f}{\partial t} \right]_{coll} = \frac{4V_0 C^2 m^3 k_0 T}{\pi M \hbar^4 \sqrt{2m}} \left[ \sqrt{E} \left\{ E f_0'' + \left( \frac{E}{k_0 T} + 2 \right) f_0' + \frac{2}{k_0 T} f_0 \right\} + \frac{\hbar \omega_0}{mu_0^2} \frac{R}{\sqrt{E}} \right. \\ \left. \times \left\{ (E f_0'' + f_0') \hbar \omega_0 + 2(E f_0' + f_0) \right\} - \frac{\sqrt{E}}{2mu_0^2} k_{\mathcal{S}'}(E) - \frac{R \sqrt{E}}{2mu_0^2} \left( 1 - \frac{\hbar \omega_0}{2E} \right) k_{\mathcal{S}}(E) \right]. \quad (32)$$

Using eqs. (32), (7) and (1), we obtain the following differential equations, which are easily seen to be the generalized equations of eqs. (11) and (12).

$$\left\{ (E f_0'' + E f_0') + \left( \frac{E^2}{k_0 T} f_0' + \frac{2E}{k_0 T} f_0 \right) \right\} + \frac{\hbar \omega_0}{mu_0^2} R \left\{ (E f_0'' + f_0') \hbar \omega_0 + 2(E f_0' + f_0) \right\} \\ = - \frac{l \sqrt{E}}{2 \sqrt{2m} \cdot u_0^2} \frac{cF}{\hbar} \left( g + \frac{2}{3} E \frac{dg}{dE} \right), \quad (33)$$

$$g(E) = \frac{\hbar l}{\sqrt{2m}} \frac{cF}{1 + R(1 - \hbar \omega_0 / E)} \frac{1}{\sqrt{E}} \frac{df_0}{dE}. \quad (34)$$

Eliminating  $g(E)$  from these equations, we obtain

$$(E^2 f_0'' + 2E f_0') + \left( \frac{E^2}{k_0 T} f_0' + \frac{2E}{k_0 T} f_0 \right) + \frac{\hbar \omega_0 R}{mu_0^2} \left\{ (E f_0'' + f_0') \hbar \omega_0 + 2(E f_0' + f_0) \right\} \\ + p k_0 T \left\{ \frac{E}{1 + R(1 - \hbar \omega_0 / 2E)} f_0'' + \frac{1}{1 + R(1 - \hbar \omega_0 / 2E)} f_0' - \right. \\ \left. \frac{R \hbar \omega_0}{2 \{ 1 - R(1 - \hbar \omega_0 / 2E) \}} f_0 \right\} = 0, \quad (35)$$

which is rewritten as

$$\frac{d}{dE} \left[ E^2 f_0' + \frac{E^2}{k_0 T} f_0 + \frac{\hbar \omega_0 R}{mu_0^2} (\hbar \omega_0 E f_0' + 2E f_0) + \frac{p k_0 T E}{1 + R(1 - \hbar \omega_0 / 2E)} f_0' \right] = 0. \quad (36)$$

It follows, after a simple integration,

$$\left\{ E + \frac{(\hbar \omega_0)^2 R}{mu_0^2} + \frac{p k_0 T}{1 + R(1 - \hbar \omega_0 / E)} \right\} f_0' + \left( \frac{E}{k_0 T} + \frac{2 \hbar \omega_0 R}{mu_0^2} \right) f_0 = 0. \quad (37)$$

When  $E$  is much larger than  $\hbar \omega_0$ , the term  $\hbar \omega_0 / 2E$  may be neglected approximately compared with unity. In that case the integration is easily performed as follows:

$$f_0(E) = N' \exp(-E/k_0 T) \left\{ E/k_0 T + p/(1 + R) + R(\hbar \omega_0)^2 / mu_0^2 k_0 T \right\}^{p/(1 + R) + R(\hbar \omega_0)^2 / mu_0^2 k_0 T} \quad (38)$$



When  $\{p/(1+R) + R(\hbar\omega_0)^2/mu_0^2k_0T\}$  is much larger than  $E/k_0T$ , (38) may be reduced approximately to

$$f_0(E) = N'' \exp[-E^2/2p'(k_0T)^2], \quad (39)$$

$$p' = p/(1+R) + R(\hbar\omega_0)^2/mu_0^2k_0T. \quad (40)$$

When  $p$  becomes very large, the average energy of the electron is much larger than  $\hbar\omega_0$ , in which case we may be allowed to assume the distribution function is approximately given by eq. (39). Then the electric current is given by

$$j/n = (1/(1+R)) (p/p')^{1/4} j_{acous}, \quad (41)$$

where  $j_{acous}$  is the current intensity in eq. (21). We see that in the case of very strong field the interaction with the optical modes of vibrations reduces the value of the electric current by about  $1/(1+R)^{1/4}$ , but the field dependence of the current is still expressed by  $\sqrt{F}$ . When the field is not so strong, the above mentioned approximation is no more valid and some other method of approximation has to be devised. As we have already obtained the distribution function valid in both regions of  $E < \hbar\omega_0$  and  $E > \hbar\omega_0$ , so we shall conventionally adopt the interpolation method of connecting both distributions at an intermediate energy  $E$  in order to obtain the approximate distribution function in all range of the electron energy. Thus it follows,

$$f_0(E) = N \exp(-x^2/2p), \quad \text{for } x \leq x_0$$

and

$$f_0(E) = N \exp[-(x_0^2/2)(1/p - 1/p')] \exp(-x^2/2p'), \quad \text{for } x > x_0,$$

where  $x = E/k_0T$ ,  $x_0 = \hbar\omega_0/k_0T$  and  $N$  is the normalization constant, which is given by the following condition:

$$k_0T^{3/2} \frac{2\pi(2m)^{2/3}}{\hbar^3} N \left[ \int_0^{x_0} \exp(-x^2/2p) \sqrt{x} dx + \exp\left[-\frac{x_0^2}{2}\left(\frac{1}{p} - \frac{1}{p'}\right)\right] \int_{x_0}^{\infty} \exp(-x^2/2p') \sqrt{x} dx \right] = 0. \quad (42)$$

Finally, the electric current is given by

$$j/n = \frac{4\pi}{3} \frac{e^2 F l m k_0 T}{\hbar^3} \frac{N}{1+R} \left[ \frac{1}{p} \int_0^{x_0} x^2 \exp(-x^2/2p) dx + \frac{1}{p'} \int_{x_0}^{\infty} \exp\left[-\frac{x_0^2}{2}\left(\frac{1}{p} - \frac{1}{p'}\right)\right] x^2 \exp(-x^2/2p') dx \right]. \quad (43)$$

It may be easily seen that this current expression contains eqs. (21) and (41) as the extreme cases. Only when  $p$  is equal to  $p'$  the above expression reduces easily to

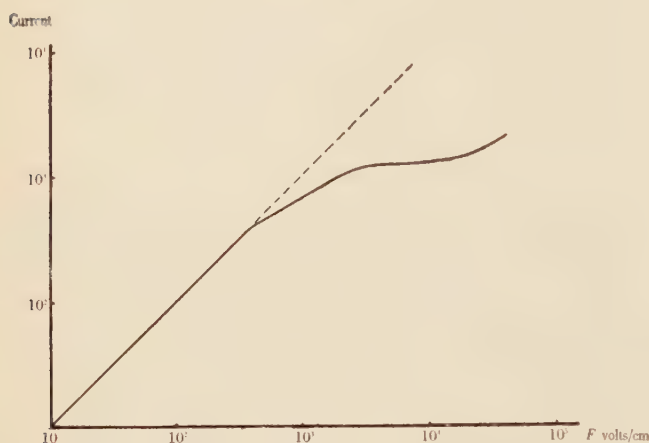
$$j/n = 1/(1+R) j_{acous}. \quad (44)$$

In other cases of  $p$ -value we have to calculate the current numerically from eqs. (42)

and (43). For example, we have worked numerically the case of

$$\hbar\omega_0/k_0T=5, \quad \hbar\omega_0/mu_0^2=40, \quad E_0=2\hbar\omega_0 \text{ and } R=1,$$

the result of which is represented in Fig. 1.



We see that our result is in good qualitative agreement with the observation.<sup>2)7)</sup> According to eq. (41) we see that the interaction ratio  $R$  may be determined by the comparison with the observed current  $j$  in the very strong field and the extrapolated value of  $j_{\text{acous}}$ . It should be stressed that Shockley also has obtained the same result as ours by the rather intuitive method.

#### § 4. The effect of the impurity scattering

For the case of the impure crystals, we have to take account of the impurity-scattering beside the lattice-one in the electronic conduction. As the scattering due to the impurity ion in non-polar crystals has been discussed by Conwell and Weisskopf,<sup>6)</sup> we shall adopt here their method of procedures. For simplicity we shall again omit the optical modes of vibrations. As is well-known, the lattice scattering is of inelastic nature, while the impurity scattering is considered approximately to be elastic one, so that the terms containing  $f_0(E)$  in  $(\partial f/\partial t)_{\text{coll}}$  remain unchanged by adding the impurity scattering term. Accordingly eq. (11) is still valid for the present case, while eq. (12) is altered by allowing for the impurity scattering as follows:

$$\frac{V_0 C^2}{4\pi M u_0} \frac{1}{k^{3/2}} \frac{k_0 T}{\hbar u_0} \sqrt{E} g(E) + \frac{\pi N_0 e^4 \ln G}{4\kappa^2 (2m)^{1/2} E^{3/2}} g(E) = - \frac{\hbar^2}{2m} \frac{eF}{\hbar} \frac{df_0}{dE} \quad (45)$$

where  $N_0$  is the number of ionized impurity centers,  $2d$  the average distance between nearest neighbour impurity ions,  $\kappa$  the dielectric constant and  $G$  is expressed as

$$G = 1 + 36 \kappa^2 d^2 (k_0 T)^2 e^{-4}. \quad (46)$$

If we put  $f_0(E) = A \exp(-E/k_0T)$  in eq. (45), we obtain the mean free path due to the impurity-scattering alone

$$l_i(E) \equiv l_0 E^2 = 4\kappa^2 E^2 / \pi N e^4 \ln G, \quad (47)$$

which of course coincides with Conwell-Weisskopf's result. Using  $l_i(E)$ ,  $l$  and the resultant mean free path  $l_r$ , which is defined as

$$1/l_i = 1/l + 1/l_i,$$

we may write down the differential equations as follows :

$$\left\{ \bar{E} + \frac{l_i}{l} p k_0 T \right\} f_0'' + \left\{ 2 + \frac{E}{k_0 T} + \frac{l_i}{l} \frac{p k_0 T}{E} \left( 1 + \frac{2l_i}{l} \right) \right\} f_0' + \frac{2}{kT} f_0 = 0, \quad (48)$$

and

$$g(E) = \hbar m / \sqrt{2m} \cdot c F / \sqrt{E} \cdot df_0 / dE. \quad (49)$$

When the field is very strong, eqs. (48) and (49) are reduced to be practically equivalent to eqs. (11) and (12) on account of  $l_0 \ll l$  and  $l_i \gg l_i$ ; which shows that the impurity effect is not very appreciable in such high field region. On the other hand, when the field is not so strong or the impurity content is so large that the conditions of  $l_0 E^2 \ll l$  and  $l_i \sim l_i$  are valid, the critical field is diminished by the factor  $(l_i/l)$ , which leads to the conclusion that the critical field is sensitive to the impurity content.<sup>7)8)</sup>

Finally we wish to express our sincere thanks to Prof. T. Muto for his continued interest and advice during the course of this work.

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**Note added in proof :** Recently many informations about the band structure of germanium crystals have been obtained. Therefore, it is very interesting to compute the current in strong field on the basis of that band structure in order to resolve the quantitative discrepancies of Shockley's and our theory. Unfortunately we have not yet succeeded in solving this problem owing to the mathematical difficulties.

# The Theory of the Dielectric Constant of Ionic Crystals, II

— *Effect of Temperature* —

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We have recently developed the theory of the dielectric constant of ionic crystals on the basis of the quantum theory of solids. In the present paper, we have worked out theoretically the temperature effect of the dielectric constant on the similar line as above, namely, we have determined the wave functions of ions so as to minimize the free energy change of the system under the electrostatic field. The numerical value of the coefficient of the temperature variation of dielectric constant of LiF crystal becomes of  $3.2 \times 10^{-4}$  in comparison with the observed value of  $3.75 \times 10^{-4}$ .

## § 1. Introduction

In the previous paper we have developed the theory of the dielectric constant of ionic crystals on the basis of the quantum theory of solids. In that case, we have determined the wave functions of ions so as to minimize the energy change of the system under the external electric field, the computed value of the dielectric constant of LiF crystal being considered as that of the absolute zero temperature. As well known experimentally, the dielectric constant of simple ionic crystals is nearly independent of temperature, as in the cases of non-polar gases and liquids. The precise observation, however, has revealed that the dielectric constant of the alkali-halide crystals increases actually with increasing temperature, while in non-polar gases and liquids it decreases with increasing temperature. In the latter case the temperature effect of dielectric constant is explained as the result of the density of the matter due to the thermal expansion. As for the ionic crystals, although the elucidation of the phenomena has been given only on the basis of the Born's formula, the detailed discussion of it from the modern theory of solids has not yet been attempted so far. By making use of our previous theory we shall here work theoretically the effect of temperature on the low frequency dielectric constant of alkali-halide crystals.

## § 2. Theoretical formulation

Now, in order to work the effect of temperature upon the dielectric constant we must determine the wave functions so as to minimize the free energy change of the system under the external electric field. Let the energy, the free energy and the entropy of the total system without external field be  $E$ ,  $A$  and  $S$ , and further the kinetic energy of the vibrating ions be  $K$ . Denoting the changes of the above quantities due to the application of external field by  $\Delta E$ ,  $\Delta A$ ,  $\Delta S$  and  $\Delta K$  respectively, we have the following relations:

$$A = E - TS = (E - K) + (K - TS), \quad (1)$$

and

$$\Delta A = \Delta(E - K) + \Delta(K - TS), \quad (2)$$

which is connected with the dielectric constant  $\kappa$  in the following way.

$$\Delta A = -1/2 \cdot \chi F^2, \quad \kappa = 1 + 4\pi\chi. \quad (3)$$

From eq. (2) and (3), it follows clearly

$$d(\Delta A)/dT = d\Delta(E - K)/dT + d\Delta(K - TS)/dT, \quad (4)$$

$$d(\Delta A)/dT = -1/2 \cdot (d\chi/dT) F^2, \quad (5)$$

and

$$d\kappa/dT = 4\pi(d\chi/dT) = -8\pi/F^2 d(\Delta A)/dT. \quad (6)$$

As for the wave function of negative ions under the influence of the external field it will be assumed

$$\phi = \phi_0(\vec{r}) (1 + \lambda r \cos \theta)$$

where  $\phi_0(r)$  is that of the unperturbed system and  $\lambda$  is the variational parameter, which is here considered as the function of temperature. We shall neglect the change of the wave function of positive ions entirely. Let  $2x$  be the displacement of the positive to the negative, then free energy is expressed as a function of  $x$ ,  $\lambda$ , and  $T$ . At given temperature the value of  $x$  and  $\lambda$  are to be determined from the following minimum conditions.

$$\partial \Delta A / \partial x = 0, \quad \partial \Delta A / \partial \lambda = 0. \quad (6')$$

Now, we shall proceed to computation of  $\Delta A$ .

As seen in (4), the temperature effect of the dielectric constant consists of two parts, each of which shall be computed separately in the following.

(I) temperature variation of  $\Delta(E - K)$

The computation of  $\Delta(E - K)$  at absolute zero temperature has been given in the previous paper. The main influence of temperature-variation of ionic distance upon  $\Delta(E - K)$  may be divided into the following three factors.

(a) The first effect which affects the dielectric constant stems from the change with temperature of the number of ion pairs per unit volume, on account of which the density of the dipole becomes smaller with increasing temperature and so the dielectric constant must decrease. Let  $a_0$  and  $a$  be the ionic distance at absolute zero temperature and  $T^\circ K$  respectively, and  $N$  be the number of ion pairs per unit volume. Thus it follows

$$a = a_0 + \delta a, \quad (7)$$

$$N = 1/2a^3 = (1/2a_0^3) (1 - 3\delta a/a_0), \quad (8)$$



in which  $\partial\alpha$  may be supposed to be very small compared to  $\alpha_0$ .

(b) The second effect is ascribed to the change of the restoring force (or energy). As the restoring force becomes smaller for larger ionic distances, it will decrease with increasing temperature, which is responsible for the decrease of the dielectric constant. If the restoring energy is assumed to be

$$[4(1/\rho - 2/a)\rho/\rho \cdot e^{-\alpha/\rho}]x^2,$$

its value at  $T^\circ K$  will be written as,

$$4[1/\rho - 2/a_0(1 - 3\partial\alpha/a_0)]\rho/\rho \cdot e^{-\alpha_0/\rho}(1 - \partial\alpha/\rho)x^2.$$

Assuming approximately that the above restoring force may be effective only between the nearest neighbouring ions, we obtain  $\rho$  from the equilibrium condition, namely,

$$\alpha/a_0^2 = (6/\rho)\rho \exp(-\alpha_0/\rho), \quad (9)$$

where  $\alpha_M$  is the Madelung's constant. Using eqs. (7) and (9) we obtain the following expression of the restoring energy at  $T^\circ K$ :

$$\{2\alpha_M/3\alpha_0^2(1/\rho - 2/a_0) - 2\alpha_M/3\alpha_0^2(1/\rho^2 - 2/a_0\rho - 2/a_0^2)\partial\alpha\}x^2. \quad (10)$$

(c) Since the decrease of the ionic distance with temperature is considered to give rise to the decrease of the restoring energy acting to prevent the ionic deformation, namely, A and B in the previous paper, the dielectric constant must increase also for this reason. If the restoring energies proportional to  $\lambda^2$  and  $\lambda x$  are supposed to be expressed by  $\rho'(-r/\rho')$ , A and B at temperature  $T$  are easily seen to take the following forms.

$$\begin{aligned} A\lambda^2 &= \rho' \exp(-a/\rho')\lambda^2 = \rho' e^{-\alpha_0/\rho'}(1 - \partial a/\rho')\lambda^2 \\ &= A_0(1 - \partial a/\rho')\lambda^2, \end{aligned} \quad (11)$$

and

$$B\lambda x = B_0(1 - \partial a/\rho'')\lambda x. \quad (12)$$

## (II) temperature variation of $\Delta(K-TS)$

If the motion of the ions is supposed always to be of periodic nature,  $(K-TS)$  is given by:

$$K-TS = -kT \sum_i \log [e^{\hbar\omega_i/kT} / (e^{\hbar\omega_i/kT} - 1)]. \quad (13)$$

When the ions are displaced in the external field, however, the vibrational frequency changes from  $\omega_{0i}$  to  $\omega_i$ , which is denoted by

$$\omega_i = \omega_{0i} + \Delta\omega_i. \quad (14)$$

Although the crystal has cubic symmetry without external field, it has no more cubic symmetry under the influence of the external field,  $\Delta\omega_i$  being assumed to become anisotropic. The substitution of (14) into (13) leads to

$$-kT \sum_i \log [e^{\hbar\omega_i/kT} / (e^{\hbar\omega_i/kT} - 1)] = -kT \sum_i \log [e^{\hbar\omega_{0i}/kT} / (e^{\hbar\omega_{0i}/kT} - 1)] \\ + \sum_i \hbar\Delta\omega_i / (e^{\hbar\omega_{0i}/kT} - 1).$$

Thus it follows

$$\Delta(K-TS) = \sum_i \hbar\Delta\omega_i / (e^{\hbar\omega_{0i}/kT} - 1). \quad (15)$$

Now it is a rather complicated task to compute these terms rigorously, because we have not yet the detailed knowledge of the vibrational spectrum of the lattice. We, therefore, shall compute them approximately under the simplified assumptions. As the temperature variation of  $\Delta(K-TS)$  is found to become quite small compared with that of  $\Delta(E-K)$ , our procedure of approximate computation may not give rise to a serious error. Since the magnitude of the frequency-change of the acoustical mode of vibration may be assumed to become much smaller than that of the optical mode, we shall neglect entirely the former change in our following computations. Fortunately the frequency of the optical mode may be considered to be nearly constant and, moreover, the frequency of the transverse wave of LiF and NaCl crystals is given by approximately

$$\omega_0 = \sqrt{\alpha_M / 6a^2 M(1/\rho - 2/a)}, \quad (16)$$

where  $M$  is the reduced mass of ions. Let us consider, then, a crystal in equilibrium under the influence of the electrostatic field along  $x$ -direction, in which case we could imagine that the negative ions remain in their original positions, while the positives are displaced a distance  $2x$ . Now, on displacing a positive ion by  $dx$ , along  $x$ -direction, from the new equilibrium position, the restoring force in such case will be computed from the following potentials,

$$A \exp [-1/\rho \cdot (a + 2x + dx)] + A \exp [-1/\rho (a - 2x - dx)] \\ + 4A \exp [-1/\rho \cdot \sqrt{a^2 + (2x + dx)^2}].$$

After the simple calculation, we obtain, for the frequency-change,

$$\omega_x = \omega_0 + \Delta\omega_x, \\ \Delta\omega_x = \omega_0 [ (1/2\rho^3 + 3/a_0^2 \rho + 3/a_0^3) / 2(1/\rho - 2/a_0) ] (2x)^2. \quad (17)$$

Similarly we have

$$\omega_y = \omega_z = \omega_0 + \Delta\omega_y, \\ \Delta\omega_y = -\omega_0 [ (1/\rho^2 - 1/\rho a_0 - 1/a_0^2) / 2a_0(1/\rho - 2/a_0) ] (2x)^2. \quad (18)$$

Inserting eqs. (17) and (18) into eq. (15) we obtain the final expression of

$$\Delta(K-TS) = V(T)x^2 \\ = \frac{N\hbar\omega_0(2x)^2}{(e^{\hbar\omega_0/kT} - 1)} (1/2\rho^3 + 5/\rho a_0^2 + 5/a_0^3 - 2/a_0\rho^2) / 2(1/\rho - 2/a_0).$$

Since, in the following computation,  $(J\omega_i + 2J\omega_y)$  is found to become of positive sign,  $J(K - TS)$  increases with increasing temperature, which leads to the decrease of the dielectric constant.

Now, summing up the above mentioned results together with (1)⋯(6) the free energy changes due to both external field and temperature-variation have been computed as follows.

$$\begin{aligned} \Delta A = & \frac{1}{2a_0^3} \left( 1 - \frac{3\partial a}{a_0} \right) \left\{ -2x_0 F - \frac{1}{2} \frac{4\pi}{3} \frac{(2x)^2}{2a_0^3} \left( 1 - \frac{3\partial a}{a_0} \right) \right. \\ & - \frac{4\pi}{3} \frac{(4\bar{r}^2 \lambda)(2\lambda)}{2a_0^3} \left( 1 - \frac{3\partial a}{a_0} \right) + \frac{2\alpha_M}{3a_0^2} \left[ \left( \frac{1}{\rho} - \frac{2}{a_0} \right) - \left( \frac{a_0}{\rho^2} - \frac{2}{\rho} - \frac{2}{a_0} \right) \frac{\partial a}{a_0} \right] x^2 \\ & - 4\bar{r}_0^2 F \lambda - \frac{1}{2} \frac{4\pi}{3} \frac{(4\bar{r}^2)^2}{2a_0^3} \left( 1 - \frac{3\partial a}{a_0} \right) \lambda^2 + 3\lambda^2 + A_0 \left( 1 - \frac{\partial a}{\rho'} \right) \lambda^2 \\ & \left. - B_0 \left( 1 - \frac{\partial a}{\rho''} \right) \lambda x \right\} + \Delta(K - TS). \end{aligned} \quad (20)$$

Now the values of  $x$  and  $\lambda$  are to be determined from eqs. (6)'. On putting  $x = x_0(1 + \eta_1)$  and  $\lambda = \lambda_0(1 + \eta_2)$ , where  $x_0$  and  $\lambda_0$  are the solution of eqs. (6)' at absolute zero temperature, eqs. (6)' lead to the following equations for  $\eta_1$  and  $\eta_2$ .

$$\begin{aligned} & \left( B_0 - \frac{4\pi}{3} \frac{8\bar{r}^2}{2a_0^3} \right) x_0 \eta_1 + \left( 6 + 2A_0 - \frac{4\pi}{3} \frac{(4\bar{r}^2)^2}{2a_0^3} \right) \lambda_0 \eta_2 \\ & = B_0 x_0 \frac{\partial a}{\rho''} + 2A_0 \lambda_0 \frac{\partial a}{\rho'} - \frac{4\pi}{3} \frac{8\bar{r}^2 x_0}{2a_0^3} \frac{3\partial a}{a_0} - \frac{4\pi}{3} \frac{(4\bar{r}^2)^2 \lambda_0}{2a_0^3} \frac{3\partial a}{a_0}, \end{aligned} \quad (21)$$

and

$$\begin{aligned} & \left\{ \frac{4\alpha_M}{3a_0^2} \left( \frac{1}{\rho} - \frac{2}{a_0} \right) - \frac{4\pi}{3} \frac{4}{2a_0^3} \right\} x_0 \eta_1 + \left( B_0 - \frac{4\pi}{3} \frac{8\bar{r}^2}{2a_0^3} \right) \lambda_0 \eta_2 \\ & = \frac{4\alpha_M}{3a_0^2} \left( \frac{a_0}{\rho^2} - \frac{2}{\rho} - \frac{2}{a_0} \right) x_0 \frac{3\partial a}{a_0} - \frac{4\pi}{3} \frac{4x_0}{2a_0^3} \frac{3\partial a}{a_0} - 2I^*(T) x_0 \\ & + B_0 \lambda_0 \frac{\partial a}{\rho''} - \frac{4\pi}{3} \frac{4\bar{r}^2}{2a_0^3} 2\lambda_0 \frac{3\partial a}{a_0}. \end{aligned} \quad (22)$$

Using the above solution the change of the free energy of the system is expressed to the first order of  $\partial a$  as follows:

$$\begin{aligned} \partial \Delta A = & \frac{1}{2a_0^3} \left\{ -2x_0 F \eta_1 - \frac{1}{2} \frac{4\pi}{3} \frac{(2x_0)^2}{2a_0^3} 2\eta_1 - \frac{4\pi}{3} \frac{(4\bar{r}^2 \lambda_0)(2x_0)}{2a_0^3} (\eta_1 + \eta_2) \right. \\ & + \frac{2\alpha_M}{3a_0^2} \left( \frac{1}{\rho} - \frac{2}{a_0} \right) x_0^2 2\eta_1 - 4\bar{r}_0^2 F \lambda_0 \eta_2 - \frac{1}{2} \frac{4\pi}{3} \frac{(4\bar{r}^2)^2}{2a_0^3} \lambda_0^2 2\eta_2 \\ & \left. + (3 + A_0) \lambda_0^2 2\eta_2 + B_0 (\lambda_0 x_0) (\eta_1 + \eta_2) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2\alpha_0^3} \left[ 3 \left\{ 2x_0 F + \frac{1}{2} \frac{4\pi}{3} \frac{(2x_0)^2}{2\alpha_0^3} + \frac{4\pi}{3} \frac{(4\bar{r}^2\lambda_0)(2x_0)}{2\alpha_0^2} - \frac{2\alpha_M}{3\alpha_0^2} \left( \frac{1}{\rho} - \frac{2}{\alpha_0} \right) x_0^2 \right. \right. \\
& + 4\bar{r}^2 F \lambda_0 + \frac{1}{2} \frac{4\pi}{3} \frac{(4\bar{r}^2)^2}{2\alpha_0^3} \lambda_0^2 - 3\lambda_0^2 - A_0 \lambda_0^2 - B_0 \lambda_0 x_0 \left. \right\} \frac{\partial a}{\alpha_0} \\
& + 3 \left\{ \frac{1}{2} \frac{4\pi}{3} \frac{(2x_0)^2}{2\alpha_0^3} + \frac{4\pi}{3} \frac{(4\bar{r}^2\lambda_0)(2x_0)}{2\alpha_0^2} + \frac{1}{2} \frac{4\pi}{3} \frac{(4\bar{r}^2)^2}{2\alpha_0^3} \lambda_0^2 \right\} \frac{\partial a}{\alpha_0} \\
& - \left\{ \frac{2\alpha_M}{3\alpha_0^2} \left( \frac{a}{\rho^2} - \frac{2}{\rho} - \frac{2}{\alpha_0} \right) x_0^2 \right\} \frac{\partial a}{\alpha_0} - \left( \frac{A_0 \alpha_0}{\rho'} \lambda_0^2 + \frac{B_0 \alpha_0}{\rho''} \lambda_0 x_0 \right) \frac{\partial a}{\alpha_0} \Big] \\
& + \frac{\partial}{\partial T} \Delta(K - TS) \delta T. \tag{23}
\end{aligned}$$

### § 3. Numerical results

In order to compare the above formula with experiment, we shall here evaluate  $d\kappa/dT$  for LiF crystal. The various kind of quantities involved in eqs. (20), (21) and (22) have been evaluated already in Part I to become

$$x_0 = 31.8F, \quad \lambda_0 = 1.91F, \quad \bar{r}^2 = 1.90, \quad 1/\rho = 1.88, \quad 1/\rho'' = 1.85,$$

$A_0 = 0.80$ ,  $B_0 = 0.503$  and  $\alpha_0 = 3.78$  in atomic units, except for the quantity of  $\rho'$ , which is supposed approximately to be  $\rho' = \rho''$ , allowing for the very small contribution from the term  $(A\alpha_0/\rho')\lambda_0^2$ . At first, we neglect the contribution from the term  $\Delta(K - TS)$ , then the solution of eqs. (21) and (22) is given as follows:

$$\eta_1 = 4.925\partial a, \quad \eta_2 = 2.96\partial a.$$

We, therefore, obtain the variation of  $\Delta(E - K)$  as follows:

$$\delta\Delta(E - K) = \frac{1}{2\alpha_0^3} [-16 + 117 + 353 - 747 - 234] \frac{\partial a}{\alpha_0} F^2 = \frac{1}{2\alpha_0^3} [-527] \frac{\partial a}{\alpha_0} F^2, \tag{23}'$$

where the numerical value of each term corresponds to that of (23). On account of the negative sign of  $\Delta(E - K)$  the dielectric constant is seen to decrease with increasing temperature. In order to obtain the absolute magnitude of (23)' we have to know the thermal expansion coefficient. According to the observation of Eucken the thermal change of lattice constant is given approximately in the temperature range of  $20^\circ C \dots \dots 794^\circ C$  by

$$a = a_0 + \delta a = a_0 + 3.33 \times 10^{-5} a_0 \delta T + 6.0 \times 10^{-8} (T - 273) a_0 \delta T. \tag{24}$$

Inserting the value of (24) into (23)' we obtain the value of  $\Delta(E - K)$  at room temperature as follows.

$$\delta\Delta(E - K) = -1/2\alpha_0^3 \cdot (18.4 \times 10^{-3}) F^2 \delta T. \tag{25}$$

If we neglect the contribution from  $\Delta(K - TS)$ , we are able to obtain, from (25), the

following estimate for the temperature variation of the dielectric constant at room temperature.

$$d\kappa/dT = 4.2 \times 10^{-3}$$

and

$$(1/\kappa)(d\kappa/dT) = 4.5 \times 10^{-4}. \quad (26)$$

Next we shall take account of the contribution from  $J(K-T'S)$ , which leads to smaller values for the solution of eqs. (21) and (22), because  $J(K-T'S)$  has a positive sign. For example,  $\eta_1$  and  $\eta_2$  become  $3.47\delta\alpha$  and  $2.58\delta\alpha$  respectively at room temperature. Including this contribution into (26) we find finally the following value for the temperature variation of the dielectric constant of LiF crystal at room temperature.

$$(1/\kappa)(d\kappa/dT) = 3.2 \times 10^{-4}.$$

The observed value<sup>(3)</sup> is  $3.75 \times 10^{-4}$  at room temperature, which lies between the results of (26) and (27). Allowing for the simplifications and approximations involved

Table I

$TK^\circ$	(a) $\frac{1}{x} \frac{dx}{dT}$	(b) $\frac{1}{x} \frac{dx}{dT}$
300	$4.5 \times 10^{-4}$	$3.2 \times 10^{-4}$
400	$5.5 \times 10^{-4}$	$3.8 \times 10^{-4}$
500	$6.3 \times 10^{-4}$	$4.3 \times 10^{-4}$
600	$7.1 \times 10^{-4}$	$4.9 \times 10^{-4}$
800	$8.8 \times 10^{-4}$	$6.3 \times 10^{-4}$
1000	$10.4 \times 10^{-4}$	$7.8 \times 10^{-4}$

(a) without  $\Delta(K-T'S)$

(b) with  $\Delta(K-T'S)$

in our theory, the agreement seems to be rather good. As seen in Table I the contribution from  $J(K-T'S)$  has a positive sign and the rapidly increasing magnitude with increasing temperature, although its absolute magnitudes are rather small compared to that of  $\Delta(E-K)$ . Also from eq. (24) and Table I the rate of the temperature-variation of dielectric constant is found not to be constant over the wide temperature-range, the general behaviour of which shows the gradual increase of  $d\kappa/dT$  with increasing temperature. Unfortunately the experimental observations have been performed so far over the rather limited temperature-range from  $90^\circ K$  to  $470^\circ K$ , with the results that  $d\kappa/dT$  increases with increasing temperature. We, therefore may conclude that our theory is in a qualitative agreement with the above experiment.

Finally the author wishes to express his sincere thanks to Prof. T. Muto for his continued interest and advice during the course of this work.

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## On the Wave Theory of Light in General Relativity, II

—— *Light as the Electromagnetic Wave* ——

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In the preceding paper\* a wave theoretical method of treating the path of light in general relativity has been developed. In this paper the wave character of light in a curved space-time is further investigated from the standpoint of the electromagnetic optics by using the generalized Maxwell equation in a curved space-time. The main problem treated in this paper is that of the spherical light wave in the static spherically symmetric space-times, and the properties of this wave are clarified by solving the wave equation of light. The wave character of light radiated from the sun is investigated by means of the results obtained by solving the generalized Maxwell equation in Schwarzschild space-time.

### § 1. Introduction

In a curved space-time the ray character of light concerning its path as null geodesic has been frequently treated, whereas its wave character has scarcely been considered. In [I] the wave theoretical significance of the path of light in a curved space-time has been clarified. In order to make research into the other wave character of light the properties of the solutions of its wave equation must be investigated. For this purpose we must begin with considerations concerning the type of the wave equation to be adopted. In the flat space-time phenomenological wave equation of Young-Fresnel's type is frequently used owing to its simplicity. In this paper, however, we shall adopt the electromagnetic field equation as the wave equation of light.\*\*

In general relativity the equation obtained by generalizing the tensorial expression of the usual Maxwell equation in Minkowski space-time into a curved space-time is usually adopted as the electromagnetic field equation. It seems suitable to use this field equation for investigating the wave character of light, therefore we shall deal with the field equation thus introduced.

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\* On the Wave Theory of Light in General Relativity, I. Prog. Theor. Phys. **10** (1953), 442. Hereafter this paper is cited as [I].

\*\* It is formally possible to generalize the Young-Fresnel's wave optics in a curved space-time, but the form of its wave equation differs from that derived from the generalized Maxwell equation. Therefore it is not suitable to discuss the wave character of light in a curved space-time using the phenomenological wave equation of the Young-Fresnel's type. See Appendix I.

Hitherto there have been made some attempts to survey the general character of the electromagnetic field equation in a curved space-time. But in such cases the field equation has scarcely been solved. In this paper the solutions of the spherical wave in the static spherically symmetric space-times are obtained and their nature are compared with those of the corresponding ones in Minkowski space-time. The reason why SSS (abbreviation of the term 'static spherically symmetric') space-times are taken is that these space-times are simple and their properties have been well studied. Moreover such physically important space-times as Schwarzschild's, de Sitter's etc. belong to them.

In § 2 the general nature of the electromagnetic field equation in a curved space-time is investigated. In § 3 the spherical wave solutions of the generalized Maxwell equation in an SSS space-time are obtained and in § 4 the wave-length and the wave-period of the above obtained waves are defined and their nature is clarified. In § 5 the spherical wave solutions in Schwarzschild space-time are obtained and the nature of the light emitted from the sun is investigated.

## § 2. The electromagnetic field equation in a curved space-time

At first we shall consider what equation must be adopted as the electromagnetic field equation in a curved space-time. It is clear that the equation must satisfy the following conditions: i) it is a tensor equation, ii) it tends to the usual Maxwell equation when the space-time reduces to the Minkowski space-time, and iii) its wave solution has, geometrically considered, the wave front corresponding to the null geodesics as the paths of light just as pointed out in [1]. The equation which satisfies the above three conditions is not unique. And probably the simplest may be the one usually adopted. Since we have no positive reason to take the other form by rejecting the ordinary one, we shall take the usual form of the equation: let  $F_{ij}$  be the electromagnetic field tensor, the equation is given by

$$\begin{aligned} \nabla_k F_{ij} + \nabla_i F_{jk} + \nabla_j F_{ki} = 0 \quad \text{or} \quad \partial F_{ij} / \partial x^k + \partial F_{jk} / \partial x^i + \partial F_{ki} / \partial x^j = 0 \\ \nabla_j F^{ij} = 0 \quad \text{or} \quad \partial (\sqrt{-g} g^{ik} g^{jl} F_{kl}) / \partial x^j = 0, * \end{aligned} \quad (2.1)$$

where  $i, j = 1, \dots, 4$ ,  $\nabla_i$  is the covariant derivative with respect to  $x^i$ , and  $g$  is the determinant  $|g_{ij}|$ . On the other hand  $g_{ij}$  in the above equation must satisfy the field equation:

$$R_{ij} - \frac{1}{2} R g_{ij} = -\kappa (T_{ij} + E_{ij}), \quad (2.2)$$

where  $T_{ij}$  is the energy-momentum tensor determined by the distribution and the motion of the matter, and  $E_{ij}$  is the energy-momentum tensor of the electromagnetic field defined by

\* In this paper we shall deal only with the radiation field in the vacuum where four-current does not exist.

$$E_{ij} = -g_{jk} F^{kl} F_{il} + \frac{1}{2} g_{ij} F^{kl} F_{kl}. \quad (2.3)$$

Even in the region where  $T_{ij}=0$ , if any electromagnetic field exists, then  $E_{ij} \neq 0$  and  $g_{ij}$  is determined depending on  $F_{ij}$ . Namely the equation (2.1) and (2.2) are not independent, but simultaneous. Since it is difficult to solve these simultaneous equations, they have scarcely been solved practically except for some simple cases most of which concern the electrostatic field. Accordingly, in the first place, we intend to solve the electromagnetic equation (2.1) regarding  $g_{ij}$  as a given function. Namely, we ignore the gravitational effect of the electromagnetic field, just as Whittaker, Laue etc. did in their papers.<sup>1)</sup> For example, in studying the properties of light propagation in the gravitational field of the sun we treat the equation (2.1) taking  $g_{ij}$  as Schwarzschild's.

In the second place we shall study the general character of the equation for the electromagnetic four-potential  $\varphi_i$ . It is easily seen from the equation (2.1) that the equation for  $\varphi_i$  takes the following form:

$$F_{ij} = \nabla_j \varphi_i - \nabla_i \varphi_j = \partial \varphi_i / \partial x^j - \partial \varphi_j / \partial x^i, \quad (2.4a)$$

$$g^{ij} \nabla_i \nabla_j \varphi_k + K^i_k \varphi_i - \nabla_k (\nabla_i \varphi^i) = 0, \quad (2.4b)$$

where  $K^i_k$  is the Ricci tensor of the space-time. Further we impose the following generalized Lorentz condition:

$$\nabla_i \varphi^i = g^{ij} \nabla_i \varphi_j = 0^*. \quad (2.5)$$

Then (2.4b) becomes

$$g^{ij} \nabla_i \nabla_j \varphi_k + K^i_k \varphi_i = 0. \quad (2.4c)$$

The propagation character of the wave solution of the equation of the type (2.4c, c') has been investigated in [1]. From the results obtained by this investigation we know that these equations satisfy the third condition proposed at the beginning of this section. And it is also to be remarked that the characteristic relation of (2.4c, c') is the same form as the characteristic equation of (2.1) for the unknown function  $F_{ij}$ , and that each component of  $F_{ij}$  satisfies the following wave equation:

$$g^{lm} \nabla_l \nabla_m F_{ij} - 2 F_{mn} K^m_{ij} - (K^i_n F_{jn} - K^j_n F_{in}) = 0, \quad (2.6)$$

where  $K^m_{ijn}$  the curvature tensor of the space-time.

In the third place the problems concerning the gauge transformation will be mentioned.

\* Ordinarily this form is taken but we can take the following alternative form:

$$K^i_k \nabla_i \varphi_k - \nabla_k (\nabla_i \varphi^i) = 0. \quad (2.5')$$

In this case the field equation (2.4b) becomes simply:

$$g^{ij} \nabla_i \nabla_j \varphi_k = 0, \quad (2.4c')$$

whereas the condition for the function  $\chi$  becomes somewhat complicated. That is,

$$K^i_k \nabla_i \chi - \nabla_k \nabla_i \chi = 0 \quad \text{or} \quad K^i_k \nabla_i \chi - \nabla_k g^{ij} \nabla_j \chi = 0. \quad (2.8')$$

In a curved space-time the gauge transformation is defined by

$$\bar{\varphi}_k = \varphi_k + \nabla_k \chi \quad (\nabla_k \text{ being covariant derivative}) \quad (2.7)$$

which keeps the electromagnetic field tensor  $F_{ij}$  invariant. The function  $\chi$  in (2.7) is required to satisfy the following equation so as the generalized Lorentz condition (2.5) may be invariant:

$$\square \chi = 0 \quad \text{or} \quad g^{ij} \nabla_i \nabla_j \chi = 0. \quad (2.8)$$

Accordingly the wave equation for  $\varphi_i$ , (2.4c), remains invariant by the gauge transformation under this condition. As is stated above the fundamental properties of gauge transformation (2.7) are a natural generalization of those in the flat space-time. But the gauge transformation in the curved space-time has some remarkable properties which are not found in the flat space-time. The detailed discussions of the properties will be made elsewhere.

### § 3. Spherical wave solution of the generalized Maxwell equation in an SSS space-time

Now we shall take the generalized Maxwell equation (2.1) as the wave equation of light in a curved space-time, and shall investigate its wave solutions. In Minkowski space-time the simplest and most fundamental wave solution of the Maxwell equation is that of the plane wave, but in a curved space-time the generalization of the concept of the plane wave is difficult, generally speaking. Therefore in the following we shall not deal with a plane wave solution.\* Another simple and physically important wave solution in the flat space-time may probably be that of the spherical wave. In an SSS space-time a spherical surface with the center at the origin has an important meaning because of its symmetry. Therefore, in the following, we shall obtain the spherical wave solutions in an SSS space-time using the coordinate system in which the line element takes the form (3.1).\*\*

$$ds^2 = -A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + C(r)dt^2, \quad (3.1)$$

where  $(r, \theta, \phi, t) = (x^1, x^2, x^3, x^4)$ . In this coordinate system the fundamental tensor is given by

$$g_{11} = -A(r), \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta, \quad g_{44} = C(r), \quad \text{other } g_{ij} = 0. \quad (3.2)$$

#### (a) Generalized Maxwell equation in an SSS space-time.

Let us rewrite the generalized Maxwell equation (2.1) using this coordinate system. In doing this we shall use the physical component of  $F_{ij}^{(1)}$ , instead of its tensor component,

\* See Appendix II.

\*\* Invariant theoretical definition and the character of SSS space-time were investigated in detail by Takeno.<sup>3)</sup>



generalizing the concept of the physical component of a tensor used in classical physics. Namely, we use the physical component of  $F_{ij}$  in the direction of  $(x^i, x^j)$ , denoting it by the usual symbol of electric vector  $\mathbf{E}$  and magnetic vector  $\mathbf{H}$ .\* As is easily seen when  $A=C=1$  this definition reduces to the usual one in the flat space-time. Using these  $\mathbf{E}$  and  $\mathbf{H}$  the generalized Maxwell equation (2.1) can be rewritten as follows :

$$\begin{aligned} \sqrt{C}/r \sin \theta \cdot \left[ \frac{\partial}{\partial \theta} (\sin \theta E_\phi) - \frac{\partial}{\partial \phi} E_\theta \right] + \frac{\partial}{\partial t} H_r &= 0, \\ \sqrt{AC}/r \sin \theta \cdot \frac{\partial}{\partial \phi} E_r - \frac{1}{r} \frac{\partial}{\partial r} (r \sqrt{C} E_\phi) + \sqrt{A} \frac{\partial}{\partial t} H_\theta &= 0, \\ \frac{1}{r} \frac{\partial}{\partial r} (r \sqrt{C} E_\theta) - \sqrt{AC}/r \cdot \frac{\partial}{\partial \theta} E_r + \sqrt{A} \frac{\partial}{\partial t} H_\phi &= 0, \\ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 H_r) + \sqrt{A}/r \sin \theta \cdot \frac{\partial}{\partial \theta} (\sin \theta H_\theta) + \sqrt{A}/r \sin \theta \cdot \frac{\partial}{\partial \phi} H_\phi &= 0, \\ \sqrt{C}/r \sin \theta \cdot \left[ \frac{\partial}{\partial \theta} (\sin \theta H_\phi) - \frac{\partial}{\partial \phi} H_\theta \right] - \frac{\partial}{\partial t} E_r &= 0, \\ \sqrt{AC}/r \sin \theta \cdot \frac{\partial}{\partial \phi} H_r - \frac{1}{r} \frac{\partial}{\partial r} (r \sqrt{C} H_\phi) - \sqrt{A} \frac{\partial}{\partial t} E_\theta &= 0, \\ \frac{1}{r} \frac{\partial}{\partial r} (r \sqrt{C} H_\theta) - \sqrt{AC}/r \cdot \frac{\partial}{\partial \theta} H_r - \sqrt{A} \frac{\partial}{\partial t} E_\phi &= 0, \\ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) + \sqrt{A}/r \sin \theta \cdot \frac{\partial}{\partial \theta} (\sin \theta E_\theta) + \sqrt{A}/r \sin \theta \cdot \frac{\partial}{\partial \phi} E_\phi &= 0. \end{aligned} \tag{3.3}$$

Here it is to be noted again that these equations agree with the usual ones in Minkowski space-time if we put  $A=C=1$ .

### (b) Three types of spherical wave solution

The general solution of the above equation (3.3) is shown in Appendix IV. Here we shall give the following three kinds of spherical wave solutions corresponding to those in Minkowski space-time<sup>7)</sup> : i.e. transverse electric wave (TE wave), transverse magnetic wave (TM wave) and transverse electromagnetic wave (TEM wave, or principal wave).

(1) TE wave. This type of wave has no longitudinal component of the electric vector  $\mathbf{E}$ , whereas that of the magnetic vector,  $H_r$ , is not zero. This type of wave is called in the flat space-time transverse electric wave. Hence we shall use the same nomenclature in the curved space-time also. If we put  $u=rH_r$ , it is easily seen that  $u$  satisfies the following equation.

\* See Appendix III.



$$\frac{1}{r\sqrt{AC}} \frac{\partial}{\partial r} \left\{ \sqrt{\frac{C}{A}} \frac{\partial}{\partial r} (ru) \right\} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} - \frac{1}{C} \frac{\partial^2 u}{\partial t^2} = 0. \quad (3.4)$$

Taking  $u = R(r, t) \Theta(\theta) \Phi(\phi)$ , (3.4) becomes

$$d^2 \Phi / d\phi^2 + n^2 \Phi = 0, \quad (3.5a)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \left[ l(l+1) - \frac{n^2}{\sin^2 \theta} \right] \Theta = 0, \quad (3.5b)$$

$$\frac{1}{r\sqrt{AC}} \frac{\partial}{\partial r} \left\{ \sqrt{\frac{C}{A}} \frac{\partial}{\partial r} (rR) \right\} - \frac{1}{C} \frac{\partial^2 R}{\partial t^2} - \frac{l(l+1)}{r^2} R = 0, \quad (3.5c)$$

where  $l$  is a positive integer and  $n$  is an integer  $|n| < l$ . The equations (3.5a) and (3.5b) are the same as in the flat space-time, but (3.5c) differs from that in the flat space-time by the factors depending on the functions  $A(r)$  and  $C(r)$ .

Now taking

$$E_r = 0, \quad E_\theta = \frac{-1}{l(l+1)r\sqrt{C}\sin\theta} \frac{\partial^2}{\partial\phi\partial t} (ru), \quad E_\phi = \frac{1}{l(l+1)r\sqrt{C}} \frac{\partial^2}{\partial\theta\partial t} (ru), \quad (3.6)$$

$$H_r = u/r, \quad H_\theta = \frac{1}{l(l+1)r\sqrt{A}} \frac{\partial^2}{\partial r\partial\theta} (ru), \quad H_\phi = \frac{1}{l(l+1)r\sqrt{A}\sin\theta} \frac{\partial^2}{\partial r\partial\phi} (ru),$$

we can show by direct substitution in (3.3) that the equations (3.3) are satisfied. These relations (3.6) coincide with the corresponding ones in the flat space-time when we put  $A=C=1$ .

(2) TM wave. This type of wave has, by definition, no longitudinal component of the magnetic vector, whereas that of the electric vector is not zero. Namely,  $H_r=0$ , but  $E_r \neq 0$ . The analogous relations hold as in the case of TE wave. That is, if we put  $rE_r = v$ ,  $v$  satisfies the same equation as  $u$ . In terms of  $v$  the other components are given by

$$E = v/r, \quad E_\theta = \frac{1}{l(l+1)r\sqrt{A}} \frac{\partial^2}{\partial r\partial\theta} (rv), \quad E_\phi = \frac{1}{l(l+1)r\sqrt{A}\sin\theta} \frac{\partial^2}{\partial r\partial\phi} (rv), \quad (3.7)$$

$$H_r = 0, \quad H_\theta = \frac{1}{l(l+1)r\sqrt{C}\sin\theta} \frac{\partial^2}{\partial\phi\partial t} (rv), \quad H_\phi = \frac{-1}{l(l+1)r\sqrt{C}} \frac{\partial^2}{\partial\theta\partial t} (rv).$$

It is easily seen that these components satisfy (3.3).

(3) TEM wave. The third type of the wave is the transverse electromagnetic wave, in which both  $E_r$  and  $H_r$  are zero. The general solution\* of this type of spherical wave is given by

$$E_\theta = \frac{1}{r\sqrt{C}\sin\theta} g_1 f_2, \quad E_\phi = \frac{1}{r\sqrt{C}\sin\theta} f_1 f_2,$$

$$H_\theta = \frac{-1}{r\sqrt{C}\sin\theta} f_1 g_2, \quad H_\phi = \frac{1}{r\sqrt{C}\sin\theta} g_1 g_2. \quad (3.8)$$

\* See Appendix IV.

Here  $f_1, g_1$  are any harmonic functions of  $(\mu, \phi)$ , determined by

$$\partial g_1 / \partial \mu = -\partial f_1 / \partial \phi, \quad \partial g_1 / \partial \phi = \partial f_1 / \partial \mu, \quad (3.9)$$

where  $\mu$  is given by

$$\mu = \int \frac{d\theta}{\sin \theta} = \log \tan \theta/2 + \text{const.} \quad (3.10)$$

And  $f_2, g_2$  are functions of  $(\sigma, t)$  given by

$$f_2 = F_1(t + \sigma) + F_2(t - \sigma), \quad g_2 = -F_1(t + \sigma) + F_2(t - \sigma), \quad (3.11)$$

where  $F_1, F_2$  are any arbitrary functions of  $(t - \sigma), (t + \sigma)$  respectively.  $\sigma$  is the function of  $r$  defined by

$$\sigma = \int \sqrt{\frac{A}{C}} dr. \quad (3.12)$$

It is seen from (3.11) that the spherical waves of the TEM type form the relative dispersionless wave family (relative verzerrungsfreie Wellenfamilie)<sup>(6)</sup>, and conversely it is easily proved that if such a wave family exists, it is necessarily composed of the TEM waves.\*

Finally it is to be noticed that the above mentioned three types of the spherical wave solutions correspond to those given by Slater<sup>(5)</sup> in Minkowski space-time.

### (c) Poynting vector and energy of the electromagnetic field in an SSS space-time

Let us consider the Poynting vector  $\mathbf{S}$  and energy  $W$  of the electromagnetic wave in an SSS space-time. We shall define these quantities in terms of the physical components of the energy-momentum tensor  $E_{ij}$  in the directions of the parameter curves:\*\*

$$\mathbf{S} = (S_r, S_\theta, S_\phi), \quad \begin{cases} S_r = -E_{rt} = E_\theta H_\phi - E_\phi H_\theta, \\ S_\theta = -E_{\theta t} = E_\phi H_r - E_r H_\phi, \\ S_\phi = -E_{\phi t} = E_r H_\theta - E_\theta H_r, \end{cases} \quad (3.13)$$

$$W = E_{tt} = 1/2 \cdot (E_r^2 + E_\theta^2 + E_\phi^2 + H_r^2 + H_\theta^2 + H_\phi^2).$$

These quantities tend to those in the flat space-time when the space-time reduces to the flat one. And moreover using the relation:<sup>(7)</sup>

$$\nabla_i E^i_j = 0, \quad (3.14)$$

we can verify the conservation of the energy for these  $W$  and  $\mathbf{S}$ .

Next we shall calculate the energy  $W$  and the Poynting vector  $\mathbf{S}$  of the out-going TEM wave. These quantities are given by

\* See Appendix V.

\*\* It is also possible to define these quantities using the tensor components of  $E_{ij}$ :  $\mathbf{S} = (E_{14}, E_{24}, E_{34})$  and  $W = E_{44}$ . But this definition will make the later treatment complicated. See Appendix III.

$$S_r = \frac{1}{r^2 C \sin^2 \theta} (g_1^2 + f_1^2) (F_2)^2, \quad S_\theta = S_\phi = 0, \quad (3.15)$$

$$W = \frac{1}{r^2 C \sin^2 \theta} (g_1^2 + f_1^2) (F_2)^2. \quad (3.16)$$

In this case it is easily seen that the direction of the energy flow is that of the radius vector and the magnitude of  $\mathbf{S}$  is equal to that of  $W$ , namely,

$$|\mathbf{S}| = S_r = W. \quad (3.17)$$

From this relation we see that the velocity of the energy flow is equal to 1.

#### § 4. Wave-length, wave-period and velocity of spherical wave in an SSS space-time

We shall introduce the concept of wave-length, wave-period and velocity of wave in an SSS space-time. Since this problem is, in general, not easy, we shall use the coordinate system (3.1) and shall confine ourselves to the spherical wave of the type:

$$\psi = k(r, \theta, \phi) f(W(r, t)), \quad (4.1)$$

where  $f$  is a periodic function of  $W$ , and  $W$  is given by

$$W = \sigma \pm t^*. \quad (4.2)$$

And we shall assume that this  $\psi$  represents a component of the electromagnetic tensor and satisfies the wave equation (2.6). Let the period of the function  $f$  be  $\omega^{**}$  (any constant independent of  $r$  and  $t$ ), that is,

$$f(W(r, t)) = f(W(r, t) + \omega). \quad (4.3)$$

Next we shall take  $\Delta r$  satisfying the relation

$$\sigma(r + \Delta r) = \sigma(r) + \omega,$$

where  $\sigma$  is given by (3.12) and is determined by  $A$  and  $C$ . For the practical cases the change of the value of these  $A$  and  $C$  corresponding to the increment of  $\Delta r$  can be ignored. Then this  $\Delta r$  is given by the following formula as the function of  $r$ :

$$\Delta r = \sqrt{C/A} \cdot \omega. \quad (4.4)$$

On the other hand, the value of the function  $f$  remains unchanged if the variable  $t$  increases by the amount  $\Delta t = \omega$ . Using these  $\Delta r$  and  $\Delta t$  we shall define the invariant wave-length  $\lambda$  and invariant wave-period  $T$  by

<sup>\*</sup> In the following we shall take only  $-$  sign of the double sign in (4.2). In the case  $+$  sign we can also discuss the problem in the same manner.

<sup>\*\*</sup> We shall call this  $\omega$  the *period of the wave function* distinguishing this from *wave-period*  $T$  which will appear later.

$$\lambda = \sqrt{A} \Delta r = \sqrt{C} \omega, \quad T = \sqrt{C} \Delta T = \sqrt{C} \omega. * \quad (4.5)$$

Then these two quantities are equal, that is,

$$\lambda = T. ** \quad (4.6)$$

Next we shall consider the phase velocity of the wave. For this purpose we shall use the invariant velocity  $v = (v^1, v^2, v^3)$ , the component of which is defined by

$$v^i = \lim_{\Delta x^4 \rightarrow 0} \frac{\text{the physical component of } \Delta x \text{ in the direction of } x^i}{\text{the physical component of } \Delta x \text{ in the direction of } x^4} \quad (i=1, 2, 3),$$

where  $\Delta x = (\Delta x^1, \Delta x^2, \Delta x^3, \Delta x^4)$  is an infinitesimal displacement vector. Then it is seen that the phase velocity of the wave given by (4.1) is equal to 1.\*\*\* The relation (4.6) is consistent with this result. Moreover, it is worth noticing that this phase velocity is equal to the ray velocity derived from the condition  $ds^2=0$  in terms of invariant velocity.\*\*\* In the last place it is to be noticed that the above results can be obtained when and only when the phase  $H$  satisfies the characteristic equation of the wave equation (2.6).

### Shift of the spectral lines

Next we shall compare the wave-lengths and wave-periods of the two waves of the type (4.1) emitted from the two atoms of the same kind situated at rest at the spatially different points,  $P_1$  and  $P_2$ . Let the spatial coordinates of  $P_1$  and  $P_2$  be  $(r_1, \theta, \phi)$  and  $(r_2, \theta, \phi)$ , and the time at which light is emitted from  $P_1$  and  $P_2$  be  $t_1$  and  $t_2$  respectively, and the wave functions of these waves be

$$\psi_1 = k_1(r, \theta, \varphi) f_1(W(r, t)), \quad \psi_2 = k_2(r, \theta, \phi) f_2(W(r, t)). \quad (4.7)$$

Here we shall assume that these two wave functions correspond to the same energy level of these two atoms. And we denote the invariant wave-periods of these waves  $T_1$  and  $T_2$  respectively. Then these are given by

$$T_1 = \sqrt{C(r_1)} \omega_1, \quad T_2 = \sqrt{C(r_2)} \omega_2, \quad (4.8)$$

where  $\omega_1$  and  $\omega_2$  are the periods of the wave function  $\psi_1$  at  $P_1$  and that of  $\psi_2$  at  $P_2$ . Since these atoms are at rest, the proper periods of the waves coincide with these wave-periods respectively and it is natural to consider that these two proper periods are of the same value. Hence we have

\* Regarding these  $\Delta r$  and  $\Delta t$  as the components of a vector we can consider  $\lambda$  and  $T$  as the physical components of the vector in the directions of  $r$  and  $t$  respectively. The same result is obtained if we define these quantities using the concepts of spatial distance and time interval proposed by Landau and Lifshitz.<sup>(5)</sup> These definitions are different from that of Laue's.<sup>(6)</sup>

\*\* Hence we consider the dimensions of  $r$  and  $t$  are equal, so that those of  $\lambda$  and  $T$  are also equal.

\*\*\* Differentiating the equation  $W = \sigma - t = \text{const.}$  with respect to  $t$  we obtain  $dr/dt = \sqrt{C/A}$ . Therefore we know that the phase velocity is equal to 1.

\*\*\*\* Putting  $ds^2=0$  and  $d\theta=d\phi=0$  in (3.1) we obtain  $dr/dt = \sqrt{C/A}$ . Hence in this case the invariant velocity of the light ray is equal to 1 also.

$$T_1 = T_2, \quad \text{or} \quad \sqrt{C(r_1)} \omega_1 = \sqrt{C(r_2)} \omega_2. \quad (4.9)$$

On the other hand the wave-period of the  $\psi_1$  wave at  $P_2$  is given by

$$T_{12} = \sqrt{C(r_2)} \omega_1. \quad (4.10)$$

If we denote  $T_{12} = T + \delta T$ ,  $T_2 = T$ , then

$$\begin{aligned} T_{12}/T_2 &= (T + \delta T)/T = \sqrt{C(r_2)} \omega_1 / \sqrt{C(r_2)} \omega_2 = \omega_1 / \omega_2 \\ &= \sqrt{C(r_2)} / \sqrt{C(r_1)}. \end{aligned} \quad (4.11)$$

In this case the ratio between the two wave-lengths are given by

$$(\lambda + \delta\lambda) / \lambda = \sqrt{C(r_2)} / \sqrt{C(r_1)}. \quad (4.12)$$

If we apply the above result to the Schwarzschild space-time (this application will appear later), we shall obtain the same shift of the spectral lines already obtained in the relativistic theory using the concept of ray optics.<sup>10)</sup>

### § 5. Properties of light in Schwarzschild space-time

Using the results obtained in the last section we can easily investigate the properties of the spherical waves with their centers at the origin. As an application, physically important, of these results, we shall examine the wave character of light radiated from the sun. Taking the space-time as that of Schwarzschild and assuming that the sun is situated at the origin of the coordinate system it may be allowed to take the radiation of the sun as the complicated superposition of the spherical waves of the above three types. As the metric of the space-time we shall adopt the following after (3.1) :

$$ds^2 = -dr^2 / (1 - 2m/r) - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + (1 - 2m/r) dt^2. \quad (5.1)$$

Therefore the results obtained in the last section hold as they are, if we take  $A(r)$  and  $C(r)$  in (3.1) as

$$A(r) = 1 / (1 - 2m/r), \quad C(r) = 1 - 2m/r. \quad (5.2)$$

#### (a) Spherical wave solutions of the generalized Maxwell equation

At first the spherical wave solutions of the equation (3.3) in Schwarzschild space-time will be considered. The solution of TEM type given by (3.9) becomes as follows :

$$\begin{aligned} E_\theta &= \frac{g_1}{r \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \sin \theta} \{F_1(t + \sigma) + F_2(t - \sigma)\}, \\ E_\phi &= \frac{f_1}{r \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \sin \theta} \{F_1(t + \sigma) + F_2(t - \sigma)\}, \end{aligned} \quad (5.3)$$



$$H_0 = \frac{-f_1}{r \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \sin \theta} \{-F_1(t+\sigma) + F_2(t-\sigma)\},$$

$$H_\phi = \frac{g_1}{r \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} \sin \theta} \{-F_1(t+\sigma) + F_2(t-\sigma)\},$$

where  $\sigma = \int \frac{dr}{1 - \frac{2m}{r}}$ .  $f_1, g_1$  are given by (3.9) and  $F_1, F_2$  are any arbitrary functions of  $(t+\sigma)$  and  $(t-\sigma)$  respectively.

Next we shall refer to the TE and TM waves. As they have many common properties, we shall examine only the TM wave taking the case when  $l=1$  for simplicity's sake. In this case, as is easily seen,  $v(=rH_r)$  becomes:

$$v = R(r, t) \cos \theta. \quad (5.4)$$

If we consider only the out-going wave this  $R$  is given from (3.5c) by

$$R = a(-\cos k(\sigma-t)/kr + \sin k(\sigma-t)/k^2 r^2) + O(1/r^4) \quad (5.5a)$$

or

$$R = b(-\sin k(\sigma-t)/kr - \cos k(\sigma-t)/k^2 r^2) + O(1/r^4), \quad (5.5b)$$

where  $a, b$  and  $k$  are any arbitrary constants. When the radiation from the sun is concerned the relation  $kr \gg 1$  holds. Neglecting the higher order of  $1/kr$  the leading term of  $R$  is also given for  $l=2, 3, \dots$  by

$$a/kr \cdot \cos k(\sigma-t) \quad \text{or} \quad b/kr \cdot \sin k(\sigma-t). \quad (5.6)$$

Therefore if we consider the nature of the wave function taking only its leading term, the only difference between the wave function in Schwarzschild space-time and that in Minkowski space-time is the phase function of the wave. Namely, in the former it is  $(\sigma-t)$ , while in the latter  $(r-t)$ . We shall evaluate the other components of the electric and magnetic vector using the approximate form of the function  $R$ :

$$R \doteq -b \sin k(\sigma-t)/kr. \quad (5.7)$$

Then we can obtain

$$E_r = -\frac{b \sin k(\sigma-t)}{kr^2} \cos \theta, \quad E_\theta = \frac{bk \cos k(\sigma-t)}{2r \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}} \sin \theta, \quad E_\phi = 0,$$

$$H_r = H_\theta = 0, \quad H_\phi = \frac{bk \cos k(\sigma-t)}{2r \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}} \sin \theta. \quad (5.8)$$

It is easily seen that almost the same results can be obtained if we take TE case.

(b) *Wave-form, wave-length and wave-period of the spherical wave*

As was mentioned above, the nature of the light emitted from the sun can be studied using the three kinds of waves. At first we shall consider the wave-form of the waves (5.3) and (5.8). The common character of these two kinds of waves is as follows: i) in their amplitude the factor  $(1-2m/r)^{-1/2}$  appears except in  $E_r$  of (5.8), ii) their phase is  $(\sigma-t)$  instead of  $(r-t)$ . Here  $\sigma$  is obtained by integration:

$$\sigma = \int \frac{dr}{1 - \frac{2m}{r}} = r + 2m \log(r-2m) + \text{const.} \quad (5.9)$$

when  $r \gg m$  and the second term is very small compared with the first, we can replace  $\sigma$  by  $r$ .

Next we shall consider the wave-length and the wave-period of the wave given by (5.3) and (5.8), putting  $F_1 \equiv 0$  and  $F_2 \equiv \sin k(\sigma-t)$  in (5.3). For such waves we can use the formula obtained in the last section. Since the period of these wave functions is  $2\pi/k$ , the wave-period  $T$  of these waves is

$$T = (1-2m/r)^{1/2} 2\pi/k. \quad (5.10)$$

And the wave-length  $\lambda$  is given by

$$\lambda = \Delta r / (1-2m/r)^{1/2} = (1-2m/r)^{1/2} 2\pi/k \quad (5.11)$$

using the relation

$$2\pi/k = \int_r^{r+\Delta r} dr / (1-2m/r) \doteq \Delta r / (1-2m/r). \quad (5.12)$$

Here it is to be remembered that the relation (4.6) can be verified directly.

The wave-forms of (5.3) and (5.8) are characterized by their phase function  $(\sigma-t)$ . Namely the wave-form of (5.8) is not the sine or cosine curves with respect to the variable  $r$ . Using the natural light of the sun such distortion of the wave-form cannot at present be detected by experiment, because the difference is very small.

(c) *Gravitational shift of the spectral lines.*

We shall investigate the shift of the spectral lines in the gravitational field of the sun. Hitherto ray optical investigations have been carried out concerning this effect, but such treatments are unsatisfactory. Now we shall adopt the wave optical standpoint taking account of the wave function of light obtained above. The wave function of the light from the sun will be expressed approximately by the superposition of the wave function of TE, TM and TEM types.\* Since the leading terms of these functions are of the type (4.1), we can apply (4.11) and (4.12) for obtaining the shift of the spectral

\* See Appendix IV.

lines of the sun. The result is as follows :

$$(T + \delta T)/T = (\lambda + \delta\lambda)/\lambda = (1 - 2m/r_1)^{1/2}/(1 - 2m/r_0)^{1/2} \simeq 1 + m/r_0, \quad (5.13)$$

where  $r_0$  and  $r_1$  are the radius of the sun and the distance from the sun to the earth respectively, and the term  $2m/r_1$  can be neglected compared with 1. The result (5.13) coincides with the one already obtained ray optically.<sup>10)</sup> But it is to be noticed that the same result will not be obtained if we take the more accurate solutions of the wave equation.

#### (d) Poynting vector and energy of radiation

The Poynting vector and energy of the electromagnetic field are given by (3.13). We shall use these results with respect to the waves of (5.3) and (5.8).

First we shall adopt the wave (5.3) putting  $f_1=0$ ,  $F_1 \equiv 0$  and  $F_2 \equiv \sin k(\sigma-t)$ . The Poynting vector  $\mathbf{S} = (S_r, S_\theta, S_\phi)$  and its time average  $\bar{\mathbf{S}}$  are given by

$$S_r = E_\theta H_\phi, \quad S_\theta = S_\phi = 0, \quad \bar{S}_r = \frac{g_1^2}{2\left(1 - \frac{2m}{r}\right)r^2 \sin^2 \theta}. \quad (5.14)$$

And the energy  $W$  and its time average  $\bar{W}$  are given by

$$W = 1/2 \cdot (E_\theta^2 + H_\phi^2), \quad \bar{W} = g_1^2 / \{2(1 - 2m/r)r^2 \sin^2 \theta\}. \quad (5.15)$$

The damping factor of these quantities are different from those in Minkowski space-time by the factor  $1/(1 - 2m/r)$ .

Next we shall calculate these quantities of TM wave (5.8). We can ignore the component  $E_r$  compared with  $E_\theta$  and  $H_\phi$  because in our case  $kr \gg 1$ , then  $\mathbf{S}$  and  $W$  are given by

$$S_r = \frac{b^2 k^2 \sin^2 \theta}{4r^2 \left(1 - \frac{2m}{r}\right)} \cos^2 k(\sigma - t), \quad S_\theta = S_\phi = 0, \quad \bar{S}_r = \frac{b^2 k^2 \sin^2 \theta}{8r^2 \left(1 - \frac{2m}{r}\right)}, \quad (5.16)$$

$$W = \frac{b^2 k^2 \sin^2 \theta}{4r^2 \left(1 - \frac{2m}{r}\right)} \cos^2 k(\sigma - t), \quad \bar{W} = \frac{b^2 k^2 \sin^2 \theta}{8r^2 \left(1 - \frac{2m}{r}\right)}, \quad (5.17)$$

where  $\bar{S}_r$  and  $\bar{W}$  are the time average of  $S_r$  and  $W$  respectively. The total radiation  $S$  is given by integration of  $\bar{S}_r$  on the spherical surface :

$$S = \int_0^\pi \bar{S}_r 2\pi r^2 \sin \theta d\theta = \frac{b^2 k^2 \pi}{4 \left(1 - \frac{2m}{r}\right)} \int_0^\pi \sin^3 \theta d\theta = \frac{b^2 k^2 \pi}{3 \left(1 - \frac{2m}{r}\right)}. \quad (5.18)$$

Similar results are obtained in the case where  $l=2, 3, \dots$  except the trivial difference of the factor depending on  $\theta$  and  $\phi$ , if we take only its leading term. The same results will be obtained also in the case of TE wave. In the Minkowski space-time the

corresponding total radiation is given by

$$S = b^2 k^2 \pi / 3. \quad (5.19)$$

Concerning the total flux of the solar radiation the similar result as (5.18) will hold approximately.

## § 6. Concluding remarks

In this paper we have obtained the spherical wave solutions of Maxwell equation in SSS space-times ignoring the gravitational effect of the electromagnetic field. Now we shall reconsider the results obtained.

In our treatment the generalized concept of the physical components of a tensor in a curved space-time has been introduced. Hitherto the application of the physical components of a tensor has been overlooked in general relativistic theories. But the usefulness of the physical component for considering the physical meanings of a tensor has been clarified in our case. In many other cases we shall obtain an invariant and convenient result by using the physical components of a tensor.

We have investigated the wave character of light in Schwarzschild space-time. The most important results are probably the gravitational shift of the spectral lines and the effect of the damping factor of the spherical light wave. In order to calculate exactly the shift of the spectral lines knowledge of the wave function is needed as was stated in § 4. Namely the ray optical treatment of the line shift hitherto employed is not valid in general because the ray velocity of light derived from the condition  $ds^2 = 0$  differs from the phase velocity of light wave in a usual case. But it has been shown that the ray optical treatment is valid in our case, because in such a case the ray velocity is equal to the phase velocity of light.

Concerning the damping factor of the spherical waves we must examine in detail the effect of the factor of  $(1 - 2m/r)$  which appears in Poynting vector and energy of the spherical light waves. But the deviation will be negligible compared with the accuracy of the present measurements.

Since we have treated only the spherical wave solutions, we cannot deal with the so-called 'Einstein effect'. Using the result obtained in [I] we can obtain the same path of light if we use either the ray optical or the wave optical method. But in studying the problems concerning the wave-form or the spectral line shift the wave theoretical considerations will be needed. In this paper we have not, however, treated such problems further, because the main purpose of this paper is to investigate the general character of spherical light wave in an SSS space-time.

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## Appendixes

### I. Scalar wave equation in a curved space-time

The scalar wave equation in a curved space-time may be written as follows :

$$g^{ij}\nabla_i\nabla_j\varphi=0 \quad \text{or} \quad g^{ij}\left[\frac{\partial^2\varphi}{\partial x^i\partial x^j}-\left\{\begin{matrix} k \\ ij \end{matrix}\right\}\frac{\partial\varphi}{\partial x^k}\right]=0, \quad (\text{I}\cdot 1)$$

generalizing the ordinary one in the flat space-time. It is to be noticed that the form of the equation differs from that of four-potentials of electromagnetic vectors given in § 2. We shall obtain the solution of (I·1) in an SSS space-time using the coordinate system given by (3·1). If we put

$$\varphi=R(r,t)\Theta(\theta)\Phi(\phi), \quad (\text{I}\cdot 2)$$

(I·1) becomes

$$\begin{aligned} d^2\Phi/d\phi^2+n^2\Phi &=0, \\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[l(l+1)-\frac{n^2}{\sin^2\theta}\right]\Theta &=0, \\ -\frac{1}{A}\frac{\partial^2 R}{\partial r^2}-\tau_1\frac{\partial R}{\partial r}+\frac{l(l+1)}{r^2}R+\frac{1}{C}\frac{\partial^2 R}{\partial t^2} &=0, \end{aligned} \quad (\text{I}\cdot 3)$$

where  $\tau_1=\frac{1}{2A}\left(\frac{A'}{A}-\frac{C'}{C}-\frac{4}{r}\right)$ ,  $l$  is an arbitrary positive integer and  $n$  is an integer satisfying  $|n|<l$ . The nature of the solution (I·1) under the condition (I·2) differs from that treated in § 3. For instance, as was shown in § 3, there exists a dispersionless light wave in any SSS space-time, but the dispersionless wave solution of (I·1) exists when and only when the following relation holds :\*

$$r/2A\cdot(-A'/A+C'/C)=l(l+1), \quad (\text{I}\cdot 4)$$

or in a tensor form using Takeno's method :<sup>3)</sup>

$$\overset{2}{\rho}+\overset{3}{\rho}+4\overset{4}{\rho}=4/\lambda^2\cdot l(l+1). \quad (\text{I}\cdot 4')$$

In Minkowski space-time the relation (I·4) holds only for S wave, i.e. when  $l=0$ . In a curved space-time, however, this relation does not hold in general. In Schwarzschild space-time (I·4) is not satisfied identically for any  $l$ . This is one of the reasons why the equation (I·1) is not suitable for investigating the wave nature of light in a curved space-time.

\* When (I·4) holds the function  $R$  is given by

$$R=a/r\cdot W_1(\sigma+t)+b/r\cdot W_2(\sigma-t), \quad (\text{I}\cdot 5)$$

where  $a, b$  are any constants and  $W_1, W_2$  are arbitrary functions of  $(\sigma+t)$  and  $(\sigma-t)$  respectively.



## II. The plane wave in a curved space-time

We shall explain the reason why it is difficult to construct the concept of plane wave in a curved space-time. In the flat space-time a plane can be expressed by a linear equation of the coordinates :

$$ax + by + cz + dt + e = 0, \quad (\text{II} \cdot 1)$$

in the coordinate system in which the metric is given by

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2. \quad (\text{II} \cdot 2)$$

But in a curved space-time, generally speaking, it is difficult to define a plane by using the expression (II·1) formally, because there is no standard coordinate system such as (II·2).<sup>1)</sup>

In a spherically symmetric space-time, however, it would be possible to take the coordinate system in which

$$ds^2 = -A(r, t) (dx^2 + dy^2 + dz^2) + C(r, t) dt^2 \quad (\text{II} \cdot 3)$$

holds as the standard coordinate system. Nevertheless such coordinate system is not unique, i.e. in some spherically symmetric space-time another coordinate system may exist which has the same type of line element.\* Therefore if we define with the equation (II·1) in one coordinate system, the form of the equation is not invariant under the coordinate transformation which connects such a coordinate system with each other. Accordingly the concept of a plane thus defined is inadequate.

When the space-time is conformally flat we can generalize the concept of plane using the coordinate system :

$$ds^2 = f(r, t) (-dx^2 - dy^2 - dz^2 + dt^2). \quad (\text{II} \cdot 4)$$

The same circumstances as in the general spherically symmetric space-time exists also in this space-time.

Moreover, though it may not be impossible to generalize the concept of the plane geometrically, we have not, at the present stage, any simple theory which we can apply for treating the plane wave. From these reasons we have not treated the plane wave solutions.

## III. Physical component of a tensor in a curved space-time

### (a) Definition

We shall introduce the concept of the physical component of a tensor in a curved space-time generalizing the one used in the tensorial expression of the classical physics as follows :

<sup>1</sup> The difficulty of this kind appears in defining the staticness of the spherically symmetric space-time.<sup>2)</sup>

\*\* In this case its covariant, contravariant and mixed components do not necessarily coincide with each other, and there exists an arbitrariness with respect to its sign of the component of the unit vector when dealing with the physical components, and therefore it is necessary to decide which component will be adopted.

Similarly we can derive the relation between the mixed (or covariant) and physical components.

(c) **Physical component of  $F_{ij}$  and  $E_{ij}$**

If we apply (III.2) and (III.3) to the electromagnetic tensor  $F_{ij} = -F_{ji}$  the following relation can be obtained :

$$\begin{cases} L_r = F_{rt} = 1/\sqrt{AC} \cdot F_{11} = -\sqrt{AC} F^{11}, & L_\theta = F_{\theta t} = 1/r \sqrt{C} \cdot F_{21} = -r \sqrt{C} F^{21}, \\ E_\phi = F_{\phi t} = 1/r \sqrt{C} \sin \theta \cdot F_{31} = -r \sqrt{C} \sin \theta F^{31}, \end{cases} \quad (\text{III.4})$$

$$\begin{cases} H_r = F_{\theta\phi} = 1/r^2 \sin \theta \cdot F_{23} = r^2 \sin \theta F^{23}, & H_\theta = F_{\phi r} = 1/r \sqrt{A} \sin \theta \cdot F_{31} = r \sqrt{A} \sin \theta F^{31}, \\ H_\phi = F_{r\theta} = 1/r \sqrt{A} \cdot F_{12} = r \sqrt{A} F^{12}. \end{cases} \quad (\text{III.5})$$

Naturally the physical components are also antisymmetric with respect to their two indices.

Next inserting the above result in (2.3) by using the relation :

$$F^{kl} F_{kl} = -2(E_r^2 + E_\theta^2 + E_\phi^2) + 2(H_r^2 + H_\theta^2 + H_\phi^2), \quad (\text{III.6})$$

we obtain :

$$\begin{cases} 2E_{rr} = (-E_r^2 + E_\theta^2 + E_\phi^2) + (-H_r^2 + H_\theta^2 + H_\phi^2), \\ 2E_{\theta\theta} = (E_r^2 - E_\theta^2 + E_\phi^2) + (H_r^2 - H_\theta^2 + H_\phi^2), \\ 2E_{\phi\phi} = (E_r^2 + E_\theta^2 - E_\phi^2) + (H_r^2 + H_\theta^2 - H_\phi^2), \end{cases} \quad (\text{III.7})$$

$$\begin{aligned} E_{\theta\phi} &= -(E_\theta E_\phi + H_\theta H_\phi), & E_{rt} &= -(E_\theta H_\phi - E_\phi H_\theta), \\ E_{\phi r} &= -(E_\phi L_r + H_\phi L_r), & L_{\theta t} &= -(L_\phi L_r - L_r H_\phi), \\ E_{r\theta} &= -(L_r L_\theta + H_r H_\theta), & L_{\phi t} &= -(L_r L_\theta - L_\theta L_r), \\ 2L_{tt} &= 2W = (E_r^2 + E_\theta^2 + E_\phi^2) + (H_r^2 + H_\theta^2 + H_\phi^2) \geq 0. \end{aligned} \quad (\text{III.8})$$

Physical components of  $L_{ij}$  as well as their tensor components are symmetric with respect to their two indices.  $L_{tt}$  and  $-(L_{rt}, L_{\theta t}, L_{\phi t}) = \mathbf{E} \times \mathbf{H}$  give the energy and Poynting vector of the electromagnetic field respectively. It is evident that these definitions of physical component of  $L_{ij}$  and  $F_{ij}$  tend to the ordinary ones when the space-time reduces to Minkowski's one.

#### IV. General solution of the generalized Maxwell equation (3.3)

Let any solution of (3.3) be  $F_{ij}$ . We shall construct the TE wave solution by (3.6) using the function  $u$  given by  $u/r = F_{21}/r \sin \theta$ , and shall denote it by  $\overset{1}{F}_{ij}$ . In the same way we shall construct TM wave solution given by (3.7) using  $v$  given by  $v/r = F_{11}/\sqrt{AC}$ , and shall denote it by  $\overset{2}{F}_{ij}$ . If we put

$$\overset{3}{F}_{ij} = F_{ij} - \overset{1}{F}_{ij} - \overset{2}{F}_{ij} \quad (\text{IV.1})$$

this  $\overset{\circ}{F}_{ij}$  is evidently a solution of (3.3) and is a TEM wave solution because  $\overset{\circ}{F}_{23}=\overset{\circ}{F}_{14}=0$ . Thus we have the following result :

*Any solution of (3.3) is given by a suitable superposition of TE, TM and TEM wave solutions.*

Next we shall determine the general form of the TEM wave solution. For this purpose we shall insert  $E_r=H_r=0$  into (3.3), and shall put

$$\begin{aligned} r\sqrt{C}\sin\theta E_\theta &= X_\theta, & r\sqrt{C}\sin\theta E_\phi &= X_\phi, \\ r\sqrt{C}\sin\theta H_\theta &= Y_\theta, & r\sqrt{C}\sin\theta H_\phi &= Y_\phi. \end{aligned} \quad (\text{V.2})$$

Moreover, using the independent variable  $\sigma$  given by (3.12),  $\mu$  given by (3.9),  $\phi$  and  $t$ , we have

$$\begin{cases} \partial X_\phi/\partial\mu - \partial X_\theta/\partial\phi = 0, & \left\{ \begin{array}{l} \partial X_\phi/\partial\sigma - \partial Y_\theta/\partial t = 0, \\ \partial X_\theta/\partial\sigma - \partial Y_\phi/\partial t = 0, \end{array} \right. \\ \partial X_\theta/\partial\mu + \partial Y_\phi/\partial\phi = 0, & \left\{ \begin{array}{l} \partial X_\theta/\partial\sigma + \partial Y_\phi/\partial t = 0, \\ \partial Y_\phi/\partial\sigma + \partial X_\theta/\partial t = 0. \end{array} \right. \end{cases} \quad (\text{IV.3})$$

If we assume that each of  $X_\theta$ ,  $X_\phi$ ;  $Y_\theta$ ,  $Y_\phi$  is a product of a function of  $(\mu, \phi)$  and that of  $(\sigma, t)$ , we obtain (3.8) as the general solution of (IV.3).

## V. The general form of the relative dispersionless wave

We shall show that any relative dispersionless spherical wave is a TEM wave. If we rewrite (3.3) using the notations :

$$X_r = r^2 \sin^2\theta E_r, \quad Y_r = r^2 \sin^2\theta H_r, \quad (\text{V.1})$$

and (IV.2), we obtain

$$\begin{cases} \frac{\partial}{\partial\mu} X_\phi - \frac{\partial}{\partial\phi} X_\theta + \frac{\partial}{\partial t} Y_r = 0, & \left\{ \begin{array}{l} -\frac{\partial}{\partial\sigma} X_\phi + \frac{\partial}{\partial t} Y_\phi + \frac{\partial}{\partial\phi} \frac{C}{r^2 \sin^2\theta} X_r = 0, \\ -\frac{\partial}{\partial t} X_\phi + \frac{\partial}{\partial\sigma} Y_\theta - \frac{\partial}{\partial\phi} \frac{C}{r^2 \sin^2\theta} Y_r = 0, \end{array} \right. \\ \frac{\partial}{\partial\phi} X_\phi + \frac{\partial}{\partial\mu} X_\theta + \frac{\partial}{\partial\sigma} X_r = 0, & \left\{ \begin{array}{l} \frac{\partial}{\partial\sigma} X_\theta + \frac{\partial}{\partial t} Y_\phi - \frac{\partial}{\partial\mu} \frac{C}{r^2 \sin^2\theta} X_r = 0, \\ \frac{\partial}{\partial t} X_\theta + \frac{\partial}{\partial\sigma} Y_\phi - \frac{\partial}{\partial\phi} \frac{C}{r^2 \sin^2\theta} Y_r = 0. \end{array} \right. \end{cases} \quad (\text{V.2})$$

We shall confine ourselves to the out-going wave only. (We can treat in the same way in the case of in-coming wave.) We put each component of **I** and **II** as follows, assuming that it is a product of a function of  $(\mu, \phi)$ , a function of  $\sigma$  and a function of  $(t-\sigma)$  :

$$\begin{aligned} X_r &= X_{r1}(\mu, \phi) X_{r2}(\sigma) X_{r3}(\omega), & X_0 &= X_{01}(\mu, \phi) X_{02}(\sigma) X_{03}(\omega). \\ Y_r &= Y_{r1}(\mu, \phi) Y_{r2}(\sigma) Y_{r3}(\omega), & (\omega &= \ell - \sigma). \end{aligned} \quad (\text{V} \cdot 3)$$

Let  $X_i^p$  be all non-vanishing functions of three independent variables  $u_i (i=1, 2, 3.)$ , then if the relation :

$$X_1^1 X_2^1 X_3^1 + X_1^2 X_2^2 X_3^2 + \cdots + X_1^p X_2^p X_3^p = 0 \quad (\text{V} \cdot 4)$$

holds, we can derive the following relation :

$$X_i^1 = a_i X_i^p, \quad X_i^2 = b_i X_i^p, \quad X_i^3 = c_i X_i^p, \quad \cdots \quad (\text{not summed for } i), \quad (\text{V} \cdot 5)$$

where  $a_i, b_i, c_i, \cdots$  are all constants.

If we apply the above general rule and insert (V·3) into (V·2) we obtain :

$$\begin{aligned} X_{\phi 1} &= a_1 Y_{01}, & X_{\phi 2} &= a_2 Y_{02}, & X_{\phi 3} &= a_3 Y_{03}, \\ Y_{\phi 1} &= b_1 Y_{01}, & Y_{\phi 2} &= b_2 Y_{02}, & Y_{\phi 3} &= b Y_{03}, \\ Y_{r1} &= c_1 \partial_\mu X_{01} / \partial \phi, & Y_{r2} &= c_2 X_{02}, & Y_{r3} &= c_3 Y_{03}, \end{aligned} \quad (\text{V} \cdot 6)$$

where  $a$ 's,  $b$ 's and  $c$ 's are any constants. Inserting these results into (V·2) again and assuming that  $Y_r \neq 0$ ,  $X_r \neq 0$ , we obtain :

$$dY_{03}/d\omega = c_1 X_{03}, \quad Y_{03} = c_2 X_{03}, \quad (\text{V} \cdot 7)$$

where  $c_1, c_2$  are any constants. From this it is seen that the functional form of  $Y_{03}$  is not arbitrary. Therefore the dispersionless wave does not exist whose  $Y_r \neq 0$  and  $X_r \neq 0$ . In the same way we can prove  $X_r = Y_r = 0$  when the wave is dispersionless.

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## Renormalization in the Covariant Treatment of Pion-nucleon Scattering\*

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The problem of renormalization for pion-nucleon scattering is treated in the Bethe-Salpeter formalism. A method to subtract divergencies, especially, overlapping-divergencies, which appear in the solution of an integral equation, is proposed and it is shown that the Salam's prescriptions for the subtraction of overlapping-divergencies can be achieved in a closed form. Finally, the Green's function for  $T=1/2$  state is obtained in a form free from all the divergencies.

### § 1. Introduction

When we treat the problems of pion-nucleon scattering, nuclear forces, etc. in the  $\not{p}s$ - $\not{p}s$  theory, difficulties occur in the renormalization of diverging quantities. In the case of the weak-coupling perturbation theory, the renormalization program is thoroughly executed in all orders of the coupling constant. But it is a well-known fact that the coupling-strength between pion and nucleon ( $f^2/4\pi$ ) is about  $10 \sim 20$ , and we are not able to resort to the perturbation theory. Many methods have been investigated to take into account higher-order effects. Of these, however, the strong coupling theory<sup>1)</sup> and the intermediate coupling theory<sup>2)</sup> are essentially based on the non-relativistic standpoint and are not accessible to the  $\not{p}s$ - $\not{p}s$  theory. In the Tamm-Dancoff method<sup>3)</sup>, nucleon recoil is easily taken into account, but the procedure to limit the number of pions and nucleon-pairs is not carried out in a covariant way and renormalization is performed only lamely. The canonical transformation method by Sawada<sup>4)</sup> is also available to investigate the contributions of pion cloud around a nucleon, taking into account the effect of nucleon recoil. But, in this case, the correspondence to the Feynman graph is difficult to see and the renormalization procedure is almost impossible to carry out. In this treatment, divergencies are avoided by a cut-off, but the results thus obtained may be in some doubt for the quantitative arguments. On the other hand, in the Bethe-Salpeter (B.S.) approximation method,<sup>5)</sup> we retain the standpoint of primary interaction and the correspondence to the Feynman graph is clear. But, in this approximation, higher-order contributions which constitute the main part of pion cloud are not taken so extensively as in the treatment by Sawada.

In the present paper, we stand on the B.S.-formalism and carry out the renormali-

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zation program, as exhaustively as possible, for the problem of pion-nucleon scattering in the symmetrical  $\rho_S$ - $\rho_S$  theory. As is well known, divergencies appear in the solution of the integral equation, even when we start from the finite kernels and it is our purpose to subtract these divergencies, especially, "overlapping-divergencies" (O. D.) in a closed form.\* Of course, on account of the limitation of integral kernels, the quantities to be renormalized in mass and coupling constant are more than one kind for each quantity,<sup>(6)</sup> but, in other respects, it is shown that the renormalization is performed consistently and all the divergencies are removed in the final expression for pion-nucleon scattering.

In the subsequent sections, we derive the integral equations for Green's functions of the pion nucleon two-body system from the  $S$ -matrix formalism (Sec. 2) and find the formal solution for  $T=1/2$  state (Sec. 3). We propose a method to subtract O. D. and give an explicit expression for the Green's function free from all the divergencies (Sec. 4).

## § 2. Integral equations for Green's functions in pion-nucleon scattering

First, we consider the scattering matrix in the B. S.-formalism. As the integral kernels of the integral equation to be satisfied by the scattering matrix, we choose the

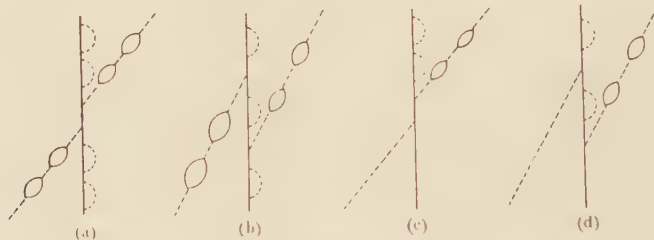


Fig. 1.

second-order irreducible kernels with corrections of the second-order self-energy parts (proper and improper) in the nucleon and pion lines. (See, Fig. 1.)

The interaction Hamiltonian density in the symmetrical  $\rho_S$ - $\rho_S$  theory is given by

$$H(x) = i f \zeta^h(x) \tau_{ij} \gamma_5 \zeta^h(x) \phi_i(x). \quad (1)$$

If we define  $p_1, k_1$  as the energy-momentum 4-vectors of the initial nucleon and pion and  $p_2, k_2$  of the final ones, the integral equation for the scattering matrix  $R(p_2, k_2; p_1, k_1)$  is easily written with the Dyson's prescriptions:

$$\begin{aligned} R(p_2, k_2; p_1, k_1) &= - (2\pi)^5 f^2 (\tau \phi_2) (\tau \phi_1) \lambda(k_2) \eta(p_2) \gamma_5 S_F(p_1 + k_1) \gamma_5 \eta(p_1) \lambda(k_1) \\ &\quad - (2\pi)^5 f^2 (\tau \phi_1) (\tau \phi_2) \lambda(k_2) \eta(p_2) \gamma_5 \eta(p_2 - k_1) S_F(p_2 - k_1) \gamma_5 \eta(p_1) \lambda(k_1) \end{aligned}$$

Recently, the author was informed that M. M. Lévy treated the same problem. Also, D. Ito and H. Tanaka treat this problem in some different way as the author.

$$\begin{aligned}
& + \frac{f^2}{2\pi} (\tau\phi_2) (\tau\phi') \lambda(k_2) \eta(p_2) \int \gamma_5 S_F(p' + k') \gamma_5 \\
& \quad \times S_F(p') \Delta_F(k') \delta(P_2 - P') R(p', k'; p_1, k_1) dp' dk' \\
& + \frac{f^2}{2\pi} (\tau\phi') (\tau\phi_2) \lambda(k_2) \eta(p_2) \int \gamma_5 \eta(p_2 - k') S_F(p_2 - k') \gamma_5 \\
& \quad \times S_F(p') \Delta_F(k') \delta(P_2 - P') R(p', k'; p_1, k_1) dp' dk', \quad (2)
\end{aligned}$$

where

$$S_F(p) = \frac{1}{2\pi i} \frac{i\gamma \not{p} - m}{p^2 + m^2 - i\epsilon}, \quad \Delta_F(k) = \frac{1}{2\pi i} \frac{1}{k^2 + \mu^2 - i\epsilon}, \quad (3)$$

$$\eta(p) = [1 - S_F(p) \Sigma_I(p)]^{-1}, \quad \lambda(k) = [1 + \Delta_F(k) \Pi(k)]^{-1}, \quad (4)$$

$$\Sigma_I(p) = \frac{3f^2}{2\pi} \int \gamma_5 S_F(p - k) \gamma_5 \Delta_F(k) dk, \quad \Pi(k) = \frac{f^2}{\pi} \int T_I[\gamma_5 S_F(p) \gamma_5 S_F(p - k)] dp, \quad (5)$$

$$P = p + k. \quad (6)$$

$\phi_1, \phi_2$  and  $\phi'$  are unit vectors in pion charge-space and correspond to pions with momenta  $k_1, k_2$  and  $k'$ , respectively.  $\Sigma_I(p)$  and  $\Pi(k)$  represent the second-order nucleon and pion self-energy parts, while  $\eta(p)$  and  $\lambda(k)$  are factors representing the effect of these self-energy parts on the nucleon and pion lines.  $P_1$  and  $P_2$  are the total energy-momentum 4-vectors to be conserved throughout the process. On the right-hand side of eq. (2), the first and second terms correspond to the Feynman diagrams shown in Fig. 1 (a) and (b), the two kernels to (c) and (d), respectively.

We introduce the Green's function of the two-body system (a system composed of one pion and one nucleon). The Green's function  $G(p_2, k_2; p_1, k_1)$  is defined by the scattering matrix as follows:

$$\begin{aligned}
G(p_2, k_2; p_1, k) &= \eta(p_2) S_F(p_2) \lambda(k_2) \Delta_F(k_2) \delta(p_2 - p_1) \delta(k_2 - k_1) \\
&- (2\pi)^{-6} S_F(p_2) \Delta_F(k_2) R(p_2, k_2; p_1, k_1) S_F(p_1) \Delta_F(k_1) \delta(P_2 - P_1), \quad (7)
\end{aligned}$$

where the first term corresponds to the Feynman graph as shown in Fig. 2. From eqs. (2), (4) and (7), the integral equation for the Green's function is easily derived:

$$\begin{aligned}
& [1 - S_F(p_2) \Sigma_I(p_2)] [1 + \Delta_F(k_2) \Pi(k_2)] G(p_2, k_2; p_1, k_1) \\
& = \delta(p_2 - p_1) \delta(k_2 - k_1) S_F(p_1) \Delta_F(k_1) \\
& + S_F(p_2) \Delta_F(k_2) \int [(\tau\phi_2) (\tau\phi') H_I(p_2, k_2; p', k') \\
& + (\tau\phi') (\tau\phi_2) H_{II}(p_2, k_2; p', k')] \delta(P_2 - P') G(p', k'; p_1, k_1) dp' dk', \quad (8)
\end{aligned}$$

where



Fig. 2.

$$H_I(p_2, k_2; p', k') = \frac{f^2}{2\pi} \gamma_5 S_F(p' + k') \gamma_5, \quad (9a)$$

$$H_{II}(p_2, k_2; p', k') = \frac{f^2}{2\pi} \gamma_5 [1 - S_F(p_2 - k') \Sigma_I(p_2 - k')]^{-1} S_F(p_2 - k') \gamma_5. \quad (9b)$$

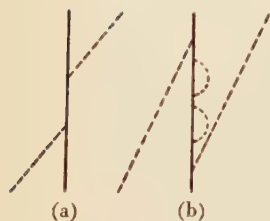


Fig. 3.

$H_I(p, k; p', k')$  and  $H_{II}(p, k; p', k')$  are integral kernels corresponding to the diagrams shown in Fig. 3 (a) and (b). Eq. (8) is the extension of the Fubini's equation,<sup>7)</sup> including the pion self-energy term in the fundamental diagrams. Following the Dyson's rules, we put

$$\Sigma_I(p) = A_I + (\gamma p - im) B_I + (\gamma p - im) S_{IC}(p), \quad (10a)$$

$$H(k) = C + (k^2 + \mu^2) D + (k^2 + \mu^2) J_C(k). \quad (10b)$$

After the removal of renormalization terms, we obtain the equation

$$\begin{aligned} G(p_2, k_2; p_1, k_1) &= \delta(p_2 - p_1) \delta(k_2 - k_1) S'_F(p_1) D'_F(k_1) \\ &+ S'_F(p_2) D'_F(k_2) \int [\tau \phi_2] (\tau \phi') H_I(p_2, k_2; p', k') \\ &+ (\tau \phi') (\tau \phi_2) H_{II}(p_2, k_2; p', k') \delta(P_2 - P') G(p', k'; p_1, k_1) dp' dk', \end{aligned} \quad (11)$$

where

$$S'_F(p) = [1 - (1/2\pi) S_{IC}(p)]^{-1} S_F(p), \quad (12a)$$

$$D'_F(k) = [1 + (1/2\pi i) J_C(k)]^{-1} J_F(k), \quad (12b)$$

and (9b) is replaced by

$$H_{II}(p_2, k_2; p', k') = f^2/2\pi \cdot \gamma_5 S'_F(p_2 - k') \gamma_5. \quad (9b)'$$

We decompose the system in charge states  $T=1/2$  and  $T=3/2$ . The projection operators for these states are

$$\mathbf{T}_{1/2} = 1/3 \cdot (\mathbf{1} - \tau\omega), \quad (13a)$$

$$\mathbf{T}_{3/2} = 1/3 \cdot (2\mathbf{1} + \tau\omega), \quad (13b)$$

where  $\omega$  represents the isotopic spin of a pion. We put

$$G(p_2, k_2; p_1, k_1) = \mathbf{T}_{1/2} \cdot G_{1/2}(p_2, k_2; p_1, k_1) + \mathbf{T}_{3/2} \cdot G_{3/2}(p_2, k_2; p_1, k_1). \quad (14)$$

Then, from eq. (11) and the relations

$$(\tau \phi_2) (\tau \phi') = \mathbf{1} - \tau\omega, \quad (15a)$$

$$(\tau \phi') (\tau \phi_2) = \mathbf{1} + \tau\omega. \quad (15b)$$

and

$$\mathbf{1} + \tau\omega = 2/3 \cdot (2\mathbf{1} + \tau\omega) - 1/3 \cdot (\mathbf{1} - \tau\omega), \quad (16)$$

we obtain the integral equations for  $G_{1/2}(p_2, k_2; p_1, k_1)$  and  $G_{3/2}(p_2, k_2; p_1, k_1)$ :

$$\begin{aligned}
 G_{1/2}(p_2, k_2; p_1, k_1) = & \delta(p_2 - p_1) \delta(k_2 - k_1) S'_F(p_1) A'_F(k_1) \\
 & + S'_F(p_2) A'_F(k_2) \int [3H_I(p_2, k_2; p', k') - H_{II}(p_2, k_2; p', k')] \\
 & \times \delta(P_2 - P') G_{1/2}(p', k'; p_1, k_1) dp' dk', \quad (17)
 \end{aligned}$$

$$\begin{aligned}
 G_{3/2}(p_2, k_2; p_1, k_1) = & \delta(p_2 - p_1) \delta(k_2 - k_1) S'_F(p_1) A'_F(k_1) \\
 & + S'_F(p_2) A'_F(k_2) \int 2H_{II}(p_2, k_2; p', k') \delta(P_2 - P') G_{3/2}(p', k'; p_1, k_1) dp' dk'. \quad (18)
 \end{aligned}$$

The integral equation for  $G_{3/2}(p_2, k_2; p_1, k_1)$  contains only the kernel  $H_{II}(p, k; p', k')$  and no divergence difficulties occur in this case. But, in the case of  $G_{1/2}(p_2, k_2; p_1, k_1)$ , divergencies appear in the solution of the integral equation on account of the kernel  $H_I(p, k; p', k')$ . We need some appropriate procedures to remove these divergencies.

### § 3. Formal solution of the equation

Hereafter, we treat the equation for  $G_{1/2}(p_2, k_2; p_1, k_1)$  only. In order to solve eq. (17) in a form suitable for renormalization, we define the resolvent  $K(p, k; p', k')$  of the kernel  $H_{II}(p, k; p', k')$  by the equation

$$\begin{aligned}
 K(p, k; p', k') = & \delta(p - p') \delta(k - k') - S'_F(p) A'_F(k) \\
 & \times \int H_{II}(p, k; p'', k'') \delta(P - P'') K(p'', k''; p', k') dp'' dk''. \quad (19)
 \end{aligned}$$



Fig. 4.

Obviously, no divergence appears in solving this equation. The resolvent  $K(p, k; p', k')$  corresponds to a scattering process as shown in Fig. 4. The integral equation for  $G_{1/2}(p_2, k_2; p_1, k_1)$

becomes

$$\begin{aligned}
 G_{1/2}(p_2, k_2; p_1, k_1) = & \int K(p_2, k_2; p', k') [\delta(p' - p_1) \delta(k' - k_1) S'_F(p_1) A'_F(k_1) \\
 & + S'_F(p') A'_F(k') \int 3H_I(p', k'; p'', k'') \delta(P' - P'') \\
 & \times G_{1/2}(p'', k''; p_1, k_1) dp'' dk''] dp' dk' = K(p_2, k_2; p_1, k_1) S'_F(p_1) A'_F(k_1) \\
 & + \delta(P_2 - P_1) \frac{3f^2}{2\pi} \int K(p_2, k_2; p', k') S'_F(p') A'_F(k') \gamma_5 S_F(P) \gamma_5 \\
 & \times G_{1/2}(p', k'; p_1, k_1) dp' dk' dp'' dk'', \quad (20)
 \end{aligned}$$

where

$$P = P_1 = P_2,$$

and in the last equation, we have used eq. (9a). This equation is easily solved and gives the solution of the form



$$\begin{aligned}
G_{1/2}(p_2, k_2; p_1, k_1) &= K(p_2, k_2; p_1, k_1) S'_F(p_1) \mathcal{A}'_F(k_1) \\
&+ \delta(P_2 - P_1) \frac{3f_\pi^2}{2\pi} \int K(p_2, k_2; p', k') S'_F(p') \mathcal{A}'_F(k') \gamma_5 [1 - S_F(P) \Sigma_{II}(P)]^{-1} \\
&\times S_F(P) \gamma_5 K(p', k''; p_1, k_1) S'_F(p_1) \mathcal{A}'_F(k_1) dp' dk' dp'' dk'', \quad (21)
\end{aligned}$$

where

$$\Sigma_{II}(P) = \frac{3f_\pi^2}{2\pi} \int \delta(P - P'') \gamma_5 K(p' k'; p'', k'') S'_F(p'') \mathcal{A}'_F(k'') \gamma_5 dp' dk' dp'' dk''. \quad (22)$$

Here, we define  $\bar{K}(p, k; p', k')$  by the relation

$$K(p, k; p', k') S'_F(p') \mathcal{A}'_F(k') = S'_F(p) \mathcal{A}'_F(k) \bar{K}(p, k; p', k'). \quad (23)$$

The integral equation for  $\bar{K}(p, k; p', k')$  follows from eqs. (21) and (25):

$$\begin{aligned}
\bar{K}(p, k; p', k') &= (p - p') \delta(k - k') - \int H_{II}(p, k; p'', k'') \\
&\times \delta(P - P'') S'_F(p'') \mathcal{A}'_F(k'') \bar{K}(p'', k''; p', k') dp'' dk''. \quad (24)
\end{aligned}$$

From eqs. (21) and (23), we have

$$\begin{aligned}
G_{1/2}(p_2, k_2; p_1, k_1) &= K(p_2, k_2; p_1, k_1) S'_F(p_1) \mathcal{A}'_F(k_1) \\
&+ \delta(P_2 - P_1) S'_F(p_2) \mathcal{A}'_F(k_2) \frac{3f_\pi^2}{2\pi} \Gamma'_a(k_2; P) [1 - S_F(P) \Sigma_{II}(P)]^{-1} \\
&\times S_F(P) \Gamma'_b(k_1; P) S'_F(p_1) \mathcal{A}'_F(k_1), \quad (25)
\end{aligned}$$

where

$$\Gamma'_a(k; P) = \int \bar{K}(p, k; p', k') \gamma_5 dp' dk', \quad (26)$$

$$\Gamma'_b(k; P) = \int \gamma_5 K(p', k'; p, k) dp' dk'. \quad (27)$$

$\Gamma'_a(k; P)$  and  $\Gamma'_b(k; P)$  represent the vertex parts with regard to the scattered and incident pions and are obtained by closing one pion line in Fig. 4. As we can see from eq. (22),  $\Sigma_{II}(P)$  represents the nucleon self-energy part obtained by closing both pion lines in the same figure.

We have obtained the formal solution of the scattering problem for  $T=1/2$ , but, as is well known, many kinds of divergencies are involved in  $\Gamma'_a(k; P)$ ,  $\Gamma'_b(k; P)$  and  $\Sigma_{II}(P)$ .

#### § 4. Subtraction of overlapping divergencies

First, we consider to remove O.D. involved in the vertex parts. For this purpose, we notice from eqs. (24) and (26) that  $\Gamma'_a(k; P)$  satisfies the following integral

equation :

$$I'_a(k; P) = \gamma_5 - \int H_{II}(P-k, k; P-k', k') S'_F(P-k') \mathcal{J}'_F(k') I'_a(k'; P) dk'. \quad (28)$$

Evidently, O.D. appear in the solution of this equation. To remove these divergencies, we follow the rules given by Salam,<sup>(1)</sup> i. e., to subtract true divergence for each subintegration. In order to perform this subtraction in a closed form, we replace the above equation by the following set of equations :

$$\begin{aligned} I'_{a1}(k; P) = & \gamma_5 - \int H_{II}(P-k, k; P-k', k') S'_F(P-k') \mathcal{J}'_F(k') I'_{a1}(k'; P) dk' \\ & + \int H_{II}(p_0, 0; p_0-k', k') S'_F(p_0-k') \mathcal{J}'_F(k') I'_{a1}(k'; p_0) dk', \end{aligned} \quad (29a)$$

$$\begin{aligned} \Gamma_{a1}(k; p_0) = & \gamma_5 - \int [H_{II}(p_0-k, k; p_0-k', k') - H_{II}(p_0, 0; p_0-k', k')] \\ & \times S'_F(p_0-k') \mathcal{J}'_F(k') \Gamma_{a1}(k'; p_0) dk', \end{aligned} \quad (29b)$$

where

$$p_0^2 + m^2 = 0, \quad \gamma p_0 - im = 0, \quad (30)$$

The above-defined  $I'_{a1}(k; P)$  is just the finite vertex contribution obtained by subtracting all the divergencies from  $I'_a(k; P)$ . One can easily verify this by solving above equations by iteration :

$$\begin{aligned} I'_{a1}(k; P) = & \gamma_5 + \gamma_5 [I'^{(1)}_a(k; P) - L_a^{(1)}] \\ & + \gamma_5 [\Gamma_a^{(2)}(k; P) - L_a^{(1)} \Gamma_a^{(1)}(k; P) - L_a^{(2)}] + \dots, \end{aligned} \quad (31)$$

where

$$\Gamma_a^{(1)}(k; P) = -\frac{f^2}{2\pi} \int S'_F(P-k-k') \gamma_5 S'_F(P-k') \gamma_5 \mathcal{J}'_F(k') dk', \quad (32)$$

$$\begin{aligned} \Gamma_a^{(2)}(k; P) = & \left(\frac{f^2}{2\pi}\right)^2 \int S'_F(P-k-k') \gamma_5 S'_F(P-k') \gamma_5 \mathcal{J}'_F(k') S'_F(P-k'-k'') \\ & \times \gamma_5 S'_F(P-k'') \gamma_5 \mathcal{J}'_F(k'') dk' dk'', \end{aligned} \quad (33)$$

$$L_a^{(1)} = -\frac{f^2}{2\pi} \int S'_F(p_0-k') \gamma_5 S'_F(p_0-k') \gamma_5 \mathcal{J}'_F(k') dk', \quad (34)$$

$$\begin{aligned} L_a^{(2)} = & \left(\frac{f^2}{2\pi}\right)^2 \int S'_F(p_0-k') \gamma_5 S'_F(p_0-k') \gamma_5 \mathcal{J}'_F(k') [S'_F(p_0-k'-k'') \gamma_5 \\ & \times S'_F(p_0-k'') \gamma_5 \mathcal{J}'_F(k'') - S'_F(p_0-k'') \gamma_5 S'_F(p_0-k'') \gamma_5 \mathcal{J}'_F(k'')] dk' dk'', \end{aligned} \quad (35)$$

$I'_a{}^{(1)}(k; P)$ ,  $I'_a{}^{(2)}(k; P)$ , ... are the original vertex correction terms of the second-, fourth-... orders and equal to each iteration term of eq. (28).  $L_a{}^{(1)}$ ,  $L_a{}^{(2)}$ , ... are true divergencies of the second-order, fourth-order, ... subintegrations and correspond to  $L_a{}^{(n)}$  in the Salam's paper. As seen clearly from eq. (31), the true divergence for each subintegration is subtracted term by term and  $I'_{a1}(k; P)$  constitutes the absolute convergent vertex contribution with regard to the scattered pion.

Here, we notice that  $L_a{}^{(1)}$ ,  $L_a{}^{(2)}$ , ... are all real numbers. This follows from the fact that, for each integration of  $dk^0(k^0$  is the energy-component of a 4-vector  $k$ ), we take residues at the poles and an imaginary unit  $i$  appears for each residue. (See eq. (3), definitions of  $S_F(p)$  and  $\mathcal{A}_F(k)$ .)

To clarify the relation between Salam's and ours, we shall find the formal solution of eq. (29). From eqs. (24) and (29a), we have the following solution:

$$\Gamma_{a1}(k; P) = \int \bar{K}(p, k; p', k') \left[ \gamma_5 + \int H_H(p_0, 0; p_0 - k'', k'') S'_F(p_0 - k'') \right. \\ \left. \times \mathcal{A}'_F(k'') \Gamma_{a1}(k''; p_0) dk'' \right] dp' dk'. \quad (36)$$

Using eqs. (9b)' and (26), we obtain

$$\Gamma_{a1}(k; P) = (1 - L_b) \Gamma_a(k; P), \quad (37)$$

where

$$L_a = -\frac{f^2}{2\pi} \int S'_F(p_0 - k') \gamma_5 S'_F(p_0 - k') \mathcal{A}'_F(k') \Gamma_{a1}(k'; p_0) dk'. \quad (38)$$

On the right-hand side of eq. (37),  $(1 - L_b)$  and  $\Gamma_a(k; P)$  are both diverging quantities and this type of solution has no meaning, but the relation to the Salam's procedure is evident in this form. We can obtain  $L_a{}^{(1)}$ ,  $L_a{}^{(2)}$ , ... by expanding eq. (38) in a power series of the coupling constant.

As for  $I'_b(k; P)$ , the same relations hold. From eq. (19), we have the reciprocity-relation

$$\int S'_F(p) \mathcal{A}'_F(k) H_H(p, k; p'', k'') \delta(P - P'') K(p'', k''; p', k') dp'' dk'' \\ = \int K(p, k; p'', k'') S'_F(p'') \mathcal{A}'_F(k'') H_H(p'', k''; p', k') \delta(P'' - P') dp'' dk''. \quad (39)$$

Using eqs. (19), (27) and (39), we have the following integral equation for  $I'_b(k; P)$ :

$$I'_b(k; P) = \gamma_5 - \int I'_b(k'; P) S'_F(P - k') \mathcal{A}'_F(k') H_H(P - k', k'; P - k, k) dk'. \quad (40)$$

As in the case of  $I'_a(k; P)$ , we replace the above equation by the following set of equations:

$$\begin{aligned} \Gamma_{b1}(k; P) = & \gamma_5 - \int \Gamma_{b1}(k'; P) S'_{\mathbf{F}}(P-k') \mathcal{A}'_{\mathbf{F}}(k') H_{II}(P-k', k'; P-k, k) dk' \\ & + \int \Gamma_{b1}(k'; p_0) S'_{\mathbf{F}}(p_0-k') \mathcal{A}'_{\mathbf{F}}(k') H_{II}(p_0-k', k'; p_0, 0) dk', \end{aligned} \quad (41a)$$

$$\begin{aligned} \Gamma_{b1}(k; p_0) = & \gamma_5 - \int \Gamma_{b1}(k'; p_0) S'_{\mathbf{F}}(p_0-k') \mathcal{A}'_{\mathbf{F}}(k') \\ & \times [H_{II}(p_0-k', k'; p_0-k, k) - H_{II}(p_0-k', k'; p_0, 0)] dk', \end{aligned} \quad (41b)$$

where  $p_0$  is the energy-momentum 4-vector of a free nucleon and satisfies eq. (30). Solving eqs. (41a) and (41b) by iteration, we have

$$\begin{aligned} \Gamma_{b1}(k; P) = & \gamma_5 + [\Gamma_b^{(1)}(k; P) - L_b^{(1)}] \gamma_5 \\ & + [\Gamma_b^{(2)}(k; P) - L_b^{(1)} \Gamma_b^{(1)}(k; P) - L_b^{(2)}] \gamma_5 + \dots, \end{aligned} \quad (42)$$

where

$$\Gamma_b^{(1)}(k; P) = -\frac{f^2}{2\pi} \int \gamma_5 S'_{\mathbf{F}}(P-k') \gamma_5 S'_{\mathbf{F}}(P-k-k') \mathcal{A}'_{\mathbf{F}}(k') dk', \quad (43)$$

$$\begin{aligned} \Gamma_b^{(2)}(k; P) = & \left(\frac{f^2}{2\pi}\right)^2 \int \gamma_5 S'_{\mathbf{F}}(P-k'') \gamma_5 S'_{\mathbf{F}}(P-k'-k'') \mathcal{A}'_{\mathbf{F}}(k'') \gamma_5 \\ & \times S'_{\mathbf{F}}(P-k') \gamma_5 S'_{\mathbf{F}}(P-k-k') \mathcal{A}'_{\mathbf{F}}(k') dk' dk'', \\ & \dots\dots\dots, \end{aligned} \quad (44)$$

$$L_b^{(1)} = -\frac{f^2}{2\pi} \int \gamma_5 S'_{\mathbf{F}}(p_0-k') \gamma_5 S'_{\mathbf{F}}(p_0-k') \mathcal{A}'_{\mathbf{F}}(k') dk', \quad (45)$$

$$\begin{aligned} L_b^{(2)} = & \left(\frac{f^2}{2\pi}\right)^2 \int [\gamma_5 S'_{\mathbf{F}}(p_0-k'') \gamma_5 S'_{\mathbf{F}}(p_0-k'-k'') \mathcal{A}'_{\mathbf{F}}(k'') - \gamma_5 S'_{\mathbf{F}}(p_0-k'') \\ & \times \gamma_5 S'_{\mathbf{F}}(p_0-k'') \mathcal{A}'_{\mathbf{F}}(k'') | \gamma_5 S'_{\mathbf{F}}(p_0-k') \gamma_5 S'_{\mathbf{F}}(p_0-k') \mathcal{A}'_{\mathbf{F}}(k') dk' dk''], \\ & \dots\dots\dots, \end{aligned} \quad (46)$$

$\Gamma_{b1}(k; P)$  gives the absolute convergent vertex contribution with regard to the incident pion.  $L_b^{(1)}, L_b^{(2)}, \dots$  are all real numbers as in the case of  $L_a^{(1)}, L_a^{(2)}, \dots$ .

From eqs. (19), (27), (39) and (41a), we obtain the formal solution of  $\Gamma_{b1}(k; P)$ :

$$\Gamma_{b1}(k; P) = (1 - L_b) \Gamma_b(k; P), \quad (47)$$

where

$$L_b = -\frac{f^2}{2\pi} \int \Gamma_{b1}(k'; p_0) S'_{\mathbf{F}}(p_0-k') \gamma_5 S'_{\mathbf{F}}(p_0-k') \mathcal{A}'_{\mathbf{F}}(k') dk'. \quad (48)$$

We shall verify the equivalence between  $L_a$  and  $L_b$ . If we put

$$\Gamma_{a1}(k; p) = \Gamma_{a1}(\mathbf{k}, k^0; \mathbf{p}, p^0), \quad (49)$$

$$\Gamma_{b1}(k; p) = \Gamma_{b1}(\mathbf{k}, k^0; \mathbf{p}, p^0), \quad (50)$$

then, from eqs. (29) and (41), we can derive the relation :

$$I'_{a1}{}^* (-\mathbf{k}, k''; -\mathbf{p}, p'') = I'_{b1}(\mathbf{k}, k''; \mathbf{p}, p''). \quad (51)$$

In this derivation, we have used the fact that when  $i\epsilon$  was replaced by  $-i\epsilon$  in the denominators of  $S_i$  and  $J_F$ , the direction of turning a pole in  $k$ -plane is reversed. Using this relation, we can easily derive the relation :

$$I_{a1}{}^* = I_{b1}. \quad (52)$$

As  $I_{a1}$  and  $I_{b1}$  are real numbers, we obtain the final result

$$I_{a1} = I_{b1}. \quad (53)$$

Next, we consider to subtract O. D. appearing in the nucleon self-energy  $\Sigma'_H(P)$ . We rewrite eq. (22) as

$$\begin{aligned} \Sigma_H(P) = & \frac{3f^2}{2\pi} \int \delta(P-P') \gamma_5 \delta(p'-p'') \delta(k'-k'') \\ & \times K(p', k''; p''', k''') S'_F(p''') \mathcal{J}'_F(k''') \gamma_5 dp' dk' dp'' dk'' dp''' dk''' \quad (54) \end{aligned}$$

and substitute for  $\delta(p'-p'')\delta(k'-k'')$  the relation obtained from eq. (19) :

$$\begin{aligned} \delta(p'-p'')\delta(k'-k'') = & K(p', k'; p'', k'') + S'_F(p') \mathcal{J}'_F(k') \\ & \times \int H_H(p', k'; p''', k''') \delta(P'-P''') K(p''', k'''; p'', k'') dp''' dk'''. \end{aligned}$$

Using eqs. (23), (26), (27) and (39), we obtain the expression :

$$\begin{aligned} \Sigma_H(P) = & \frac{3f^2}{2\pi} \int dp' dk' \delta(P-P') [ \Gamma_b(k'; P) S'_F(p') \mathcal{J}'_F(k') \Gamma_a(k'; P) \\ & + \int dp'' dk'' \Gamma_b(k''; P) S'_F(p'') \mathcal{J}'_F(k'') H_H(p'', k''; p', k') \\ & \times \delta(P''-P') S'_F(p') \mathcal{J}'_F(k') \Gamma_a(k'; P) ]. \quad (55) \end{aligned}$$

In order to subtract O. D., we make the following replacements :

$$I'_a(k; P) \longrightarrow I'_{a1}(k; P),$$

$$I'_b(k; P) \longrightarrow I'_{b1}(k; P).$$

Then, the O. D.-free self-energy  $\Sigma'_H(P)$  is given by

$$\begin{aligned} \Sigma'_H(P) = & \frac{3f^2}{2\pi} \int dp' dk' \delta(P-P') [ I'_{b1}(k'; P) S'_F(p') \mathcal{J}'_F(k') \Gamma_{a1}(k'; P) \\ & + \int dp'' dk'' \Gamma_{b1}(k''; P) S'_F(p'') \mathcal{J}'_F(k'') H_H(p'', k''; p', k') \\ & \times \delta(P''-P') S'_F(p') \mathcal{J}'_F(k') I'_{a1}(k'; P) ]. \quad (56) \end{aligned}$$

In order to see that no O. D. appear in (56), we expand the right-hand side in a power



series of the coupling constant. Using eqs. (31) and (42), we have

$$\begin{aligned}\Sigma'_{II}(P) = & \Sigma^{(1)}(P) + [\Sigma^{(2)}(P) - L_a^{(1)}\Sigma^{(1)}(P) - L_b^{(1)}\Sigma^{(1)}(P)] \\ & + [\Sigma^{(3)}(P) - L_a^{(1)}\Sigma^{(2)}(P) - L_b^{(1)}\Sigma^{(2)}(P) \\ & - L_a^{(2)}\Sigma^{(1)}(P) - L_b^{(2)}\Sigma^{(1)}(P) + L_a^{(1)}L_b^{(1)}\Sigma^{(1)}(P)] + \dots,\end{aligned}\quad (57)$$

where

$$\Sigma^{(1)}(P) = 3 \frac{f^2}{2\pi} \int \gamma_5 S'_F(P-k') \gamma_5 A'_F(k') dk', \quad (58)$$

$$\begin{aligned}\Sigma^{(2)}(P) = & -3 \left( \frac{f^2}{2\pi} \right)^2 \int \gamma_5 S'_F(P-k') \gamma_5 A'_F(k') \\ & \times S'_F(P-k'-k'') \gamma_5 S'_F(P-k'') \gamma_5 A'_F(k'') dk' dk'',\end{aligned}\quad (59)$$

$$\begin{aligned}\Sigma^{(3)}(P) = & 3 \left( \frac{f^2}{2\pi} \right)^3 \int \gamma_5 S'_F(P-k') \gamma_5 A'_F(k') S'_F(P-k'-k'') \gamma_5 S'_F(P-k'') \\ & \times \gamma_5 A'_F(k'') S'_F(P-k''-k''') \gamma_5 S'_F(P-k''') \gamma_5 A'_F(k''') dk' dk'' dk''',\end{aligned}\quad (60)$$

$\Sigma^{(1)}(P)$ ,  $\Sigma^{(2)}(P)$ ,  $\Sigma^{(3)}(P)$ ,  $\dots$  are the original self-energy parts of the second-, fourth-, sixth-,  $\dots$  orders, respectively. As seen clearly from eq. (57) and the discussions by Salam, the true divergence for each subintegration is correctly subtracted in every orders and  $\Sigma'_{II}(P)$  contains only the final true divergence of a self-energy.

From eqs. (37), (47), (53), (55) and (56), we obtain

$$\Sigma'_{II}(P) = (1-L)^2 \Sigma_{II}(P), \quad (61)$$

where

$$L = L_a = L_b. \quad (62)$$

Eq. (61) has the same form as Salam derived in the perturbation theory.

To separate the final divergence from (56), we put

$$\Sigma'_{II}(P) = A_{II} + (\gamma P - im) B_{II} + (\gamma P - im) S_{II0}(P). \quad (63)$$

$A_{II}$  and  $B_{II}$  represent the mass and coupling constant renormalization quantities. After these divergencies are removed, we have the final expression for  $G_{1/2}(p_2, k_2; p_1, k_1)$ :

$$\begin{aligned}G_{1/2}(p_2, k_2; p_1, k_1) = & K(p_2, k_2; p_1, k_1) S'_F(p_1) A'_F(k_1) \\ & + \delta(P_2 - P_1) S'_F(p_2) A'_F(k_2) \frac{3f^2}{2\pi} \Gamma_{a1}(k_2; P) \left[ 1 - \frac{1}{2\pi} S_{II0}(P) \right]^{-1} \\ & \times S_F(P) \Gamma_{b1}(k_1; P) S'_F(p_1) A'_F(k_1).\end{aligned}\quad (64)$$

Obviously, all quantities in this expression are convergent and, thus, we have completely

executed the renormalization program for pion-nucleon scattering in the B. S.-formalism.\*

With eq. (64), we can perform a partial wave analysis and obtain the phase-shifts for pion-nucleon scattering. Here, we notice that, in the center-of-mass system, the second term of (64) contributes only to  $S$ - and  $P_{1/2}$ -waves and not to higher angular-momentum ones. This is because the variables of the incident and scattered particles are separated in this term. If we apply  $\bar{u}(p_2)$  and  $u(p_1)$  on the left and on the right hand side of the term  $(\bar{u}(p_2) \text{ and } u(p_1))$  are spinor functions of the final and initial nucleons, where  $|\mathbf{p}_2| = |\mathbf{p}_1|$  and notice the relations

$$\begin{aligned} i\gamma p_1 u(p_1) &= -m u(p_1), \\ i\gamma k_1 u(p_1) &= (m - i\gamma_4 H^*) u(p_1), \\ \bar{u}(p_2) i\gamma p_2 &= -m \bar{u}(p_2), \\ \bar{u}(p_2) i\gamma k_2 &= \bar{u}(p_2) (m - i\gamma_4 H^*), \\ &\dots\dots\dots \end{aligned} \quad (65)$$

where

$$H = P_1' = P_2'',$$

the terms which contain the angle between the incident and scattered particles are only in the forms of  $\bar{u}(p_2) u(p_1)$  and  $\bar{u}(p_2) \gamma_4 u(p_1)$ . But, from the expressions

$$\bar{u}(p_2) u(p_1) = \frac{p^0 + m}{2p^0} \left[ 1 - \frac{1}{(p^0 + m)^2} (\mathbf{p}_2 \cdot \mathbf{p}_1 + i\boldsymbol{\sigma} \cdot \mathbf{p}_2 \times \mathbf{p}_1) \right], \quad (66)$$

$$\bar{u}(p_2) \gamma_4 u(p_1) = \frac{p^0 + m}{2p^0} \left[ 1 + \frac{1}{(p^0 + m)^2} (\mathbf{p}_2 \cdot \mathbf{p}_1 + i\boldsymbol{\sigma} \cdot \mathbf{p}_2 \times \mathbf{p}_1) \right], \quad (67)$$

we can just see that the second term of (64) contributes only to  $S$ - and  $P_{1/2}$ -waves.

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### Note to eqs. (29) and (41)

The solutions of eqs. (29) and (41) should be finite, but the expressions given by eqs. (37) and (47) are in the forms of infinity minus infinity. The meaningful solutions are obtained in the following way:

From eqs. (29a) and (29b), we have

$$\begin{aligned} I'_{a1}(k; P) - I'_{a1}(k; p_0) &= I'_a(p, k; p_0) - \int II(p, k; p', k') S'_F(p') \\ &\quad \times J'_F(k') \delta(P - P') [I'_{a1}(k'; P) - I'_{a1}(k'; p_0)] dp' dk', \quad (\text{N} \cdot 1) \end{aligned}$$

where

\* We have applied the above method of renormalization to the Tamm-Dancoff approximation and calculated the phase-shifts for pion-nucleon scattering. The details will appear in the same journal.

$$\Gamma_a(p, k; p_0) = - \int [H_{II}(p, k; p', k') S'_F(p') \mathcal{A}'_F(k') \delta(P - P') - H_{II}(p_0 - k, k; p', k') S'_F(p') \mathcal{A}'_F(k') \delta(p_0 - P')] \Gamma_{a1}(k'; p_0) dp' dk'. \quad (\text{N} \cdot 2)$$

The above equation is an integral equation for  $\Gamma_{a1}(k; P) - \Gamma_{a1}(k; p_0)$ , and, as  $\Gamma_a(p, k; p_0)$  is a finite quantity, the solution of this equation is also obtained in a finite form. Using  $\bar{K}(p, k; p', k')$  defined by eq. (24), we obtain the solution of eq. (N·1):

$$\Gamma_{a1}(k; P) = \Gamma_{a1}(k; p_0) + \int \bar{K}(p, k; p', k') \Gamma_a(p', k'; p_0) dp' dk'. \quad (\text{N} \cdot 3)$$

In the same way, we have the solution of eq. (41):

$$\Gamma_{b1}(k; P) = \Gamma_{b1}(k; p_0) + \int \Gamma_b(p_0; p', k') K(p', k'; p, k) dp' dk', \quad (\text{N} \cdot 4)$$

where  $K(p', k'; p, k)$  is defined by eq. (19) and

$$\Gamma_b(p_0; p, k) = - \int \Gamma_{b1}(k'; p_0) [S'_F(p') \mathcal{A}'_F(k') \delta(P - P') H_{II}(p', k'; p, k) - S'_F(p') \mathcal{A}'_F(k') \delta(p_0 - P') H_{II}(p', k'; p_0 - k, k)] dp' dk'. \quad (\text{N} \cdot 5)$$

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## On the Renormalization in the Tamm-Dancoff Approximation for One-nucleon Problem, I

— *A Covariant Generalization of the Tamm-Dancoff Method* —

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To carry out the consistent subtraction of divergences in Tamm-Dancoff method, a covariant formalism is proposed for the one nucleon problem. That is, neglecting all the nucleon closed loops and paying an attention to the "development of events along a nucleon line" rather than the "time-like development of events", the simultaneous integral equations for the Feynman kernels which can be regarded as a covariant generalization of T-D equations are derived. The meaning of the word "meson number" in this theory is also generalized, though it agrees with the usual one when the nucleon pairs are entirely neglected. How to carry out the subtraction technique in this generalized T-D method will be investigated in the next paper (Part II).

### § 1. Introduction

Because of the failure of the perturbation theory for the problems concerning with the meson fields, the various approximation methods have been proposed by many authors. Among them, the Tamm-Dancoff method<sup>1)</sup> seems to be very interesting. For, if we adopt the assumption that the pion-nucleon coupling is still not so strong, it is natural to consider the average number of mesons around a nucleon to be rather small, so that the contributions of the configurations with many mesons are negligible. However, the T-D method has been formulated in a non-covariant form, and therefore the consistent subtraction of divergences was very difficult. Thus, it is desirable to get a covariant generalization of the T-D approximation so as to make consistent renormalization technique applicable.

The renormalization in the T-D method has already been investigated by Cini<sup>2)</sup> and Fubini<sup>3)</sup>. But, the method of Cini is essentially based on the notion of the "irreducible kernels" of Bethe-Salpeter<sup>4)</sup>, so that, in the higher order approximation, his method resembles to the B-S approximation rather than the T-D approximation. In addition to this, it should be noted that the solutions of the equations for problems such as meson-nucleon scattering would contain the divergences, even if the kernels of the integral equations had been properly renormalized. For, the "final" meson of the kernel can be absorbed by the nucleon in the next stage, and therefore the solution of the integral equations generally contains the integration with respect to that meson. This circumstance would easily be realized by the example shown in Fig. 1. Fig. 1a) contains obviously the self-energy type divergences, though its "irreducible" kernel Fig. 1b), itself, is quite

divergence-free. Moreover, in the formalism of Cini, the space-like surface  $\sigma$  appears explicitly in the boundary of integration, and therefore the new type of divergences necessarily occurs owing to the sharpness of the integration domain<sup>5)</sup>. On the other hand, the work of Fubini is free from the defect above mentioned, but he treated only the case where the meson numbers are restricted within two, and his method

rests on the specially simple circumstances of this case (see Part II). So it will be shown in the present work how to get a covariant formalism which, in any order of approximation, can be regarded as the generalization of T-D approximation in the corresponding order. How to carry out the subtraction procedure in this generalized T-D method will be discussed in Part II.

Since the concept of meson number plays an important role in the original T-D method, the straightforward generalization of this concept would necessarily contain the sharply defined domains of integrations, and therefore give rise to a new type of divergences as in the formalism of Cini. To avoid this new difficulty, it is necessary to give up the concept of "number" and to adopt the more general one. In this respect, the method used by Feynman<sup>6)</sup> in his derivation of a covariant theory from the old quantum electrodynamics would be very instructive. Thus, we pay an attention to the "nucleon line" and trace the "development of events along the nucleon line" instead of the "time-like development of events."

It would be rather difficult to treat by such a method the problems which have two or more separated nucleon lines. Hence, in the following, we shall confine to the "one nucleon problem" and, moreover, neglect all the diagrams that contain the nucleon closed loops so that we may have only one connected nucleon line.

## § 2. Analysis of Feynman-Dyson diagrams

The definition of the Feynman kernel for the pion-nucleon scattering

$$N + \pi^3 \rightarrow N' + \pi'^3$$

is given as follows:

$$\begin{aligned} K(x_1, y_1^3; x_0, y_0^3) &= \epsilon(x_1, x_0) \langle \Psi_0, P[\phi(x_1)\phi_\alpha(y_1)\bar{\psi}(x_0)\phi_\beta(y_0)] \Psi_0 \rangle \\ &= \frac{\epsilon(x_1, x_0) \langle \Phi_0, P[U(\infty, -\infty)\phi(x_1)\phi_\alpha(y_1)\bar{\psi}(x_0)\phi_\beta(y_0)] \Phi_0 \rangle}{\langle \Phi_0, U(\infty, -\infty) \Phi_0 \rangle}. \end{aligned} \quad (1)$$

In eq. (1), the notations are same as those of Gell-Mann and Low<sup>7)</sup>, except that  $\psi$  and  $\phi$  are the nucleon and meson field variables in the interaction representation, respectively,

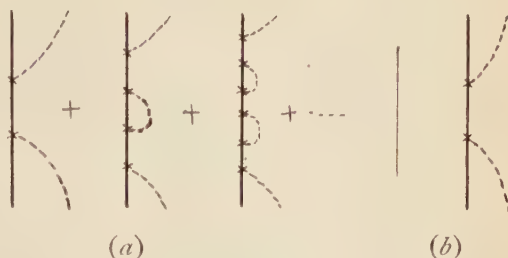
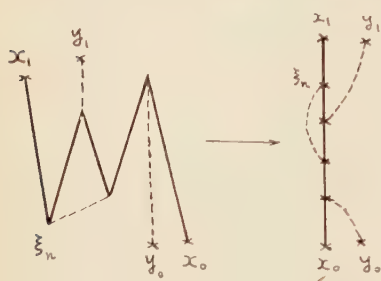


Fig. 1. In the problems where the real meson exists, it is not sufficient to renormalize only the kernel of the integral equation.



and the bold-face letters represent the operators in the Heisenberg representation.  $i$  and  $\alpha$  denote the charge states of the "initial" and "final" mesons, respectively. Of course, the words "initial", "final", etc. in this article must be taken, not chronologically, but along the nucleon line in question. The variables  $x_1, y_1$ , etc. before the semi-colon in the



arguments of  $K$ -functions denote the space-time variables of the "final" nucleon or mesons, while  $x_0, y_0$  after the semi-colon denote those of the "initial" ones. It may happen that the "final" particles are created before in time than the "initial" particles. (cf. Fig. 2.)  $\epsilon(x_1, x_0)$  is  $\pm 1$  according to whether the permutation of the times  $(1, 0)$  induced by  $P$  is even or odd. For simplicity, we shall omit the common factor

$$\left\{ \begin{array}{l} x_1, y_1 : \text{"final"} \\ x_0, y_0 : \text{"initial"} \\ \xi_n : \text{"last" interaction vertex} \end{array} \right.$$

Fig. 2. The meaning of the words "initial" and "final".

$$(\Phi_0, U(\infty, -\infty) \Phi_0)$$

and write the expression  $(\Phi_0, A \Phi_0)$  as  $\langle A \rangle$ . Then, eq. (1) can be written as follows:

$$K(x_1, y_1^a; x_0, y_0^b) = \epsilon(x_1, x_0) \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_n \times \langle P[H(\xi_1), \dots, H(\xi_n), \psi(x_1), \phi_a(y_1), \bar{\psi}(x_0), \phi_b(y_0)] \rangle. \quad (2)$$

Similarly, the Feynman kernels for the other processes can easily be written down. For example, corresponding to the process

$$N + \pi^0 \rightarrow N',$$

$$K(x_1; x_0, y_0^0) = \epsilon(x_1, x_0) \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_n \times \langle P[H(\xi_1), \dots, H(\xi_n), \psi(x_1), \bar{\psi}(x_0), \phi_0(y_0)] \rangle, \quad (3)$$

and for the process

$$N + \pi^0 \rightarrow N' + \pi'^0 + \pi''^0,$$

$$K(x_1, y_1^0, y_2^0; x_0, y_0^0) = \epsilon(x_1, x_0) \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_n \times \langle P[H(\xi_1), \dots, H(\xi_n), \psi(x_1), (\phi_a(y_1) \phi_r(y_2)), \bar{\psi}(x_0), \phi_0(y_0)] \rangle, \quad (4)$$

where  $(\phi_a(y_1), \phi_r(y_2))$  in the  $P$ -bracket in the right hand side of eq. (4) means that the operators  $\phi_a(y_1)$  and  $\phi_r(y_2)$  refer to different particles and therefore they are commutable. In general, the expression such as eq. (2) represents not only the process  $N + \pi \rightarrow N' + \pi'$  but includes also the processes  $N \rightarrow N' + \pi + \pi'$  and  $N + \pi + \pi' \rightarrow N'$ . However, in the following, we consider only the processes where one and only one meson

exists in the "initial" stage, so that it would not cause any confusion to say that the expression (2) represents the pion-nucleon scattering  $N + \pi \rightarrow N' + \pi'$ . In the perturbation theory, these  $K$ -functions could be computed by the help of the Feynman-Dyson diagrams. According to the usual technique, we shall omit the numerical factor  $(n!)^{-1}$  in each term of the defining equations (2), (3) and (4) of the Feynman kernels and, as the price for that, always take the order of the vertices along the nucleon line as  $(x_0, \xi_1, \xi_2, \dots, \xi_n, x_1)$ .

Now, let us consider the process  $N + \pi^a \rightarrow N' + \pi'^a$ . Since no term with odd  $n$  in eq. (2) contributes to this process, it is sufficient to consider only the terms with even  $n$ . Similarly, no term with even  $n$  in eqs. (3) and (4) contributes to the corresponding processes. As we have fixed the order of the vertices, the number of the Feynman-Dyson diagrams corresponding to the term with a given  $n$  (even) in the summation in eq. (2) is

$$(n+1)(n-1)(n-3) \times \dots \times 5 \times 3, \quad (5)$$

i. e.  $(n-1)(n-3) \times \dots \times 5 \times 3$  corresponding to the processes where the "initial" meson propagates without interaction with the nucleon, and  $n(n-1)(n-3) \times \dots \times 5 \times 3$  corresponding to the others.

These diagrams can be classified into two kinds:

- a) The diagrams where the "final" meson line  $(\phi_\alpha(y_1))$  starts from the "last" vertex  $\xi_n$ . There are  $(n-1)(n-3) \times \dots \times 5 \times 3$  diagrams of this kind.
- b) The diagrams other than those belonging to kind a), the number being  $n(n-1)(n-3) \times \dots \times 5 \times 3$  in all.

The diagrams of type a) and b) are closely related to the diagrams of order  $(n-1)$  corresponding to the processes  $N + \pi \rightarrow N'$  and  $N + \pi \rightarrow N' + \pi' + \pi''$ , respectively. That is, if we omit the interaction at the vertex  $\xi_n$  in the diagrams of kind b), then we shall get the diagrams corresponding to the terms of order  $(n-1)$  in the summation in eq. (4), and similarly we shall get the diagrams corresponding to the terms of order  $(n-1)$  in eq. (3) from the diagrams of type a). Moreover, it is clear that the correspondences are one to one. In fact, the numbers of the diagrams corresponding to the terms of order  $(n-1)$  in eqs. (3) and (4) are

$$(n-1)(n-3) \times \dots \times 5 \times 3$$

and

$$n(n-1)(n-3) \times \dots \times 5 \times 3,$$

respectively. The same relation holds in any order  $n(-\frac{1}{2}, 0)$ , and the similar analysis could be carried out for any other processes. Thus, it is easily supposed that there is an intimate relation between the  $K$ -functions for the processes, whose "final" meson numbers are different by one. In the next section we shall derive these relations in the analytic form.

### § 3. The simultaneous integral equations for the $K$ -functions

We consider the symmetrical pseudoscalar meson theory with the pseudoscalar coupling.

$$H(x) = i\bar{\psi}\psi(x)\gamma_5\tau_\alpha\psi(x)\phi_\alpha(x). \quad (6)$$

Then, taking into account the fact that we have fixed the order of vertices along the nucleon line, eqs. (2), (3) and (4) can be written as follows:

$$\begin{aligned} K(x_1, y_1^\alpha; x_0, y_0^\beta) = & \epsilon(x_1, x_0) \sum_{n=0}^{\infty} g^n \int_{-\infty}^{\infty} d\hat{\xi}_1 \cdots d\hat{\xi}_n \epsilon(x_1, \hat{\xi}_n) \epsilon(x_1, x_0) \epsilon(\hat{\xi}_n, x_0) \\ & \times \langle P[\psi(x_1), \bar{\psi}(\hat{\xi}_n)\gamma_5\tau_{\alpha_n}] \rangle \epsilon(\hat{\xi}_n, \hat{\xi}_{n-1}) \epsilon(\hat{\xi}_n, x) \epsilon(\hat{\xi}_{n-1}, x_0) \\ & \times \langle P[\psi(\hat{\xi}_n), \bar{\psi}(\hat{\xi}_{n-1})\gamma_5\tau_{\alpha_{n-1}}] \rangle \cdots \times \epsilon(\hat{\xi}_2, \hat{\xi}_1) \epsilon(\hat{\xi}_2, x_0) \epsilon(\hat{\xi}_1, x_0) \\ & \times \langle P[\psi(\hat{\xi}_2), \bar{\psi}(\hat{\xi}_1)\gamma_5\tau_{\alpha_1}] \rangle \langle P[\psi(\hat{\xi}_1), \bar{\psi}(x_0)] \rangle \\ & \times \langle P[\phi_{\alpha_1}(\hat{\xi}_1), \cdots, \phi_{\alpha_n}(\hat{\xi}_n), \phi_\alpha(y_1), \phi_\beta(y_0)] \rangle, \end{aligned} \quad (7)$$

$$\begin{aligned} K(x_1; x_0, y_0^\beta) = & \epsilon(x_1, x_0) \sum_{n=0}^{\infty} g^n \int_{-\infty}^{\infty} d\hat{\xi}_1 \cdots d\hat{\xi}_n \\ & \times [\epsilon\text{-factors and } \psi\text{-parts same as in (7)}] \times \langle P[\phi_{\alpha_1}(\hat{\xi}_1), \cdots, \phi_{\alpha_n}(\hat{\xi}_n), \phi(y_0)] \rangle \end{aligned} \quad (8)$$

and

$$\begin{aligned} K(x_1, y_1^\alpha, y_2^\tau; x_0, y_0^\beta) = & \epsilon(x_1, x_0) \sum_{n=0}^{\infty} g^n \int_{-\infty}^{\infty} d\hat{\xi}_1 \cdots d\hat{\xi}_n \\ & \times [\epsilon\text{-factors and } \psi\text{-parts same as in (7)}] \times \langle P[\phi_{\alpha_1}(\hat{\xi}_1), \cdots, \phi_{\alpha_n}(\hat{\xi}_n), \\ & \times (\phi_\alpha(y_1), \phi_\tau(y_2)), \phi_\beta(y_0)] \rangle. \end{aligned} \quad (9)$$

Now,  $P$ -bracket of  $\phi$ 's in eq. (7) can be rewritten as follows:

$$\begin{aligned} & \langle P[\phi_{\alpha_1}(\hat{\xi}_1), \cdots, \phi_{\alpha_{n-1}}(\hat{\xi}_{n-1}), \phi_{\alpha_n}(\hat{\xi}_n), \phi_\alpha(y_1), \phi_\beta(y_0)] \rangle \\ & = \langle P[\phi_{\alpha_n}(\hat{\xi}_n), \phi_\alpha(y_1)] \rangle \langle P[\phi_{\alpha_1}(\hat{\xi}_1), \cdots, \phi_{\alpha_{n-1}}(\hat{\xi}_{n-1}), \phi_\beta(y_0)] \rangle \\ & + \langle P[\phi_{\alpha_1}(\hat{\xi}_1), \cdots, \phi_{\alpha_{n-1}}(\hat{\xi}_{n-1}), (\phi_{\alpha_n}(\hat{\xi}_n), \phi_\alpha(y_1)), \phi_\beta(y_0)] \rangle. \end{aligned} \quad (10)$$

Therefore, if we write each term in the summation in eqs. (7), (8) and (9) as  $\hat{K}_n(x_1, y_1^\alpha; x_0, y_0^\beta)$ ,  $\hat{K}_n(x_1; x_0, y_0^\beta)$  and  $\hat{K}_n(x_1, y_1^\alpha, y_2^\tau; x_0, y_0^\beta)$ , respectively, it would be clear that the following relation holds for the  $\hat{K}_n$ -functions. (Remember the analysis of the preceding section regarding with the relations between the Feynman-Dyson diagrams for various processes.)

$$\begin{aligned} & \hat{K}_n(x_1, y_1^\alpha; x_0, y_0^\beta) \\ & = g \int_{-\infty}^{\infty} d\hat{\xi}_n \epsilon(x_1, \hat{\xi}_n) \langle P[\psi(x_1), \bar{\psi}(\hat{\xi}_n)\gamma_5\tau_{\alpha_n}] \rangle \langle P[\phi_{\alpha_n}(\hat{\xi}_n), \phi_\alpha(y_1)] \rangle \\ & \quad \times \hat{K}_{n-1}(\hat{\xi}_n; x_0, y_0^\beta) \\ & + g \int_{-\infty}^{\infty} d\hat{\xi}_n \epsilon(x_1, \hat{\xi}_n) \langle P[\psi(x_1), \bar{\psi}(\hat{\xi}_n)\gamma_5\tau_{\alpha_n}] \rangle \hat{K}_{n-1}(\hat{\xi}_n, y_1^\alpha, \hat{\xi}_n^{\alpha_n}; x_0, y_0^\beta), \quad (n \neq 0). \end{aligned} \quad (11)$$

Taking into account the fact that  $K_n(x_1; x_0, y_0^B)$ ,  $K_n(x_1, y_1^A; y_2^T; x_0, y_0^B)$  with even  $n$  and  $K(x_1, y_1^A; x_0, y_0^B)$  with odd  $n$  do not give any contribution, we get the following equation for the  $K$ -functions by summing up the both side of eq. (11).

$$\begin{aligned} K(x_1, y_1^A; x_0, y_0^B) &= K_0(x_1, y_1^A; x_0, y_0^B) \\ &+ g \int_{-\infty}^{\infty} d\tilde{\xi} \epsilon(x_1, \tilde{\xi}) \langle P[\phi(x_1), \bar{\psi}(\tilde{\xi}) \gamma_5 \tau_T] \rangle \langle P[\phi_T(\tilde{\xi}), \phi_A(y_1)] \rangle K(\tilde{\xi}; x_0, y_0^B) \\ &+ g \int_{-\infty}^{\infty} d\tilde{\xi} \epsilon(x_1, \tilde{\xi}) \langle P[\phi(x_1), \bar{\psi}(\tilde{\xi}) \gamma_5 \tau_T] \rangle K(\tilde{\xi}, y_1^A, \tilde{\xi}^T; x_0, y_0^B), \end{aligned} \quad (12)$$

where

$$\begin{aligned} K_0(x_1, y_1^A; x_0, y_0^B) \\ = \epsilon(x_1, x_0) \langle P[\phi(x_1), \bar{\psi}(x_0)] \rangle \langle P[\phi_A(y_1), \phi_B(y_0)] \rangle. \end{aligned} \quad (13)$$

By the similar analysis, the relations between the other  $K$ -functions can easily be derived. For example,

$$K(x_1; x_0, y_0^B) = g \int_{-\infty}^{\infty} d\tilde{\xi} \epsilon(x_1, \tilde{\xi}) \langle P[\phi(x_1), \bar{\psi}(\tilde{\xi}) \gamma_5 \tau_\delta] \rangle K(\tilde{\xi}, \tilde{\xi}^A; x_0, y_0^B), \quad (14)$$

$$\begin{aligned} K(x_1, y_1^A, y_2^T; x_0, y_0^B) \\ = g \int_{-\infty}^{\infty} d\tilde{\xi} \epsilon(x_1, \tilde{\xi}) \langle P[\phi(x_1), \bar{\psi}(\tilde{\xi}) \gamma_5 \tau_\delta] \rangle \langle P[\phi_\delta(\tilde{\xi}), \phi_A(y_1)] \rangle K(\tilde{\xi}, y_2^T; x_0, y_0^B) \\ + g \int_{-\infty}^{\infty} d\tilde{\xi} \epsilon(x_1, \tilde{\xi}) \langle P[\phi(x_1), \bar{\psi}(\tilde{\xi}) \gamma_5 \tau_\delta] \rangle \langle P[\phi_\delta(\tilde{\xi}), \phi_T(y_2)] \rangle K(\tilde{\xi}, y_1^A; x_0, y_0^B) \\ + g \int_{-\infty}^{\infty} d\tilde{\xi} \epsilon(x_1, \tilde{\xi}) \langle P[\phi(x_1), \bar{\psi}(\tilde{\xi}) \gamma_5 \tau_\delta] \rangle K(\tilde{\xi}, y_1^A, y_2^T, \tilde{\xi}^B; x_0, y_0^B), \end{aligned} \quad (15)$$

and so on.

#### § 4. Momentum representation of the integral equations

In the preceding section, we have derived the simultaneous integral equations for the Feynman kernels. Now, we shall transform these equations into the momentum representation. For this purpose, we introduce the Fourier transform of the various quantities.

$$\begin{aligned} K(x_1, y_1^A; x_0, y_0^B) &= \int \frac{dp_1}{(2\pi)^2} \int \frac{dk_1}{(2\pi)^2} \int \frac{dp_0}{(2\pi)^2} \int \frac{dk_0}{(2\pi)^2} \\ &\times \exp[i(p_1 x_1 + k_1 y_1 - p_0 x_0 - k_0 y_0)] K(p_1, k_1^A; p_0, k_0^B), \text{ etc.}, \end{aligned} \quad (16)$$

$$\begin{aligned} \epsilon(x_1, \tilde{\xi}) \langle P[\phi(x_1), \bar{\psi}(\tilde{\xi}) \gamma_5 \tau_T] \rangle &= -\frac{1}{2} S_F(\tilde{\xi} - x_1) \gamma_5 \tau_T \\ &= \lim_{\epsilon \rightarrow +0} \frac{i}{(2\pi)^4} \int d\tilde{p} e^{-i\tilde{p}(\tilde{\xi} - x_1)} \frac{i\tilde{p} - M}{\tilde{p}^2 + M^2 - i\epsilon} \gamma_5 \tau_T = \frac{1}{(2\pi)^4} \int d\tilde{p} e^{-i\tilde{p}(\tilde{\xi} - x_1)} S_F(\tilde{p}) \gamma_5 \tau_T, \end{aligned} \quad (17)$$

and

$$\langle P[\phi_\alpha(y_1), \phi_\beta(y_0)] \rangle = \frac{1}{2} \mathcal{A}_F(y_0 - y_1) \delta^{\alpha\beta} \\ = \lim_{\epsilon \rightarrow +0} \frac{-i}{(2\pi)^4} \int dk e^{-ik(y_0 - y_1)} \frac{1}{k^2 + \mu^2 - i\epsilon} \delta_{\alpha\beta} \equiv \frac{1}{(2\pi)^4} \int dk e^{-ik(y_0 - y_1)} \mathcal{A}_F(k) \delta_{\alpha\beta}, \quad (18)$$

where  $M$  and  $\mu$  are the masses of the nucleon and the meson, respectively. Then, our simultaneous integral equations can be written in the following form.

$$K(p_1; p_0, k_0^\beta) = \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_a \int dk_1 K(p_1 - k_1, k_1^\alpha; p_0, k_0^\beta), \quad (19)$$

$$K(p_1, k_1^\alpha; p_0, k_0^\beta) = \delta(p_1 - p_0) \delta(k_1 - k_0) \delta_{\alpha\beta} S_F(p_1) \mathcal{A}_F(k_1) \\ + \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_a \mathcal{A}_F(k_1) K(p + k_1; p_0, k_0^\beta) \\ + \frac{g}{(2\pi)^2} S(p_1) \gamma_5 \tau_\tau \int dk_2 K(p_1 - k_2, k_1^\alpha, k_2^\tau; p_0, k_0^\beta), \quad (20)$$

$$K(p_1, k_1^\alpha, k_2^\tau; p_0, k_0^\beta) = \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_a \mathcal{A}_F(k_1) K(p_1 + k_1, k_2^\tau; p_0, k_0^\beta) \\ + \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_\tau \mathcal{A}_F(k_2) K(p_1 + k_2, k_1^\alpha; p_0, k_0^\beta) \\ + \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_\delta \int dk_3 K(p_1 - k_3, k_1^\alpha, k_2^\tau, k_3^\delta; p_0, k_0^\beta), \quad (21)$$

$$K(p_1, k_1^\alpha, k_2^\tau, k_3^\delta; p_0, k_0^\beta) \\ = \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_a \mathcal{A}_F(k_1) K(p_1 + k_1, k_2^\tau, k_3^\delta; p_0, k_0^\beta) \\ + \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_\tau \mathcal{A}_F(k_2) K(p_1 + k_2, k_1^\alpha, k_3^\delta; p_0, k_0^\beta) \\ + \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_\delta \mathcal{A}_F(k_3) K(p_1 + k_3, k_1^\alpha, k_2^\tau; p_0, k_0^\beta) \\ + \frac{g}{(2\pi)^2} S(p_1) \gamma_5 \tau_\epsilon \int dk_4 K(p_1 - k_4, k_1^\alpha, k_2^\tau, k_3^\delta, k_4^\epsilon; p_0, k_0^\beta), \quad (22)$$

and so on.

The correspondence between each term of these equations and the diagram is obvious and it is very easy to write down the similar equations for the Feynman kernels with the "final" meson number larger than 3 or with the "initial" meson number other than 1. (see Fig. 3)

Watanabe<sup>9)</sup> has derived the similar simultaneous integral equations for the Green's functions with various "particle" numbers. And recently, Matthews and Salam<sup>10)</sup> have also proposed the similar equations for their new Feynman amplitudes which seemed to be the covariant analogues of Dyson's new Tamm-Dancoff amplitudes<sup>10)</sup>. Their methods and



resultant formulas are more general than ours in that they include also the nucleon closed loops and are applicable also to the many nucleon problems. However, apart from the fact that the physical meaning of the new Feynman amplitudes of Matthews and Salam is not so clear, the equations in question become non-linear and very difficult to treat owing to the inclusion of the closed nucleon loops. Of course, it might be possible to treat approximately the contributions of the nucleon closed loops to the meson propagators in the equations of Matthews and Salam. But we would not concern with those problems in this article. Moreover, it is the essential difference between the

formulas of the authors quoted above and ours that we have here distinguished the "initial" meson state  $k_0^3$  from the "final" meson states, i. e. we have always considered the Feynman kernels referred to the "initial" state  $(p_0, k_0^3)$  and derived the equations for the  $K$ -functions by paying attention to the change of the meson "numbers" along the nucleon line. It was also for this purpose that we have modified the definition of  $K(x_1, y_1^\alpha, y_2^\alpha; x_0, y_0^3)$  as eq. (4). It would become apparent in the next section that such a distinction of "initial" and "final" mesons is convenient and essential for the generalized T-D approximation (in our nomenclature).

## § 5. A covariant generalization of Tamm-Dancoff approximation

For the practical treatment of the simultaneous integral equations derived in the preceding section, we assume that the  $K$ -functions concerning with the numbers of the "final" meson larger than a certain critical number are identically zero. For example, in the approximation where the meson "numbers" are restricted within three, our simultaneous equations reduce to 4 equations in all, i. e. eqs. (19), (20), (21) and (22'), the last one being eq. (22) with omission of the kernel

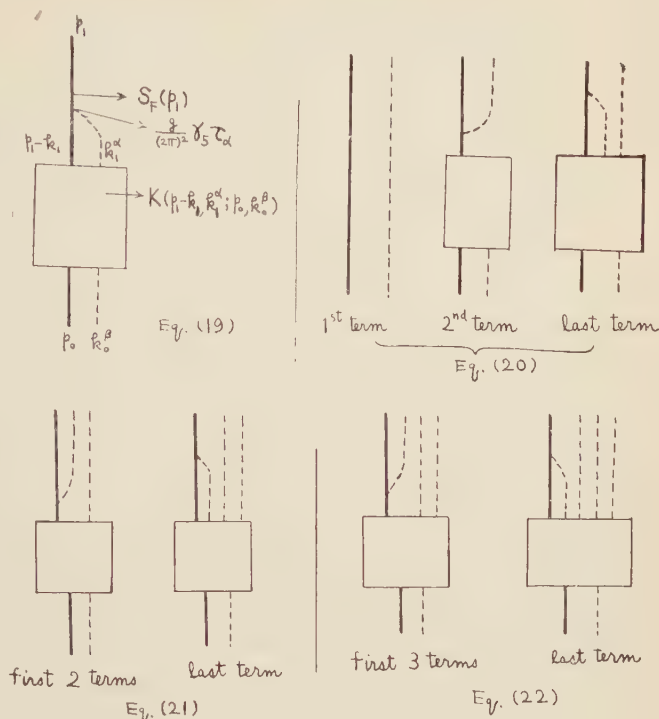


Fig. 3 The correspondences between eqs. (19)-(22) and the diagrams.

$K(p_1 - k_1, k_1, k_2, k_3, k_4; p_0, k_0)$ , i. e.

$$\begin{aligned}
 & K(p_1, k_1^\alpha, k_2^\gamma, k_3^\delta; p_0, k_0^\beta) \\
 &= \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_\alpha \Delta_F(k_1) K(p_1 + k_1, k_2^\gamma, k_3^\delta; p_0, k_0^\beta) \\
 &+ \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_\gamma \Delta_F(k_2) K(p_1 + k_2, k_1^\alpha, k_3^\delta; p_0, k_0^\beta) \\
 &+ \frac{g}{(2\pi)^2} S_F(p_1) \gamma_5 \tau_\delta \Delta_F(k_3) K(p_1 + k_3, k_1^\alpha, k_2^\gamma; p_0, k_0^\beta). \quad (22')
 \end{aligned}$$

From this viewpoint, the case studied by Fubini corresponds to the approximation where the meson numbers are restricted within two.

If we represent the propagation of the nucleon by the straight line, the meaning of the word "meson number" in this article is the number of the meson lines which intersect the straight line drawn perpendicularly to the nucleon line. (This straight line in our theory is the substitute for the space-like surface of the usual theory.) Therefore, the approximation introduced above can be regarded as a covariant generalization of the T-D approximation, and reduces to the usual T-D approximation when the pair creations and annihilations of the nucleons are entirely excluded, though our theory includes also the nucleon pair effects except the closed nucleon loops.

Of course, the main purpose of such a covariant generalization of the T-D method is to carry out the consistent subtraction procedure of divergences in the theory. This problem will be investigated in Part II of this article.

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## On Nuclear Force for $^3S_1 + ^3D_1$ State

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Deuteron problem and 90 Mev  $n$ - $p$  scattering are discussed according to the Taketani prescription. L shape potentials are used for the outside potentials, which are nearly equivalent to the meson theoretical potentials. In the inside region the phenomenological potentials are used which are composed of a hard core surrounded by square well so as to give the deuteron binding energy correctly. The depths of these inside central and tensor square well potentials are varied extensively as two parameters, and effects of these variations on the electric quadrupole moment and  $D$  state probability of deuteron, triplet effective range of neutron-proton scattering and 90 Mev neutron-proton scattering phase shifts are examined. The results are summarized in various graphs and tables. The depth of the inside central potential does not affect the physical quantities of deuteron, which, however, depend on the depth of the inside tensor force. Asymptotic forms of wave functions of neutron-proton system at high energy are mainly determined by the depth of the inside central potential, if the sign of the inside tensor potential is negative. On the contrary, if the inside tensor potential is positive and higher than about 80 Mev the high energy data depend strongly on the both inside parameters. " $J$  wave" of the deuteron has one node when the height of the inside tensor potential is larger than about 80 Mev. It is also found that the calculated value of the total cross section of the high-energy  $n$ - $p$  scattering is reduced by introducing a hard core, and can be fit to the experimental values without violating the angular distribution.

### § 1. Introduction

Some meson theoretical treatments<sup>1),2)</sup> have been made recently to obtain nuclear potentials including fourth order terms. These results are partially at variance and not warranted to be correct in the region inner than about half the meson Compton wave length<sup>2),3)</sup>. (This region will be called as "inside region".) In this region, approximations which are hitherto made in the course to derive the potentials are not adequate, because higher order effects<sup>3)</sup>, multiple-scattering terms<sup>2)</sup>, recoil effects, relativistic effects, etc., are expected to be large enough. However we know little about these effects, and the inside nuclear potentials are not finally determined yet. On the other hand, the main character of the outside potentials, especially in the triplet even state, seems to have already been determined. The outside potentials of the second and fourth order are comparatively large in the triplet even state, and contribute additively. The correction to be added to these outside potentials from higher order terms may be relatively small and negligible.

At these circumstances, it is promising to take Taketani's prescription<sup>4)</sup> for attacking nuclear force problem. This prescription assumes the meson theoretical potentials in the outside region and some phenomenological potentials in the inside region. According to

this prescription, Matsumoto and Watari<sup>†</sup> and Fujii *et. al.*<sup>10</sup> treated the nucleon scattering at high energy. The former (M.W.) extensively investigated the relations between phase shifts of partial wave and shapes of potential assuming spherical symmetry. The tensor potential has not been considered at all. They show that the inside potential does not affect  $P$ -wave and higher angular momentum waves so far as the collision energy is below 100 Mev, whereas only  $S$ -wave phase shift is sensitive to the details of the inside region of potential. Since the nature of the tensor force is unfamiliar to us, the same treatments for cases with tensor force are desirable. In the latter paper (Fujii *et. al.*) the validity of the potentials used by Taketani, Machida and Ohnuma<sup>11\*</sup> have been tested for the high energy nucleon-nucleon scattering. These potentials consist of the meson potentials from  $\rho$ - $\rho$  theory in the outside region and (tentative) phenomenological potentials in the inside region. The angular distribution of 90 Mev neutron-proton scattering is well explained by these potentials. However, the calculated value of total cross section is much larger than the experimental one (about 25%). This tendency is common with all potentials adjusted to the low energy scattering data<sup>7</sup>: It is commonly believed that the large theoretical  $n$ - $p$  total cross section is due to the large value of  $^3S$  phase shift.

In view of the above situation, the following question is arisen: Is there any potential model which can give both reasonable values of low-energy parameters (triplet effective range and deuteron parameters) and the high-energy scattering data (angular distribution and total cross section) at the same time? Thereby the outside potentials derived from meson theory should be left unchanged and suitable inner potentials should be assumed.

One of the purpose of the present paper is to answer this problem. It is also to be regarded as a extension of the analysis made earlier by the present authors<sup>7)</sup> to the cases including the tensor force for the triplet even state. The effects of details of the inside potential on deuteron parameters and 90 Mev neutron-proton scattering phase shifts are extensively examined. Results are summarized in various tables and graphs. Properties of tensor potential are hard to be understood intuitively, so that efforts are made to get as intuitive pictures for the tensor force as possible.

In the next section, potentials to be used in this analysis are explained in detail and methods of calculation are also illustrated. The deuteron quadrupole moment, triplet effective range and  $D$  state probability are calculated and their dependencies on the inside potential parameters are discussed in Section 3. It is found that the depth of the inside central potential does not affect the deuteron parameters, while depth of the inside tensor potential does especially when its sign is positive. With these potentials 90 Mev neutron-proton scattering is treated in section 4. Dependences of eigen phase shifts  $\delta_a$ ,  $\delta_T$  and "mixture parameter"  $\eta$  on the inside potentials are examined. When the sign of the inside tensor potential is negative, these parameters are nearly independent of the depth of the tensor potential and mainly depend of the depth of the central potential. When the

<sup>†</sup> In what follows, these potential are abbreviated as TMO potential.



sign of the inside tensor potential is positive, these parameters depend largely upon all potential parameters. Radius of a hard core is determined by the condition that potentials give the deuteron binding energy correctly: It is shown that this quantity can not be the principal parameter as it is in the case of the spherically symmetrical potentials<sup>(5)</sup>. A hard core of fairly large radius seems to be necessary to reduce " ${}^3S$ " phase shift at high energy in order to reproduce the experimental total cross section. The introduction of such a large hard core does not seem to contradict with the angular distribution. Finally, in appendix, the behaviours of the deuteron wave function in the case when inside tensor potential is positive are examined in detail.

## § 2. Triplet even potentials

### i) Outside potentials

Only the nuclear force for  ${}^3S_1+{}^3D_1$  state is considered. The region where inter-nucleon distance is larger or smaller than 0.6 times meson Compton wave length ( $1/\kappa$ ) is called outside or inside region. Throughout this paper, we use the unit  $1/\kappa=1.4\times 10^{-13}$  cm for length and Mev for energy. Following Taketani's prescription<sup>1)</sup>, meson theoretical nuclear potentials are taken in the outside region. So far the representative meson theoretical potentials have been given by Taketani, Machida and Ohnuma<sup>1)</sup> (TMO potentials),

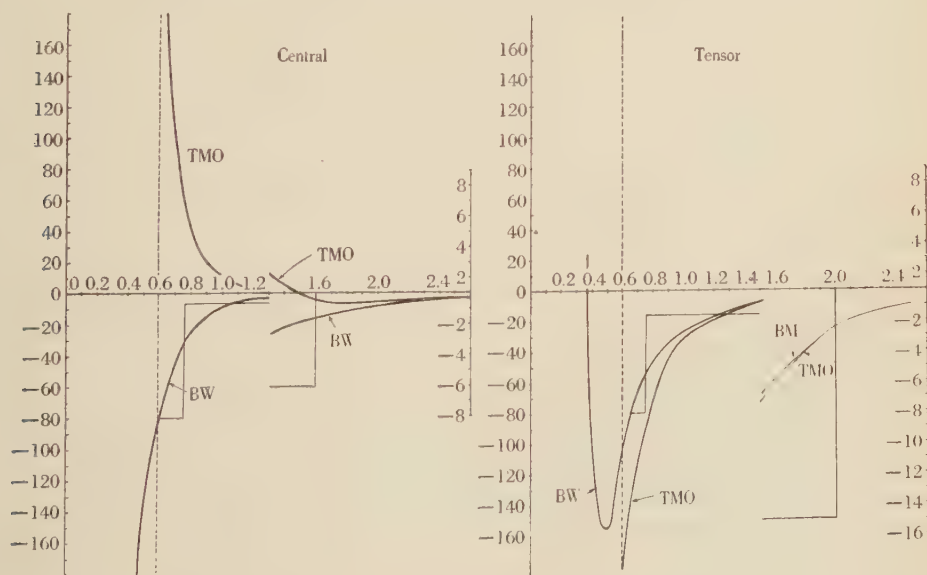


Fig. 1.

Outside potentials. Curves labeled as TMO are the potentials derived by Taketani, Machida and Ohnuma from the  $\rho$ - $\pi$ - $\nu$  meson theory (with  $g^2/4\pi=0.08$ ). Curves labeled as BW are the potentials including BW corrections. L shape potentials are one used in present calculations, and effectively nearly equal to BW potentials in their effects. Abscissa: Inter-nucleon distance in units of meson Compton wave length; Ordinate: Strength of the potential in units of Mev.



which are derived from  $\rho S(\rho\pi\pi)$  meson theory including up to the fourth order by the perturbation calculation. In this calculation, some fourth order terms are treated incorrectly and become very large for small inter-nucleon distance. These terms may be interpreted to be proportional to the dissociation probability of a clothed nucleon into mesons and a bare nucleon<sup>11</sup>. Corrections to TMO potentials are proposed by Brueckner and Watson<sup>12</sup>. These are equivalent to take the dissociation probability as zero according to the later analysis performed by Fukuda, Swada and Taketani<sup>13</sup>. (These corrections are referred to as BW corrections in this paper.) The "true" outside potentials may be found between these two extreme case and is conjectured to be rather near to meson potentials including BW corrections in triplet even state. These two potentials are plotted in Fig. 1.\* However, the exact meson theoretical potentials are not used here, but so called "L shape" potentials are used for the sake of reducing the labour of numerical calculations. L shape potential consists of a sequence of square wells with different depth and range. Potential of this type was extensively used earlier by the present authors<sup>14</sup>. It is shown there that in the case of spherically symmetrical potentials any given potential is well approximated by an appropriate L shape potential in a wide energy range. It is assumed here these are also true for the case of tensor potentials. Outside L shape potentials used here which are approximately equivalent to the meson theoretical potentials (including BW corrections) are shown in Fig. 1. The depth of the central potential between  $x=0.60$  and  $x=0.75$  is taken to be 80 Mev but it might be better to make its depth a little more shallow (by about 20%).\*\* Also it might be better either to make the tensor potential more shallow between  $x=0.75$  and  $x=2.00$  about 30% or to cut its tail about at  $x=1.80$  in order to be honest to potentials given by Brueckner and Watson. Nevertheless these potentials may represent the main feature of the meson theoretical potentials. This will be approved by comparing values of deuteron parameters calculated from our potentials with the values obtained by Brueckner and Watson (cf. table at p. 511).

## ii) Inside potentials

The purpose of this paper is to investigate phenomenologically the nature of the inside potentials which are assumed to have the form of "hard core surrounded by square well". Depths of the central potential and the tensor potential are varied extensively. Especially, cases where the sign of the tensor potential is positive, i.e. the tensor potential changes its sign at  $x=0.60$  are carefully studied, since BW tensor potential changes its sign at about  $x=0.40$ . The radius of the hard core is determined by the condition that our potentials give the correct deuteron binding energy,  $|\epsilon|=2.227$  Mev.

The wave function may be written as a linear combination of  $S$  wave ( $u(x)$ ) and  $D$  wave ( $w(x)$ )\*\*\*

\* Potentials labeled by BW are calculated according to equations next to (60) of Brueckner and Watson's paper with  $(G^2/4\pi)(\mu^2/4M^2)=0.08$ . These curves seem somewhat different from the corresponding curves of Fig. 6 of Brueckner and Watson's.

\*\*  $x$  is the inter-nucleon distance in units of meson Compton wave length.

\*\*\* For the detail of the notation cf. reference 1.

$$\phi = 1/x \cdot [u(x) + 1/\sqrt{8} \cdot S_{12} w(x)] \chi_m, \quad (1)$$

where  $\chi_m$  is the spin function with magnetic quantum number  $m$ .  $u(x)$  and  $w(x)$  are the solutions of the two simultaneous differential equations

$$\begin{aligned} u''(x) &= M/\kappa^2 \cdot [V_c(x) - E] u(x) + 2\sqrt{2} M/\kappa^2 \cdot V_t(x) w(x), \\ w''(x) &= M/\kappa^2 \cdot [V_c(x) - 2V_t(x) - E] w(x) + 6/x^2 \cdot w(x) + 2\sqrt{2} M/\kappa^2 \cdot V_t(x) u(x). \end{aligned} \quad (2)$$

For the deuteron ground state,  $u(x)$  and  $w(x)$  must vanish for  $x \rightarrow \infty$ . Then, at the outside of potentials  $u(x)$  and  $w(x)$  have following forms:

$$\begin{aligned} u(x) &= N \exp(-x/\xi) \\ w(x) &= N' \exp(-x/\xi) [3(\xi/x)^2 + 3(\xi/x) + 1] \end{aligned} \quad \text{for } x \gg \kappa^{-1}, \quad (3)$$

where  $\xi$  is a dimensionless constant determined from the binding energy of deuteron. The simultaneous equations (2) have two independent solutions which satisfy the boundary conditions (3). These two solutions are denoted by  $\phi_1 \equiv (u_1, w_1)$  and  $\phi_2 \equiv (u_2, w_2)$ . The boundary conditions (3) determine values of  $u'/u$  and  $w'/w$  at  $x \gg \kappa^{-1}$ , but they do not give the value of  $w/u$ . Any two different values of  $w/u$  correspond to two linearly independent solutions. Any solution of eqs. (2) satisfying (3) is given by  $\phi_1 + \alpha \phi_2 \equiv (u_1 + \alpha u_2, w_1 + \alpha w_2)$ . On the surface (and inside) of the hard core\*,  $S$  wave and  $D$  wave of deuteron have to vanish. If  $x_0$  is core radius,

$$\begin{aligned} u_1(x_0) + \alpha u_2(x_0) &= 0, \\ w_1(x_0) + \alpha w_2(x_0) &= 0, \end{aligned} \quad (4)$$

therefore,  $x_0$  is the zero point of

$\begin{vmatrix} u_1(x) & u_2(x) \\ w_1(x) & w_2(x) \end{vmatrix}^{**}$ , and  $\alpha$  (for deuteron) is given by  $\alpha = -u_1(x_0)/u_2(x_0)$ . Thus  $x_0$  is given

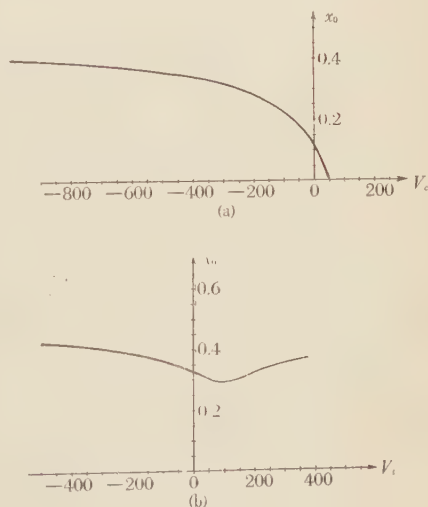


Fig. 2.

Hard core radius  $x_0$  is a function of  $V_c$  and  $V_t$ . In figure (a),  $x_0$  is plotted as a function of  $V_c$  with  $V_t = 0$ . In figure (b),  $x_0$  is plotted as a function of  $V_t$  with  $V_c = -300$  Mev. Abscissa is the potential strength in units of Mev. Ordinate is the hard core radius in units of meson Compton wave length.

\* The term "hard core" means here that in the region  $x \leq x_0$  the amplitude of the wave function  $\phi$  is zero. This is equivalent to let  $V_c \rightarrow \infty$ . But it is undesirable to think that  $V_t \rightarrow +\infty$  (or  $V_t \rightarrow -\infty$ ), because a tensor potential has "attractive character" regardless of its sign.

\*\* There may be many zero points of  $\begin{vmatrix} u_1(x) & u_2(x) \\ w_1(x) & w_2(x) \end{vmatrix}$ . In these cases,  $x_0$  must be the largest zero point. If other zero points are taken as the hard core radius, potentials thus obtained are more attractive, and wave functions obtained now are those of the excited state of these potentials whose binding energy is  $\epsilon$ . These wave functions may have some "true" nodes. ("true" node means at which  $\phi = 0$  (i.e.  $u = 0, w = 0$ )).

as the function of the heights of the inside central and tensor potentials. In Fig. 2 this function is plotted.

For some inside potentials which give the same  $\alpha$  their wave functions are the same in the region  $x \leq 0.60$ , since potentials are fixed there. Thus the depth of the inside central and tensor potentials are chosen as two parameters and the effects of the change of these parameters on the deuteron quadrupole moment,  $D$ -state probability, triplet neutron-proton effective range, and 90 Mev  $n$ - $p$  scattering phase shifts are examined. All results are obtained by direct numerical integration of the differential equations (2).

### § 3. Deuteron

As is well known<sup>8)</sup>, the wave function of deuteron is extended and has rather large amplitude even outside the nuclear potentials ( $x=3\sim 4$ ). The proportion of the wave function in the inner region ( $x \leq 0.60$ ) to that in the outer region is very small. Therefore, any change of the inside potential act very little on the wave function as a whole. Of course, these circumstances are essentially due to the fact that the asymptotic form of the wave function is determined by the deuteron binding energy which is small. The contribution from the inside part of the wave function to the value  $\int w^2 dx$  or  $\int x^2 w^2 dx$  is usually one or two percents in the present calculation, and the ratio of the inner part contribution of  $\int x^2 w^2 dx$  or of  $\int x^2 u w dx$  to the total integral is of negligible order of 0.1 percents.

Hereafter,  $V_c$  and  $V_t$  mean the height (depth) of the central and tensor potential in the inside region respectively (including with the sign, if  $V_c, V_t > 0$ , they are termed "repulsive", and if  $V_c, V_t < 0$ , "attractive"). The following values are obtained.

i)  $V_t = 0$

contribution to	$\int u^2 dx$	$\int w^2 dx$	$\int x^2 u w dx$	$\int x^2 w^2 dx$
$x_0 = 0.4423$				
$V_c = -1355 \text{ Mev}$	1.3%	1.4%	0.1%	0.05%
$x_0 = 0.3814$				
$V_c = -531 \text{ Mev}$	1.0%	1.0%	0.2%	0.1%

These values are almost constant for  $V_c > -531 \text{ Mev}$  ( $x_0 < 0.381$ ).

ii)  $V_t = -500 \text{ Mev}$

contribution to	$\int u^2 dx$	$\int w^2 dx$	$\int x^2 u w dx$	$\int x^2 w^2 dx$
$x_0 = 0.444$				
$V_c = -500 \text{ Mev}$	0.6%	2.6%	0.2%	0.1%

These values are almost constant for any value  $V_c > -500 \text{ Mev}$ .

iii)  $V_c = 500$  Mev

contribution to	$\int u^2 dx$	$\int w^2 dx$	$\int x^2 u w dx$	$\int x^2 w^2 dx$
$x_0 = 0.421$				
$V_c = -500$ Mev	2.0%	18.8%	-2%	2%

When  $V_c$  is larger than about 80 Mev, the  $D$  wave has a node and its amplitude is negative in the inside region. Negative part of the  $D$  wave increases with increasing  $V_c$ , and the contribution from this part to the above mentioned integrals becomes large. These circumstances which seem curious at the first glance will be discussed in detail in the appendix.

By the above argument, it is concluded that effects of the inside potentials are exclusively represented by a single parameter  $\alpha$  defined in the last section if  $V_c$  is not so large as 80 Mev. This parameter depends on the choice of  $\zeta'_1$  and  $\zeta'_2$ , but if the values of  $w_1/u_1$  and  $w_2/u_2$  at a point outside the potential range, for example at  $x=2.00$ , are once determined, (and these same values are used throughout the calculation,) then  $\alpha$  is uniquely determined from the inside potential parameters. The physical quantities of deuteron—electric quadrupole moment  $Q$ ,  $D$ -state probability  $p_D$ , and triplet effective range for neutron-proton scattering  ${}^3r$ —are determined by  $\alpha$  and the outside potentials alone. They depend on the detail of the inside potential implicitly through  $\alpha$ .

In general these quantities are given by the from

$$X = (A\alpha^2 + 2B\alpha + C) / (a\alpha^2 + 2b\alpha + c),$$

where  $A, B, C, a, b, c$  are some integrals quadratic of wave functions, specifically

$$a = \int (u_2^2 + w_2^2) dx, \quad b = \int (u_1 u_2 + w_1 w_2) dx, \quad c = \int (u_1^2 + w_1^2) dx.$$

Solve this equation with respect to  $\alpha$ , then one will get

$$\alpha = [- (bX - B) \pm \sqrt{D}] / (aX - A),$$

$$D = X^2 (b^2 - ac) - X(2bB - Ac - aC) + (B^2 - AC).$$

The region of  $X$  for which  $D \geq 0$  are limited because  $b^2 - ac < 0$ . This means that the assumed outside potentials confine the physical quantity  $X$  to some limited region. If the experimental value of  $X$  is out of this region, any choice of inside potentials joined to these outside potentials is of no use. When  $V_c > 80$  Mev, the exactly calculated value of  $Q$  is somewhat smaller than the one estimated using the outside potentials only, and exact values for  $p_D$  and  ${}^3r$  are larger than estimated ones.

The inside potentials are assumed, for the moment, to be given by  $V_c f(x)$  for the central part and by  $V_t g(x)$  for the tensor part, where  $f(x)$  and  $g(x)$  are non-negative functions of  $x$ , and are appropriately normalized (for example  $f(r=0.60) = 1$  and  $g(x=0.60) = 1$ ).  $V_c$  and  $V_t$  represent the strength of the central and tensor forces including their signs. The hard core is added to the central potential so as to give the



deuteron binding energy correctly in the manner described in the last section. In this way the differential equations (2) of deuteron are completely integrated, and one can calculate values of the quantities  $Q$ ,  $p_D$  and  ${}^3r$ . Then one can draw the curves  $Q=const.$  on the  $V_c-V_t$  plane as shown in Fig. 5. The family of curves  $Q=const.$  are practically the same as that of  $p_D=const.$  or of  $\alpha=const.$  by the reasons already mentioned.

Deuteron quadrupole moment is given by

$$Q = \frac{1}{\kappa^2} \int \left( \frac{x}{2} \right)^2 (3 \cos \theta - 1) \psi^4 \psi^4 d\tau,$$

where  $x$  is the inter-nucleon distance,  $\theta$  is the angle between deuteron axis and spin axis,  $\psi$  is the normalized wave function for the ground state of deuteron with  $m=1$ , and the integration is extended over the spacial coordinates and spin coordinates. We can see that  $Q$  is large when  $\theta$  is small. The tensor part of the potential energy is expressed as follows:

$$V_{ij}(x) \left( 3 \frac{\sigma^{(1)} \cdot x \sigma^{(2)} \cdot x}{x^3} - 1 \right),$$

where  $\sigma^{(1)} \cdot x \sigma^{(2)} \cdot x$  is an increasing function of  $\cos \theta$ . Therefore, if  $V_t < 0$  the state which has smaller  $\theta$  ([A] in Fig. 3) has the lower potential energy than the state of larger  $\theta$  ([B] in Fig. 3). From this consideration, one may conclude that the negative tensor potential of large absolute value gives large  $Q$  for the ground state.

According to the results of calculation,  $V_c$ -axis (i.e.,  $V_t=0$ ) is a member of the family of curves  $\alpha=const.$  ( $Q=const.$  and  $p_D=const.$ ). The radius of the hard core changes with the variation of the  $V_c$ , but these changes of the inside central potentials (with  $V_t=0$ ) scarcely change the shape of the wave functions on the whole but only change the wave functions near the origin. These situation is illustrated in Fig. 4. In fact following values are obtained with  $f(x) \equiv 1$ :

$V_c$ (Mev)		-530.8	-300	-86.8	-9.8	-  $\epsilon$
$x_0$		0.365	0.323	0.211	0.12	0.11
		0.172	0.171	0.171	0.17	0.17
$p_D$ (%)		6.38	6.38	6.38		
${}^3r$ ( $10^{-13}$ cm)		1.96	1.96	1.96		
$Q$ ( $10^{-27}$ cm <sup>2</sup> )		3.17	3.17	3.18		

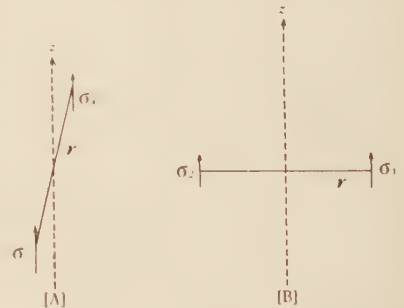


Fig. 3

In view of these facts, it is safely concluded that the form of  $f(x)$  is immaterial at low energy. These circumstances are similar to those at low energy in the case of spherically symmetrical potential alone.



For  $V_i \neq 0$ , the effective strength of the tensor potential changes, in general, with the variation of the hard core, or  $V_c$ , and so does the value of  $Q$  (and also  $\alpha$ ,  $p_D$  and  ${}^3r$ ). In other words, curves for  $\alpha = \text{const.}$  are generally not parallel to  $V_c$ -axis except for  $V_c$ -axis itself. (Fig. 5b). But if the effective depth of tensor force is not changed with  $x_0$ , (i.e., if  $g(x) \equiv 1$ ), then curves  $\alpha = \text{const.}$  are all parallel to  $V_c$ -axis (Fig. 5c). When  $g(x) \equiv 1$ , the "shape independence" is always satisfied for the inside central potential. Therefore, only the height of the inside tensor potential  $V'_i$  is the primary parameter for the deuteron state. This parameter determines the values of  $Q$ ,  $p_D$  and  ${}^3r$ , but the depth of the inside

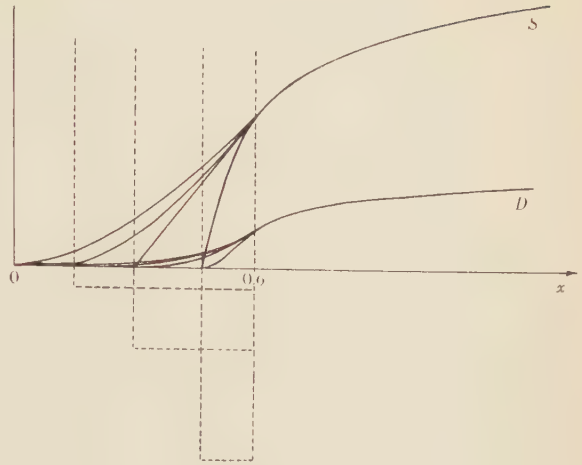


Fig. 4.

Behaviours of the wave function of the deuteron with variations of the inside central potential are schematically represented.

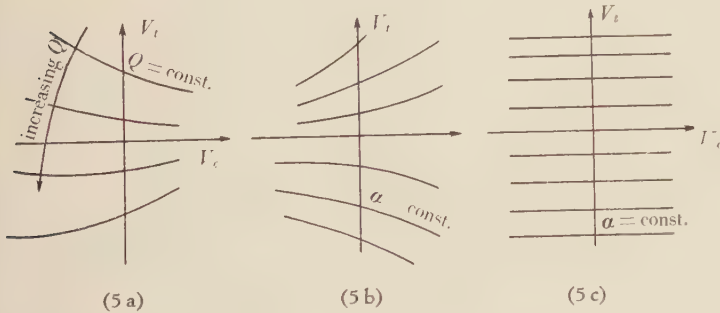


Fig. 5.

central potential (and a radius of the hard core) does not.

Numerical results are given below with  $V_c = -500$  Mev. There the results of Brueckner and Watson and experimental values are also listed for the sake of comparison.

$V_c$ (Mev)	-500	0	300	Results of B W.	Experimental results.
$x_0$	0.437	0.363	0.370		
$p_D$ (%)	7.60	6.38	5.10	6.12	4—9
${}^3r$ ( $10^{-13}$ cm)	1.97	1.96	1.86	1.73	1.70
$Q$ ( $10^{-27}$ cm <sup>2</sup> )	3.25	3.17	2.77	2.83	2.74

The values in the case of  $V_c = -|\epsilon|$  are also given and they will show the fact of "shape independence" with respect to  $V_c$ .

$V_t(\text{Mev})$	-500	300
$x_0$	0.404	0.310
$p_D$	7.58	5.09
$^3r$	1.97	1.86
$Q$	3.25	2.78

As is seen from these tables, the radius of hard core increases when  $|V_t|$  becomes large. This means that the tensor force has attractive nature irrespective of its sign. But the value of  $Q$  depends on the sign of  $V_t$ . The large value of  $Q$  corresponds to the large absolute value of (negative) tensor potential. As the value of  $V_t$  approaches zero and then changes to positive, the value of  $Q$  decreases and finally becomes negative. On the other hand, the ratio of the wave functions,  $w/u$  in the inside region decrease with increasing  $V_t$ , and change the sign about  $V_t \simeq 80$  Mev. For  $V_t > 80$  Mev,  $D$  wave function  $w$  has a node. The amplitude of the negative part of  $w$  increase as  $V_t$  increase, and the contribution of the inside wave function to the value of  $Q$ ,  $p_D$  and  $^3r$  becomes large and is not negligible as is the case of  $V_t < 80$  Mev. But it is also correct in this case that  $\alpha = \text{const.}$  means  $Q = \text{const.}$  or  $p_D = \text{const.}$

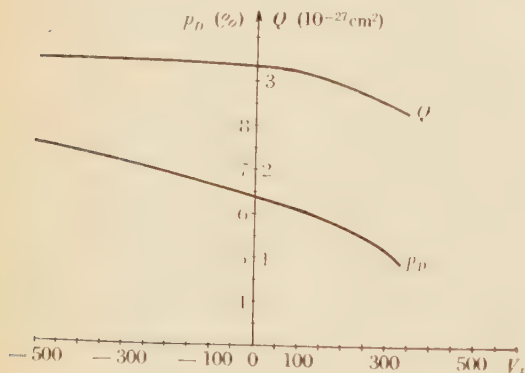


Fig. 6.

The deuteron quadrupole moment and the  $D$  state probability of deuteron are plotted as the function of  $V_t$ .

When  $V_t > 80$  Mev, it may appear that the state which corresponds to this nodal wave function\* is not the ground state, and that there may be a lower state which corresponds to nodeless  $D$  wave function as the ground state. However this is certainly not true. The detailed

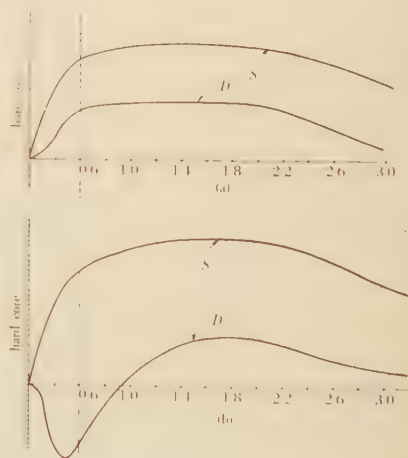


Fig. 7.

Wave functions of deuteron. (a) for  $V_t < 0$ , (b) for  $V_t > 80$  Mev. Abscissa is the inter-nucleon distance in units of meson Compton wave length.

\* Strictly speaking, the term "nodal wave function" is not correct, since a wave function  $\psi$  does not have the node. The node of the  $D$  wave is mere appearance.

consideration about this problem will be postponed till Appendix.  $Q$  and  $p_n$  are plotted as the functions of  $V_t$  in Fig. 6.

In order to adjust the quadrupole moment to the experimental value,\* one must have the positive inside tensor force if combined with the outside potentials of the present work. The situation may also be the same in the case including BW correction. For the TMO potentials, these things do not occur<sup>(9)</sup>. This is due to the fact that the outside central potential is repulsive in TMO potentials. Correct meson potentials will be between TMO potentials and BW potentials<sup>(9)</sup>, and it is possible that for "true" outside potentials the "repulsive nature" of the inside tensor potential will be reduced or vanish.

In Fig. 7 gross feature of wave functions for  $V_t < 0$  and  $V_t > 80$  Mev are shown. The meson theoretical potentials are very strong for small  $x$  as compared with commonly used phenomenological potentials, and so these wave functions become more concentrated than the phenomenological ones.

#### § 4. Neutron proton scattering at 90 Mev

The neutron proton scattering at 90 Mev is examined using the same potentials as the preceeding section. The phase shifts  $\delta_a$ ,  $\delta_T$  and mixture parameter  $\eta = \tan \epsilon^{**}$  are varied as the functions of the inside potential depths  $V_c$  and  $V_t$  as to be shown in Figs. 8, 9 and 10 respectively. In these figures, abscissa is  $V_c$  and ordinate is  $V_t$ , and curves are the traces of the points  $(V_c, V_t)$  which give the same values of phase shifts or mixture parameter. Phase shifts are plotted with the intervals of  $2^\circ$ .

The situations are remarkably different according to either  $V_t > 0$  or  $V_t < 0$ . When  $V_t < 0$ , eigen phase shift  $\delta_a$  (which tends to  $S$  wave phase shift when tensor force vanishes) seems to be almost constant with respect to  $V_t$ , and is a function of  $V_c$  only. The value of  $\delta_a$  is fairly small and confined between  $28^\circ$  and  $42^\circ$  (i.e.,  $0.49$ — $0.75$  radian). On the other hand,  $\delta_T$  varies very slowly with  $V_t$

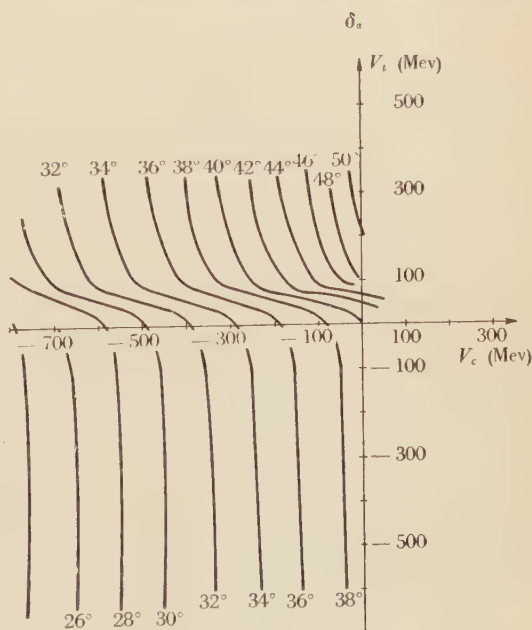


Fig. 8.

Phase shift  $\delta_a$  is plotted as the function of  $V_c$  and  $V_t$ .  $V_c$  and  $V_t$  means height of the inside central and tensor potential.

\* Experimental value to be adjusted is commonly taken as  $2.74 \times 10^{-27} \text{cm}^2$ . But there are some questions to adjust to this value. Cf., for example, I. Sato and K. Iwabashi.<sup>(10)</sup>

\*\* The same notations are used as that of Rohrlch and Eisenstein<sup>(9)</sup>, here we use real phase shifts.

and  $V_c$ , and can be regarded as having almost constant value over the region considered, (about  $-18^\circ \pm 1^\circ$ ). Then, in this case, the value of  $\sin^2 \delta_\alpha + \sin^2 \delta_\tau$  falls between 0.300—0.555, and this value is small compared with the value  $\sin^2 \delta_\alpha + \sin^2 \delta_\tau = 0.781$  given by TMO potentials<sup>6)</sup>. Therefore total cross section of  $n\text{-}p$  scattering may be reduced by our potentials. The value of the mixture parameter  $\eta$  is mainly determined by the depth of central potential  $V_c$ . Thus for  $V_t < 0$ , the asymptotic form of wave functions are mainly determined by  $V_c$  but not by  $V_t$ .

When the sign of the inside tensor potential is positive, the above situation does not occur. Phase shifts  $\delta_\alpha$  and  $\delta_\tau$  now strongly depend on the depth of the tensor force as much as the depth of the central force, especially for  $0 < V_t < 200$  Mev. Both phase shifts become large with increasing  $V_t$  and  $V_c$ . The value of  $\sin^2 \delta_\alpha + \sin^2 \delta_\tau$  varies extensively. A mixture parameter which usually considered as a positive quantity<sup>9)</sup> becomes negative when  $V_t$  and  $V_c$  are sufficiently large.

Angular distribution of  $n\text{-}p$  scattering can be written as<sup>9)</sup>

$$k^2 d\sigma = \sum_{n=0}^{\infty} c_n \cos^n \theta.$$

Here  $c_0, c_2, \dots$  (with even  $n$ ) are independent of the sign of  $\eta$ . Therefore the scattering in the state  $J=1$  is independent of the sign of  $\eta$ . But  $c_1, c_3, \dots$  ( $n=\text{odd}$ ) depend on the sign of  $\eta$ , which are due to the

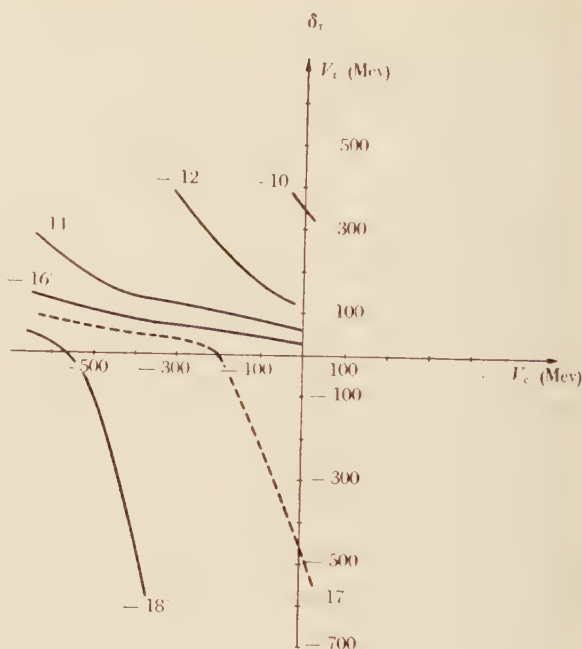


Fig. 9.  
Map of phase shift  $\delta_\tau$ .

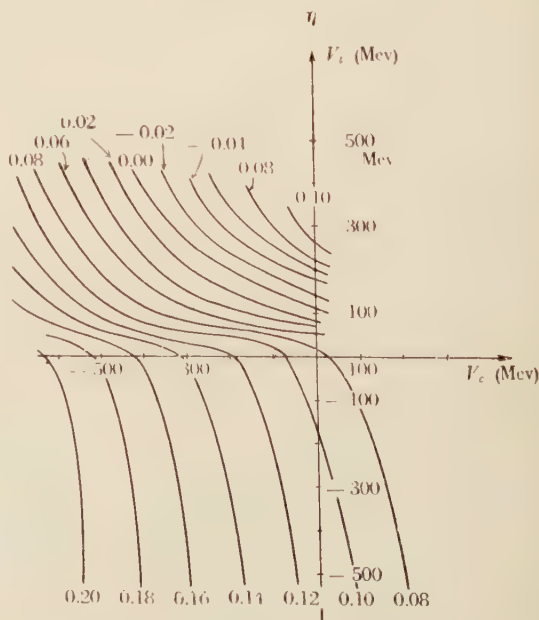


Fig. 10.  
Map of mixture parameter  $\eta$ .

interference of the triplet even states with odd states. Then, the deviation of the angular distribution from the  $90^\circ$  symmetry may considerably depend on the sign of  $\eta$ . If one use the phase shifts obtained by Fujii et. al.<sup>(6)</sup> for the states except that of  $J=1$ , then the angular distribution is the V shape having the minimum shifted forwards from  $90^\circ$  for  $\eta > 0$ , and backwards for  $\eta < 0$ . But since the angular distribution of the singlet states is large for backward scattering, total angular distributions have the minimum shifted forwards from  $90^\circ$  irrespective of the sign of  $\eta$ . At any rate, such discussions depend delicately on the potentials of the triplet odd state. Moreover, to compare with experimental data, one must consider the scattering of the singlet states. More inquisitive discussions are not given here.

Finally, gross feature of the wave functions for 90 Mev scattering system are shown in Figs. 11 and 12. On the left half of the figures, the wave functions for  $x \leq 2.2$  are given, and on the right half plane, asymptotic forms of these functions are shown schematically. Scales of the figures are not always equal on the left and on the right sides. The case in which the inside tensor potential is negative is shown in Fig. 11. In Fig. 12 the wave functions when inside tensor potential is positive are shown with negative  $\eta$ . When a inside central potential is strongly attractive, then even for positive  $V_t$ , the case with  $\eta > 0$  may occur. In this case  $D$  wave has the form which is shown by dotted line.

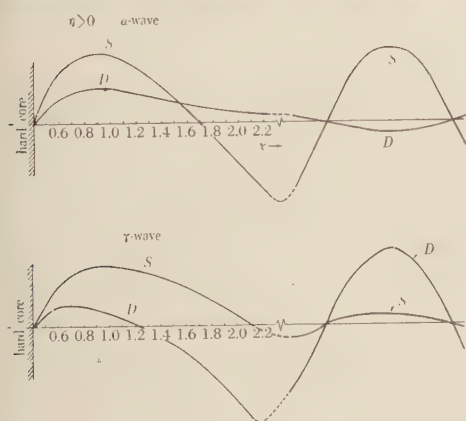


Fig. 11.

Gross feature of wave functions for 90 Mev scattering system for  $V_t < 0$ . On the left half of the figure, the wave functions for  $x \leq 2.2$  are given, and on the right half plane, asymptotic forms of these functions are shown schematically. Scales of the figures are not always equal for the left side and for the right side.

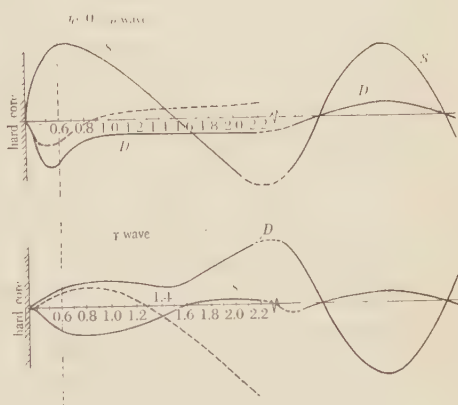


Fig. 12.

Gross feature of wave functions for 90 Mev scattering system for  $V_t > 90$  Mev with negative  $\eta$ . When  $\eta$  is positive,  $D$  wave has the form which is shown by dotted line.

## § 5. Conclusion and summary

Under the condition that the deuteron binding energy is given correctly, inside phenomenological region  $x \leq 0.6$  of the triplet even potential for the neutron-proton



system are extensively varied, and effects of these variations on the deuteron quadrupole moment,  $D$ -state probability, triplet effective range and 90 Mev  $n$ - $p$  phase shifts are examined. Following results are obtained:

1) If the height of the inside tensor potential is smaller than 80 Mev (including negative value), then physical quantities of deuteron are almost determined by the outside potentials only. Contributions from the inside regions to these quantities are almost few per cents.

Even in high energy scatterings the asymptotic forms of the wave functions are not much affected by the change of the inside potentials. As regards to the role of the inside potentials, the depth of the inside tensor potential is primarily important for the bound state, while for the scattering state the depth of the inside central potential is the essential parameter.

2) If the height of the inside tensor potential is positive, (strictly speaking, for deuteron, if the height of the inside tensor potential is greater than 90 Mev), the two components of the wave function have different sign near the origin both in the deuteron and in the 90 Mev scattering states: i.e.,  $w/u < 0$  for  $x \ll 1/\kappa$ . For the high energy scattering, the case where mixture parameter  $\eta$  is negative may occur with sufficiently large value of  $V_c$ . But in these circumstances the results can not be compared with experiment. Physical quantities of deuteron and high energy  $n$ - $p$  scattering phase shifts depend strongly on the inside potential parameters  $V_c$  and  $V_t$ .

Potentials so far used for the analysis of the high  $n$ - $p$  scattering, whether it is drawn from the meson theory or assumed phenomenologically, give larger values for total cross section than the experimental one, provided that potential parameters are adjusted to low energy experimental data correctly. It is commonly believed that these things are mainly due to the fact that the phase shifts of the triplet even state of these potentials are too large. From the present calculation, especially referring to Figs. 8, 9 and 10, it is shown that these phase shifts are reduced by introducing "hard core" to the triplet even state potentials, since negative large value of  $V_c$  means large value of the radius of hard core  $x$ .

In the analysis of 90 Mev  $n$ - $p$  scattering using TMO potentials performed by one of us (W.W.) and others<sup>(1)†</sup>, theoretical value of the total cross section is about 25% greater than the experimental value.

If one assume the energy independence of the potentials at least up to the scattering energy about 100 Mev, one can see from Fig. 6 that with the present outside potentials the value of  $V_c$  must be about 300 Mev to give the deuteron quadrupole moment correctly. When one take  $V_c$  about -500 Mev, one get from Figs. 8, 9 and 10 the value of positive  $\eta$  of the order of about 0.1 (this value is of the same order as that given by the usual potentials),\*\*  $\delta_a$  about  $30^\circ \pm 4^\circ$  and  $\delta_T$  about  $-16^\circ \pm 2^\circ$ . Then  $\sin^2 \delta_a + \sin^2 \delta_T$  are about  $0.3 \pm 0.1$ . Thus triplet cross section is reduced about 27% compared with the value obtained in I, and the total cross section is reduced about 23% compared with the

† Hereafter, this paper is referred to as I.

\*\* Cf. Note added to proof.

results of I. Therefore one get the total cross section nearly equal to the experimental one. The angular distributions given by these potentials is nearly equal to the results given in I, but the value of  $\delta_r$  is fairly large compared with that of I, consequently the ratio  $\sigma(90^\circ)/\sigma(180^\circ)$  becomes somewhat smaller. The radius of the hard core of the potential under the consideration is  $0.37 \times 1/\kappa = 0.52 \times 10^{-13}$  cm, (or somewhat larger). The reduction of the total cross section is mainly due to the repulsion of this fairly large hard core.

The authors wish to thank S. Tani and S. Otsuki for their frequent suggestions and friendly help, and Professor M. Taketani and all members of the research group of nuclear force for their illuminating discussions. They also indebted to Professor S. Hayakawa and T. Kikuta for many stimulating and informative discussions on the subject of the nodal wave in  $D$ -state.

## Appendix

As mentioned in the text, if the inside tensor potential is strongly "repulsive", the  $D$ -state wave function of the deuteron has a opposite sign to that of the  $S$  wave in the inside region. But for large  $x$ , it has the same sign. Consequently  $D$  wave has (at least) one node as shown in Fig. 7b. As the ground state wave function is nodeless in the case of pure central force, it might appear that in these curious cases the bound state with the binding energy  $|\epsilon| = 2.227$  Mev considered in the text should not be the ground state for this potential. In the text, the binding energy is first fixed, and the inside potentials are determined to give this binding energy, and so there is no guarantee that this state is the ground state of the so determined potentials. In usual case, both  $D$  wave and  $S$  wave have no node, therefore one may think it is certainly the ground state of these potentials. But in our case the  $D$  wave has a node and a question arises whether it is the ground state or not.

The system considered is described by the simultaneous second order differential equations (2) (which is equivalent to an ordinary fourth order differential equation) with the boundary conditions (3) and

$$u(x_0) = 0, \quad w(x_0) = 0,$$

where  $x_0$  is the radius of the hard core. Binding energy is determined from the value of  $\xi$ . Unfortunately there is no satisfactory theory concerning these simultaneous differential equations as far as we know, and we can not answer above question fully. Nevertheless, one may conjecture that there is no bound state with a larger binding energy than  $|\epsilon|$ , on the basis of the following consideration:

1) First, the behaviour of the ( $D$ ) wave function is examined in detail with increasing  $V_t$  having  $V_c$  (for example  $V_c = -300$  Mev) and binding energy (for example  $|\epsilon|$ ) fixed. Then  $x_0$  is given as a function of  $V_t$  which is shown in Fig. 2b. This function has a minimum about at  $V_t = 80$  Mev, and has an almost constant value in the

neighbourhood of this point. This point will be denoted as  $V_t^0$ . The sign of  $dw/dx$  at  $x=x_0$  is positive when  $V_t < V_t^0$  and negative when  $V_t > V_t^0$ . These situations are shown in Fig. 13, where  $D$  wave is plotted as the function of  $x$ , and numbers attached to each wave function are the corresponding values of  $V_t$ .  $S$  wave for  $V_t=90$  Mev is also shown for comparison.  $S$  wave for  $V_t=30, 60$  or  $120$  Mev are almost the same as that for  $90$  Mev.

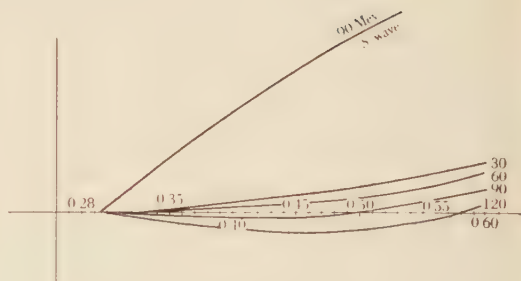


Fig. 13.

Behaviours of  $D$  wave function near the hard core are shown for  $V_t=30, 60, 90, 120$  Mev.

2) Next if  $x_0$  is fixed to the value  $x_0(V_t^0)$  and  $V_t$  is changed about the vicinity of  $V_t^0$ , then binding energy varies as the function of  $V_t$ , which has a minimum at  $V_t=V_t^0$ . Only very small increase of the binding energy over this minimum value is caused as  $V_t$  deviates from  $V_t^0$  provided that  $(V_t-V_t^0)$  is small irrespective of its sign; this will be expected also from the discussions of the last paragraph. In fact, the increase of the binding energy is so small that it can not be detected against the increase of  $V_t$  about  $10$  Mev. These facts may show that the binding energy is a continuous function of  $V_t$  having a minimum at  $V_t=V_t^0$ . However, if  $V_t > V_t^0$  the  $D$  wave has a node in the same manner as mentioned in the last paragraph, and if  $V_t < V_t^0$  it has not.

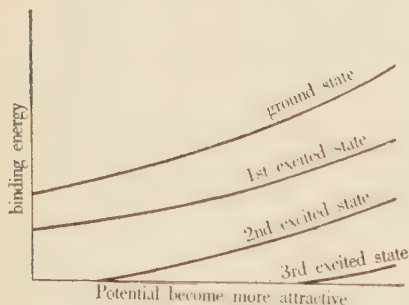


Fig. 14.

3) It is commonly believed that when some potential parameters are continuously varied, the binding energies of the potential change as continuous functions of these parameters. If the potential becomes more attractive and begin to have one more bound state, this new bound state must be borne anew from the zero energy states.\*\*

In the other words, the energy curves have common features as shown in Fig. 14.\*\*\*

4) As was shown in paragraph 2), if  $V_t$  is a little smaller than  $V_t^0$ ,  $D$  wave has no node and therefore this state is certainly the ground state. When  $V_t$  increase above  $V_t^0$  the binding energy changes continuously and at the same time  $D$  wave function varies continuously as shown in Fig. 13 and becomes to have a node. This nodal wave function also corresponds to the ground state because there is no sudden change to be expected in the energy level curve through  $V_t^0$ . If this nodal wave function was not of the ground

\*  $V_t^0$  may be, in general, the function of  $V_c$ , but this is out of the problem.

\*\* This state is a degenerate one, when the box of closure for the system becomes infinitely large.

\*\*\* In this case, a crossing of energy curves does not occur.

state, it must be continued with a corresponding second energy level curve which would arise from zero energy. But actually this energy curve is continuous with the curve of the lowest level at  $V_t < V_t^0$ .

5) For  $V_t > V_t^0$ , if the state which has the nodal  $D$  wave function as shown in Fig. 13 does not happen to be the ground state, then there must be the ground state with a nodeless wave function. This function will be denoted as  $\psi'_0 = (u_0, w_0)$  and the nodal wave function as  $\psi' = (u, w)$ . Then these two functions must be orthogonal each other. If one assume  $u$  and  $u_0$  have the same sign (positive), then  $w_0$  must have either of the following alternative characters, since  $\int u_0 u dx = -\int w_0 w dx$ .

i)  $w_0$  has the same sign as  $u_0$  and its amplitude is very large in the region where  $w$  is negative and very small in the region where  $w$  is positive as shown in Fig. 15a, or

ii)  $w_0$  has the opposite sign to  $u_0$  and its amplitude is large for the region where  $w$  is positive as shown in Fig. 15b.

Thus main part of  $w_0$  must have the same sign as the tensor potential. Total energy is given by the following form

$$E = \int u P u dx + \int w R w dx + \beta \int u V_t w dx,$$

where  $P$  and  $R$  include the kinetic energy operator and a ("effective") central potential term, and  $\beta$  is a positive constant.  $V_t$  is a tensor potential inclusive of its sign. Then, if other things are the same, the wave function

for which the sign of  $w$  is the opposite to that of  $V_t$  has lower energy. Therefore  $\psi'_0$  has rather larger potential energy than  $\psi'$ . The kinetic energy also seems to become larger for  $\psi'_0$  than for  $\psi'$  since  $w_0$  (and also  $u_0$ ) behaves in a more steeply varying manner than  $\psi'$ . So it seems impossible that  $\psi'_0$  has lower energy than  $\psi'$ ; and these facts support the conjecture that  $\psi$  is the wave function of the ground state.

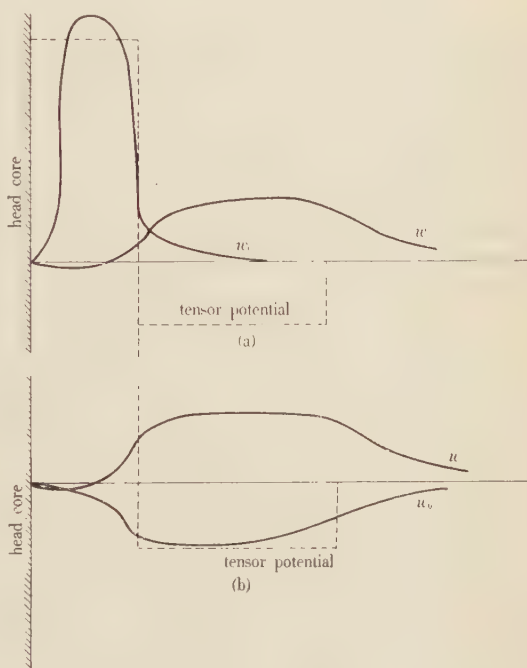


Fig. 15.



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**Note added to proof.** According to the recent experiment performed at Berkeley (O. Chamberlain and J. W. Easley, Phys. Rev. **94** (1954), 208), the  $n$ - $p$  angular distribution at 90 Mev shows good symmetry about  $80^\circ$ . The results calculated by TMO potentials<sup>(1)</sup> did not show such a good symmetry. This may be also the case in our calculation if we take the value of  $\eta$  about 0.1. But this value is tentative one, and can be varied by changing inside potential parameters without violating the restriction to fit the experimental values of the total cross section and the quadrupole moment. But angular distribution depends delicately on the details of the  $P$ -state potentials. Symmetry at 90 Mev will be due to the balance of the asymmetric effects of  $P$ -states and  $\eta$ . But such a balancing will not hold at 135 Mev.<sup>(2)</sup>



# Phenomenological Analysis of the Meson Theory of Nuclear Force

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Taking into account new development of meson theory of nuclear force both from field theoretical and from phenomenological point of view,  $p$ - $p$  scattering upto 100 Mev is analyzed. Namely, adopting the pseudoscalar meson theoretical potential in the singlet states and subtracting their effect from the experimental data, magnitudes of the central and tensor potential in the triplet odd state are obtained. This analysis shows that the central potential must be very weak and that the tensor potential is not so strong as potentials of the other states. These features are also favoured to  $n$ - $p$  scattering. The meson theoretical potentials of this state are very similar to this phenomenological one and consequently it is concluded that no corrections are requisite that alter the characteristics of the meson theoretical potential of the triplet odd state. The properties of the meson theoretical potentials of the other states are also discussed.

## § 1. Introduction

The problem of the meson theory of nuclear forces has recently been attacked by several authors. On the one hand many attempts<sup>1)2)3)4)</sup> have been made to derive the nuclear forces from the symmetrical pseudoscalar meson theory by improving the approximation method and taking into account various corrections. On the other hand such meson theoretical potentials have been shown to reproduce low energy nucleon-nucleon scattering data including deuteron parameters provided physically reasonable prescriptions such as the introduction of phenomenological potentials when the two nucleons are close together or the reduction of the meson pair terms are made.<sup>1)2)5)</sup> Also the nucleon-nucleon scattering upto 100 Mev has been investigated by us and others<sup>6)</sup> (hereafter referred to as II) adopting the static potentials of the second and fourth order derived from the symmetrical pseudoscalar meson theory with pseudovector coupling<sup>1)</sup> (hereafter referred to as TMO potentials), where they have been shown to be satisfactory on the whole for such high energy phenomena. In reference II, however, we have pointed out two problems to be investigated anew. One is concerning to the  $S$ -wave: Because only  $S$ -waves are affected by the phenomenological potentials of the region where two nucleons are close together, it is necessary to reinvestigate this phenomenological character more in detail adopting the meson theoretical potentials far from the origin. The other is concerning to the triplet odd state: Owing to the cancellation of many terms derived from the meson theory, the potentials of this state have very delicate features. So it is rather desirable to find out the potentials of this state from the phenomena.

Brueckner and Watson<sup>2)</sup> have also derived nuclear potentials of the pseudoscalar meson

theory reducing the effects of the meson pair terms which arises in the non-relativistic approximation to the pseudoscalar coupling (hereafter referred to as BW). The nuclear potentials thus obtained are different from TMO's in the treatment of non-adiabatic correction to the second order terms.

Let us call attention to the region  $x \leq 0.6$ ,\* where the sixth order potentials<sup>(6)</sup> or multiple scattering effect<sup>(7)</sup> are not important. The difference between the potentials of TMO and those of BW in this region is quite negligibly small in the singlet states irrespective of their spacial parity, while it is remarkable in the triplet states. In the triplet even state the non-adiabatic correction term of BW yields the attractive central force instead of the repulsive one of TMO and makes the tensor force less strong. In the triplet odd state this correction term is numerically one ninth of the even state.

Putting all that is mentioned above together, it seems reasonable for us at this stage to attack the problem of  $p$ - $p$  scattering in the following way: Confine ourselves to the nucleon-nucleon scattering of the limited region of energy upto 100 Mev for the reasons discussed in detail in II. Divide the inter-nucleon distance significantly into two regions, i. e., the outside region (which means the region of  $x \geq 0.6$ ). and the inside one ( $x \leq 0.6$ ). Take for the singlet state TMO's potentials in the outside region regarding them to represent the true features of the meson theoretical potentials, while in the inside region various types of phenomenological potentials are taken. Recalculate the low energy parameters and  $^1S$ -wave phase shift whereby paying attention to the effect of the inside phenomenological potentials because only  $^1S$ -wave is affected by them (Sec. 2). The inside phenomenological potentials are assumed to be energy independent as the first approximation. Then subtract this singlet even scattering from the experimental result of  $p$ - $p$  scattering at 18.3 Mev to get the triplet odd scattering, which is analyzed according to the procedure explained below assuming that only central and tensor forces are present in the two-nucleon interaction as predicted by the meson theory (Sec. 3).

This analysis shows that the central potential of this state must be very weak and that the tensor potential is not so strong as potentials of the other states. Such salient features of the phenomenological triplet odd potentials are favourable not only for both very low and high energy  $p$ - $p$  scattering but also for  $n$ - $p$  scattering (Sec. 4). It is to be emphasized that weak triplet odd central force, which is favourable in  $n$ - $p$  scattering, is the only one that is allowed by  $p$ - $p$  scattering.

To find out the triplet odd potentials,  $P$ -waves are required to play a certain role to determine the angular distribution of  $p$ - $p$  scattering. However, as energy goes higher,  $F$ -waves begin to take part in the angular distribution mainly through the mixture parameter  $\tan \epsilon_2$  of the two channeled odd state with  $J=2$  and the situation becomes very delicate. This is the reason why we should choose the 18.3 Mev experiment as the object of our analysis.

The triplet odd state potentials due to both TMO and BW treatment are very similar to our phenomenological ones as far as the region  $x \gtrsim 1$  concerns and it seems

\*  $x$  is the inter-nucleon distance in the unit of the meson Compton wave length  $\hbar/\mu c = 1.40 \times 10^{-13}$  cm.

quite possible for them to fit the experimental data very well if the proper value of the coupling constant is chosen. Consequently no corrections that alter the characteristics of the TMO or BW potentials of this state are requisite.

Of late, a new method treating the nuclear force problem of the meson theory has been proposed by Fukuda, Sawada and Taketani<sup>31</sup> (hereafter referred to as FST) in which the dissociation probability of nucleons is properly taken into account. Their result goes to that of BW when the dissociation probability is put identically zero while the treatment of TMO gives the divergent dissociation probability. As to the potentials of FST, they lie between those of TMO and of BW. This result on the one hand supports our adoption of the singlet meson potential of TMO (which is quite the same as BW's) in the region  $x \geq 0.6$  and on the other hand agrees with our conclusion that the correction to the triplet odd state potentials of TMO or BW is not so large as to change its characteristic features.

In the triplet even state the potentials of FST are close to BW's rather than TMO's. A phenomenological research to this state assuming the potentials similar to BW's has been made by Matsumoto and Watari<sup>71</sup> reporting that such potentials are favourable to give the  $n$ - $p$  total cross section of the magnitude known experimentally at high energies. Neither phenomenological potentials<sup>9,10</sup> nor meson theoretical one<sup>51</sup> have ever been successful to reduce the  $n$ - $p$  total cross section to the experimental values. In view of all these phenomenological properties of the meson theoretical potentials, one may conclude that the symmetrical pseudoscalar meson theory is consistent with the main features of nucleon-nucleon scattering upto 100 Mev concerning both angular distributions and total cross sections (Sec. 5).

## § 2. Singlet even state

We adopt the potential of TMO in the outside region with various values of coupling constant  $g^2/4\pi$ . The results are summarized in Table I and II. In Table I we take as the phenomenological inside potential hard core surrounded by square well, the core radius being a parameter, while in Table II we subdivide the inside region into two parts at  $x=0.3$ , the depth of the potential for  $0 \leq x < 0.3$  being a parameter.

It is to be noted that the magnitude of  $^1S$ -wave phase shifts obtained here is not so large as in II. As mentioned above, we can expect for the potentials of BW and FST not to yield results so different from Table I and II.

### 2—1. Low energy scattering

In Table I and II, we show the values of effective range  $r_e$  corresponding to scattering length  $a = -23.69 \pm 0.06 \times 10^{-13}$  cm, which is obtained from the  $n$ - $p$  experimental data<sup>10</sup>. The experimental values of some of the low energy parameters such as singlet  $n$ - $p$  and  $p$ - $p$  effective range  $r_{enp}$  and  $r_{epn}$  and singlet  $p$ - $p$  scattering length  $a_{pp}$  depend more or less on the shapes of the potential themselves assumed to analyze the data. Taking this fact into consideration, the experimental data available at present are, including the experimental errors :

$$r_{en-p} = 1.9 \sim 2.7 \times 10^{-13} \text{cm.}^{11)}$$

$$r_{ep-p} = 2.5 \sim 2.8 \times 10^{-13} \text{cm.}^{12)}$$

In this paper, we expect for meson theoretical potential also to yield effective range of the values above, which is quite reasonable, since the shape factor  $I'$  of the meson theoretical potential is as small as those of the usual phenomenological potentials as can be seen from the phase shifts at 18.3 Mev in Table I and II.

Table I. Properties of the singlet even TMO potential together with phenomenological inside potentials such as

$${}^1I_e^-(x) = \bar{I}', \text{ for } 0.6 > x \geq x_0,$$

$${}^1I_e^-(x) = \infty, \text{ for } x_0 > x.$$

They give the scattering length  $a = -23.7 \times 10^{-13} \text{cm.}$

$g^2/4\pi$	$x_0$	$\bar{I}' (\text{Mev})$	$r_e (10^{-13} \text{cm})$	${}^1\delta_0(18.3 \text{Mev})$	${}^1\delta_0(40 \text{Mev})$	${}^1\delta_0(90 \text{Mev})$	${}^1\delta_2(18.3 \text{Mev})$	${}^1\delta_2(40 \text{Mev})$	${}^1\delta_2(90 \text{Mev})$
0.08	0.0	3.5	2.04		52.4°	41.4°		1.0°	2.7°
	0.3	$-2.8 \times 10^2$	2.15		49.6°	36.1°			
	0.5	$-3.5 \times 10^3$	2.27		47.2°	31.8°			
0.10	0.0	$1.4 \times 10^2$	2.46	57.4°	$\sim 48^\circ$	$\sim 33^\circ$			
	0.3	$-1.1 \times 10^2$	2.52	55.6°					
	0.5	$-3.5 \times 10^2$	2.58	53.7°	$\sim 43^\circ$	$\sim 27^\circ$			
0.12	0.0	$4.0 \times 10^2$	2.76	51.5°			0.1°	0.5°	2.1°
	0.3	$2.2 \times 10^2$	2.77	51.5°					
	0.5	$-2.8 \times 10$	2.81	51.5°					

Table II. Properties of the singlet even TMO potential with  $g^2/4\pi=0.08$ , together with phenomenological inside potentials such as

$${}^1I_e^-(x) = \bar{I}'', \text{ for } 0.6 > x \geq 0.3,$$

$${}^1I_e^-(x) = \bar{I}'', \text{ for } 0.3 > x.$$

They give the scattering length  $a = -23.7 \times 10^{-13} \text{cm.}$

$\bar{I}'' (\text{Mev})$	$\bar{I}' (\text{Mev})$	$r_e (10^{-13} \text{cm})$	${}^1\delta_0(40 \text{Mev})$	${}^1\delta_0(90 \text{Mev})$
3.5	3.5	2.04	52.4°	41.1°
$1.0 \times 10$	$-7.6 \times 10$	2.10	51.1°	37.2°
$\infty$	$-2.8 \times 10^2$	2.15	49.6°	36.1°

Table III. Phase shifts yielded by the phenomenological potentials fitted to the low energy parameters, taken from references 21 and 23.

potential shape	${}^1\delta_0(18.3 \text{Mev})$	${}^1\delta_0(20 \text{Mev})$	${}^1\delta_0(32 \text{Mev})$	${}^1\delta_2(18.3 \text{Mev})$	${}^1\delta_2(20 \text{Mev})$	${}^1\delta_2(32 \text{Mev})$
square <sup>21)</sup>		48.5°	41.99°		0.26°	0.770°
Yukawa <sup>21)</sup>			47.54°			1.20°
singular Yukawa <sup>21)</sup>		54.2°	51.15°		0.7°	1.40°
Lévy <sup>23)</sup>	52.8°		44.85°	0.35°		0.97°



2-2. 18.3 Mev  $p$ - $p$  scattering

The singlet  $S$ -wave phase shift  ${}^1\delta_0$  and  $D$ -wave phase shift  ${}^1\delta_2$  are given\* in Table I and II.

${}^1\delta_0$  increases as effective range  $r_e$  or coupling constant  $g^2/4\pi$  decreases as can easily be seen from the shape independent approximation

$$k \cot {}^1\delta_0 = -(1/\alpha) + (1/2)r_e k^2. \quad (1)$$

It is necessary to modify the values of phase shift  ${}^1\delta_0$  in Table I and II if we adopt them to analyze the 18.3 Mev  $p$ - $p$  scattering data. The first modification comes from the difference  $a_{p-p}$  and  $a_{n-p}$ . This is estimated according to the fact that  $a_{n-p} < a_{p-p} \lesssim -16 \times 10^{-13}$  cm.<sup>13)</sup> The second modification comes from the presence of Coulomb force which amount, however, small. Thus, the resultant lower bound for true  ${}^1\delta_0$  of  $p$ - $p$  scattering due to the meson theoretical potential is estimated to be about  $49^\circ$ . The upper bound for true  ${}^1\delta_0$  for  $p$ - $p$  scattering is determined from the differential cross section of  $p$ - $p$  scattering and is about  $54^\circ$ . (See Fig. 2). The same modifications to  ${}^1\delta_2$  are quite negligible.

## 2-3. High energy scattering

The phase shifts  ${}^1\delta_0$  of 40 Mev and 90 Mev are also given in Table I and II, which are apparently smaller than those obtained in II, i. e.,  $64.8^\circ$  and  $47.3^\circ$  respectively. The origin for this differences seems to consist in the two facts that, first, we have adopted the rather small value of  $r_e = 2.26 \times 10^{-13}$  cm in II and second, obtained the core radius corresponding to the coupling constant  $g^2/4\pi = 0.08$  by the linear interpolation from Table 2b in the reference 1.

As can be seen in Fig. 1, the potential of this state derived from the meson theory is characterized by weak attractive force in the region  $x < 1.2$  and by rapidly varying strong attractive force in the region  $0.6 < x < 1.0$ . It suggests the existence of a interaction of very strong repulsive nature in the region  $x < 0.6$ , as it

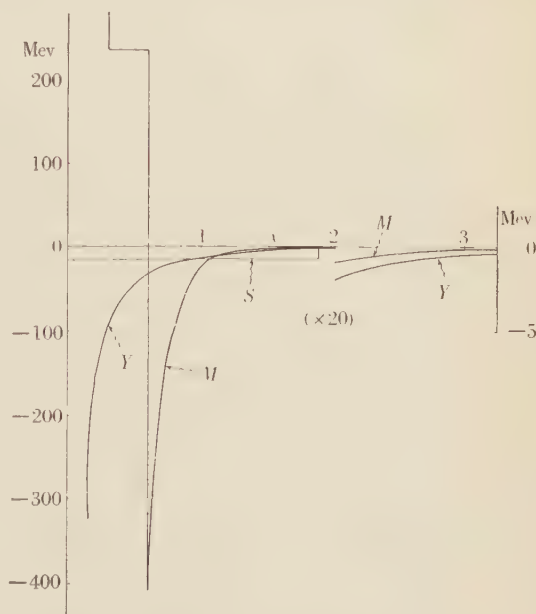


Fig. 1. Singlet even potentials that are equivalent at low energies:

$$a = -23.7 \times 10^{-13} \text{ cm and } r_e = 2.77 \times 10^{-13} \text{ cm.}$$

M: meson theoretical potential of TMO with  $g^2/4\pi = 0.12$ ,

S: square well potential,

Y: Yukawa potential.

\* We denote the singlet and the triplet phase shift by  ${}^1\delta_L$  and  ${}^3\delta_L$  respectively.



must reproduce experimental effective range. If the concept "potential" were allowed to keep its validity above the energy of 100 Mev, such a strong repulsive inside potential might begin to contradict with scattering data, which forces us to introduce energy dependent interaction. (Note that the core radius of Jastrow's potential in the singlet state is 0.43 in our unit.) But this might be only a conjecture because the behavior of the potential in the neighbourhood of  $x=0.6$  is not finally determined, for example, on account of the multiple scattering effect.

As pointed out in II, the magnitude of  ${}^1\partial_2$  is exceedingly small compared with the phase shifts  ${}^1\partial_2$  given by the usual phenomenological potentials, since the meson potential is extremely weak in the region  $x \sim 1.2$ . (See Table III and Fig. 1.) This situation is very desirable for isotropic  $p$ - $p$  scattering data.

### § 3. Phenomenological analysis of the triplet odd state

#### 3—1. Analysis of the experimental data of $p$ - $p$ scattering at 18.3 Mev

The differential cross section of  $p$ - $p$  scattering can be regarded to consist of the five terms :

$$\begin{aligned}\sigma_{pp}(\theta) = & \sigma_{MOTT} \text{ (due to Coulomb force only)} \\ & + {}^1\sigma_N \text{ (due to nuclear force of the singlet even state only)} \\ & + {}^1\sigma_{CN} \text{ (due to interference of Coulomb and singlet nuclear force)} \\ & + {}^3\sigma_N \text{ (due to nuclear force of the triplet odd state only)} \\ & + {}^3\sigma_{CN} \text{ (due to interference of Coulomb and triplet nuclear force).}\end{aligned}\quad (2)$$

Neglecting  $F$ -waves and waves with higher angular momentum,<sup>(14)</sup>

$${}^3\sigma_N = (1/k^2) (3C_0 + 3C_2 \cos^2 \theta), \quad (3)$$

$$\begin{aligned}C_0 = & \frac{1}{3} (\sin {}^3\partial_1^0)^2 + \frac{3}{4} (\sin {}^3\partial_1^1)^2 + \frac{13}{12} (\sin {}^3\partial_1^2)^2 - \frac{2}{3} \sin {}^3\partial_1^0 \sin {}^3\partial_1^2 \cos({}^3\partial_1^0 - {}^3\partial_1^2) \\ & - \frac{3}{2} \sin {}^3\partial_1^1 \sin {}^3\partial_1^2 \cos({}^3\partial_1^1 - {}^3\partial_1^2),\end{aligned}\quad (3')$$

$$\begin{aligned}C_2 = & \frac{3}{4} (\sin {}^3\partial_1^1)^2 + \frac{7}{4} (\sin {}^3\partial_1^2)^2 + 2 \sin {}^3\partial_1^0 \sin {}^3\partial_1^2 \cos({}^3\partial_1^0 - {}^3\partial_1^2) \\ & + \frac{9}{2} \sin {}^3\partial_1^1 \sin {}^3\partial_1^2 \cos({}^3\partial_1^1 - {}^3\partial_1^2).\end{aligned}\quad (3'')$$

$${}^3\sigma_{CN} = (1/k^2) (\eta/2) \cos \theta \{ f(\theta) \mathcal{F} + g(\theta) \mathcal{G} \}, \quad (4)$$

$$\mathcal{F} = \sin 2 {}^3\partial_1^0 + 3 \sin 2 {}^3\partial_1^1 + 5 \sin 2 {}^3\partial_1^2, \quad (4')$$

$$\mathcal{G} = 9 - (\cos 2 {}^3\partial_1^0 + 3 \cos 2 {}^3\partial_1^1 + 5 \cos 2 {}^3\partial_1^2). \quad (4'')$$

where  $k$  is the wave number of the relative motion and  $\eta = e^2/\hbar v$ ,  $v$  being the relative velocity. The functions  $f(\theta)$  and  $g(\theta)$  are independent of the nuclear phase shifts and

their numerical values are easily to be obtained at an arbitrary energy. After both  $\sigma_{MOTT}$  and  ${}^1\sigma_N + {}^1\sigma_{CN}$  calculated assuming  ${}^1\delta_0$  and  ${}^1\delta_2$  in Sec. 2 are subtracted from  $\sigma_{pp}(\theta)$ , the rest terms  ${}^3\sigma_N + {}^3\sigma_{CN}$  are compared with the experimental result at 18.3 Mev.<sup>15)</sup> Three conditions below follow immediately. (See Fig. 2.)

i) Compare the  $\sigma_{pp}(90^\circ)$  :

$$\text{the allowed } 3C_0 = 0.08 \rightarrow 0 \text{ corresponding } {}^1\delta_0 = 49^\circ \rightarrow 54^\circ. \quad \text{Cond. (I)}$$

ii) Compare the  $\sigma_{pp}(30^\circ)$  taking the experimental error into account :

$$0.006 > \mathcal{A} > -0.018, \quad \text{Cond. (II)}$$

$$\begin{aligned} \text{where } \mathcal{A} &\equiv [\{ {}^3\sigma_N(30^\circ) + {}^3\sigma_{CN}(30^\circ) \} - \{ {}^3\sigma_N(90^\circ) + {}^3\sigma_{CN}(90^\circ) \}] \cdot k^2 \\ &= 2.250C_2 - 0.111 \mathcal{B} - 0.0001 \mathcal{C} \\ &\approx 2.250C_2 - 0.111 \mathcal{B}. \end{aligned}$$

iii) Compare the  $\sigma_{pp}(\theta)$  in the region  $40^\circ < \theta < 80^\circ$ , where the  ${}^3\sigma_{CN}$  is negligibly small :

$$0.01 > C_2 > 0. \quad \text{Cond. (III)}$$

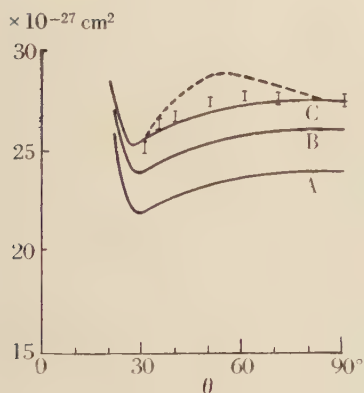


Fig. 2.  $p$ - $p$  scattering at 18.3 Mev

— Differential cross section in the absence of triplet odd potentials ( $\sigma_{MOTT} + {}^1\sigma_N + {}^1\sigma_{CN}$ ) with  ${}^1\delta_2 = 0.1^\circ$  and, A: with  ${}^1\delta_0 = 49.0^\circ$ , B: with  ${}^1\delta_0 = 51.5^\circ$ , C: with  ${}^1\delta_0 = 54.0^\circ$ .  
 ..... Differential cross section when  $C_2 = 0.02$  with adjusted  $C_0$  to reproduce the experimental  $\sigma_{pp}(90^\circ)$  and with  $\mathcal{A} = 0$ .

Experimental data are taken from reference 15.

### 3—2. Phenomenological phase shifts

According to Wigner and Eisenbud,<sup>16)</sup> types of the nuclear two-body non-central interaction are as follows if we limit ourselves to expressions involving no higher powers of the relative momenta  $p$  than the first :

$$S_{12} \equiv 3 \frac{(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{x})(\boldsymbol{\sigma}^{(2)} \cdot \boldsymbol{x})}{x^3} - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}),$$

$$\boldsymbol{L} \cdot \boldsymbol{S} \equiv (\boldsymbol{x} \times \boldsymbol{p}) \cdot (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}),$$

$$(\boldsymbol{x} \times \boldsymbol{p}) \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}), \quad (\boldsymbol{x} \times \boldsymbol{p}) \cdot (\boldsymbol{\sigma}^{(1)} \times \boldsymbol{\sigma}^{(2)}).$$

The latter two of which, however, can be effective only between neutron-proton interaction. Consequently we adopt tensor and  $\boldsymbol{L} \cdot \boldsymbol{S}$  type as non-central interaction.

Following the procedure below, one can generally find all sets of phase shifts  $({}^3\delta_1)^0$ ,

${}^3\delta_1^1, {}^3\delta_1^2$ ) allowed by  $p-p$  scattering experiment assuming the type of the non-central interaction in the triplet state, if one knows the singlet phase shifts by some means. (Note that the force of the singlet state is purely central and is easy to be treated.) Conversely one can find the type and magnitude of the non-central interaction in the triplet odd state if one knows the values of the three phase shifts ( ${}^3\delta_1^0, {}^3\delta_1^1, {}^3\delta_1^2$ ).

Putting

$$\left. \begin{aligned} \sin {}^3\delta_1^0 &= {}^3\delta_1^0 = x, \\ \sin {}^3\delta_1^1 &= {}^3\delta_1^1 = y, \\ \sin {}^3\delta_1^2 &= {}^3\delta_1^2 = z, \end{aligned} \right\} \quad (5)$$

$$x - z = X, \quad y - z = Y, \quad (6)$$

then retaining terms quadratic in  $X$ ,  $Y$  and  $z$ , it follows that

$$C_0 = (1/3)X^2 + (3/4)Y^2; \quad (7)$$

$$C_2 = (3/4)Y^2 + 6YZ + 2XZ + 9z^2, \quad (7')$$

$$\phi = 2X + 6Y + 18z, \quad (8)$$

$$\mathcal{C} = 2X^2 + 6Y^2 + 18z^2 + 4XZ + 12YZ. \quad (8')$$

Plotted on the  $X-Y$  plane (Fig. 3), condition (I)  $C_0 = \text{a constant}$  corresponds to an ellipse with its center at the origin (for example the curve  $a$  and  $b$  of the Fig. 3). As allowed  $3C_0 = 0.08 \sim 0$ , an arbitrary point inside the ellipse  $a$  satisfies the condition (I) independently of  $z$ . The condition (II)  $J = \text{a constant}$  corresponds to a parabola with its axis parallel to the  $X$ -axis if  $z$  is fixed (Fig. 3, curve  $c, d, e$  and  $f$ , for example). Therefore, if  $z$  is fixed, an arbitrary point on the corresponding parabola inside the ellipse  $a$  represents the phase shifts ( ${}^3\delta_1^0, {}^3\delta_1^1, {}^3\delta_1^2$ ) satisfying the condition (I) and (II). The condition (III)  $C_2 = \text{a constant}$  corresponds also to a parabola with its axis parallel to the  $X$ -axis if  $z$  is fixed. In the region of interest, the leftward open parabolas (Fig. 3,  $f$ , for example) satisfying the condition (II) make  $C_2 \sim 0.1$  and must be discarded by the condition (III)  $C_2 < 0.01$ .

Assume only the tensor type as the non-central interaction. Then according to the Born approximation\*

$$\begin{aligned} x &\propto {}^3V_0^e - 4{}^3V_0^t, \\ y &\propto {}^3V_0^e + 2{}^3V_0^t, \\ z &\propto {}^3V_0^e - 0.4{}^3V_0^t. \end{aligned} \quad (9)$$

Therefore

$$x - z : y - z = X : Y = -3.6{}^3V_0^t : 2.4{}^3V_0^t = -3 : 2, \quad (10)$$

\* We denote, for example, the triplet central odd potential as  ${}^3V_0^e$ , and the singlet even potential  ${}^1V_0^e$ .

$$\text{i. e., } Y = -(2/3)X. \quad (11)$$

Eq. (11) means that any set of the phase shifts ( ${}^3\delta_1^0, {}^3\delta_1^1, {}^3\delta_1^2$ ) yielded by central plus tensor force exists near the straight line  $Y = -(2/3)X$  (Fig. 3, straight line  $s$ ). In addition, the distance between the origin and the point representing the phase shifts gives the measure of the magnitude of the tensor force as  $X$  and  $Y$  are proportional to  ${}^3I_0^t$  independently of  ${}^3V_0^c$ . According to  ${}^3V_0^t > 0$  or  $< 0$ , this point lies in the second or fourth quadrant respectively. The magnitude of the central force presents itself through the parameter  $z$  though indirectly and a little complicatedly. In the absence of tensor force the point representing the phase shifts lies on the origin, which means that the pure central force in the triplet odd state gives rise no isotropic differential cross section, namely  $C_0 = 0$ .

From the relation between the exact phase shifts and those of Born approximation, one can see that the point representing the exact phase shifts lies in general in the upper side of  $Y = -(2/3)X$ , i. e., in the shadowed region of Fig. 3.

What is mentioned above holds also true when a hard core is adopted as the inside potential, because  $X$  and  $Y$  are unaffected by the presence of a hard core so far as the first approximation is concerned.

If one assumes only  $L \cdot S$  type as non-central interaction, the point representing the phase shifts lies near and in the upper side of the straight line

$$Y = +(2/3)X,$$

(See Fig. 3, straight line  $t$ ) (12)

instead of  $Y = -(2/3)X$  that corresponds to the tensor force.

Conversely, if  ${}^3\delta_1^0, {}^3\delta_1^1$  and  ${}^3\delta_1^2$  are known by some means, the point representing them on the  $X$ - $Y$  plane shows the relative weight of the types of the non-central interaction—by the

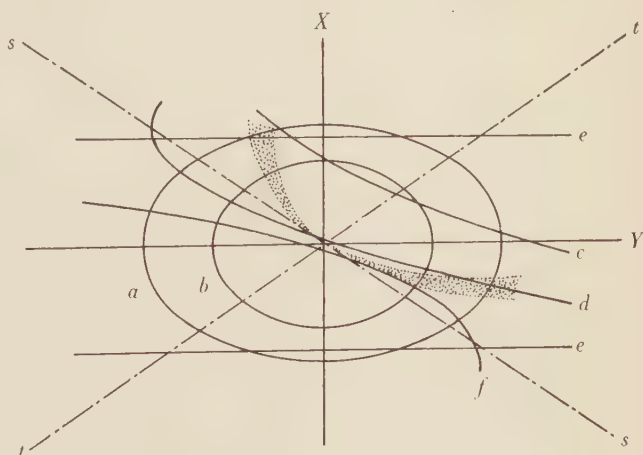


Fig. 3. Phenomenological triplet odd phase shifts represented on the  $X$ - $Y$  plane.

$a$ :	$3C_0 = 0.08$ corresponding to ${}^1\delta_0 = 49.0^\circ$ .	} Cond. (I)
$b$ :	$3C_0 = 0.03$ corresponding to ${}^1\delta_0 = 51.5^\circ$ .	
$c$ :	$\Delta = -0.006$ when $z = -0.05$ .	} Cond. (II)
$d$ :	" $z = 0.0$ .	
$e$ :	" $z = 0.049$ .	
$f$ :	" $z = 0.1$ .	

$s$ : Phase shifts by the Born approximation when central and tensor force are present.

$t$ : Phase shifts by the Born approximation when central and  $L \cdot S$  force are present.

: The region where the exact phase shifts lie when central and tensor force are present.

tangent of the straight line through this point and the origin—and the magnitude of the interaction—by the distance between this point and the origin—.

The pseudoscalar meson theory predicts negligibly small  $\mathbf{L} \cdot \mathbf{S}$  force between two nucleons as far as the outside region concerns so that we assume here only tensor force as the non-central interaction, at least upto 100 Mev\*. It is impossible to determine the potential uniquely even if the phase shifts are known at certain energies. Therefore, adopting simply square well potentials both for central and tensor part with the range  $x=2$  which corresponds to the considerably long range that the meson potentials for this state seem to have, we obtain the depth allowed by the experimental data and recalculate the phase shifts yielded by them. The results are summarized in Table IV.

Table IV. Phenomenological potentials (square well of the range  $x=2$ ) and phase shifts allowed by 18.3 Mev  $p$ - $p$  scattering experiment when only central and tensor force are assumed.  ${}^3V_0^c$  and  ${}^3V_0^t$  are their depth respectively.  $C_0$ ,  $C_2$  and  $A$  determine the angular distribution.

${}^3V_0^c$ (Mev)	${}^3V_0^t$ (Mev)	${}^3\delta_1^0$	${}^3\delta_1^1$	${}^3\delta_1^2$	$C_0$	$C_2$	$A$
2.15	5.35	14.3°	-4.3°	0.0°	0.024	0.004	0.014
0.5	5.0	14.9	-3.4	0.6	"	5	7
-0.65	4.85	15.5	-3.1	1.1	"	8	— 18
1.5	3.8	8.6°	-2.9°	0.0°	0.010	0.002	0.014
-0.1	3.55	9.2	-2.7	0.6	"	3	— 1
-0.25	3.2	9.7	-1.7	1.7	"	9	— 16
1.8	0.0	-0.6°	-0.6°	-0.6°	0.000	0.002	0.022
0.0	0.0	0.0	0.0	0.0	"	0	— 0
-2.6	0.0	1.1	1.1	1.1	"	3	— 32
3.55	-6.55	-7.4°	5.7°	-2.3°	0.017	0.003	0.022
1.9	-6.0	-6.9	5.7	-1.7	16	2	— 6
-0.2	-5.0	-6.3	6.9	-0.6	15	8	— 18
4.9	-8.0	-8.3°	6.6°	-2.9°	0.028	0.003	0.025
1.45	-7.1	-8.0	8.0	-1.7	26	5	— 9
0.05	-7.0	-7.7	8.9	-0.6	24	7	— 30

#### § 4. Potentials of the triplet odd state

##### 4 -1. Properties of the phenomenological potentials

From Table IV we can see that the experimental results of the  $p$ - $p$  scattering at 18.3 Mev require the phenomenological potentials of the triplet odd state such as

$${}^3V_0^c \text{ is very weak,} \quad (13-1)$$

\* Recently, K. Nishijima and M. Shindo obtained  $\mathbf{L} \cdot \mathbf{S}$  force of the fourth order by canonical transformation method. It is very small compared with tensor force in the region  $x > 1$ .



$${}^3V_0^t = +5 \sim -8 \text{ Mev.} \quad (13-2)$$

First, the central part of them is very weak. It is much weaker than the tensor part except the case of extremely large  ${}^1\partial_0$ . Second, the tensor part itself is not so strong as the potentials of the other states. These phenomenological potentials are continuous with  ${}^3V_0^t$  where the larger  $|{}^3V_0^t|$  means that the contribution of the triplet odd interaction to the isotropic angular distribution is the larger, which is required when  ${}^1\partial_0$  is the smaller.

Physically, possibility of strong attractive central force is excluded as it gives rise to large  $C_2$ , when the angular distribution gives a minimum at  $90^\circ$ . This excluded possibility corresponds to the curves such as *e* and *f* in Fig. 3, which have been laid aside in 3-2 already. These curves require instead of eqs. (13-1) and (13-2)

$${}^3V_0^c \sim -8 \text{ Mev,} \quad (14-1)$$

$${}^3V_0^t = +4 \sim -6 \text{ Mev.} \quad (14-1)$$

Another possibility of strong repulsive central force is also ruled out, as first, it gives large  $C_2$ , and second, it interferes with Coulomb force constructively in the region  $\theta \sim 30^\circ$  and definitely contradicts with experiment, since the interference term is determined mainly  $\delta = (1/9) (\sin 2^3\partial_1^0 + 3 \sin 2^3\partial_1^1 + 5 \sin 2^3\partial_1^2)$  which is proportional to  ${}^3V_0^c$  according to the Born approximation. Therefore the central force of the triplet odd state must be weak indispensably.

The properties of the required phenomenological potentials (13-1) and (13-2) are as follows:

i) Some of them are not in disagreement with the information of low energy *p-p* scattering experiment<sup>17)</sup> which tells that the averaged *P*-wave phase shift is negative, for example, at 3.899 Mev,  $(1/9) ({}^3\partial_1^0 + 3{}^3\partial_1^1 + 5{}^3\partial_1^2) = -0.109 \pm 0.20^\circ$ . At such a low energies *P*-wave phase shifts are determined mainly by the features of the potential far from the origin (the wave length  $\lambda \gtrsim 3 \times \hbar/\mu c$ ) so that the square well potential model adopted here is not adequate for quantitative discussion. (See also 4-3 and Note Added in Proof.)

ii) Not only at 18.3 Mev but also at the higher energies upto 100 Mev, there remains the relation  $C_2 \ll C_0$  and it is satisfactory for the isotropic *p-p* angular distribution. Actually we have performed the same prescription at 40 Mev. The result is quite similar to (13-1) and (13-2) or Table IV, that is,  ${}^3V_0^c = +1 \sim -2 \text{ Mev}$  and  ${}^3V_0^t = +4 \sim -5 \text{ Mev}$ . For the quantitative comparison, however, attention must be called to the fact that: In the presence of tensor force, the mixture parameter  $\tan \epsilon_2$  of the two channeled odd state with  $J=2$  ( ${}^3P_2 - {}^3F_2$  state) makes a delicate effect on *p-p* differential cross section. For example, even at 90 Mev, the *F*-wave phase shift is in general indeed much smaller than the *P*-wave phase shift but according to our estimation, the mixture parameter  $\tan \epsilon_2 \sim 0.3$  do have the possibility to change the differential *p-p* cross section due to the triplet odd interactions to the extent of about 30%. Unfortunately, while we scarcely know about the nature of this mixture parameter (What is responsible for determining the sign and magnitude of it? Or which region of  $x \sim 0.6$  or  $-0.6$  is more responsible for it?), the large tensor part of the phenomenological potentials above suggests the importance of this

parameter. Therefore, the higher the energy goes, the more suspicious it is to discuss quantitatively adopting these phenomenological potentials in place of meson theoretical potentials. At low energies  $\tan \epsilon_2$  decreases like  $k^2$  and the  $\gamma$ -wave phase shift (that goes to the  $P$ -wave phase shift in the absence of tensor force) decreases like  $k^{1/2}$  and it is certainly unnecessary to take above fact into account.

iii) From the angular distribution of high energy  $n$ - $p$  scattering which is fairly symmetrical about  $90^\circ$ ,<sup>\*</sup> the next inequality was concluded by Christian and Hart<sup>8</sup> assuming the same shape and range of potentials for all states:

$$0 < (1/4)^1V_0 + (3/4)^3V_0^e < -(1/5)^3V_0^e. \quad (15)$$

The singlet odd phenomenological square well potential with the range  $r=2$  that gives the same phase shift as the meson theoretical  ${}^1I_0'$  of TMO or BW has the depth of about +16 Mev. Combined with this phenomenological  ${}^1I_0'$ , our phenomenological  ${}^3I_0'^e$  gives

$$(1/4)^1V_0 + (3/4)^3V_0^e = 2 \sim 7 \text{ Mev}$$

and is quite satisfactory.

As was stated before, the conditions (II) and (III) depend on the value of  ${}^1\delta_2$ . Qualitatively, for large  ${}^1\delta_2 \rightarrow 0$ ,  $J$  must be smaller because of the interference between  ${}^1S$ - and  ${}^1D$ -waves. Quantitatively, however, even for  ${}^1\delta_0 = 0.5^\circ$ ,  $J \sim -0.035$  and so the characteristic features of the phenomenological potentials (13-1) and (13-2) are not affected.

#### 4-2. Comparison with the other phenomenological potentials

Upto date several  $p$ - $p$  phenomenological potentials have been proposed which consist of central and tensor part. We will discuss here briefly the characteristic features of their triplet odd state and compare them with our phenomenological ones (13-1) and (13-2).

##### i) Jastrow's potentials<sup>9)</sup>

$$\begin{aligned} {}^1I_0'^e &= 0, \\ {}^3I_0'^e &= + (50.8 \text{ Mev}) e^{-x/x_t}, \quad x_t = 0.535. \end{aligned} \quad (16)$$

Clearly they satisfy (13-1) and owing to the exchange character attached to the tensor part,  $(0.3 + 0.7P_x)$ , it is weak compared with the potentials of the other states.

##### ii) Christian and Noyes' potentials<sup>21)</sup>

$$\begin{aligned} {}^3V_0^e &= 0, \\ {}^3I_0'^e &= \pm (23.5 \text{ Mev}) e^{-x/x_t} / (x/x_t)^2, \quad x_t = 1.14. \end{aligned} \quad (17)$$

Also they satisfy (13-1).  ${}^3I_0'^e$  is rather large because their singlet central force is square well and gives the small  ${}^1\delta_0$ . (See Table III.)

##### iii) Lévy's potentials<sup>22)</sup>

\* As far as we know, there is one exceptional experimental result that gives the small forward scattering at 145 Mev,<sup>19)</sup> but recent results<sup>20)</sup> again give the symmetrical curve about  $90^\circ$  in the region  $5^\circ$  to  $180^\circ$ .

$$\begin{aligned}
{}^3V_0^c &= \frac{1}{3} \left( \frac{G^2}{4\pi} \right) \left( \frac{\mu}{2M} \right)^2 \mu c^2 \frac{e^{-x}}{x} \\
&\quad - 3 \left( \frac{G^2}{4\pi} \right) \left( \frac{\mu}{2M} \right)^2 \mu c^2 \frac{1}{x^2} \left\{ \frac{2}{\pi} K_1(2x) + \frac{\mu}{2M} \left( \frac{2}{\pi} K_1(x) \right)^2 \right\}, \\
{}^3V_0^t &= \frac{1}{3} \left( \frac{G^2}{4\pi} \right) \left( \frac{\mu}{2M} \right)^2 \mu c^2 \left( 1 + \frac{3}{x} + \frac{3}{x^2} \right) \frac{e^{-x}}{x}.
\end{aligned} \tag{18}$$

These potentials have been shown to reproduce 18.3 Mev  $p$ - $p$  scattering data well.<sup>23</sup> According to Martin and Verlet,<sup>23</sup> we take the value  $(g^2/4\pi)=10.36$  corresponding to  $g^2/4\pi=0.058$ . The numerical estimate shows that  ${}^3V_0^t=|{}^3V_0^c|$  at  $x\sim 0.8$ ,  $|{}^3V_0^c|\ll {}^3V_0^t$  outside. Furthermore  ${}^3V_0^c=0$  at  $x\sim 2.0$  owing to the cancellation of the two terms in the central force. So eq. (13-1) is satisfied as far as the outside region is concerned.  ${}^3V_0^t$  is the usual symmetrical pseudoscalar meson potential of the second order and so eq. (13-2) is also satisfied as will be discussed in 4-3. However, since  ${}^3V_0^c$  is attractive in the region  $x\lesssim 2$  and rather strong near the origin, Lévy's potentials might give rise too large  $C_2$  at higher energies.

As a summary, the salient feature (13-1) is common to all phenomenological potentials and (13-2) depends on the values of  ${}^1\hat{O}_0$  more or less. It is to be emphasized that the weak central force of the triplet odd state is necessary and sufficient for  $p$ - $p$  angular distribution, while it is only sufficient for the  $n$ - $p$  angular distribution, as eq. (15) can be satisfied if the central forces of the singlet and triplet odd states have opposite sign and cancel out each other, however strong they may be.

#### 4-3. Comparison with the meson theoretical potentials

The TMO potentials of the triplet odd state with  $g^2/4\pi=0.08$  and the BW potentials with the equivalent coupling constant  $G^2/4\pi(\mu/2M)^2=0.08$  with reduced meson pair terms are plotted in Fig. 4. They are very similar with the phenomenological potentials (13-1) and (13-2) with the positive tensor part,  ${}^3V_0^t\sim 0$  in the region  $x>1$ . Therefore, we may surely say that no corrections altering the characteristic features of both tensor and central part of the triplet odd state potentials are necessary from the phenomena and that it is possible for the meson theoretical potentials to reproduce the  $p$ - $p$  scattering data fairly well if the value of  $g^2/4\pi$  is adjusted and suitable inside phenomenological potentials are adopted.

On the other hand, from the field theoretical point of view, no corrections to this state are to be put to the potentials in the region  $x>1$  whose dominant part is of the second order. The potentials of TMO and BW are markedly different from each other only in the region  $x<1$ .

The tensor part of the meson theoretical potentials in the region  $x>1$  is repulsive and mainly proportional to  $g^2/4\pi$ . So that for the larger  $g^2/4\pi$  we have the smaller  ${}^1\hat{O}_0$  in the singlet even state and the stronger tensor force of the triplet odd state. This tendency is consistent with the  $p$ - $p$  scattering phenomena because the decrease of the singlet cross section due to the smaller  ${}^1\hat{O}_0$  must be covered with the tensor part of the triplet odd

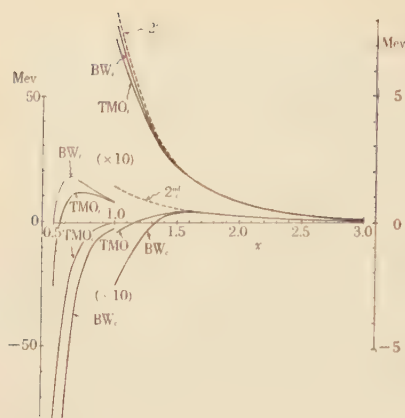


Fig. 4. Symmetrical pseudoscalar meson theoretical potentials of the triplet odd state of

TMO with  $g^2/4\pi = 0.08$ .

BW with equivalent coupling constant  $G^2/4\pi(\mu/2M)^2 = 0.08$ .

the second order alone with

$$g^2/4\pi = (G^2/4\pi)(\mu/2M)^2 = 0.08.$$

The suffices *c* and *t* mean central and tensor part respectively.

potentials as pointed out in 4-1.

The fact that the central part of the meson theoretical potential is slightly repulsive in the region  $r > 1$  is favourable to the low energy data suggesting the repulsive potential effective to the averaged *P*-wave phase shift. Note that the wave length  $\lambda > 3\hbar/(\mu c)$ . (See 4-1, i). (See also Note Added in Proof.)

## § 5. Summary of the properties of the meson theoretical potentials

### 5-1 The triplet even state

The potential of TMO and those of BW are markedly different from each other only in the triplet even state, that is, the central part of TMO is strongly repulsive and in consequence the phenomenological inside potential of attractive nature is necessary for the deuteron to be bound<sup>5)</sup>, while the central part of BW is attractive. The potentials of FST is between those of TMO and BW but close to the latter<sup>1)</sup>. Matsumoto and Watari<sup>7)</sup> have adopted the "A-shaped" potential similar to those of BW in the region  $r < 0.6$  and obtained many interesting results. According to them, if suitable energy independent inside potentials are assumed, not only low energy *n-p* scattering data including deuteron parameters are reproduced, but also the total *n-p* triplet cross section at 90 Mev is smaller than that of II by about 27%. Consequently, the *n-p* total cross section decreases by about 23%. This is very satisfactory because both the meson theoretical<sup>5)</sup> and the phenomenological potentials<sup>8, 9)</sup> proposed so far have given the larger *n-p* total cross sections than experimental ones by about 20~30%. Their success is attributed partly to the particular choice of the inside phenomenological potentials, that is, for  $0 \leq r < 0.4$  infinitely repulsive hard core and for  $0.4 < r < 0.6$  the square well of the depth  ${}^3V_c^r = -500$  Mev,  ${}^3V_c^t = +300$  Mev. As we have pointed out previously, meson theoretical results for nucleon-nucleon scattering have some phenomenological nature through *S*-wave phase shift even at the energies lower than 100 Mev, which, in the presence of tensor force, is itself complicated threefold through  $\alpha$ -wave phase shift,  $\gamma$ -wave phase shift and mixture parameter  $\tan \epsilon_1$ . So, the problems left in future are first to reexamine the triplet even state potentials derived from pseudoscalar meson theory with FST treatment and then to connect the above



inside potentials with the field theoretical one in the region  $0.4 \leq x < 0.6$ .

### 5—2. *The triplet odd state*

The only type of the potentials that the  $p$ - $p$  scattering experiment upto 100 Mev requires is such that the central part is very weak and the tensor part is not so strong as the potentials of the other states. The meson theoretical potentials are very similar to these phenomenological ones and it is surely possible to adjust the coupling constant  $g^2/4\pi$  and assume suitable phenomenological inside potentials to fit the experimental data very well from 0 to 100 Mev. It is to be noted that as energy goes higher, the details of the potential shape will perhaps present themselves through the mixture parameter  $\tan \epsilon_2$ . Quantitatively, it is necessary to reexamine the  $p$ - $p$  scattering upto 100 Mev assuming the FST potentials of various values of  $g^2/4\pi$  in the region  $x \sim 0.6$  and several types of phenomenological potentials in the region  $x \leq 0.6$ .

### 5—3. *The singlet states*

There is no difference between the potentials of TMO and those of BW in these states.

The strong repulsive force in the singlet odd state satisfies, combined with the triplet odd central potential, the Christian-Hart inequality (15) which is necessary to reproduce the  $n$ - $p$  angular distribution.

The exceedingly small  ${}^1\partial_2$  is favourable for  $p$ - $p$  angular distribution. The phenomenological inside interaction of strongly repulsive nature in the singlet even state which is required from the low energy experimental data might contradict with the experiments at the energies higher than 100 Mev, if one assumed their energy independence.

### 5—4. *Conclusions*

The nuclear force derived from the symmetrical pseudoscalar meson theory both with pseudovector coupling and with pseudoscalar coupling (if the meson pair term which arises in the non-relativistic approximation is reduced) is consistent with the main features of nucleon-nucleon scattering upto 100 Mev including both angular distributions and total cross sections.

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## Note added in proof :

Recently, one of the author (S.O.) and collaborator have shown that the meson theoretical potentials actually reproduce negative averaged  $f$ -wave phase shift at low energies and that the slightly repulsive central potential of the second order in the region  $x > 1.5$  is important for it. The coupling constant that fits the experimental data is  $g^2/4\pi = 0.08 \sim 0.06$  and agrees with that predicted from the singlet even state data in this paper, i.e.  $0.08 \sim 0.12$ . We think that this fact gives strong support to the meson theory of nuclear forces. S. Otsuki and R. Tamagaki, Prog. Theor. Phys. **12** (1954), to be published.

# Generalization of Feynman's Theory of Energy Levels of Liquid Helium

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Feynman's method<sup>1)</sup> of finding the lower energy levels of liquid helium is generalized to include energy levels corresponding to a number of excitons. The expectation values of momentum in these levels are calculated. Our extension is valid so long as the number of excitons is small compared with that of atoms, and no pair of excitons with wave numbers equal in magnitude but opposite in sign co-exists. The excitons derived according to our scheme are found to behave as bosons. The interaction potential between helium and the surrounding wall can be regarded as a creation and an annihilation operator of the excitons. The usual hypothesis<sup>2)</sup> for the critical velocity is discussed in this connection.

## 1. Introduction

Feynman<sup>1)</sup> found by his ingenious method approximate energy levels and eigenfunctions near the ground state of liquid helium. Thanks to him we can now have a clue to the atomic theoretical understanding of liquid helium near 0° K. He gave the energy levels corresponding to phonons and rotons which Landau introduced.<sup>3)</sup>

But according to his result the energy levels are restricted to those corresponding to a single exciton, and it was not clear to what statistics the exciton should obey. So, we shall generalize his method and find states of a number of excitons.

He estimated the character of the ground state eigenfunction  $\varphi$  by the physical argument that the eigenfunctions of slightly excited states are expressed as

$$\psi = \sum_{i=1}^N f(\mathbf{R}_i) \varphi, \quad (1.1)$$

since this form is likely to give the state which is the nearest to the ground state in energy and orthogonal to it. Indeed, so long as the orthogonal relations are maintained, it seems that the forms like this are more appropriate to lower levels than eigenfunctions corresponding to a locally excited state which might be approximated by a cage model, because a considerable amount of kinetic energy might be associated with the latter case.

We shall try to generalize by putting

$$\psi = \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_l=1}^N f_{k_1}(\mathbf{R}_{i_1}) f_{k_2}(\mathbf{R}_{i_2}) \cdots f_{k_l}(\mathbf{R}_{i_l}) \varphi \quad (l \leq N), \quad (1.2)$$

where  $f_k(\mathbf{R}_i)$  is a continuous bounded function, and  $\mathbf{R}_i$  is the coordinate of the  $i$ -th atom. Feynman used the variation method to get the explicit form of  $f(\mathbf{R}_i)$ . As the

system under consideration is liquid, there are salient characteristics which made Feynman's calculation simple:

(1) When  $\varphi^2$  is integrated with the coordinates of all atoms except one, the result is a constant.

(2) When  $\varphi^2$  is integrated with the coordinates of all atoms except two, the result is a function of only the distance between the two atoms.

It will be natural to assume, moreover, that

(3) When  $\varphi^2$  is integrated with all the coordinates of atoms except some atoms, the result (which we shall denote by  $W(\mathbf{R}_{i_1}, \dots, \mathbf{R}_{i_l})$ ) becomes almost constant when all of their mutual distances are sufficiently large.

This fact permits us to adopt the Ursell's expansion method:<sup>4)</sup>

$$W(\mathbf{R}_{i_1}, \dots, \mathbf{R}_{i_l}) = \sum_{(p_k)} S U_{p_1} U_{p_2} \dots U_{p_s} \quad (l \leq N), \quad (1.3)$$

where  $U_{p_k}$  is a function of coordinates of  $p_k$  particles and vanishes when any pair of these  $p_k$  particles is separated sufficiently from each other.  $S$  means the sum taken by substituting  $\mathbf{R}_{i_1}, \dots, \mathbf{R}_{i_l}$  as the arguments of  $U_{p_k}$ 's in all possible ways; and  $\sum_{(p_k)}$  is taken with all sets of partition of  $l$ , such that

$$l = p_1 + p_2 + \dots + p_s.$$

Then  $U_1$  is equal to  $1/V$  ( $V$  is the volume of the container), and  $U_2(\mathbf{R}_1, \mathbf{R}_2)$  is a function of  $|\mathbf{R}_1 - \mathbf{R}_2|$  alone.

Especially,

$$W(\mathbf{R}_1, \mathbf{R}_2) = 1/V^2 + U_2(\mathbf{R}_1, \mathbf{R}_2). \quad (1.3)'$$

We see that

$$U_s = O(1/V^s) \quad (1.4)$$

$$\int f_{k_1}(\mathbf{R}_{i_1}) f_{k_2}(\mathbf{R}_{i_2}) \dots f_{k_l}(\mathbf{R}_{i_l}) U_l(\mathbf{R}_{i_1}, \dots, \mathbf{R}_{i_l}) d\mathbf{R}_{i_1} \dots d\mathbf{R}_{i_l} = O(1/V^{l-1}). \quad (1.5)$$

Therefore, if we calculate something like

$$\int f_{k_1}(\mathbf{R}_{i_1}) f_{k_2}(\mathbf{R}_{i_2}) \dots f_{k_l}(\mathbf{R}_{i_l}) W(\mathbf{R}_{i_1}, \mathbf{R}_{i_2}, \dots, \mathbf{R}_{i_l}) d\mathbf{R}_{i_1} \dots d\mathbf{R}_{i_l}$$

substituting (1.3) for  $W(\mathbf{R}_{i_1}, \dots, \mathbf{R}_{i_l})$ , the dominant term is that for the largest value of  $s$  (the number of factors  $U_{p_k}$  of a term in (1.3)) which gives non-zero value to the integral; the other terms being neglected when the domain of integration tends to infinity. Note that  $l \leq N$ .

## § 2. Energy eigenfunctions and eigenvalues

We set a trial function for the excited eigenfunction in the form

$$\begin{aligned} \underbrace{\psi_{k_1, \dots, k_1}}_{l(1)} \underbrace{\psi_{k_2, \dots, k_2}}_{l(2)} \dots \underbrace{\psi_{k_s, \dots, k_s}}_{l(s)} = \sum_{i_{11}=1}^N \sum_{i_{12}=1}^N \dots \sum_{i_{1l(1)}=1}^N \dots \sum_{i_{sl(s)}=1}^N \dots \sum_{i_{sl(s)}=1}^N f_{k_1}(\mathbf{R}_{i_{11}}) f_{k_1}(\mathbf{R}_{i_{12}}) \times \dots \\ \dots \times f_{k_1}(\mathbf{R}_{i_{1l(1)}}) \dots f_{k_p}(\mathbf{R}_{i_{pl}}) \dots f_{k_s}(\mathbf{R}_{i_{sl(s)}}) \varphi, \end{aligned} \quad (2.1)$$

The following relations are obtained :

(1) from the orthogonality between  $\phi_k$  and  $\varphi$ ,

$$\int f_k(\mathbf{R}) U_1(\mathbf{R}) d\mathbf{R} = 0. \quad (2.2)$$

(2) from the orthogonality between  $\phi_{k,k'}$  and  $\varphi$ ,

$$\int f_k(\mathbf{R}) f_{k'}(\mathbf{R}) U_1(\mathbf{R}) d\mathbf{R} + (N-1) \int f_k(\mathbf{R}) f_{k'}(\mathbf{R}') U_2(\mathbf{R}, \mathbf{R}') d\mathbf{R} d\mathbf{R}' = 0. \quad (2.3)$$

(3) from the orthogonality between  $\phi_k$  and  $\phi_{k'}$

$$\int f_k^*(\mathbf{R}) f_{k'}(\mathbf{R}) U_1(\mathbf{R}) d\mathbf{R} + (N-1) \int f_k^*(\mathbf{R}) f_{k'}(\mathbf{R}') U_2(\mathbf{R}, \mathbf{R}') d\mathbf{R} d\mathbf{R}' = S(\mathbf{K}) \delta_{k,k'}. \quad (2.4)$$

Then, noting that

$$\sum_{\mu=1}^s l(\mu) \equiv n \ll N,$$

we obtain

$$\int |\phi_{k_1 \dots k_s}|^2 d\tau = \prod_{\mu=1}^s N^{\mu} l(\mu)! S(\mathbf{K}_{\mu})^{l(\mu)}, \quad (2.5)$$

where  $d\tau$  denotes the volume element in the configuration space of all the particles. In the above calculation, the terms including any  $U_l$  ( $l \geq 3$ ) as a factor have been completely neglected, since the number of those terms is  $O(N^{2n})$ , which is the same order of magnitude as that of terms including  $U_2$ 's only. On the other hand, we have

$$\int f_{k_1}(\mathbf{R}_{i_1}) \dots f_{k_l}(\mathbf{R}_{i_l}) U_l(\mathbf{R}_{i_1}, \dots, \mathbf{R}_{i_l}) d\mathbf{R}_{i_1} \dots d\mathbf{R}_{i_l} = O(1/V^{l-1}) \quad (2.6)$$

and

$$\int f_k^*(\mathbf{R}) f_k(\mathbf{R}') U_2(\mathbf{R}, \mathbf{R}') d\mathbf{R} d\mathbf{R}' \geq O(1/V). \quad (2.6)'$$

The hamiltonian of the  $N$  atoms when energy is measured from that of ground state  $E_0$  is

$$\mathbf{H} = -\hbar^2/2m \sum_{j=1}^N \Delta_j^2 + V(\mathbf{R}_1, \dots, \mathbf{R}_N) - E_0 \quad (2.7)$$

Thus,

$$\begin{aligned} \mathbf{H} \phi_{k_1, \dots, k_s} = & -\hbar^2/2m \sum_{j=1}^s \sum_{\{i_{jp}\}} [\varphi \nabla_j^2 \{f_{k_1}(\mathbf{R}_{i_{11}}) \dots f_{k_s}(\mathbf{R}_{i_{s,l(s)}})\} \\ & + 2\nabla_j \{f_{k_1}(\mathbf{R}_{i_{11}}) \dots f_{k_s}(\mathbf{R}_{i_{s,l(s)}})\} \nabla_j \varphi]. \end{aligned} \quad (2.8)$$

Using integration by parts we easily obtain

$$\begin{aligned} & \int \phi_{k_1, \dots, k_s}^* \mathbf{H} \phi_{k_1, \dots, k_s} d\tau \\ = & \hbar^2/2m \sum_{\{i_{jp'}\}} \sum_{\{i_{jp}\}} \sum_j \int \{f_{k_1}^*(\mathbf{R}_{i_{11}'}) \dots f_{k_s}^*(\mathbf{R}_{i_{s,l(s)'})\} \nabla_j \{f_{k_1}(\mathbf{R}_{i_{11}}) \dots f_{k_s}(\mathbf{R}_{i_{s,l(s)}})\} \varphi^2 d\tau \\ = & \frac{\hbar^2}{2m} \sum_{\mu=1}^s l(\mu)^{2j-1} \frac{\prod_{j=1}^s S(\mathbf{K}_j)^{l(j)} l(j)!}{S(\mathbf{K}_{\mu}) l(\mu)} \int f_{k_{\mu}}^*(\mathbf{R}) \nabla f_{k_{\mu}}(\mathbf{R}) U_1 d\mathbf{R} \end{aligned} \quad (2.9)$$

neglecting the higher order terms in  $1/V$ . Note that the terms with  $K_{\mu} \neq K_{\nu}$  become

higher order.

The expectation value of energy is

$$E_{\underbrace{k_1 \dots k_1}_{l(1)}, \underbrace{k_2 \dots k_2}_{l(2)}, \dots, \underbrace{k_s \dots k_s}_{l(s)}} = \frac{\int \psi_{k_1 \dots k_s}^* H \psi_{k_1 \dots k_s} d\tau}{\int |\psi_{k_1 \dots k_s}|^2 d\tau} \\ = -\frac{\hbar^2}{2m} \sum_{\mu=1}^s \frac{l(\mu) \int \nabla f_{k_\mu}^* (\mathbf{R}) \nabla f_{k_\mu} (\mathbf{R}) d\mathbf{R} / V}{S(\mathbf{K}_\mu)}. \quad (2 \cdot 10)$$

As Feynman pointed out, this becomes stationary if

$$f_{k_\mu} = e^{i\mathbf{k}_\mu \cdot \mathbf{R}} \quad (2 \cdot 11)$$

and

$$E_{k_1, \dots, k_s} = \hbar^2 / 2m \cdot \sum_{\mu=1}^s l(\mu) \mathbf{K}_\mu^2 / S(\mathbf{K}_\mu), \quad (2 \cdot 12)$$

$$(\mathbf{K}_\mu)_x = 2\pi / L_x \cdot k_{\mu x}, \quad (\mathbf{K}_\mu)_y = 2\pi / L_y \cdot k_{\mu y}, \quad (\mathbf{K}_\mu)_z = 2\pi / L_z \cdot k_{\mu z}, \quad (2 \cdot 12)'$$

where  $L_x$ ,  $L_y$ ,  $L_z$  are the edge-lengths of the container and  $k_{\mu x}$ ,  $k_{\mu y}$  and  $k_{\mu z}$  are integers.

It is easily verified that (2.11) satisfies (2.2), (2.3), and (2.4), so long as there is no pair of  $\mathbf{k}_p$ ,  $\mathbf{k}_q$  such that  $\mathbf{k}_p = -\mathbf{k}_q$  among  $\mathbf{k}_1, \dots, \mathbf{k}_s$ . This means that the level for which there is a pair of excitons with  $\mathbf{k}_p$  and  $\mathbf{k}_q$  such that  $\mathbf{k}_p = -\mathbf{k}_q$  is not permissible in our scheme. Practically this restriction is not so serious, as the value for  $\mathbf{K}$  is so densely distributed. Further significances of this fact will be discussed in the subsequent paper.

We can normalize the eigenfunction  $\psi_{k_1 \dots k_s}$  with the normalization factor

$$1 / \sqrt{\prod_{j=1}^s N^{l(j)} e(j)! S(\mathbf{K}_j)^{l(j)}}.$$

The normalized one will be referred to as  $\phi_{\underbrace{k_1 \dots k_1}_{l(1)}, \dots, \underbrace{k_s \dots k_s}_{l(s)}}$ , where

$$\phi_{\underbrace{k_1 \dots k_1}_{l(1)}, \dots, \underbrace{k_s \dots k_s}_{l(s)}} = \frac{1}{\sqrt{\prod_{j=1}^s N^{l(j)} l(j)! S(\mathbf{K}_j)^{l(j)} \{l(j)\}}} \sum \exp i[\mathbf{k}_1 \cdot \mathbf{R}_{i(1)} + \dots + \mathbf{K}_s \cdot \mathbf{R}_{i(s) l(s)}] \varphi. \quad (2 \cdot 13)$$

It is also verified that the orthogonality between the different normalized eigenfunctions holds in the limit  $V \rightarrow \infty$  with  $N/V$  fixed. For, if  $(\mathbf{k}_1, \dots, \mathbf{k}_m) \neq (\mathbf{k}'_1, \dots, \mathbf{k}'_l)$ ,

$$\int \psi_{k'_1 \dots k'_l}^* \psi_{k_1 \dots k_m} d\tau = O(N^{l+m} / V^{-(l+m)/2}) = O(N^{l+m/2}) \quad (2 \cdot 14)$$

whereas

$$\sqrt{\int |\psi_{k_1 \dots k_m}|^2 d\tau} \sim \sqrt{\int |\psi_{k'_1 \dots k'_l}|^2 d\tau} \geq O(N^{l+m/2})$$

Now from (2.3) and (2.11),

$$S(\mathbf{K}_\mu) = 1 + (N-1) \int e^{i\mathbf{K}_\mu \cdot (\mathbf{R}' - \mathbf{R})} U_2(\mathbf{R}, \mathbf{R}') d\mathbf{R} d\mathbf{R}' \\ = 1 + (N-1) \int e^{i\mathbf{K}_\mu \cdot (\mathbf{R}' - \mathbf{R})} \{W(\mathbf{R}, \mathbf{R}') - 1/V^2\} d\mathbf{R} d\mathbf{R}'$$



$$= 1 + (N-1) \int e^{i\mathbf{K}_\mu \cdot (\mathbf{R}' - \mathbf{R})} W(\mathbf{R}, \mathbf{R}') d\mathbf{R} d\mathbf{R}'. \quad (2.15)$$

The correlation function (the probability per unit volume of finding an atom at  $\mathbf{R}'$  if one is known to be at  $\mathbf{R}$ ) is defined by

$$p(\mathbf{R} - \mathbf{R}') = \delta(\mathbf{R} - \mathbf{R}') + (N-1) VW(\mathbf{R}, \mathbf{R}'). \quad (2.16)$$

Then,

$$S(\mathbf{K}_\mu) = \int e^{i\mathbf{K}_\mu \cdot \mathbf{R}} p(\mathbf{R}) d\mathbf{R}. \quad (2.17)$$

This is nothing but  $S(\mathbf{K}_\mu)$  in Feynman's paper.<sup>1)</sup>

With Feynman we see that

$$E_K = \hbar^2 / 2m \cdot \mathbf{K}^2 / S(\mathbf{K}) \simeq c\hbar |\mathbf{K}| \quad (2.18)$$

for smaller  $|\mathbf{K}|$ ,

$$E_K = \hbar^2 / 2m \cdot \mathbf{K}^2 / S(\mathbf{K}) \simeq J + (\mathbf{K} - \mathbf{K}_0)^2 / 2\mu$$

for moderately large  $|\mathbf{K}|$ ; where  $c$  is the velocity of sound in liquid helium, and  $J(>0)$ ,  $\mathbf{K}_0$  and  $\mu$  are some constants.

### § 3. Expectation values of momentum

We can also calculate the expectation values of momentum for these levels.

$$\begin{aligned} & -i\hbar \sum_{j=1}^N \int \psi_{\mathbf{k}_1 \dots \mathbf{k}_s}^* \frac{\partial}{\partial \mathbf{R}_j} \psi_{\mathbf{k}_1 \dots \mathbf{k}_s} d\tau \\ &= \sum_{\mu} l(\mu) \hbar \mathbf{K}_\mu \prod_{j=1}^s N^{l(j)} l(j)! S(\mathbf{K}_j)^{l(j)} \\ &= 1/2 \cdot i\hbar \sum_{\{i_{j\mu'}\}} \sum_{\{i_{j\mu}\}} \sum_j \sum_{(\mu, \nu)} \{ \exp(-i\mathbf{K}_1 \cdot \mathbf{R}_{i_{11}'} \dots \{ \partial / \partial \mathbf{R}_j \exp(-i\mathbf{K}_\mu \cdot \mathbf{R}_{\mu\nu}) \} \\ & \quad \times \exp(-i\mathbf{K}_s \cdot \mathbf{R}_{i_{s, l(s)}}) \exp i \{ \mathbf{K}_1 \cdot \mathbf{R}_{i_{11}} + \dots + \mathbf{K}_s \cdot \mathbf{R}_{i_{s, l(s)}} \} \varphi^2 d\tau \\ & + 1/2 \cdot i\hbar \sum_{\{i_{j\mu'}\}} \sum_{\{i_{j\mu}\}} \sum_j \sum_{(\mu, \nu)} \{ \exp \{ -i(\mathbf{K}_1 \cdot \mathbf{R}_{i_{11}'} + \dots + \mathbf{K}_s \cdot \mathbf{R}_{i_{s, l(s)}}) \} \\ & \quad \times \exp(i\mathbf{K}_1 \cdot \mathbf{R}_{i_{11}}) \dots \{ \partial / \partial \mathbf{R}_j \exp(i\mathbf{K}_\mu \cdot \mathbf{R}_{\mu\nu}) \} \dots \exp(i\mathbf{K}_s \cdot \mathbf{R}_{i_{s, l(s)}}) \varphi^2 d\tau. \quad (3.1) \end{aligned}$$

The last two terms just cancel each other, and the expectation values of momentum becomes

$$\mathbf{P}_{\underbrace{\mathbf{k}_1 \dots \mathbf{k}_1}_{l(1)}, \dots, \underbrace{\mathbf{k}_s \dots \mathbf{k}_s}_{l(s)}} = \sum_{\mu=1}^s l(\mu) \hbar \mathbf{K}_\mu. \quad (3.2)$$

We can regard the state specified by (1.2) as one in which there are  $l$  excitons specified by wave number vectors  $\mathbf{K}_1, \mathbf{K}_2, \dots, \mathbf{K}_l$ , respectively. Thus, we can see that (1.2) or (2.1) corresponds to the so called phonon or roton levels in their form of energy levels and expectation values of momentum so long as the number of these excitons is small enough compared to the total number of atoms. It goes without saying that these excitons behave as bosons.

#### § 4. Creation and annihilation of excitons due to the wall

We consider the system of liquid helium and the wall surrounding it, the total hamiltonian of which being

$$H = H_{He}(\{R\}) + H_{wall}(\{Q\}) + H_{int}(\{R\}, \{Q\}), \quad (4.1)$$

where  $\{R\}$  and  $\{Q\}$  denote the coordinates of all the particles of helium and the wall respectively. We suppose  $H_{int}$  as a perturbing energy which may be written in the form

$$H_{int} = \sum_K \sum_{i=1}^N \alpha_K(\{Q\}) \exp i\mathbf{K} \cdot \mathbf{R}_i. \quad (4.2)$$

We see that

$$\int \psi^*_{\mathbf{k}_1, \dots, \mathbf{k}_l} H_{int} \psi_{\mathbf{k}'_1, \dots, \mathbf{k}'_m} d\tau \approx 0$$

when and only when  $l=m$  and we can arrange such that

$$\mathbf{K}_1 = \mathbf{K}'_1, \dots, \mathbf{K}_{l-1} = \mathbf{K}'_{l-1}, \quad \mathbf{K}_l = \mathbf{K}'_l \pm 1.$$

Indeed,  $\sum_{i=1}^N \exp i\mathbf{K} \cdot \mathbf{R}_i$  serves as a creation operator of an exciton with wave number  $\mathbf{K}$  when it is operated from left to the eigenfunction which has no exciton with wave number  $\mathbf{K}$ . On the other hand, if it is operated from left to the eigenfunction which has excitons with  $-\mathbf{K}$ , it serves as an annihilation operator of an exciton with  $-\mathbf{K}$ . For, if none of  $\mathbf{K}'_1, \dots, \mathbf{K}'_l$  is equal to  $\mathbf{K}$ , we have

$$\begin{aligned} & \int \psi^*_{\mathbf{k}'_1, \dots, \mathbf{k}'_l} \sum_{i=1}^N \exp(i\mathbf{K} \cdot \mathbf{R}_i) \psi_{\mathbf{k}_1, \dots, \mathbf{k}_m} d\tau \\ &= \int \psi^*_{\mathbf{k}'_1, \dots, \mathbf{k}'_{l-1}, -\mathbf{K}} \psi_{\mathbf{k}_1, \dots, \mathbf{k}_m, -\mathbf{K}} d\tau. \end{aligned} \quad (4.3)$$

This is different from zero only when

$$\{\mathbf{K}'_1, \dots, \mathbf{K}'_l\} = \{\mathbf{K}_1, \dots, \mathbf{K}_m\},$$

because of the orthogonality. If there were some  $\mathbf{K}_{p'} = \mathbf{K}$  among  $\mathbf{K}'_1, \dots, \mathbf{K}'_l$ ,

$$\int \psi^*_{\mathbf{k}'_1, \dots, \mathbf{k}'_l} \sum_{i=1}^N \exp(i\mathbf{K} \cdot \mathbf{R}_i) \psi_{\mathbf{k}_1, \dots, \mathbf{k}_m, -\mathbf{K}} d\tau = \int \psi^*_{\mathbf{k}'_1, \dots, \mathbf{k}'_{l-1}, \mathbf{k}_{p'+1}, \dots, \mathbf{k}_{l'-\mathbf{K}}} \psi_{\mathbf{k}_1, \dots, \mathbf{k}_m, -\mathbf{K}, -\mathbf{K}} d\tau = 0. \quad (4.4)$$

for there should be no  $-\mathbf{K}$  among  $\mathbf{K}'_1, \dots, \mathbf{K}'_l$ .

#### § 5. Discussion on the critical velocity

If we assume the periodic boundary condition on the eigenfunction of the liquid helium under question, and there is no external field, the hamiltonian is separable into the following two parts: the one for the motion of the center of mass and the other for the motion relative to the center of mass. That is

$$H = H_1 + H_2,$$

$$H_1 = P^2/M, \quad H_2 = 1/2m \left\{ \sum_{i=1}^{N-1} p_i^2 + (p_1 + \dots + p_{N-1})^2 \right\} + V(q_1, \dots, q_{N-1}). \quad (5.1)$$

$$\begin{aligned}
 M &= Nm, & \mathbf{Q} &= \sum m \mathbf{R}_i / M, & \mathbf{P} &= M \dot{\mathbf{Q}}, \\
 \mathbf{q}_i &= \mathbf{R}_i - \mathbf{Q}, & \mathbf{p}_i &= m \dot{\mathbf{q}}_i.
 \end{aligned}
 \quad (5.2)$$

Since  $\mathbf{H}_1$  and  $\mathbf{H}_2$  commute, we can write the energy eigenfunction as a product of the eigenfunctions of  $\mathbf{H}_1$  and  $\mathbf{H}_2$ . The eigenfunction of  $\mathbf{H}_1$  is

$$\gamma_k(\mathbf{Q}) \propto e^{i\mathbf{k} \cdot \mathbf{Q}} \quad (5.3)$$

the lowest of which being for  $\mathbf{K} = 0$ . As  $\varphi$  is the lowest eigenfunction of the system,

$$\begin{aligned}
 \varphi(\mathbf{R}_1, \dots, \mathbf{R}_N) &= \varphi_1(\mathbf{q}_1, \dots, \mathbf{q}_{N-1}) \gamma_0(\mathbf{Q}) \\
 &\propto \varphi_1(\mathbf{q}_1, \dots, \mathbf{q}_{N-1}),
 \end{aligned}
 \quad (5.4)$$

where  $\varphi_1$  is the lowest eigenfunction of  $\mathbf{H}_2$ .

We have

$$\begin{aligned}
 \psi_{\mathbf{K}_1 \dots \mathbf{K}_s} &= \sum_{i_1=1}^N \dots \sum_{i_s=1}^N \exp i(\mathbf{K}_1 \cdot \mathbf{R}_{i_1} + \dots + \mathbf{K}_s \cdot \mathbf{R}_{i_s}) \varphi \\
 &= \sum_{i_1=1}^N \dots \sum_{i_s=1}^N \exp i(\mathbf{K}_1 \cdot \mathbf{q}_{i_1} + \dots + \mathbf{K}_s \cdot \mathbf{q}_{i_s}) \varphi_1 \exp i(\mathbf{K}_1 + \dots + \mathbf{K}_s) \cdot \mathbf{Q},
 \end{aligned}
 \quad (5.5)$$

where  $\exp i(\mathbf{K}_1 + \dots + \mathbf{K}_s) \cdot \mathbf{Q}$  is an eigenfunction of  $\mathbf{H}_1$ . Hence,  $\sum_{i_1=1}^N \dots \sum_{i_s=1}^N \exp i(\mathbf{K}_1 \cdot \mathbf{q}_{i_1} + \dots + \mathbf{K}_s \cdot \mathbf{q}_{i_s}) \varphi_1$  may be regarded as the eigenfunction of  $\mathbf{H}_2$ . We see that choosing  $\mathbf{K}$  for  $\gamma_k(\mathbf{Q})$  in  $\sum_{i_1=1}^N \dots \sum_{i_s=1}^N \exp i(\mathbf{K}_1 \cdot \mathbf{q}_{i_1} + \dots + \mathbf{K}_s \cdot \mathbf{q}_{i_s}) \varphi_1 \gamma_k(\mathbf{Q})$  different from  $\mathbf{K}_1 + \dots + \mathbf{K}_s$ , we can construct another system of energy eigenfunctions, which corresponds to the translational motion of the center of mass.

Usually, it is said that the critical velocity results from the energy and momentum conservation laws when these excitons are created. This would not be true if it were not for some reasons to exclude the transition from  $\varphi_1$  to  $\varphi_1 \gamma_k(\mathbf{Q})$ , the process of transferring momentum from the wall to helium without creating excitons. For, in that case the mere conservation laws could not give the non-zero critical velocity. As we have seen in § 4 such transition will practically be inhibited because of the form of  $\mathbf{H}_{int}$ .

Then, assuming that the internal energy of the wall does not decrease by the transition, we have

$$\begin{aligned}
 \hbar \mathbf{K} &= \Delta \mathbf{P} \\
 c \hbar \mathbf{K} &\geq a \Delta \mathbf{p} \quad (= \text{loss of translational energy of the wall}) \\
 \therefore v &\leq c,
 \end{aligned}
 \quad (5.6)$$

in the case of phonon level, where  $\Delta \mathbf{p}$  is the loss of momentum of the wall,  $v$  is the velocity of the wall relative to the center of mass of liquid helium, and  $c$  is the sound velocity in helium.

If we take roton level,

$$J + \hbar^2 (\mathbf{K} - \mathbf{K}_0)^2 / 2\mu \leq v\hbar |\mathbf{K}|$$

$$\therefore v \leq \sqrt{2J/\mu} - \hbar |\mathbf{K}_0| / \mu. \quad (5.7)$$

In any case it is well known that the critical velocity thus obtained is too large compared with the experimental values.<sup>5)</sup> Therefore, it is expected that either it is essentially wrong to regard  $H_{int}$  as a small perturbation, or there are energy levels with another type of energy-momentum relations.

Reviewing the method of derivation of phonon and roton levels given above, one must suppose that the approximation becomes worse and worse as the number of excitons becomes larger, while the number of Landau's rotons are generally assumed to attain the order of the number of atoms  $N$  near the lambda point. In spite of the resemblance of the form of energy levels, there is not a little doubt in identifying rotons derived here with Landau's.

It is possible that there are wave functions which are orthogonal to anyone we have obtained so far still giving lower expectation values of energy than some of those we have obtained. There seems to be a close connection between these wave functions and those we have omitted as impermissible in our scheme. We hope we shall discuss on them in the subsequent paper.

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**Note added in proof.** After the completion of this paper the full text of Feynman's theory of liquid helium [Phys. Rev. **94** (1954), 262] arrived, in which he discussed the multiple excitation very briefly.

## Letters to the Editor

### A Remark on Tamm-Dancoff Approximation

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September 1, 1954

Tamm-Dancoff's method gives a convenient formulation introducing non-adiabatic potentials, and is used to solve the problems of nucleon-meson scattering and nuclear forces. However, the characteristic feature of this method as an approximation method has not been noticed. In this note, we want to call readers' attention to this point. We take up two-nucleons-system, interacting with meson field through the ordinary tri-linear interaction term, as an example, and use the notation of the Levy's paper<sup>1)</sup> for economy. Of course, the technique used here is available to the more general interactions and other problems.

We expand the solution  $\psi(E)$  of the Schrödinger's equation for the stationary problem

$$E\psi(E) = (H_0 + H')\psi(E),$$

in terms of the solutions of the free equation  $\phi_{\lambda}^{(m, n)}$  ( $m$  and  $n$  are the numbers of virtual mesons and nucleon pairs, respectively, and  $\lambda$  specifies the state of the system), i.e.,

$$\psi(E) = \sum_{m, n} \sum_{\lambda} \alpha_{\lambda}^{(m, n)} \phi_{\lambda}^{(m, n)}.$$

The equations for  $\alpha_{\lambda}^{(m, n)}$ 's are

$$(E - E_{\lambda}^{(m, n)}) \alpha_{\lambda}^{(m, n)}$$

$$= \sum_{q=n-1}^{n+1} \sum_{p=m \pm 1} \sum_{\mu} (\lambda, m, n | H' | \mu, p, q) \alpha_{\mu}^{(p, q)}.$$

At first, let us consider the simplest case considering no nucleon pairs. Using  $K_{\mu\nu}^{(m)}$  defined by

$$(E - E_{\mu}^{(m, 0)}) \alpha_{\mu}^{(m, 0)} = \sum_{\nu} K_{\mu\nu}^{(m)} \alpha_{\nu}^{(m-1, 0)}$$

we can get as the equation determining the potential

$$(E - E_{\mu}^{(0, 0)}) \alpha_{\mu}^{(0, 0)} = \sum (\mu, 0, 0 | H' | \nu, 1, 0) \alpha_{\nu}^{(1, 0)} \\ = \sum \frac{(\mu, 0, 0 | H' | \nu, 1, 0) K_{\nu\sigma}^{(1)}}{E - E_{\nu}^{(1, 0)}} \alpha_{\sigma}^{(0, 0)}. \quad (1)$$

Integral equation for  $K_{\mu\nu}^{(m)}$  is

$$K_{\mu\nu}^{(m)} = (\mu, m, 0 | H' | \nu, m-1, 0)$$

$$+ \sum \frac{(\mu, m, 0 | H' | \lambda, m+1, 0) K_{\lambda\sigma}^{(m+1)}}{(E - E_{\lambda}^{(m+1, 0)})(E - E_{\sigma}^{(m, 0)})} K_{\sigma\nu}^{(m)}. \quad (2)$$

In the approximation in which the maximum of mesons is  $N$ , we must solve eq. (2) with the initial condition

$$K_{\lambda\sigma}^{(N+1)} = 0. \quad (3)$$

Now, to solve eq. (2), let us break up the interaction operator  $H'$  into

$$H' = H_1 + H_1^*$$

where  $H_1$  and  $H_1^*$  are such operators that the non-vanishing matrix elements of them are only  $(\mu, m, 0 | H_1 | \nu, m+1, 0)$  and  $(\mu, m, 0 | H_1^* | \nu, m-1, 0)$ , respectively, and introduce operator  $K^{(m)}$  satisfying

$$K_{\mu\nu}^{(m)} = (\mu, m, 0 | K^{(m)} | \nu, m-1, 0).$$

Making use of those convention, eq. (2) can be considered as the matrix element between the states  $(\mu, m, 0)$  and  $(\nu, m-1, 0)$  of the operator equation

$$K^{(m)} = H_1^* + H_1 \frac{1}{E - H_0} K^{(m+1)} \frac{1}{E - H_0} K^{(m)} \quad (4)$$

and this equation defines all elements  $(\mu, n, 0 | K^{(m)} | \nu, n-1, 0)$  for not only  $n=m$ , but also  $n \neq m$ . Eq. (4) can be solved at once, getting

$$K^{(m)} = \frac{1}{1 - H_1(1/E - H_0) K^{(m+1)}(1/E - H_0)} H_1^*. \quad (5)$$

According to eq. (1), we can write the potential as

$$(\mu, 0, 0 | H_1 \frac{1}{E - H_0} K^{(1)} | \nu, 0, 0).$$

Let us define potential operator  $V_N$ , specifying the maximum number of mesons considered, by

$$V_N = H_1 \frac{1}{E - H_0} K_N^{(1)}$$

which is diagonal in the occupation number of mesons. Now, eqs. (3) and (5) give the operator relations, for all  $m (\leq N)$ ,

$$K_{N+1}^{(m+1)} = K_N^{(m)}$$

Making use of this relation, we can get the following relation between  $V_{N+1}$  and  $V_N$ .

$$V_{N+1} = H_1 \frac{1}{E - H_0} K_{N+1}^{(1)}$$



$$\begin{aligned}
&= H_1 \frac{1}{E - H_0} \frac{1}{1 - H_1(1/E - H_0)K_{N+1}^{(2)}(1/E - H_0)} H_1^{\dagger} \\
&= H_1 \frac{1}{E - H_0} \frac{1}{1 - H_1(1/E - H_0)K_N^{(1)}(1/E - H_0)} H_1^* \\
&= H_1 \frac{1}{E - H_0} \frac{1}{1 - V_N(1/E - H_0)} H_1^* \\
&= H_1 \frac{1}{E - H_0 - V_N} H_1^*.
\end{aligned}$$

Further, we can write this relation, as follows,

$$V_{N+1} = H_1 \omega_N \frac{1}{E - H_0} H_1^* \quad (7a)$$

where  $\omega_N$  is the solution of the integral equation

$$\omega_N = 1 + \frac{1}{E - H_0} V_N \omega_N \quad (7b)$$

That is, if we can solve the integral equation (7b) using known  $V_N$ , we can get the next approximation  $V_{N+1}$  simply by integration procedure. In this connection, we notice the fact that the  $(N+1)$ -th meson is emitted into the distorted state  $\omega_N$ , and that  $V_2$  contains not only the fourth order graphs, but also higher order graphs to infinite order), as can be seen if we expand the denominator of eq. (6). Tamm-Dancoff's approximation differs from the perturbation method and Brueckner-Watson's<sup>2)</sup> approximation, and gives a kind of bound-state approximation<sup>3)</sup>.

In the case containing nucleon pairs, the resulting expressions and their interpretation are very complicated, although we can apply the above technique to this case, too. This is due to the fact that we cannot increase a nucleon pair, keeping the number of mesons, by single operator.

The simplest case is that the maximum numbers of nucleon pairs and mesons are arbitrary and one, respectively, neglecting the matrix elements of  $H'$  creating and annihilating one nucleon pair and one meson at the same time. In this case the connected states constitute a chain, and the final result is

$$V_{(1, N+1)} = H_1 \frac{1}{E - H_0 - H_2^* \omega_{(1, N)} (1/E - H_0) H_2} H_1^{\dagger}$$

where  $N$  is the maximum number of nucleon pairs, and  $\omega_{(1, N)}$  satisfies

$$\omega_{(1, N)} = 1 + \frac{1}{E - H_0} V_{(1, N)} \omega_{(1, N)},$$

and  $H_2, H_2^*$  are the operator having only non-zero matrix elements  $\langle \mu, m, n | H_2 | \nu, m+1, n-1 \rangle$  and  $\langle \mu, m, n | H_2^* | \nu, m-1, n+1 \rangle$ , respectively. This

equation reflects the situation that in order to increase one nucleon pair, keeping the number of mesons, we must increase one meson by  $H_1^*$  at first, and then annihilate one meson and create one nucleon pair at the same time by  $H_2$ , and the nucleon pair is emitted in the state  $\omega_{(1, N)}$ .

The more general cases are further complicated, and we do not give them here. In our opinion, it is convenient to eliminate the terms changing the nucleon pairs in the original Hamiltonian.

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## On the Multiple Scattering Phase Shifts and the Nuclear Force (Scalar Pair Theory)

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September 3, 1954

The relation existing between the self energy of a fixed nucleon source and the scattering phase shifts was discussed by Sawada<sup>1)</sup> for several examples. He showed that the nucleon self energy is connected with phase shifts of scattered meson wave by the following relation

$$iE = -(2\pi)^{-1} \int_0^\infty dk (k/\epsilon_k) \sum_l \delta_l(k), \quad \epsilon_k = \sqrt{\mu^2 + k^2}, \quad (1)$$

where  $\delta_l$  is the phase shift of the  $l$ -th partial wave and  $\mu$  is meson's rest mass. A formal discussion was given by Okubo<sup>2)</sup> using  $S$  matrix formalism. Schwinger<sup>3)</sup> also discussed the similar problem for a electron in the external electromagnetic field.

Partly for the reason to ascertain the above relation and partly as its application, we consider the case where there are two fixed nucleon sources, in the scalar pair theory with scalar coupling, and show that there exists also the same relation as (1) bet-

ween the multiple scattering phase shifts and the self energy of two nucleon system including the nuclear force which has been evaluated rigorously by Wentzel<sup>4)</sup>; meson-nucleon scattering problem was also exactly solved by Blatt<sup>5)</sup>.

We assume the sources with delta function form for simplicity and renormalize the divergence into charge. The total hamiltonian and the canonical commutation relation are given by

$$H=1/2 \cdot \int [(\nabla\varphi)^2 + \mu^2\varphi^2 + \pi^2] d\mathbf{x} + 1/2 \cdot g \sum_{i=1,2} (\varphi(\mathbf{x}_i))^2, \quad (2)$$

$$i[\pi(\mathbf{x}), \varphi(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}'),$$

where  $\mathbf{x}_i$  is the position of the nucleon. The scattering problem is solved exactly and the  $S$  matrix is given by<sup>6)</sup>

$$\langle \mathbf{k} | S | \mathbf{k}' \rangle = k^{-2} \delta(k - k') \{ \delta(\mathbf{n} - \mathbf{n}') + (\mathbf{n} | T(k) | \mathbf{n}') \},$$

$$(\mathbf{n} | T(k) | \mathbf{n}') = (-igk/4\pi^2) \{ f(k) \cos [r(u - u')/2]$$

$$- (g/4\pi) (e^{ikr}/r) \cos [r(u + u')/2] \} / \Delta,$$

$$f(k) = 1 + i(g/4\pi)k,$$

$$\Delta = (f(k))^2 - [(g/4\pi)(e^{ikr}/r)],$$

$$u = \cos \theta, \quad u' = \cos \theta',$$

$$r = |\mathbf{x}_1 - \mathbf{x}_2|, \quad \mathbf{n} = \mathbf{k}/k, \quad k = |\mathbf{k}'|, \quad (3)$$

where we have taken  $(\mathbf{x}_1 + \mathbf{x}_2)/2$  as origin,  $z$ -axis parallel to  $\mathbf{x}_1 - \mathbf{x}_2$ .

In the representation which makes  $S$  diagonal,

$$\langle m | S(k) | m' \rangle = \delta_{mm'} \exp [2i\delta_m(k)]. \quad (4)$$

As suggested by Sawada and Okubo, the energy shift of our total system from the interaction-free state will be given by

$$\Delta E = - (2\pi)^{-1} \int_0^\infty dk (k/\epsilon_k) \sum_m \delta_m(k). \quad (5)$$

$\Delta E$  should be equal to the potential energy between the two nucleons plus self energies of the single nucleons. As the value of spur is independent of the representation, we have

$$2i \sum_m \delta_m(k) = S \rho [\log S(k)] = \int d\mathbf{r} (\mathbf{n} | \log S(k) | \mathbf{n}). \quad (6)$$

This expression is evaluated as follows

$$S [\log S \rho(k)] = \int_0^g dg d/dg S \rho [\log S(k)], \quad (7)$$

$$d/dg S \rho [\log S(k)] = S \rho [(d/dg S(k)) \tilde{S}(k)]. \quad (8)$$

By substituting the explicit expression (3) for  $S$  and after somewhat lengthy calculations, we get

$$S \rho [(d/dg S(k)) \tilde{S}(k)] = -d/dg \log (\Delta/\Delta^*) \quad (9)$$

Finally we get

$$\Delta E = (4\pi i)^{-1} \int_0^\infty dk (k/\epsilon_k) \log (\Delta/\Delta^*). \quad (10)$$

This is divided into two parts

$$\Delta E = U + 2\Delta E_{se} \quad (11)$$

where

$$U(r=\infty) = 0, \quad U(r) = (4\pi i)^{-1} \times$$

$$\int_0^\infty dk \frac{k}{\epsilon_k} \log \frac{1 - [(g/4\pi)(e^{ikr}/f(k)r)]^2}{1 - [(g/4\pi)(e^{-ikr}/f^*(k)r)]^2},$$

$$\Delta E_{se} = (4\pi i)^{-1} \int_0^\infty dk (k/\epsilon_k) \log [f(k)/f^*(k)]. \quad (13)$$

$U$  just agrees with the nuclear force deduced by Wentzel, except for his omission of  $i(g/4\pi)k$  term in the integrand, inclusion of which is necessary for the unitarity of  $S$  matrix. Besides it can be shown easily that

$$-1/2i [f(k)/f^*(k)] \quad (14)$$

is equal to the scattering phase shift by a single nucleon, therefore (13) is identical to the nucleon self-energy which Sawada deduced.

In present stage these relations are of mere academic interest, and our intension was to make first step to their practical application. It may give a kind of approximation method when applied for problems which concern with energy of the total system, but its extension for more arbitrary type of theory will have to be done carefully. Full account including the scalar pair theory with derivative coupling<sup>7)</sup> will be published soon later elsewhere.

We wish to express our sincere thanks to Professors K. Nakabayasi and I. Sato and also to Dr. K. Sawada for their kind interest and encouragement.

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# On the Analysis of the Anomalous Magnetic Moment of the Nucleon in the Second Order Calculations

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September 6, 1954

As is well-known, the second order perturbation calculation of the anomalous magnetic moment of the nucleon in the symmetrical p.s. meson theory with p.s. (or p.v.) coupling shows rather serious discrepancy with the experimental facts. According to Case<sup>1)</sup> the discrepancy is due to the fact that the virtual meson cloud effective to the a.m.m. spreads only over a considerably small range compared with the Compton wave length of  $\pi$ -meson, thus giving too small meson contribution to the a.m.m., and that feature originates in the meson-nucleon interaction through the odd Dirac operator  $\gamma_5$ .

In order to manifest these speculations more concretely we have analyzed the second order calculation, classifying the covariant Feynman's diagrams into the conventional perturbation theoretic ones both for nucleon and meson contributions to the a.m.m., as shown in Fig. 1. Each contribution from the diagrams in Fig. 1 was conveniently represented as an integral over virtual meson momentum  $k=|k'|$ , the integrand as a function of  $k$  then showed what values of  $k$  are essential in the process. This analysis shows in turn clearly the spatial distribution of virtual meson cloud effective to the a.m.m. The main conclusions thus obtained are:

1) The nucleon contributions from the processes (a) and (b) are almost of the same magnitude; those from (b) and (c) are nearly equal to each other. The former contributions are about one half of the latter ones. The same holds for the meson contributions, when (a), (b), (c) and (d) are replaced by (a'), (b'), (c') and (d') respectively.

2) There are marked differences according as whether or not a nucleon pair is involved in the intermediate state. The integrand for the process

(a) increases rapidly as  $k$ , reaches to a sharp maximum at  $0.4K$  ( $K=Mc/\hbar$ ) and then decreases rapidly taking the half value at  $0.85K$ . On the other hand, the integrand for (b) has a flat maximum at  $0.8K$  and descends much more slowly than for (a), reaching to the half value at  $1.5K$ . Thus, virtual mesons with momenta larger than  $K$  play essential roles in the process (b), but not in (a). The behavior for the processes (c) and (d) resembles that for (b), but is suppressed in low energy region. The same situation holds for the corresponding meson contributions except for (b') which makes a similar behavior as (a').

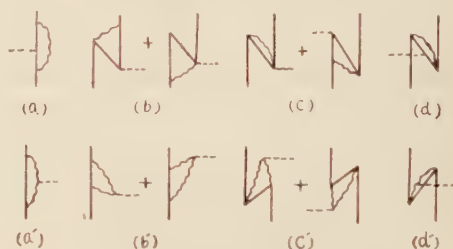


Fig. 1. (1) Nucleon current's contributions. (2) Meson current's contributions.

We notice that the pair formation processes are indispensable to the a.m.m. and that there the considerably high momenta of meson are essential.

It will be interesting to compare these features in p.s. meson theory with those of scalar meson theory with scalar coupling which were obtained applying the analogous technique for this case. In scalar meson theory most of the contribution to the a.m.m. comes from the processes which do not involve nucleon-pair in the intermediate states. For instance, about 80% of the whole nucleon contribution comes from the process (a) only. Also, for the meson contribution the processes (c') and (d') can be safely neglected compared with (a') and (b'). As for the momentum distribution of virtual meson cloud effective to the a.m.m. the significant difference between the two theories is that while all the meson contributions remain finite in the limit of  $\kappa=\mu c/\hbar \rightarrow 0$  in p.s. meson theory, the contributions from (a') and (b') in scalar meson theory diverge in the same limit. This already indicates that the spread of the scalar meson cloud is comparable with  $\kappa^{-1}$ .

We have known that the results of the non-covariant Tamm-Dancoff calculation of meson-nucleon scattering and the a.m.m. of nucleon<sup>2)</sup> in p.s. meson theory depend crucially on the cut-off momenta. The reason for this may be attributed to the same situa-

tion as clarified from the present analysis, i.e. the importance of higher meson momenta in p.s. theory.

Finally we add the two comments; firstly, if we take into account the higher order corrections to the a.m.m. sufficiently in any calculation, the large effects of higher momenta of virtual meson might be depressed considerably. But according to the results of the 4-th order corrections<sup>3)</sup> of Feldman's calculation by use of a modified nucleon propagator<sup>4)</sup>, such possibility is rather unlikely. Secondly, if the effects of higher momentum transfer are important, it might be insufficient to discuss the meson phenomena without taking into account of heavy mesons as emphasized by N. Fukuda<sup>5)</sup>.

The associated arguments will be published soon in the Science Reports of Tohoku University. Independently of us the same analysis as ours were applied to the a.m.m. of electron as well as nucleon, also to  $\pi^0-2\gamma$  process by S. Goto.

In conclusion the authors express their cordial thanks to Prof. K. Nakabayasi, Drs. K. Sawada, N. Fukuda and S. Tani for suggesting this work and to Dr. I. Sato for valuable discussions.

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## Elastic Scattering of Alpha-particles by Heavy Nuclei

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September 8, 1954

Recently the precise experiment has been reported on the elastic scattering of alpha-particles by gold, for intermediate energy range, by G. Farwell and H. Wegner<sup>1)</sup>. As shown in Fig. 1 and Fig. 2 the differential cross section decreases exponentially

above the minimum incident energy which alpha-particles ride over the Coulomb barrier, and the closest approach distance from experiment ( $12.5 \times 10^{-13}$  cm) is larger than the nuclear radius of gold plus alpha-particle radius ( $10.6 \times 10^{-13}$  cm) calculated by  $1.42 A^{1/3} \times 10^{-13}$ . We tried to explain this queer experimental facts by means of optical model in this letter. Now optical model has gained many successes on high energy and low energy neutron scatterings<sup>2)3)</sup>, but Chase-Rohlich<sup>4)</sup> have concluded that this model could not apply for intermediate energy range. However, it seems too haste for conclusion, since they have fixed the relation between imaginary part of potential and the neutron mean free path in nucleus, and what is worse, the calculation of mean free path can be put no great reliance. So we also studied for intermediate energy range if complicated reactions, such as alpha-gold, could be explained by means of the idea of optical model. The applying of partial wave analysis is difficult for large nuclear radius of target nuclei and 20~45 Mev incident energy, so that we pick out Montroll-Greenberg's variational method<sup>5)</sup> which gives better approximation.

From the standpoint of optical model we assume that the alpha-nuclear interaction can be written by the following two body square well potential,

$$\left. \begin{aligned} V(r) &= V_0 + iV_1; \quad r < a \\ &= -2Ze^2/r; \quad r > a \end{aligned} \right\} \quad (1)$$

Here,  $a$  is nuclear radius and  $V_0$ ,  $W_0$  and  $2Ze^2/r$  are respectively real part, imaginary part of potential and Coulomb potential. Applying Montroll Greenberg's variational method we get straightforwardly following differential cross section using potential (1).

$$\sigma(\theta) = |A_s|^2 \left| \int_0^\infty [k^2(r) - k_0^2] r^2 \left\{ \frac{\sin \omega r}{\omega r} - \alpha \frac{\sin \nu r}{\nu r} \times \right. \right. \\ \left. \left. \times \exp(2iRk_1) \right\} dr \right|^2, \dots \quad (2)$$

$$\left. \begin{aligned} A_s &= \frac{2k_1}{k_1 + k_0} \cdot \frac{\exp i[R(k_1 - k_0)]}{[1 - \alpha^2 \exp(4ia k_1)]}, \\ \omega^2 &= k_1^2 + k_0^2 - k_1 k_0 \cos \theta, \\ \nu^2 &= k_1^2 + k_0^2 + 2k_1 k_0 \cos \theta, \\ k^2(r) &= 2m/\hbar^2 \cdot (E_\alpha - 2Ze^2/r); \quad r > R, \\ &= 2m/\hbar^2 \cdot (E_\alpha + V_0 + iV_1); \quad r < R, \\ k_0^2 &= 2m/\hbar^2 \cdot E_\alpha, \quad \alpha = (k_1 - k_0)/(k_1 + k_0), \end{aligned} \right\} \quad (3)$$

$R$  and  $k_1$  are respectively a radius and a wave number which are decided by variational principle, and  $E_\alpha$  is the incident energy of alpha-particles. Fixing  $a$



to  $1.42 \cdot A^{1/3} \times 10^{-13}$  cm, we calculated  $\sigma(\theta)$  for four  $V_0$  values; 20 Mev, 15 Mev, 10 Mev and 5 Mev. The results agree very closely with experiment as shown in Fig. 1 and Fig. 2. In either case if  $W_0$  increases linearly with incident energy of alpha-particles,  $E_\alpha$ , the queer experimental fact that  $\sigma(\theta)$  decreases exponentially with  $E_\alpha$  can be explained. Still more, applying Montrolli-Greenberg's method, the radius  $R$  which has real effect in calculation must be taken larger value than  $a$ . It seems to explain the fact that the closest approach from the experiment is larger than  $a$ . These good agreements with experiment can be interpreted that it proves that optical model can be applied these complicated reactions and is useful at intermediate energy range if imaginary part of

potential,  $W_0$ , is freely changed as a parameter. It seems to be perplexed that  $W_0$  must have different  $E_\alpha$  dependence at  $\theta=60^\circ$  and  $95^\circ$ . But  $\theta$  dependent  $W_0$  is rather natural because  $W_0$  includes the contribution from very complicated processes in alpha-nuclear interaction.  $|W_0|$  is as large as 4~9 times of  $|V_0|$ , and it confirms easy decomposition of alpha-particle in nucleus. We can't assert strongly for the sake of few examples, but from this calculation we can conclude that the idea of optical model is useful for the case of complicated reactions. We desire more precise experiments on alpha-heavy nuclei scattering are set about. The details will be published soon.

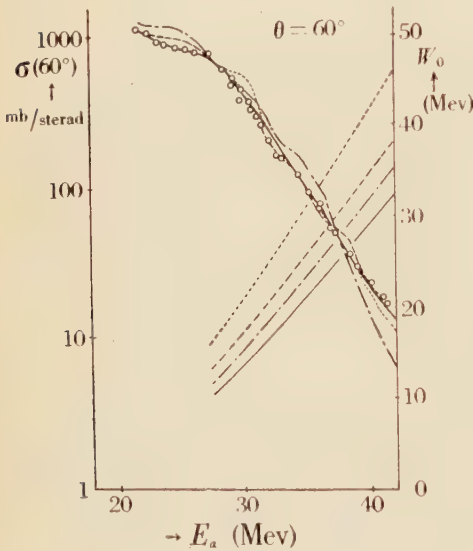


Fig. 1.

The differential cross section of alpha-Au elastic scattering at  $\theta=60^\circ$ , vs the incident energy of alpha particles. The right side is the curve of imaginary part of potential.

- ; Theoretical curve of  $V_0=20$  Mev.
- - -;  $V_0=15$  Mev.
- ....;  $V_0=10$  Mev.
- .....;  $V_0=5$  Mev.

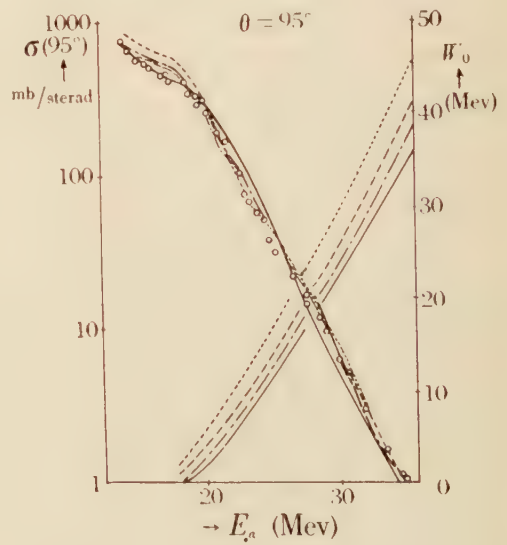


Fig. 2.

The differential cross section of alpha-Au elastic scattering at  $\theta=95^\circ$ .

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## On Some Notes on the so-called G. Feldman's Modified Propagator

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September 8, 1954

When I was going to state my opinion about the "rule" derived by G. Feldman<sup>1)</sup>, I received the paper discussed by R. Utiyama and T. Imamura<sup>2)</sup>, which I write as U-I for brevity, on that rule and so I will state something other than U-I's view.

First, I think that contributions from the new type of poles of modified propagator comes when its series expansion does not satisfy the condition of summation and integration. As the order of summation and integration of infinite series cannot always be changed even if it converges uniformly when the range of integration of the series is infinite, the answer to U-I's apprehension, "if the new type of poles of modified propagators appears in course of calculation and at the same time if the perturbation expansion of modified propagators converges uniformly with respect to momentum variables we could scarcely understand from what origin these new poles result" is as follows; if the new type of poles of modified propagators appears in course of calculation and at the same time if the perturbation expansion of modified propagators converges uniformly with respect to momentum variables, these new poles result from that the order of summation and integration of the perturbation expansion of modified propagators cannot

be changed in spite of its uniform convergence.

Now, as stated above, the rule of Feldman on the modified propagator seems to me to be wrong at least in the way of reasoning; first, on the assumption of uniform convergence, he integrated the series term by term which, in fact, does not converge uniformly, and standardized it; second, calculating with the modified propagator summed up formally, he unconditionally omitted the contributions from new type of poles so as to coincide with the above standard. But as a non-uniformly converged series cannot be integrated term by term, we must not standardize the calculation with use of such a series integrated term by term, so that his idea, "the contributions from new type of poles of modified propagator shall be omitted unconditionally", seems to be wrong.

An especially important thing is as follows; contributions from the new type of poles of modified propagators, as Feldman himself said, are not always infinite but may be finite. Such finite contributions may perhaps not be omitted...most likely I can say so from the "distribution analysis" but it will be discussed later...and so if we calculate with only first several terms of series which does not converge uniformly we would ignore such contributions as resulting from the new type of poles of modified propagators.

Such an apprehension may also occur, for instance in the derivation of Salpeter-Bethe's equation<sup>3)</sup>.

- 1) G. Feldman, Proc. Roy. Soc. **223** (1954), 112.
- 2) R. Utiyama and T. Imamura, Prog. Theor. Phys. **11** (1954), 606.
- 3) E. E. Salpeter and H. A. Bethe, Phys. Rev. **84** (1951), 1232.

## Corrigenda

### On the Excited States of Even-even Nuclei

M. Nagasaki and T. Tamura

Prog. Theor. Phys. **12** (1954), 248

p. 249, Left column, 11 line from the bottom,	for 0-2-0	read 0-2-2
" , " , 7	" , for 0-2-4	read 0-2-2
" , " , 5	" , for 0-2-2	read 0-2-4
" , Right column, 4	" , for 5/2	read 7/2



Progress of Theoretical Physics, Vol. 12, No. 5, November 1954

# On the Effects of Excited States of Nucleons upon Static Nuclear Potential in Symmetrical Pseudoscalar Meson Theory, II

—Numerical Results on Deuteron Problem—

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Numerical results of the deuteron problem which were carried out by Illiac computer at the University of Illinois are presented for our potential derived in our former paper (part I) and the one derived by Taketani et al. These potentials, and also the one recently derived by Brueckner and Watson have the tensor force of right sign of almost the same magnitude, but they are quite different from each other with respect to their central parts: Taketani's is strongly repulsive, ours weakly attractive and Brueckner and Watson's strongly attractive. The radius of hard core was chosen as 0.3, 0.383, and 0.466 in units of meson Compton wave length. Firstly the numerical factor multiplying the potential is adjusted so as to give the correct binding energy. Then other quantities are calculated. The numerical results show that the relative weight of central force to tensor one in our potential is too small because ours gives too large quadrupole moment and  $D$ -state probability of approximately three or four times as large as the empirical ones. In Taketani's case the deuteron cannot bind, because the repulsive central force is too large compared with the tensor one. Thus we can conclude that the attractive central force at least of the same order of magnitude as the tensor one is definitely necessary in order to obtain the agreement with empirical data in deuteron problem.

## § 1. Introduction

In the former part of this paper<sup>1)</sup>, we calculated the effects of the possible nucleon isobars upon static nuclear potential in symmetrical pseudoscalar theory, assuming that nucleon isobars behave as if they were new elementary particles of both spin and isotopic spin  $3/2$ . As the results, we obtained a large effect consisting mainly of strong singular and attractive central potential for the ground state of the deuteron, which is just large enough to cancel out the unfavorable strong central repulsive part of the potentials derived by Taketani et al.<sup>2)</sup> for ps-pv coupling and by Klein<sup>3)</sup> for ps-ps coupling. Therefore, our total potential consists of a tensor force of right sign and of the same order of magnitude and almost of the same shape as the usual ones, and a very weak central force which has large values only near the repulsive core.

Afterwards, Brueckner and Watson<sup>4)</sup> found another correction (not due to nucleon isobars) to the above potentials<sup>2)3)</sup> consisting mainly of so large an attractive central force, that the central part of the resulting potential is attractive and its magnitude is even larger than the tensor part. Furthermore, they obtained reasonable agreement with em-

pirical data in the deuteron problem. We cannot, however, at present say definitely what is the most reasonable potential from the meson theoretical standpoint, because there still remain some unsolved problems in the nuclear force problems.

Phenomenologically, on the other hand, we can say to some extent definitely what kind of potential is promising. Indeed one of the authors<sup>5)</sup> concluded previously that, in order to get agreement with empirical data, a large central attractive force of at least the same order of magnitude as the tensor one of right sign is definitely necessary, because otherwise the values of quadrupole moment and  $D$ -state probability would become too large. In the above investigation, however, the possible existence of hard core and the very singular behavior of potential near the origin were not taken into accounts. At present, it is generally believed that the nuclear potential behaves very singularly near the origin, though it has no singularity at the origin owing to the existence of hard core. It is, therefore, interesting to investigate whether the previous conclusion<sup>5)</sup> still holds in the present situations of nuclear force problem.

Furthermore, according to a private communication to the authors, the calculation of Taketani et al.<sup>2)</sup> was proven to be erroneous by Taketani et al. themselves. So the right answer of the deuteron problems for their potential characterized by the very large repulsive central force is not given either.

Therefore, it is very interesting also from phenomenological point of view to compare the numerical results of deuteron problem for our potential and Taketani's with those of Brueckner and Watson<sup>1)</sup>; this will be done in the following. The numerical calculations were all carried out by the Illiac computer at the University of Illinois by courtesy of Professors Blatt and Snyder. As a result it was shown that the deuteron can never bind for Taketani's potential, and the values of quadrupole moment and  $D$ -state probability for our potential are too large, indicating that relative weight of tensor force to central one is too large. Thus, we can definitely say, as an important conclusion of this paper, that the central attractive force of at least the same order of magnitude as the tensor one is certainly necessary in order to get agreement with the empirical facts in deuteron problem, thus justifying the conclusions of previous investigations<sup>5)</sup> from more general point of view.

## § 2. The numerical results of deuteron problem for our potential

The exact expression of our potentials are given in the first part of this paper.<sup>1)</sup> Their behaviors are shown in Fig. 1 for ps-ps case and in Fig. 2 for ps-pv case. The various dimensionless coupling constants and the excitation energy of nucleon isobar  $\Delta E$  are chosen so as to get the best agreement with experiments of pion-nucleon scattering, whose analysis were done by Kanazawa and Sugawara<sup>6)</sup>. These values are given as follows:

$$\left. \begin{aligned} f^2/4\pi\hbar c &= 9, \quad (f^2/4\pi\hbar c)_{\text{pair}} = 1.5, \quad G^2/4\pi\hbar c = 0.1, \\ \Delta E &= 2.4 \, m_\pi c^2, \quad \text{for ps-ps case,} \end{aligned} \right\} \quad (1)$$

and

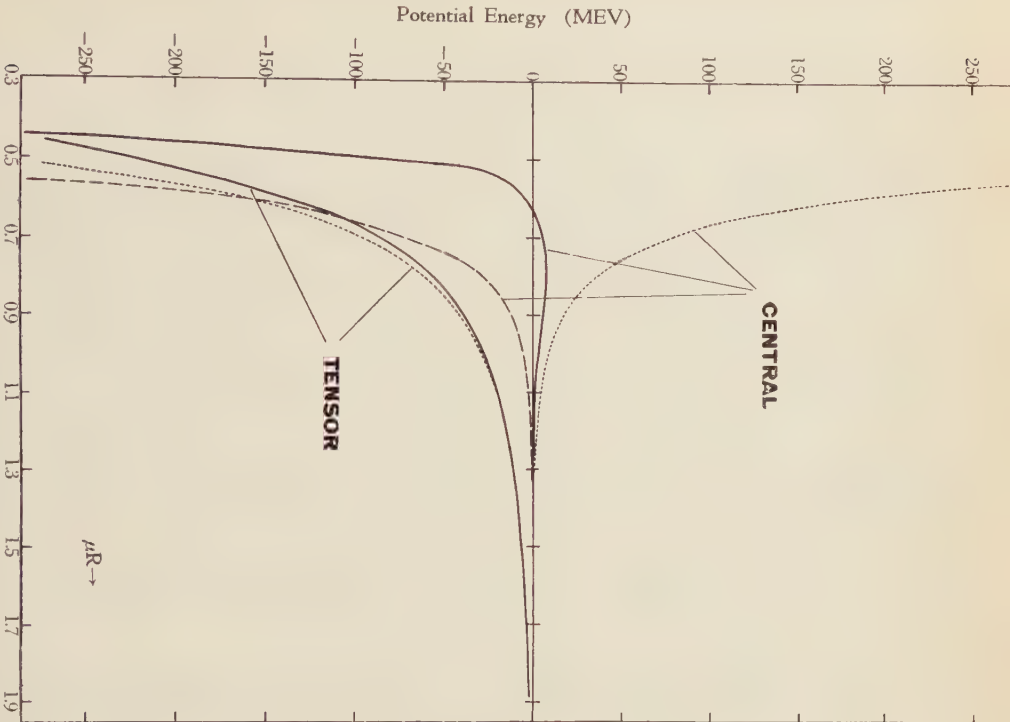


Fig. 1. The curves of our potential (solid lines) together with the isobar effect (dashed line) and the usual ones without correction (dotted lines) for ps-ps theory.

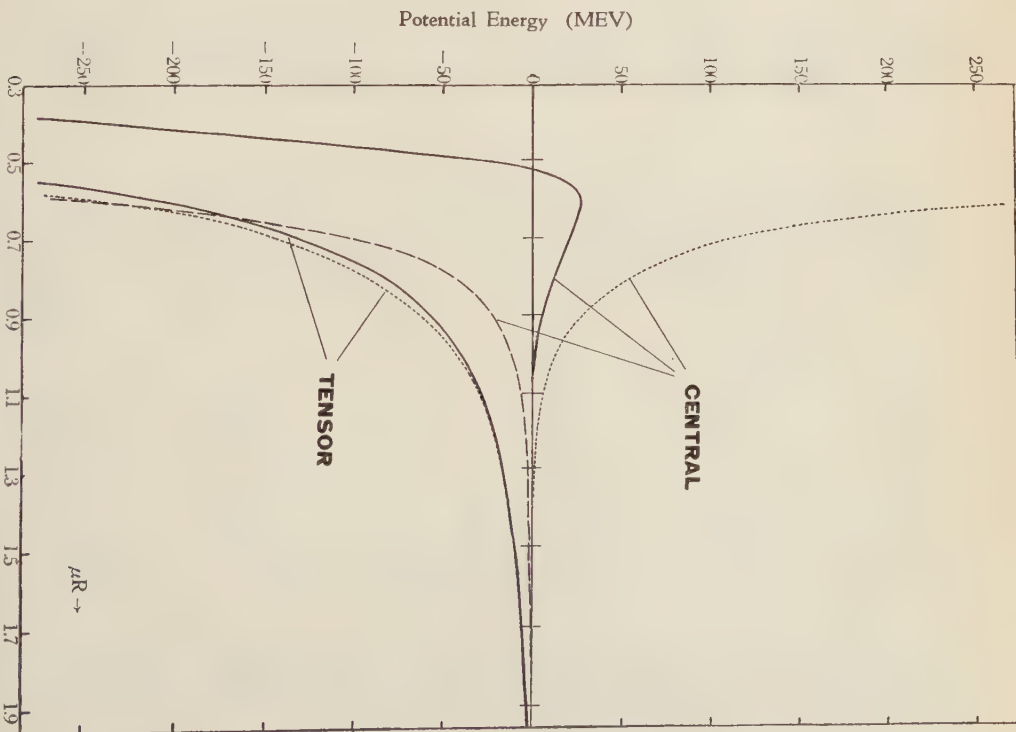


Fig. 2. The curves of our potential (solid lines) together with the isobar effect (dashed line) and the usual ones without correction (dotted lines) for ps-pv theory.



$$\left. \begin{aligned} g^2/4\pi\hbar c &= 0.07 & G^2/4\pi\hbar c &= 0.1 \\ \Delta E &= 2.4 \, m_\pi c^2, & \text{for ps-pv case,} & \end{aligned} \right\} \quad (2)$$

where the subscript "pair" means that we have taken into account the damping of the characteristic pair term in ps-ps theory and the coupling constant  $G^2/4\pi\hbar c$  is the phenomenological parameter which determines the coupling of nucleon isobar with its ground state and other coupling constants have usual meanings<sup>1)</sup>. In the figures dotted lines represent the usual potentials without the isobar effects, dashed ones the central part of the isobar correction and the solid ones our total potentials. The dotted lines (the usual ones without isobar effects) represent nothing but the potentials given by Klein<sup>1)</sup> for ps-ps case and by Taketani et al.<sup>2)</sup> for ps-pv case, both of which are characterized by a very strong repulsive central force, while ours by very weak attractive one. On the other hand, the tensor parts of these potentials are approximately the same. In the figures plus and minus on the ordinates mean repulsion (central) or wrong sign (tensor) and attraction (central) or right sign (tensor) respectively.

The results are shown in Table I, where  $\lambda$  is an adjustable parameter multiplying only the tensor part of our potential, the expression of which, therefore, is given by

$$V = V(\text{central}) + \lambda V(\text{tensor}) \, S_{12} \quad (3)$$

where  $S_{12}$  is the usual tensor operator.<sup>1)</sup> This parameter  $\lambda$  is determined firstly so as to give the correct binding energy. Then, for each  $\lambda$  thus determined, Illiac gives automatically the values of the quantities in the Table I and the  $S$ - and  $D$ -wave functions at prescribed points. Units are chosen respectively such as indicated in the Table I. The core radius  $R_0$  is chosen as 0.3, 0.383 and 0.466 in units of  $\mu^{-1}$ , the meson Compton wave length.

Table I. The calculated values for our potential of parameter  $\lambda$ ,  $D$ -state probability, quadrupole moment, and triplet effective range.

	Core radius $\mu R_0$	Parameter $\lambda$	$D$ -state probability	Quadrupole moment	Triplet effective range
Observed values				+ 2.73	1.70
PS 1	0.300	+1.58	11.9%	+ $8.173 \times 10^{-27} \text{cm}^2$	$2.792 \times 10^{-13} \text{cm}$
PS 2	0.383	+2.27	13.5	+ 9.734	2.997
PS 3	0.466	+2.96	23.3	+17.629	3.580
PV 1	0.300	+1.14	12.2	+ 8.226	2.814
PV 2	0.383	+1.62	13.5	+ 9.752	3.012
PV 3	0.466	+2.12	52.3	+39.771	3.989

As can be seen from the table, our potential gives too large values for  $D$ -state probability, quadrupole moment, and triplet effective range, which are about three or four times larger than the empirical ones. Thus it can be concluded that the relative weight of central to tensor force is too small in our potential.

Fig. 3. PS 1.  $\mu R_0=0.300$ ,  $\lambda=+1.58$ .

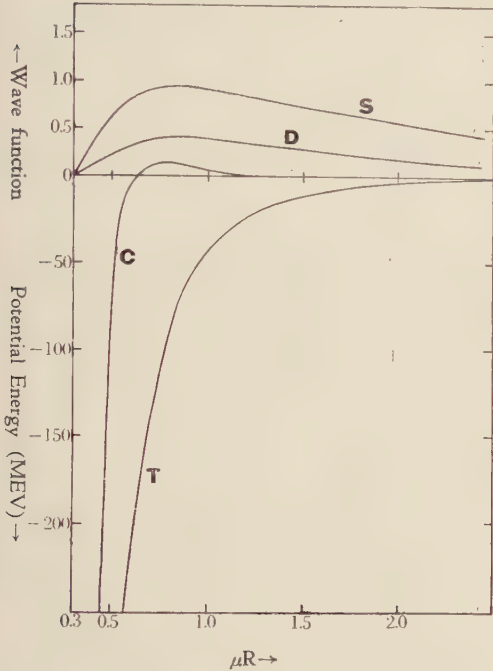


Fig. 4. PS 2.  $\mu R_0=0.383$ ,  $\lambda=+2.27$ .

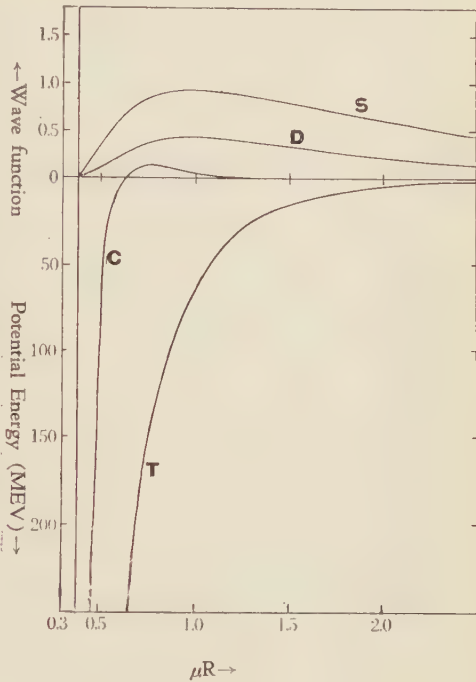


Fig. 5. PS 3.  $\mu R_0=0.466$ ,  $\lambda=+2.96$ .

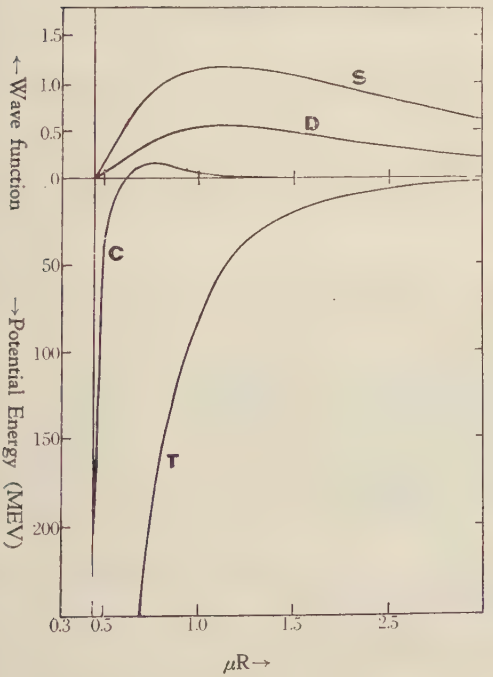


Fig. 6. PV 1.  $\mu R_0=0.300$ ,  $\lambda=+1.14$ .

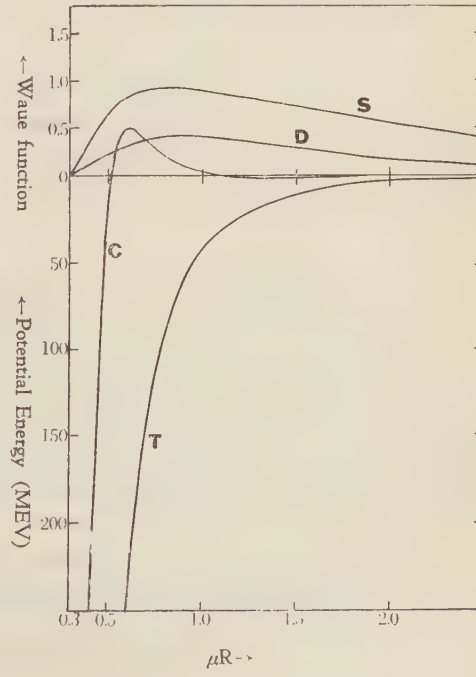


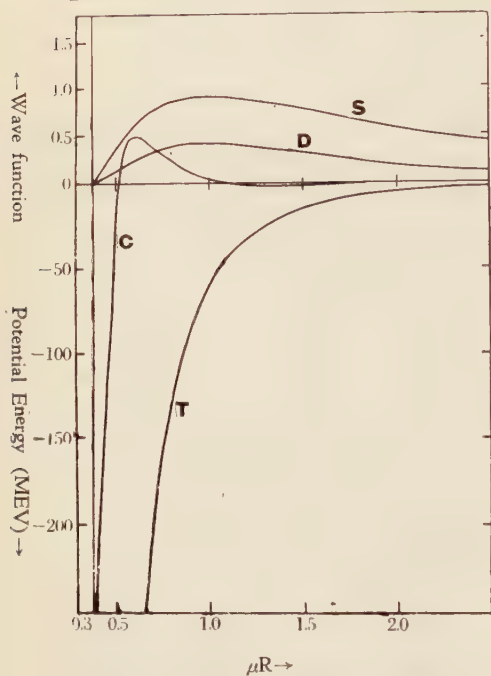
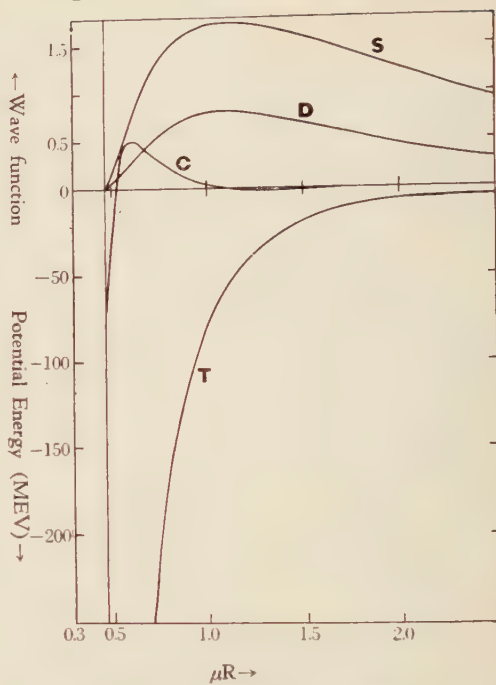
Fig. 7. PV 2.  $\mu R_0=0.383$ ,  $\lambda=+1.62$ .Fig. 8. PV 3.  $\mu R_0=0.466$ ,  $\lambda=+2.12$ .

Fig. 3~8. The potentials adjusted to give the correct binding energy and the corresponding  $S$ - and  $D$ -wave functions of deuteron ground state in ps-ps case (Fig. 3~5) and ps-pv case (Fig. 6~8). Hard core radius  $\mu R_0$  and the parameter  $\lambda$  are chosen as indicated in respective figures.

Fig. 3~8 show the behaviors of our potential whose tensor parts are multiplied by the factor  $\lambda$  and the corresponding behaviors of the  $S$ - and  $D$ -state wave functions of deuteron ground state for the above considered six cases. In Fig. 3~10,  $S$ ,  $D$ ,  $C$ , and  $T$  refer to the  $S$ - and  $D$ -wave function, central and tensor parts of adjusted potentials, respectively.

### § 3. The numerical results of deuteron problem for Taketani potential

We carried out at the same time the calculations for the so-called Taketani potential, by which is meant here the one which can be obtained from our potential (solid line in Fig. 1 and 2) by dropping simply the isobar correction. The so-called Taketani potential, therefore, means in this paper exactly the same one as was given by Klein<sup>3)</sup> for ps-ps case and by Taketani et al.<sup>2)</sup> for ps-pv case. These potentials are given by dotted lines in Fig. 1 and 2.

In this case, we adjusted parameter  $\lambda$  so that the total potential is multiplied by the factor  $\lambda$ . The potential in this case, therefore, has the form

$$V = \lambda [V(\text{central}) + V(\text{tensor}) S_{12}], \quad (4)$$

and Illiac computer determines the values of parameter  $\lambda$  so as to give the correct binding energy under fixing the relative weight of central to tensor force. The results are shown in Table II, where branks mean that it was not successful even to adjust the parameter  $\lambda$  to give the correct binding energy, because of the singular behavior of these potentials, especially because of the strong repulsive central force. Thus we can conclude that the deuteron cannot even bind for such types of potential, characterized by the strong repulsive central force.

Table II. The numerical results for the potentials given by Klein and Taketani of parameter  $\lambda$  D-state probability, quadrupole moment and triplet effective range.

Core radius $\mu R_0$		Parameter $\lambda$	D-state probability	Quadrupole moment	Triplet effective range
Observed values				+2.73	1.70
PS 1	0.300	-0.84	13.2%	$-9.622 \times 10^{-27} \text{cm}^2$	$2.427 \times 10^{-13} \text{cm}$
PS 2	0.383	—	—	—	—
PS 3	0.466	-1.85	21.1	-14.935	2.748
PV 1	0.300	-0.56	12.1	- 8.898	2.374
PV 2	0.383	—	—	—	—
PV 3	0.466	—	—	—	—

For two cases given in Table II, both of which are the cases of the core radius 0.3, the modified potentials by the factor  $\lambda$  and the corresponding S- and D-wave functions are shown in Fig. 9 and Fig. 10.

Fig. 9. Modified Klein's potential by the factor  $\lambda$  and the corresponding S- and D-wave functions. Core radius  $\mu R_0=0.300$  and the parameter  $\lambda=-0.84$ .

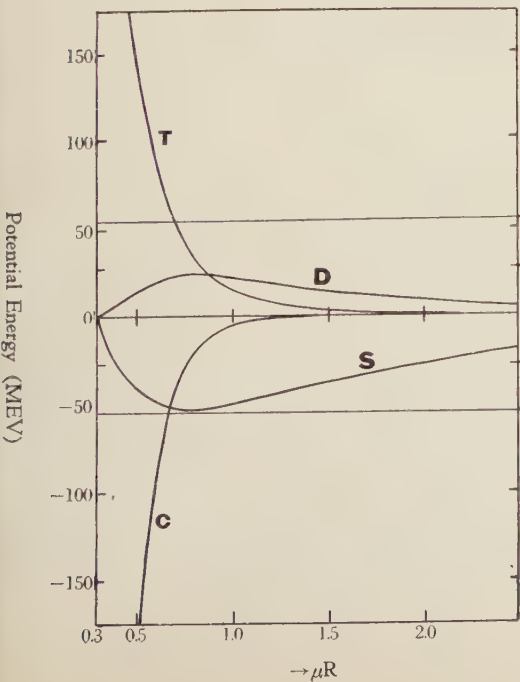
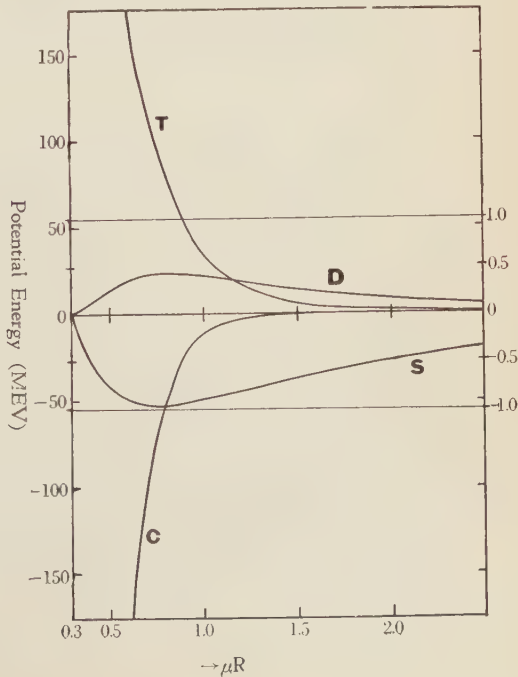


Fig. 10. Modified Taketani's potential by the factor  $\lambda$  and the corresponding S- and D-wave functions. Core radius  $\mu R_0=0.300$  and the parameter  $\lambda=-0.56$ .



#### § 4. Conclusions and acknowledgments

Comparing the results of §§ 2 and 3 with those of Brueckner and Watson<sup>4)</sup>, which are characterized by the very strong attractive central force, we can conclude as follows. The large central attractive force, which is at least of the same order of magnitude as the tensor one of right sign, is definitely necessary in order to give the correct values, especially of D-state probability and quadrupole moment. For such types of potential as were given by Taketani et al.<sup>2)</sup> and Klein<sup>3)</sup>, deuteron cannot even bind, because of the strong repulsive central force. Our potential, whose central part is very small in magnitude as compared with its tensor force, gives in general too large values (three or four times larger) of D-state probability and quadrupole moment, which is due mainly to too small central force. The reasonable agreement obtained by Brueckner and Watson<sup>4)</sup> can be said as mainly due to the approximate equality of central with tensor one, thus justifying the previous conclusions<sup>5)</sup> from more general point of view.

Finally we should very much like to express our deep gratitudes to Professor Blatt and Professor Snyder for helping us in carrying out all the numerical calculations done in this paper by the Illiac automatic computer at the University of Illinois. The authors are also indebted to Professor Y. Nambu for his cordial encouragements. They owe some of the numerical calculations to Miss H. Nitta.

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## Beta-ray Spectra, I

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Probabilities of  $\beta$ -decay transitions forbidden to any degree have been calculated in the case of the linear combination of the Fermi interactions, with some corrections taken into account.

### § 1. Introduction

Recently, according to the progress of experimental technique, many types of  $\beta$ -ray spectra have been found, and measured accurately. At present, it is known that all these spectra can be explained by the Fermi theory. Even if some spectra which cannot be explained by the usual procedure are found, it is necessary to examine whether we can overcome the difficulties by developing the Fermi theory in detail before making the next step, such as the meson theory of  $\beta$ -decay or others<sup>1)</sup>.

Usually, in order to explain the forbidden type spectra in terms of a linear combination of the Fermi interactions, we had to adjust the ratios of the products of nuclear matrix elements and coupling constants. In many cases, cancellation between main terms in the correction factor went so far that the small terms in it became important. It is known that in such cases several hitherto neglected effects might be effective.

Therefore, it is necessary to make the formula for transition probability, or correction factor, given by many authors<sup>2)-7)</sup>, more accurate. That is, although these authors have obtained elegant formulas by performing various approximations, it is desirable to return to the starting point, and to derive an expression with least approximations. After this derivation, we can make appropriate approximations according to individual situations.

This paper is the first step of such an attempt. We have calculated the correction factor by making use of the method originated by Racah<sup>7)(8)</sup> and developed by Wigner<sup>10)</sup>, Schwinger<sup>11)</sup>, Jahn and Hope<sup>12)</sup>, Arima et al.<sup>13)</sup> and so on.

This correction factor is applicable to the linear combination of the five Fermi interactions forbidden to any degree. In this calculation we have i) considered the deviations of the electron wave functions from those for pure Coulomb field, ii) retained the lepton part in the integral, without putting it out of the integral, iii) not discarded the higher order terms in  $pR$  or  $qR$ , where  $R$  is a constant, which has the order of the magnitude of the nuclear radius, and  $p$  and  $q$  are the momenta of the electron and the neutrino, respectively, moreover iv) not discarded the terms which have hitherto been considered as negligible corrections to the transition forbidden to lower degree. We have not considered the radiative<sup>14)15)</sup>

and mesonic<sup>16)</sup> corrections.

§ 2. Matrix element

As is well known, the probability of the emission of an electron with energy between  $W$  and  $W+dW$  is :

$$P(W)dW=\frac{2}{\pi}\cdot\frac{1}{2J'+1}\sum_{\kappa,\mu,\mu',M}|\int Hd\tau|^2\frac{W}{p}dW. \tag{1}$$

In this expression  $H$  is the interaction Hamiltonian for  $\beta$ -decay,  $p$  is the momentum of the electron,  $J'$  is the spin value of the initial nucleus,  $M'$  and  $M$  are the magnetic quantum numbers of the initial and the final nuclei,  $\kappa=\pm(j+1/2)$  for  $j=l\mp 1/2$  for the electron with the total angular momentum  $j$  and orbital angular momentum  $l$ ,  $\mu$  is the magnetic quantum number of the electron, and the suffix  $\nu$  represents the neutrino.

If  $\Psi$  and  $\Phi$  are the wave functions of the final and the initial nuclei, and  $\psi$  and  $\phi$  are those of the electron and the neutrino normalized per unit sphere,  $H$  can be generally written as follows :

$$H=\sum_s\lambda_s\Psi^*\psi^*I_s\Phi\phi, \tag{2}$$

with

$$\begin{aligned} I_s &= \sum_q \omega_{ps} \times \mathcal{Y}_k^q(\sigma_p) Q_p \cdot \omega_s \times \mathcal{Y}_k^{-q}(\sigma) (kko|q-qo) \\ &= (-1)^k (2k+1)^{-1/2} \sum_q (-1)^q \omega_{ps} \times \mathcal{Y}_k^q(\sigma_p) Q_p \cdot \omega_s \times \mathcal{Y}_k^{-q}(\sigma), \end{aligned} \tag{3}^*)$$

Table I

$s$	$S$	$V$		$T$		$A$		$P$
	1	2	3	4	5	6	7	8
Operator	$\beta$	1	$\alpha$	$\beta\sigma$	$\beta\alpha$	$\sigma$	$r_5$	$\beta r_5$
$k$	0	0	1	1	1	1	0	0
$\epsilon$	-1	1	1	-1	-1	1	1	-1
$\lambda$	$\lambda_1$	$\lambda_2$	$\lambda_3$	$\lambda_4$	$\lambda_5$	$\lambda_6$	$\lambda_7$	$\lambda_8$
		$\lambda_2=\sqrt{3}\lambda_3$		$\lambda_4=\lambda_5$		$\sqrt{3}\lambda_6=\lambda_7$		

where  $s$  specifies the sort of the operator for  $\beta$ -decay as is shown in Table I. The value of  $k$  is specified by  $s$ , but for the sake of simplicity  $k$  or  $k'$  shall be used instead of  $k_s$  or  $k'_s$ .  $Q_p$  is an operator which replaces  $\Phi$  with a wave function describing a nucleus in which the  $p$ -th neutron is replaced by a proton.  $\sigma$  is the usual  $2\times 2$  Pauli spin matrix.

\*  $(j_1 j_2 j | m_1 m_2 m)$  is the vector addition coefficient, and is equal to  $(j_1 j_2 m_1 m_2 | j_1 j_2 j m)$  of Condon and Shortley<sup>17)</sup>.

$\mathcal{Y}_k^q(\sigma)$  is defined as follows:

$$\left. \begin{aligned} \mathcal{Y}_0^0 &= \sigma \sqrt{1/4\pi} \mathbf{1} \quad (\mathbf{1} \text{ is the } 2 \times 2 \text{ unit matrix}), \\ \mathcal{Y}_1^1(\sigma) &= -\sqrt{3/4\pi} (\sigma_x + i\sigma_y)/\sqrt{2}, \quad \mathcal{Y}_1^0(\sigma) = \sqrt{3/4\pi} \sigma_z, \\ \mathcal{Y}_1^{-1}(\sigma) &= \sqrt{3/4\pi} (\sigma_x - i\sigma_y)/\sqrt{2}. \end{aligned} \right\} \quad (4)$$

$\omega$  and  $\omega_p$  represent the even-odd character of the operator for  $\beta$ -decay;  
if this is even

$$\omega \times \mathcal{Y}_k^q(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} \times \mathcal{Y}_k^q(\sigma) = \begin{pmatrix} \mathcal{Y}_k^q(\sigma) & \mathbf{0} \\ \mathbf{0} & \varepsilon \mathcal{Y}_k^q(\sigma) \end{pmatrix}, \quad (5)$$

and if the operator is odd

$$\omega \times \mathcal{Y}_k^q(\sigma) = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \times \mathcal{Y}_k^q(\sigma) = \begin{pmatrix} \mathbf{0} & \mathcal{Y}_k^q(\sigma) \\ \varepsilon \mathcal{Y}_k^q(\sigma) & \mathbf{0} \end{pmatrix}, \quad (6)$$

where  $\varepsilon = \pm 1$  and  $\mathbf{0}$  is the  $2 \times 2$  zero matrix. By making use of the Racah and associated coefficients<sup>(7)-(13)(18)-(26)</sup>,  $H$  can be written as the sum of the following two parts:

i) for odd  $\omega$

$$\begin{aligned} H_0 &= (-1)^{j+j_v+l+l_v} \sum_{\varepsilon u v} \lambda_s (2k+1)^{-1/2} (1/2 \|\mathcal{Y}_k\| 1/2) \begin{pmatrix} j & j_v & u \\ -\mu & -\mu_v & \mu + \mu_v \end{pmatrix} \\ &\times [f_{\kappa} f_{\kappa_v} B_{uvk}(-\kappa, -\kappa_v) + \varepsilon_s g_{\kappa} g_{\kappa_v} B_{uvk}(\kappa, \kappa_v)] \\ &\times \Psi^* \omega_{sp} \times \mathcal{Y}_{u(k, v)}^{-\mu_v - \mu}(\sigma_p, \mathbf{r}) Q_p \Phi, \end{aligned} \quad (7a)$$

ii) for even  $\omega$

$$\begin{aligned} H_e &= (-1)^{j+j_v+l+l_v+1} i \sum_{\varepsilon u v} \lambda_s (2k+1)^{-1/2} (1/2 \|\mathcal{Y}_k\| 1/2) \begin{pmatrix} j & j_v & u \\ -\mu & -\mu_v & \mu + \mu_v \end{pmatrix} \\ &\times [f_{\kappa} g_{\kappa_v} B_{uvk}(-\kappa, \kappa_v) - \varepsilon_s g_{\kappa} f_{\kappa_v} B_{uvk}(\kappa, -\kappa_v)] \\ &\times \Psi^* \omega_{sp} \times \mathcal{Y}_{u(k, v)}^{-\mu_v - \mu}(\sigma_p, \mathbf{r}) Q_p \Phi, \end{aligned} \quad (7b)$$

where

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} = (-1)^{j_2 - j_1 + \mu_3} (2j_3 + 1)^{-1/2} (j_1 j_2 j_3 | \mu_1 \mu_2 - \mu_3)$$

is the Wigner  $3j$ -coefficient<sup>(10)</sup> or the Schwinger  $X$ -coefficient<sup>(11)</sup>.  $(1/2 \|\mathcal{Y}_k\| 1/2)$  and  $B_{uvk}(\kappa, \kappa')$  are defined by

$$(\chi_{1/2}^{\tau}, \mathcal{Y}_k^q \chi_{1/2}^{\tau'}) = (1/2k 1/2 | \tau' q \tau) (1/2 \|\mathcal{Y}_k\| 1/2), \quad (8)$$

$(\chi_{1/2}^{\tau}$ : the spin wave function) and

$$\begin{aligned} B_{uvk}(\kappa, \kappa') &\equiv \left[ \frac{(2l+1)(2l'+1)(2j+1)(2j'+1)(2u+1)(2v+1)}{2\pi} \right]^{1/2} \\ &\times \begin{pmatrix} l & l' & v \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & v \\ j & j' & u \\ 1/2 & 1/2 & k \end{pmatrix}, \end{aligned} \quad (9)$$

where the Wigner  $9j$ -coefficient<sup>10)</sup>

$$\left\{ \begin{array}{ccc} a & b & c \\ c & d & e' \\ f & f' & g \end{array} \right\} = \sum_{\lambda} (2\lambda + 1) W(fybd; f'\lambda) W(cgcd; e'\lambda) W(fcbe; a\lambda)$$

is the same with the  $U$ -coefficient of Arima et al<sup>11)</sup>, or with the Schwinger  $S$ -coefficient<sup>11)</sup> multiplied by a phase factor.  $f_e$  and  $g_e$  are the radial wave functions of the electron, and those of the neutrino  $f_{\nu}$  and  $g_{\nu}$  can be represented by spherical Bessel function<sup>3)20)</sup> \*.

Spherical tensor  $\mathcal{Y}_{u(k,v)}^{-\mu_{\nu}-\mu}(\sigma, \nu)$  is given by:

$$\mathcal{Y}_{u(k,v)}^{-\mu_{\nu}-\mu}(\sigma, \nu) = \sum_q \langle kvu | -q \ q \ -\mu_{\nu} - \mu \ -\mu_{\nu} - \mu \rangle \mathcal{Y}_k^{-q}(\sigma) Y_r^{q-\mu_{\nu}-\mu}(\nu) \quad (10)$$

where  $Y_l^M(\nu)$  is the ordinary spherical harmonics<sup>17)</sup>.

If we further define  $(J \| Q_{\rho} \omega_{\rho} \times \mathcal{Y}_{u(k,v)}(\sigma_{\rho}, \nu) \| J')$  by

$$\begin{aligned} & \int \Psi^* \omega_{\rho} Q_{\rho} \mathcal{Y}_{u(k,v)}^{-\mu_{\nu}-\mu}(\sigma_{\rho}, \nu) \Psi d\Omega \\ & \equiv (J \| J \| M' - \mu_{\nu} - \mu \ M) (J \| Q_{\rho} \omega_{\rho} \times \mathcal{Y}_{u(k,v)}(\sigma_{\rho}, \nu) \| J'), \end{aligned} \quad (11)$$

(  $\int \dots d\Omega$ : integration over the direction )

we can express  $\int H d\tau$  in eq. (1) by

$$\begin{aligned} \int H d\tau = & \sum_{S, K, \nu} (-1)^{J+j+j_{\nu}+l+l_{\nu}-M} (2J+1)^{1/2} \begin{pmatrix} j & j_{\nu} & u \\ -\mu & -\mu_{\nu} & \mu + \mu_{\nu} \end{pmatrix} \begin{pmatrix} J & J' & u \\ M & M' & M \end{pmatrix} \\ & \times \mathfrak{R}_{\text{step}}(\kappa, \kappa_{\nu}, J, J') \end{aligned} \quad (12)$$

with

$$\begin{aligned} \mathfrak{R}_{\text{step}}(\kappa, \kappa_{\nu}, J, J') &= (2k+1)^{-1/2} (1/2 \| \mathcal{Y}_k \| 1/2) \\ & \times \int [f_{\kappa} f_{\kappa_{\nu}} B_{uvk}(-\kappa, -\kappa_{\nu}) + \varepsilon_s g_{\kappa} g_{\kappa_{\nu}} B_{uvk}(\kappa, \kappa_{\nu})] \\ & \times (J \| Q_{\rho} \omega_{sp} \times \mathcal{Y}_{u(k,v)}(\sigma_{\rho}, \nu) \| J') r^2 dr \quad \text{for odd } \omega_s \end{aligned} \quad (13a)$$

and

$$\begin{aligned} \mathfrak{R}_{\text{step}}(\kappa, \kappa_{\nu}, J, J') &= -i(2k+1)^{-1/2} (1/2 \| \mathcal{Y}_k \| 1/2) \\ & \times \int [f_{\kappa} g_{\kappa_{\nu}} B_{uvk}(-\kappa, \kappa_{\nu}) - \varepsilon_s g_{\kappa} f_{\kappa_{\nu}} B_{uvk}(\kappa, -\kappa_{\nu})] \\ & \times (J \| Q_{\rho} \omega_{sp} \times \mathcal{Y}_{u(k,v)}(\sigma_{\rho}, \nu) \| J') r^2 dr \quad \text{for even } \omega_s. \end{aligned} \quad (13b)$$

\* For the sake of simplicity, we write  $f_{\kappa_{\nu}}$  and  $g_{\kappa_{\nu}}$  instead of  $f_{\nu_{\kappa_{\nu}}}$  and  $g_{\nu_{\kappa_{\nu}}}$ .

## § 3 Transition probability

From eq. (12), one can easily verify that

$$\frac{1}{2J'+1} \sum_{\kappa \nu \mu M'} |\{Hd\tau\}|^2 = \frac{2J+1}{2J'+1} \sum_{\kappa \nu s' v'} \lambda_{s'}^* \left[ \frac{1}{2u+1} \right]^2 \times \Re_{suv}(\kappa, \kappa_\nu, J, J') \Re_{s'uv'}^*(\kappa, \kappa_\nu, J, J'), \quad (14)$$

and  $v+v'$  is even (odd) when both the even-odd characters of  $\omega$  and  $\omega'$  are the same (different). This restriction corresponds to the well known fact that the matrix elements with different parity do not interfere with each other.

We shall further define  $\mathfrak{T}_{suv}(J, J')$  by the following formulas:

$$\Re_{suv}(\kappa, \kappa_\nu, J, J') = \int [f_\kappa f_{\kappa_\nu} B_{uvk}(-\kappa, -\kappa_\nu) + \varepsilon_s g_\kappa g_{\kappa_\nu} B_{uvk}(\kappa, \kappa_\nu)] \times \mathfrak{T}_{suv}(J, J') r^2 dr \quad (15a)$$

for odd  $\omega_s$ , and

$$\Re_{suv}(\kappa, \kappa_\nu, J, J') = \int [f_\kappa g_{\kappa_\nu} B_{uvk}(-\kappa, \kappa_\nu) - \varepsilon_s g_\kappa f_{\kappa_\nu} B_{uvk}(\kappa, -\kappa_\nu)] \times \mathfrak{T}_{suv}(J, J') r^2 dr$$

for even  $\omega_s$ .

Now, eq. (1) can be written as

$$P(W) dW = (2/\pi) \cdot \{ (2J+1)/(2J'+1) \} (2p^2 F_1(R)) \sum' \lambda_{s'}^* [1/(2u+1)]^2 \times \int \int dx dy \mathfrak{T}_{suv}(J, J', x) \mathfrak{T}_{s'uv'}^*(J, J', y) (xy)^{\kappa+1} \times [a_{ss'uvuv'}^{(L)}(\kappa, \kappa_\nu) L_\kappa(\varepsilon_s \varepsilon_{s'}; xy) \varphi_{ss'}^{(L)}(\kappa_\nu; xy) + a_{ss'uvuv'}^{(M)}(\kappa, \kappa_\nu) M_\kappa(\varepsilon_s \varepsilon_{s'}; xy) xy \varphi_{ss'}^{(M)}(\kappa_\nu; xy) + a_{ss'uvuv'}^{(N1)}(\kappa, \kappa_\nu) N_\kappa(\varepsilon_s \varepsilon_{s'}; xy) x \varphi_{ss'}^{(N1)}(\kappa_\nu; xy) + a_{ss'uvuv'}^{(N2)}(\kappa, \kappa_\nu) N_\kappa(\varepsilon_s \varepsilon_{s'}; yx) y \varphi_{ss'}^{(N2)}(\kappa_\nu; xy)] \times (W/p) \cdot dW. \quad (16)$$

$\sum'$  is the summation over  $\kappa > 0$ , and over all possible values of  $\kappa_\nu$ ,  $s$ ,  $s'$ ,  $u$ ,  $v$  and  $v'$ . Here

$$L_\kappa(\varepsilon_s \varepsilon_{s'}; xy) = \frac{1}{2p^2 F_1(R)} \frac{f_\kappa(x) f_\kappa(y) + \varepsilon_s \varepsilon_{s'} g_{-\kappa}(x) g_{-\kappa}(y)}{(xy)^{\kappa-1}}, \quad (17a)$$

$$M_\kappa(\varepsilon_s \varepsilon_{s'}; xy) = \frac{1}{2p^2 F_1(R)} \frac{f_{-\kappa}(x) f_{-\kappa}(y) + \varepsilon_s \varepsilon_{s'} g_\kappa(x) g_\kappa(y)}{(xy)^\kappa}, \quad (17b)$$

$$N_\kappa(\varepsilon_s \varepsilon_{s'}; xy) = \frac{1}{2p^2 F_1(R)} \frac{f_{-\kappa}(x) g_{-\kappa}(y) - \varepsilon_s \varepsilon_{s'} g_\kappa(x) f_\kappa(y)}{x^\kappa y^{\kappa-1}}, \quad (17c)$$



$$F_1(R) = \frac{4}{\Gamma^2(2\gamma + 1)} (2\phi R)^{2\gamma-2} e^{\pi\alpha ZW/p} |\Gamma(\gamma + i\alpha ZW/p)|^2, \quad (17d)$$

$$\gamma = (\kappa^2 - \alpha^2 Z^2)^{1/2}, \quad (17e)$$

$\alpha$  = fine structure constant.

$\alpha_{ss'l}^{(L)}(\kappa, \kappa_v)$  etc. and  $\varphi_{ss'l}^{(L)}(\kappa_v; xy)$  etc. are shown in Table II.

Table II

$S(x_v)$  is the sign of  $x_v$ .

$(s; s')$	even; even	even; odd	odd; odd
$\alpha^{(L)}$	$B_{uvk}(-x, x_v) B_{uv'l k'}(-x, x_v)$	$S(x_v) B_{uvk}(-x, x_v) B_{uv'l k'}(-x, -x_v)$	$B_{uvk}(-x, -x_v) B_{uv'l k'}(-x, -x_v)$
$\alpha^{(M)}$	$B_{uvk}(x, -x_v) B_{uv'l k'}(x, -x_v)$	$-S(x_v) B_{uvk}(x, -x_v) B_{uv'l k'}(x, x_v)$	$B_{uvk}(x, x_v) B_{uv'l k'}(x, x_v)$
$\alpha^{(N1)}$	$\varepsilon_{st} S(x_v) B_{uvk}(x, -x_v) B_{uv'l k'}(-x, x_v)$	$\varepsilon_{st} B_{uvk}(x, -x_v) B_{uv'l k'}(-x, -x_v)$	$- \varepsilon_{st} S(x_v) B_{uvk}(x, x_v) \times B_{uv'l k'}(-x, -x_v)$
$\alpha^{(N2)}$	$\varepsilon_{st} S(x_v) B_{uvk}(-x, x_v) B_{uv'l k'}(x, -x_v)$	$- \varepsilon_{st} B_{uvk}(-x, x_v) B_{uv'l k'}(x, x_v)$	$- \varepsilon_{st} S(x_v) B_{uvk}(-x, -x_v) \times B_{uv'l k'}(x, x_v)$
$\varphi^{(L)}(x_v; xy)$	$f_{-x_v}(x) f_{-x_v}(y)$	$f_{-x_v}(x) f_{x_v}(y)$	$f_{x_v}(x) f_{x_v}(y)$
$\varphi^{(M)}(x_v; xy)$	$f_{x_v}(x) f_{x_v}(y)$	$f_{x_v}(x) f_{-x_v}(y)$	$f_{-x_v}(x) f_{-x_v}(y)$
$\varphi^{(N1)}(x_v; xy)$	$f_{x_v}(x) f_{-x_v}(y)$	$f_{x_v}(x) f_{x_v}(y)$	$f_{-x_v}(x) f_{x_v}(y)$
$\varphi^{(N2)}(x_v; xy)$	$f_{-x_v}(x) f_{x_v}(y)$	$f_{-x_v}(x) f_{-x_v}(y)$	$f_{x_v}(x) f_{-x_v}(y)$

#### § 4. $L_x$ , $M_x$ and $N_x$

Usually, in the calculation of the correction factor for  $\beta$ -decay, we assume that the nuclear charge is concentrated in the centre of the nucleus. That is, we use for  $\psi$  the solution of the Dirac equation in the pure Coulomb field. Moreover, to facilitate the calculation, the lepton part of the transition matrix element is evaluated at  $R$ . Since the potential for the electron is not pure Coulombian because of the charge distribution over the nuclear volume, the current procedure is not strictly correct. Rose and Holmes<sup>(27)</sup> and Malcolm<sup>(28)</sup> have calculated the corrections introduced by this effect of the finite size of the nucleus in  $\beta$ -decay, and discussed its energy dependence. But they have evaluated the correction at the nuclear radius and not investigated the  $r$ -dependence. In such a treatment, the effect of the nuclear size appears mainly in the normalization factor of the radial wave function inside the nucleus, which is determined by the continuity of the inside and the outside wave functions. On the other hand, the  $r$ -dependence of this correction has been considered first by Yamada<sup>(29)</sup> and recently by Nataf<sup>(30)</sup>. But they have not considered the energy dependence of the normalization factor precisely. It is desirable to consider this finite size effect making no approximation to the normalization factor and not evaluating the lepton wave functions at the nuclear radius. In the following we shall try to calculate such a correction by means of the method of Rose and Holmes<sup>(27)</sup>.

We write following them,

$$\begin{aligned}\mathfrak{F}_\kappa(r) &= rf_\kappa^c(r), & \mathfrak{G}_\kappa(r) &= rg_\kappa^c(r), \\ \bar{\mathfrak{F}}_\kappa(r) &= r\bar{f}_\kappa^c(r), & \bar{\mathfrak{G}}_\kappa(r) &= r\bar{g}_\kappa^c(r), \\ \Phi_\kappa(r) &= rf_\kappa(r), & \Gamma_\kappa(r) &= rg_\kappa(r), \\ \Delta_\kappa &= [1 + U^2 + 2U \cos(\delta - \bar{\delta})]_{\kappa, r=R}^{-1/2},\end{aligned}$$

$$U = \{ (D^{(i)} - D) / (\bar{D} - D^{(i)}) \} \cdot \mathfrak{F} / \bar{\mathfrak{F}},$$

$f_\kappa^c(r)$ ,  $g_\kappa^c(r)$  : the regular radial wave functions for the pure Coulomb field<sup>31)</sup>,

$\bar{f}_\kappa^c(r)$ ,  $\bar{g}_\kappa^c(r)$  : the irregular radial wave functions for the pure Coulomb field,

$\mathfrak{F}_\kappa^{(i)}(r)$ ,  $\mathfrak{G}_\kappa^{(i)}(r)$  : the regular solutions multiplied by  $r$  for the potential  $V_i(r)$  appropriate to the charge distribution inside the nucleus, with arbitrary normalization,

$\delta$ ,  $\bar{\delta}$  : the phase at infinity for the regular and irregular solutions,

and further

$$D_\kappa = \mathfrak{G}_\kappa(R) / \mathfrak{F}_\kappa(R), \quad \bar{D}_\kappa = \bar{\mathfrak{G}}_\kappa(R) / \bar{\mathfrak{F}}_\kappa(R) \quad \text{and} \quad D_\kappa^{(i)} = \mathfrak{G}_\kappa^{(i)}(R) / \mathfrak{F}_\kappa^{(i)}(R),$$

In the following we shall omit suffix  $\kappa$ , when no confusion will occur.

When  $r < R$ ,

$$\Phi(r) = \{ (\bar{D} - D) / (\bar{D} - D^{(i)}) \} \mathfrak{F}(R) (\mathfrak{F}^{(i)}(r) / \mathfrak{F}^{(i)}(R)) \cdot \mathcal{A}, \quad (18a)$$

and

$$\Gamma(r) = \{ (1/D - 1/\bar{D}) / (1/D^{(i)} - 1/\bar{D}) \} \mathfrak{G}(R) (\mathfrak{G}^{(i)}(r) / \mathfrak{G}^{(i)}(R)) \cdot \mathcal{A}. \quad (18b)$$

$D$ ,  $\bar{D}$  and  $D^{(i)}$  can be evaluated by the iterational methods<sup>27,31)</sup>. The series for the ratio  $\mathfrak{F}^{(i)}(r) / \mathfrak{F}^{(i)}(R)$  and  $\mathfrak{G}^{(i)}(r) / \mathfrak{G}^{(i)}(R)$  can be obtained by expanding  $\mathfrak{F}^{(i)}(r)$  or  $\mathfrak{G}^{(i)}(r)$  as a power series of  $(r - R)$ .

$$\mathfrak{F}^{(i)}(r) / \mathfrak{F}^{(i)}(R) = \sum_{n=0}^{\infty} A_n (r - R)^n = \sum_{n=0}^{\infty} (1/n!) (\mathfrak{F}^{(i)[n]} / \mathfrak{F}^{(i)})_{r=R} (r - R)^n, \quad (19a)$$

$$\mathfrak{G}^{(i)}(r) / \mathfrak{G}^{(i)}(R) = \sum_{n=0}^{\infty} B_n (r - R)^n = \sum_{n=0}^{\infty} (1/n!) (\mathfrak{G}^{(i)[n]} / \mathfrak{G}^{(i)})_{r=R} (r - R)^n. \quad (19b)$$

$\mathfrak{F}^{(i)[n]}(\mathfrak{G}^{(i)[n]})$  means the  $n$ -th derivative of  $\mathfrak{F}^{(i)}(r)$  ( $\mathfrak{G}^{(i)}(r)$ ) concerning  $r$ . One can obtain  $A_n$  and  $B_n$  by the successive differentiation of the differential equations for  $\mathfrak{F}^{(i)}(r)$  and  $\mathfrak{G}^{(i)}(r)$ . (See Appendix A.)

Substituting (19a,b) into (17a,b,c), we obtain

$$2p^0 F_1(R) (xy)^\kappa L_\kappa(\mathcal{E}_s \mathcal{E}_{s'}; xy) = \sum_{nm} \mathfrak{L}_{\kappa nm} (x - R)^n (y - R)^m, \quad (20a)$$

$$2p^0 F_1(R) (xy)^{\kappa+1} M_\kappa(\mathcal{E}_s \mathcal{E}_{s'}; xy) = \sum_{nm} \mathfrak{M}_{\kappa nm} (x - R)^n (y - R)^m, \quad (20b)$$

$$2p^0 F_1(R) x^{\kappa+1} y^\kappa N_\kappa(\mathcal{E}_s \mathcal{E}_{s'}; xy) = \sum_{nm} \mathfrak{N}_{\kappa nm} (x - R)^n (y - R)^m, \quad (20c)$$

where

$$\begin{aligned}\mathfrak{L}_{\kappa n m} &\equiv X_{\kappa}^2 \mathfrak{F}_{\kappa}^2(R) A_{\kappa n} A_{\kappa m} + \varepsilon_s \varepsilon_{s'} Y_{-\kappa}^2 \mathfrak{G}_{-\kappa}^2(R) B_{-\kappa n} B_{-\kappa m} \\ &= (Y_{\kappa}^2 A_{\kappa n} A_{\kappa m} + \varepsilon_s \varepsilon_{s'} Y_{-\kappa}^2 B_{-\kappa n} B_{-\kappa m}) l_{\kappa}^{(L)} \\ &\quad + (X_{\kappa}^2 A_{\kappa n} A_{\kappa m} - \varepsilon_s \varepsilon_{s'} Y_{-\kappa}^2 B_{-\kappa n} B_{-\kappa m}) l_{\kappa}^{(S)},\end{aligned}\quad (21a)$$

$$\begin{aligned}\mathfrak{M}_{\kappa n m} &\equiv X_{-\kappa}^2 \mathfrak{F}_{-\kappa}^2(R) A_{-\kappa n} A_{-\kappa m} + \varepsilon_s \varepsilon_{s'} Y_{\kappa}^2 \mathfrak{G}_{\kappa}^2(R) B_{\kappa n} B_{\kappa m} \\ &= (X_{-\kappa}^2 A_{-\kappa n} A_{-\kappa m} + \varepsilon_s \varepsilon_{s'} Y_{\kappa}^2 B_{\kappa n} B_{\kappa m}) m_{\kappa}^{(L)} \\ &\quad + (X_{-\kappa}^2 A_{-\kappa n} A_{-\kappa m} - \varepsilon_s \varepsilon_{s'} Y_{\kappa}^2 B_{\kappa n} B_{\kappa m}) m_{\kappa}^{(S)},\end{aligned}\quad (21b)$$

$$\begin{aligned}\mathfrak{N}_{\kappa n m} &\equiv X_{-\kappa} Y_{-\kappa} \mathfrak{F}_{-\kappa}(R) \mathfrak{G}_{-\kappa}(R) A_{-\kappa n} B_{-\kappa m} \\ &\quad - \varepsilon_s \varepsilon_{s'} X_{\kappa} Y_{\kappa} \mathfrak{F}_{\kappa}(R) \mathfrak{G}_{\kappa}(R) A_{\kappa n} B_{\kappa m} \\ &= (X_{-\kappa} Y_{-\kappa} A_{-\kappa n} B_{-\kappa m} + \varepsilon_s \varepsilon_{s'} X_{\kappa} Y_{\kappa} A_{\kappa n} B_{\kappa m}) n_{\kappa}^{(L)} \\ &\quad + (X_{-\kappa} Y_{-\kappa} A_{-\kappa n} B_{-\kappa m} - \varepsilon_s \varepsilon_{s'} X_{\kappa} Y_{\kappa} A_{\kappa n} B_{\kappa m}) n_{\kappa}^{(S)},\end{aligned}\quad (21c)$$

$$X = \mathcal{A}(\bar{D} - D) / (\bar{D} - D^{(S)}), \quad Y = \mathcal{A}(1/D - 1/\bar{D}) (1/D^{(S)} - 1/\bar{D}). \quad (22)$$

As for  $l_{\kappa}^{(L)}$  etc., see the Appendix B.

If all the lepton wave functions are evaluated at  $r=R$ , (20a, b, c) become  $\mathfrak{L}_{\kappa 00}$ ,  $\mathfrak{M}_{\kappa 00}$  and  $\mathfrak{N}_{\kappa 00}$ , respectively. If we further neglect the effect of the nuclear size ( $X=Y=1$ ), (20a) etc. give

$$\begin{aligned}2R^{2\kappa} p^2 F_1(R) L_{\kappa}(\varepsilon_s, \varepsilon_{s'}; R) &= \mathfrak{L}_{\kappa 00} (X=Y=1) \\ &= \mathfrak{F}_{\kappa}^2(R) + \varepsilon_s \varepsilon_{s'} \mathfrak{G}_{-\kappa}^2(R) \\ &= \begin{cases} 2l_{\kappa}^{(L)}, & \text{for } \varepsilon_s \varepsilon_{s'} = 1, \\ 2l_{\kappa}^{(S)}, & \text{for } \varepsilon_s \varepsilon_{s'} = -1, \end{cases}\end{aligned}$$

and so on. Namely,  $L_{\kappa}(\varepsilon_s \varepsilon_{s'}; X, Y)$  etc. reduce to the customarily used  $L_{\kappa}^{\pm}$ ,  $M_{\kappa}^{\pm}$  and  $N_{\kappa}^{\pm}$ .

## § 5. Correction factor

The correction factor  $C$  is defined by

$$C = \frac{1}{2J'+1} \frac{4\pi^2}{p^2 q^2 F_1(R)} \sum |\{Hd\tau\}|^2. \quad (22)$$

We expand  $\varphi_{ss'}^{(L)}(\kappa_\nu; X, Y)$  etc. in powers of  $X$  and  $Y$  (cf. Table II and Appendix C):

$$\varphi_{ss'}^{(L)}(\kappa_\nu; X, Y) = \varphi_{L, ss', \kappa_\nu} X^{\eta(L, ss', \kappa_\nu)} Y^{\eta(L, ss', \kappa_\nu)} \sum_{ij} b_{ij}^{(L, ss', \kappa_\nu)} X^i Y^j, \text{ etc.} \quad (23)$$

By virtue of (16) and (23),  $C$  is

$$C = (4\pi^2/p^2 q^2 F_1(R)) \cdot \{(2J+1/2J'+1)\} \sum_{\substack{\kappa\kappa' \\ ss', vv' \\ ijmn}} \lambda_s \lambda_{s'}^* [1/(2u+1)]^2$$

$$\begin{aligned}
& \times [a_{ss'uvv'}^{(L)}(\kappa, \kappa_\nu) \varphi_{Lss'\kappa_\nu} \mathfrak{L}_{\kappa nm} b_{ij}^{(Lss'\kappa_\nu)} \\
& \quad \times \mathfrak{S}_{su v}^{m, \mathfrak{E}(Lss'\kappa_\nu)+i+1}(J, J') \mathfrak{S}_{s'uv'}^{m, \mathfrak{I}(Lss'\kappa_\nu)+j+1}(J, J') * \\
& + a_{ss'uvv'}^{(M)}(\kappa, \kappa_\nu) \varphi_{Mss'\kappa_\nu} \mathfrak{M}_{\kappa nm} b_{ij}^{(Mss'\kappa_\nu)} \\
& \quad \times \mathfrak{S}_{su v}^{m, \mathfrak{E}(Mss'\kappa_\nu)+i}(J, J') \mathfrak{S}_{s'uv'}^{m, \mathfrak{I}(Mss'\kappa_\nu)+j+1}(J, J') * \\
& + a_{ss'uvv'}^{(N1)}(\kappa, \kappa_\nu) \varphi_{N1ss'\kappa_\nu} \mathfrak{N}_{\kappa nm} b_{ij}^{(N1ss'\kappa_\nu)} \\
& \quad \times \mathfrak{S}_{su v}^{m, \mathfrak{E}(N1ss'\kappa_\nu)+i+1}(J, J') \mathfrak{S}_{s'uv'}^{m, \mathfrak{I}(N1ss'\kappa_\nu)+j+1}(J, J') * \\
& + a_{ss'uvv'}^{(N2)}(\kappa, \kappa_\nu) \varphi_{N2ss'\kappa_\nu} \mathfrak{N}_{\kappa nm} b_{ij}^{(N2ss'\kappa_\nu)} \\
& \quad \times \mathfrak{S}_{su v}^{m, \mathfrak{E}(N2ss'\kappa_\nu)+i+1}(J, J') \mathfrak{S}_{s'uv'}^{m, \mathfrak{I}(N2ss'\kappa_\nu)+j+1}(J, J') *] \quad (24)
\end{aligned}$$

where

$$\mathfrak{S}_{su v}^{a, b}(J, J') = \int (r-R)^a r^b \mathfrak{T}_{su v}(J, J') dr. \quad (25)$$

(24) is the most general form of the correction factor which we have requested. For the applications to experimental results,  $a_{ss'uvv'}^{(L)}$ , etc. can be obtained by referring to the Tables of Racah or the  $Z$ -coefficients<sup>(22)(23)(25)</sup>.  $\varphi_{Lss'\kappa_\nu}$  etc.,  $\mathfrak{L}_{\kappa nm}$  etc. and  $b_{ij}^{(Lss'\kappa_\nu)}$  etc. should be calculated by the methods given in the Appendix.  $\mathfrak{S}_{su v}^{a, b}(J, J')$ 's are to be considered as parameters. As we have not evaluated the lepton wave functions at the nuclear radius, the number of parameters have considerably increased. The approximation to evaluate the lepton part at the nuclear radius is only permissible when the main terms of these are  $r$ -independent. But, if large cancellations occur between main terms, the small  $r$ -dependent terms might be important. In such cases, this approximation should be discarded even if that might make the number of the unknown parameters increase. By the same reason, in such cases, the effect of the finite size of the nucleus might play an important role in the explanation of the  $\beta$ -spectra and should be taken into account. Discussions on  $\mathfrak{S}_{su v}^{a, b}(J, J')$  and the application of (24) to the experimental results shall be given in the following papers.

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## Appendix A

In eqs. (19a, b),

$$A_{\kappa 1} = B_{\kappa 1} = 1. \quad (\text{A1})$$

$\mathcal{F}^{(i)}(r)$  and  $\mathcal{G}^{(i)}(r)$  satisfy the following differential equations

$$\left. \begin{aligned} \mathcal{F}^{(i)'}(r) &= \kappa r^{-1} \mathcal{F}^{(i)}(r) - (W-1 - V^{(i)}(r)) \mathcal{G}^{(i)}(r), \\ \mathcal{G}^{(i)'}(r) &= (W+1 - V^{(i)}(r)) \mathcal{F}^{(i)}(r) - \kappa r^{-1} \mathcal{G}^{(i)}(r). \end{aligned} \right\} \quad (\text{A2})$$

From these equations, we obtain immediately

$$A_{\kappa 1} = \left( \frac{\mathcal{F}^{(i)'}}{\mathcal{F}^{(i)}} \right)_{r=R} = \frac{\kappa}{R} - \left( W-1 + \frac{\alpha Z}{R} \right) D^{(i)}, \quad (\text{A3a})$$

$$B_{\kappa 1} = \left( \frac{\mathcal{G}^{(i)'}}{\mathcal{G}^{(i)}} \right)_{r=R} = \left( W+1 + \frac{\alpha Z}{R} \right) \frac{1}{D^{(i)}} - \frac{\kappa}{R} \quad (\text{A3b})$$

for arbitrary charge distribution inside the nucleus.

Differentiating eqs. (A2) with respect to  $r$ ,  $A_{\kappa 2}$  and  $B_{\kappa 2}$  are given as :

$$A_{\kappa 2} = \frac{1}{2} \left( \frac{\mathcal{F}^{(i)''}}{\mathcal{F}^{(i)'}} \right)_{r=R} = \frac{1}{2} \left[ \frac{\kappa(\kappa-1)}{R^2} - \left( W-1 + \frac{\alpha Z}{R} \right) \left( W+1 + \frac{\alpha Z}{R} \right) - \left( W-1 - \frac{\alpha Z}{R^2} \right) D^{(i)} \right], \quad (\text{A4a})$$

$$B_{\kappa 2} = \frac{1}{2} \left( \frac{\mathcal{G}^{(i)''}}{\mathcal{G}^{(i)'}} \right)_{r=R} = \frac{1}{2} \left[ \frac{\kappa(\kappa+1)}{R^2} - \left( W+1 + \frac{\alpha Z}{R} \right) \left( W-1 + \frac{\alpha Z}{R} \right) + \left( W+1 - \frac{\alpha Z}{R^2} \right) \frac{1}{D^{(i)}} \right]. \quad (\text{A4b})$$

$A_{\kappa n}$  and  $B_{\kappa n}$  for arbitrary  $n$  can be obtained by successive differentiation of eqs. (A2).

## Appendix B

In general,

$$\begin{aligned} a \mathcal{F}_{-\kappa}^2(R) + b \mathcal{G}_{-\kappa}^2(R) &= (2\rho R)^{2|\kappa|} (16W)^{-1} F_{\kappa}(R) \\ &\times \left[ (a+b) \{S_{\kappa}(\rho R) + WT_{\kappa}(\rho R)\} \right. \\ &\quad \left. - (a-b) \{T_{\kappa}(\rho R) + WS_{\kappa}(\rho R)\} \right], \end{aligned} \quad (\text{B1})$$

and

$$\begin{aligned} a \mathcal{F}_{-\kappa}(R) \mathcal{G}_{-\kappa}(R) - b \mathcal{F}_{\kappa}(R) \mathcal{G}_{\kappa}(R) &= (2\rho R)^{2|\kappa|} (1-W^2)^{1/2} (8W)^{-1} F_{\kappa}(R) \\ &\times \left[ (a-b) i(\alpha Z/\rho) \mathcal{R} \{ \gamma + i\alpha ZW/\rho \} h^2(\rho R) \right. \\ &\quad \left. + (a+b) i\kappa \mathcal{J} \{ (\gamma + i\alpha ZW/\rho) h^2(\rho R) \} \right], \end{aligned} \quad (\text{B2})$$



where

$$F_{\kappa}(R) = 4(2pR)^{2(\gamma+|\kappa|)} e^{\pi\alpha ZW/p} |\Gamma(\gamma + i\alpha ZW/p)|^2 / \Gamma^2(2\gamma + 1), \quad (\text{B3a})$$

$$S_{\kappa}(pR) = \{h^2(pR)(\gamma + i\alpha ZW/p) - h^{*2}(pR)(\gamma - i\alpha ZW/p)\} i\alpha Z/p, \quad (\text{B3b})$$

$$T_{\kappa}(pR) = \kappa h^2(pR)(\gamma + i\alpha ZW/p) + \kappa h^{*2}(pR)(\gamma - i\alpha ZW/p) + 2|h(pR)(\gamma + i\alpha ZW/p)|^2, \quad (\text{B3c})$$

$$h(pR) = e^{-ipR} I^{\gamma}(\gamma + 1 + i\alpha ZW/p, 2\gamma + 1; 2ipR), \quad (\text{B3d})$$

$\mathcal{R}\{X\}$ ,  $\mathcal{I}\{X\}$  are the real and imaginary part of  $X$ .

From (21a, b, c) and (B1, 2),  $l_{\kappa}^{(L)}$ ,  $l_{\kappa}^{(S)}$  etc. are given by

$$l_{\kappa}^{(L)} = (2pR)^{2|\kappa|} (16W)^{-1} F_{\kappa}(R) \{S_{\kappa}(pR) + WT_{\kappa}(pR)\}, \quad (\text{B4a})$$

$$l_{\kappa}^{(L)} = - (2pR)^{2|\kappa|} (16W)^{-1} F_{\kappa}(R) \{T_{\kappa}(pR) + WS_{\kappa}(pR)\}, \quad (\text{B4b})$$

$$m_{\kappa}^{(L)} = (2pR)^{2|\kappa|} (16W)^{-1} F_{\kappa}(R) \{S_{-\kappa}(pR) + WT_{-\kappa}(pR)\}, \quad (\text{B4c})$$

$$m_{\kappa}^{(S)} = - (2pR)^{2|\kappa|} (16W)^{-1} F_{\kappa}(R) \{T_{-\kappa}(pR) + WS_{-\kappa}(pR)\}, \quad (\text{B4d})$$

$$n_{\kappa}^{(L)} = - (2pR)^{2|\kappa|} (8W)^{-1} F_{\kappa}(R) \kappa p \mathcal{I}\{(\gamma + idZW/p) h^2(pR)\}, \quad (\text{B4e})$$

$$n_{\kappa}^{(S)} = - (2pR)^{2|\kappa|} (8W)^{-1} F_{\kappa}(R) \alpha Z \mathcal{R}\{(\gamma + i\alpha ZW/p) h^2(pR)\}. \quad (\text{B4f})$$

In order to apply to the experimental results, it is necessary to calculate the explicit forms of (21a, b, c), namely, those of (B4a-f). The factor  $cS(pR) + dT(pR)$  (with  $c, d = W$  or 1) which appears in (B4a-d) and the factor  $(\gamma + i\alpha ZW/p) h^2(pR)$  in (B4e, f) are

$$\begin{aligned} cS_{\kappa}(pR) + dT_{\kappa}(pR) = & -2c\alpha^2 Z^2 \frac{W}{p^2} + 2d \left\{ \kappa(\gamma + \kappa) + \frac{\alpha^2 Z^2}{p^2} \right\} \\ & - \frac{4}{2\gamma + 1} \frac{\alpha Z}{p} \left[ c(\gamma - 2\alpha^2 Z^2 - 2 \frac{\alpha^2 Z^2}{p^2}) + dW \left\{ 2\kappa^2 + (2\gamma + 1)\kappa + 2 \frac{\alpha^2 Z^2}{p^2} \right\} \right] (pR) \\ & + \frac{4}{(2\gamma + 1)^2 (\gamma + 1)} \left[ c\alpha^2 Z^2 \frac{W}{p^2} \left\{ - (2\gamma + 1)(2\gamma^2 - 1) + (4\gamma + 3)(\kappa^2 + \gamma - 2\alpha^2 Z^2 - \frac{\alpha^2 Z^2}{p^2}) \right\} \right. \\ & \left. + d \left\{ -\kappa\gamma(2\gamma + 1)^2 (\gamma + 1) + \left( \kappa^2 + \frac{\alpha^2 Z^2}{p^2} \right) (-\kappa^2 - \gamma + 4\alpha^2 Z^2 (\gamma + 1) + (4\gamma + 3)(\kappa(\gamma + 1) + \frac{\alpha^2 Z^2}{p^2})) \right\} \right] (pR)^2 \\ & + \dots \dots \dots \end{aligned} \quad (\text{B5a})$$

$$\begin{aligned}
& \left( \gamma + i\alpha Z \frac{W}{p} \right) h^2(pR) \\
&= \gamma + i\alpha Z \frac{W}{p} + \frac{2}{2\gamma+1} \left\{ - (2\gamma+1) \alpha Z \frac{W}{p} + i(\gamma - \alpha^2 Z^2 \frac{W}{p^2}) \right\} (pR) \\
&- \frac{2}{(2\gamma+1)^2(\gamma+1)} \left[ (\gamma+1) \left\{ \gamma(\gamma+1) - (4\gamma+3) \alpha^2 Z^2 \frac{W^2}{p^2} \right\} \right. \\
&\quad \left. + i\alpha Z \frac{W}{p} \left\{ 5\gamma^2 + 5\gamma + 1 - (4\gamma+3) \alpha^2 Z^2 \frac{W^2}{p^2} \right\} \right] (pR)^2 \\
&+ \dots\dots\dots
\end{aligned} \tag{B5b}$$

By Appendices A and B, we can obtain the explicit forms of  $\mathfrak{L}_{\kappa\nu m}$ ,  $\mathfrak{M}_{\kappa\nu m}$  and  $\mathfrak{N}_{\kappa\nu m}$ .

### Appendix C

The neutrino radial wave function  $f_{\kappa\nu}(qr)$  is given by

$$\begin{aligned}
f_{\kappa\nu}(qr) &= \sqrt{\pi q/2r} J_{l(-\kappa_\nu)+1/2}(qr) \\
&= \sqrt{\frac{\pi q}{2r}} \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+l(-\kappa_\nu)+3/2)} \left( \frac{qr}{2} \right)^{l(-\kappa_\nu)+1/2+2m}.
\end{aligned} \tag{C1}$$

Therefore,

$$\begin{aligned}
& f_{\kappa_\nu}(qx) f_{\kappa_\nu''}(qy) \\
&= \pi \left( \frac{q}{2} \right)^{l(-\kappa_\nu') + l(-\kappa_\nu'') + 2} x^{l(-\kappa_\nu')} y^{l(-\kappa_\nu'')} \frac{1}{\Gamma(l(-\kappa_\nu') + 3/2) \Gamma(l(-\kappa_\nu'') + 3/2)} \\
&\quad \times \left[ 1 - \frac{1}{l(-\kappa_\nu') + 3/2} \frac{q^2}{4} x^2 - \frac{1}{l(-\kappa_\nu'') + 3/2} \frac{q^2}{4} y^2 + \dots \right].
\end{aligned} \tag{C2}$$

Namely,

$$\varphi_{X\kappa_\nu} = \frac{\pi}{\Gamma(l(-\kappa_\nu') + 3/2) \Gamma(l(-\kappa_\nu'') + 3/2)} \left( \frac{q}{2} \right)^{l(-\kappa_\nu') + l(-\kappa_\nu'') + 2}, \tag{C3a}$$

$$\hat{\kappa}(X\kappa_\nu) = l(-\kappa_\nu'), \quad \eta(X\kappa_\nu) = l(-\kappa_\nu''), \tag{C3b}$$

$$\delta_{00}^{(X\kappa_\nu)} = 1, \tag{C3c}$$

$$\delta_{10}^{(X\kappa_\nu)} = \delta_{01}^{(X\kappa_\nu)} = \delta_{11}^{(X\kappa_\nu)} = 0, \tag{C3d}$$

$$\delta_{20}^{(X\kappa_\nu)} = -\frac{1}{l(-\kappa_\nu') + 3/2} \frac{q^2}{4}, \quad \delta_{02}^{(X\kappa_\nu)} = -\frac{1}{l(-\kappa_\nu'') + 3/2} \frac{q^2}{4}. \tag{C3e}$$

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## Beta-ray Spectra, II

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The relation between the many-particle and the single-particle matrix element is calculated in the  $jj$  coupling scheme.

## § 1. Introduction

In the previous paper<sup>1)\*</sup>, the correction factor for the transitions forbidden to any degree has been obtained. This correction factor contains the unknown parameters  $\mathfrak{S}_{sur}^{a,b}(J, J')$ . In the present paper, the transition probability for many-nucleon configuration is reduced to that of the single-nucleon by making use of the  $jj$  coupling model. By this reduction, all the ambiguities of  $\mathfrak{S}_{sur}^{a,b}(J, J')$  on the ground of the nuclear configuration are removed, except those from the unknown weights of the configuration mixing, and the only remaining ambiguity is that of the single nucleon wave function.

For the allowed transition, such a reduction has been given by Talmi<sup>2)</sup> with isotopic spin formalism. For the two nucleon configuration, Brysk<sup>3)</sup> and Rose and Osborn<sup>1)</sup> have given this reduction. In this paper we have not restricted the number of the nucleons present and the degree of forbiddenness.

## § 2. Calculation

As the  $jj$  coupling shell model seems to be a good approximation for heavy nuclei, we shall adopt this model in this calculation. For heavy nuclei the total isotopic spin quantum number cannot be a good quantum number by virtue of the Coulomb field. Therefore, the total isotopic spin quantum number shall not be used in the following.

We shall consider the transition of a neutron with angular momentum  $j_N$  and in the shell ( $N$ ) to a proton with angular momentum  $j_P$  and in the shell ( $P$ ). The shell ( $P$ ) is not saturated in the initial state. The numbers of the nucleons in the shell ( $P$ ) (shell ( $N$ )) in the initial and the final states shall be denoted by  $p'$  and  $p$  ( $n'$  and  $n$ ), and the spins by  $J_{P'}$  and  $J_P$  ( $J_{N'}$  and  $J_N$ ). The initial and the final state wave functions are given by

$$\Phi(\alpha' \{[(j_{P'}^{p'}) J_{P'}, (j_{N'}^{n'}) J_{N'}] J_{T'} J_C\} J' M'), \quad (1a)$$

$$\Psi(\alpha \{[(j_P^p) J_P, (j_N^n) J_N] J_T J_C\} J M). \quad (1b)$$

The coupling of  $J_P$  and  $J_N$  ( $J_{P'}$  and  $J_{N'}$ ) gives  $J_T$  ( $J_{T'}$ ). The coupling of the spins of

\* This will be referred to as I.

all the nucleons except those in the shell ( $P$ ) and ( $N$ ) gives a resultant spin  $J_R(J'_R)$ . The spin of the final (initial) nucleus ( $J'$ ) consists of  $J_T$  and  $J_R$  ( $J'_T$  and  $J'_R$ ).  $M(M')$  is the magnetic quantum number of the final (initial) nucleus.  $\alpha$  and  $\alpha'$  stand for the additional quantum numbers necessary to define the states.

The transition matrix element  $\int H d\tau$  is the matrix element of the linear combination of the tensors of degree  $u$  (I. 12, 13) :

$$\omega_{sp} \times \mathcal{O}_{u(k_i v)}^{-\mu \nu - \mu} (\sigma_p, \mathbf{p}) \mathcal{Q}_p. \quad (2)$$

For the sake of simplicity, we write this linear combination as follows :

$$\sum_{su\nu p} A_{su\nu p}^m \quad m = u, u-1, \dots, -u,$$

where  $A_{su\nu p}^m$  is the  $m$ th component of the tensor and operates on the  $p$ th nucleon.  $s$  specifies the sort of the operator for  $\beta$ -decay (I. Table I). The parity of this tensor is determined by  $v$  and  $\omega$  (I. 5, 6, 10).

If the wave functions (1a) and (1b) are antisymmetrical in the protons and the neutrons separately, the matrix element of the operator (3) between these states is<sup>3)</sup>:

$$\begin{aligned} \int H d\tau &= (\alpha \{[(j_P^p) J_P, (j_N^n) J_N] J_T J_R\} J M | \sum_{su\nu p} A_{su\nu p}^m | \alpha' \{[(j_P^{p'}) J'_P, (j_N^{n'}) J'_N] J'_T J'_R\} J' M') \\ &= \{(N+1)Z\}^{1/2} \sum_{su\nu} (\alpha \{[(j_P^p) J_P, (j_N^n) J_N] J_T J_R\} J M | A_{su\nu p}^m | \\ &\quad \times \alpha' \{[(j_P^{p'}) J'_P, (j_N^{n'}) J'_N] J'_T J'_R\} J' M') \end{aligned}$$

where  $N$  and  $Z$  are the numbers of the neutrons and the protons in the final nucleus.

We shall define the reduced matrix element  $M(J_P J_N J_T J_R J; u; J'_P J'_N J'_T J'_R J')$  and the reduced single-nucleon matrix element  $[j_P | A_{su\nu} | j_N]$  as follows :

$$\int H d\tau = (J' u J | M' m M) M(J_P J_N J_T J_R J; u; J'_P J'_N J'_T J'_R J'), \quad (5a)$$

$$(j_P | A_{su\nu}^m | j_N) = (j_P u j_N | m_P m_N - m_P m_N) [j_P | A_{su\nu} | j_N]. \quad (5b)$$

We have omitted the suffix  $\rho$  for the sake of simplicity. As the nucleon transforms from the shell ( $N$ ) to the shell ( $P$ ),  $J_R$  does not change :

$$J_R = J'_R.$$

Making use of Racah's method<sup>(4)</sup>, we can obtain the following relation between  $M(J_P J_N J_T J_R J; u; J'_P J'_N J'_T J'_R J')$  and  $[j_P | A_{su\nu} | j_N]$  :

$$\begin{aligned} M(J_P J_N J_T J_R J; u; J'_P J'_N J'_T J'_R J') \\ &= (-)^{J'_P - J_P + J - J_R + J_T - j_P} [(N+1)Z]^{1/2} \\ &\quad \times \sum_{su\nu} \sum_{\alpha^{(1)} \alpha^{(1)'}} (\alpha (j_P^p) J_P \{ | \alpha^{(1)} (j_P^{p-1}) J'_P j_P J_P \} (\alpha^{(1)'} (j_N^{n-1}) J_N j_N J'_N) | \alpha' (j_N^n) J_N) \\ &\quad \times [(2J_P+1)(2J'_N+1)(2j_P+1)(2J'_T+1)(2J_T+1)(2J'+1)]^{1/2} \\ &\quad \times W(J J_T J' J'_T; J_R u) \begin{Bmatrix} J_P j_P J'_P \\ J_N j_N J'_N \\ J_T u J'_T \end{Bmatrix} [j_P | A_{su\nu} | j_N]. \quad (6) \end{aligned}$$



References on the Racah and the  $9j$ -coefficients are given in I.

*Special case (i) :  $J_R=0$ .* In this case,

$$J_T=J, \quad J'_T=J',$$

and the initial and the final states are :

$$\Phi(\alpha'[(j_P^{p'})J'_P, (j_N^{n'})J'_N]J'M'), \quad (1a')$$

$$\Psi(\alpha[(j_P^p)J_P, (j_N^n)J_N]JM). \quad (1b')$$

(6) becomes as follows :

$$\begin{aligned} & M(J_P J_N J; u; J'_P J'_N J') \\ &= (-)^{J'_P - J_P + J - J' + u - j_P} [(N+1)Z]^{1/2} \\ & \times \sum_{sup} \alpha^{(1)} \alpha^{(1)'} (\alpha(j_P^p)J_P \{|\alpha^{(1)}(j_P^{p-1})J'_P j_P J_P\rangle (\alpha^{(1)'}(j_N^{n-1})J'_N j_N J'_N)\} \alpha'(j_N^{n'})J'_N) \\ & \times [(2j_P+1)(2J_P+1)(2J'_N+1)(2J'+1)]^{1/2} \\ & \times \left\{ \begin{matrix} J_P j_P J'_P \\ J_N j_N J'_N \\ J \quad u \quad J' \end{matrix} \right\} [j_P | A_{sup} | j_N]. \end{aligned} \quad (7)$$

*Special case (ii) : two-nucleon configuration.* If an odd-odd nucleus with odd neutron in  $j_N$  and an odd proton in  $j_P$  and total angular momentum  $J'$  undergoes a  $\beta$ -decay to the state  $J$  of the configuration  $j_P$  and  $j'_P$  in the daughter nucleus, the initial and the final states are :

$$\Phi(\alpha' j'_P j_N J' M'), \quad (1a'')$$

$$\Psi(\alpha j'_P j_P J M). \quad (1b'')$$

In this case,

$$\begin{aligned} J_P &= j_P, \quad J_N=0, \quad J_T=j_P, \quad J_R=j'_P, \\ J'_P &= 0, \quad J'_N=j'_N, \quad J'_T=j'_N, \quad J'_R=j'_P, \end{aligned}$$

and (6) becomes :

$$\begin{aligned} & M(j_P j'_P J; u; j_P j_N J') \\ &= (-1)^{j_N - j'_P + J - u} [(N+1)Z]^{1/2} [(2J'+1)(2j_P+1)]^{1/2} \\ & \times W(J' j_N J j_P; j'_P u) [j_P | A_{sup} | j_N]. \end{aligned} \quad (8)$$

### § 3. Further remarks

From (I. 12) we get

$$[j_P | A_{sup} | j_N] = \lambda_g (-1)^{j+j_v+l+l(z_v)} \begin{pmatrix} j & j_v & u \\ -\mu & -\mu & \mu + \mu_v \end{pmatrix} \bar{\mathfrak{R}}_{sup}(\kappa \kappa_v K K') \quad (9)$$

where  $\bar{\mathfrak{R}}_{sup}(\kappa \kappa_v K K')$  is  $\mathfrak{R}_{sup}(\kappa \kappa_v J J')$  of (I. 13a, b) for single nucleon transition.  $K = \pm (J + \frac{1}{2})$  for  $J = L \mp \frac{1}{2}$ . From (6) and (9) we obtain

$$\begin{aligned}
\int H d\tau = & \sum_{sum} \lambda_s \sum_{\alpha(1)} \lambda_{\alpha(1)'} [(N+1)Z]^{1/2} (-1)^{J'_P - J_P - J_R + J_T - j_P + j + j_v + l + l(x_v) + M} \\
& \times (2J+1)^{1/2} \begin{pmatrix} j & j_v & u \\ -\mu & -\mu_v & \mu + \mu_v \end{pmatrix} \begin{pmatrix} J & J' & u \\ M & -M' & M' - M \end{pmatrix} \\
& \times (\alpha(j_P^p) J_P \{ |\alpha^{(1)}(j_P^{p-1}) J_P' j_P j_P \} (\alpha^{(1)'}(j_N^{n-1}) J_N j_N j_N' \} \alpha'(j_N^{n'}) J_N') \\
& \times [(2J_P+1)(2J_N'+1)(2j_P+1)(2J_T'+1)(2J_T+1)(2J'+1)]^{1/2} \\
& \times W(JJ_T J' J_T'; J_R u) \left\{ \begin{matrix} J_P & j_P & J_P' \\ J_N & j_N & J_N' \\ J_T & u & J_T' \end{matrix} \right\} \mathfrak{R}_{sum}(\kappa \kappa_v K_P K_N). \quad (10)
\end{aligned}$$

As to the transition probability, or the correction factor, the phase factor of (10) is irrelevant\*. Therefore, we can obtain the transition probability of  $\beta$ -decay in terms of single nucleon transition matrix elements by substituting  $\lambda_s B(u) \mathfrak{R}_{sum}^{a,b}(\kappa \kappa_v K_P K_N)$  for  $\lambda_s \mathfrak{R}_{sum}(\kappa \kappa_v J J')$  in (I. 14), where

$$\begin{aligned}
B(u) = & [(N+1)Z]^{1/2} \sum_{\alpha(1)} \sum_{\alpha(1)'} (\alpha(j_P^p) J_P \{ |\alpha^{(1)}(j_P^{p-1}) J_P' j_P j_P \} (\alpha^{(1)'}(j_N^{n-1}) J_N j_N j_N' \} \\
& \times \alpha'(j_N^{n'}) J_N') [(2J_P+1)(2J_N'+1)(2j_P+1)(2J_T'+1)(2J_T+1)(2J'+1)]^{1/2} \\
& \times W(JJ_T J' J_T'; J_R u) \left\{ \begin{matrix} J_P & j_P & J_P' \\ J_N & j_N & J_N' \\ J_T & u & J_T' \end{matrix} \right\}. \quad (11)
\end{aligned}$$

As for the correction factor, we should put  $\lambda_s B(u) \mathfrak{R}_{sum}^{a,b}(K_P K_N)$  in place of  $\lambda_s \mathfrak{R}_{sum}^{a,b}(J J')$  in (I. 24). Here  $\mathfrak{R}_{sum}^{a,b}(K_P K_N)$  is  $\mathfrak{R}_{sum}^{a,b}(J J')$  of (I. 25) for single nucleon transition.

In this paper we have considered only the transition between the pure states (1a) and (1b). If there is a configuration mixing<sup>7)</sup> in these states, (11) or (12) should be replaced by a linear combination of  $A$  or  $B$ . The coefficients of the linear combination is determined by the mixing ratio.

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\* Here the phase factor means  $(-)^{J'_P - J_P - J_C + J_T - j_P + j + j_v + l + l(x_v) + M}$ , which is common for all terms in the summation.

## Gamma Ray Energy Spectrum from Iron Bombarded by 14-Mev Neutrons

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The energy spectrum of gamma rays from iron when bombarded by 14-Mev neutrons is calculated on the basis of the compound nucleus formalism. The cross sections for various competing processes are found to be considerably dependent upon the magnitude of temperature. We have assumed three values of  $a$ , the quantity closely related to temperature. Although they are based upon the experimental evidences of different sources, none of them is found to be able to account for the experimental energy spectrum. There seems to remain a difficulty in the statistical theory pointed out by Cohen.

### § 1. Introduction

In the previous paper<sup>1)</sup>, cited as I, we have investigated the yield of gamma rays excited by fast neutrons, aiming at the study of the mass number dependence of nuclear temperature. We were able to show that the statistical theory could account for the over-all behaviour of the gamma ray yields for various nuclei, provided that the empirical values of temperatures as well as compound nucleus formation cross sections were assumed. Although this result may be thought to support the statistical description of nuclear reactions, the material<sup>2)</sup> we employed is not always suitable to our purpose by several reasons. Firstly, the energy of incident neutrons was not monochromatic, but had a broad width around 10 Mev. Secondly, the number of gamma quanta was not directly counted, but was observed with a counter whose efficiency is assumed to be proportional to the energy of gamma-rays. Thirdly, we were only concerned with the over-all  $A$ -dependence, but ignored the detailed properties of individual nuclei, except for three cases where the detailed properties would be exhibited. These drawbacks are forced by the shortage of material available in those days, while our analysis seemed enough to prove the essential correctness of the statistical theory.

Recent experiments with a monochromatic beam have facilitated one to make a closer examination of the statistical theory by observing the energy spectrum of emitted neutrons.<sup>3,4)</sup> The nuclear temperature thus obtained shows the  $A$ -dependence similar to that obtained by Blatt and Weisskopf,<sup>5)</sup> as employed in I, but its absolute magnitude is

greatly different, depending upon the method to be based on. The difference in the absolute magnitude of temperature is so great that it seems worth while to examine this point on a little different basis, namely on the analysis of the gamma-ray spectrum from iron excited by neutrons of nearly monochromatic energy 14 Mev.<sup>6)</sup> Such a study is intended to make up the drawbacks mentioned above in reference to our previous work.

The method we used is essentially the same as that in I. We are concerned only with the most abundant isotope of iron,  $^{56}\text{Fe}$  (93%), and calculate the energy spectrum of gamma rays by assuming three values of temperatures, i. e., those adopted by Blatt and Weisskopf,<sup>7)</sup> by Gugelot<sup>8)</sup> and by Graves and Rosen<sup>4)</sup>. Since the temperature depends upon the excitation energy, we rather use, as before, constant  $\alpha$  appearing in the level density formula (I. 2). The interpolation of Blatt and Weisskopf's  $\alpha$  values gives us for  $^{56}\text{Fe}$

$$\alpha = 1.7 \text{ Mev}^{-1}. \quad (\text{I})$$

On the other hand, Gugelot obtained

$$\alpha = 4.75 \text{ Mev}^{-1} \quad (\text{II})$$

by observing the energy spectrum of neutrons from iron bombarded by protons of 16 Mev, while Graves and Rosen obtained

$$\alpha = 24.5 \text{ Mev}^{-1} \quad (\text{III})$$

by observing the energy spectrum of neutrons excited by neutrons of 14 Mev. Since the condition in the last one is exactly the same as that we refer to, one might expect that the third choice of  $\alpha$  should give such a spectrum of gamma rays as to be in good agreement with the experiment so long as  $\alpha$  were really an energy independent constant. This is the case, as will be shown below, in the shape of the energy spectrum, but the absolute intensity calculated is too large. Further unsatisfactory points are observed, for example, in the  $(n; p)$  cross section. This shows the inconsistency of the statistical theory as far as our analysis is uncritically adopted. We may, however, say that the inconsistency is due to the energy dependence of  $\alpha$ , as can be expected in the nuclear model different from the Fermi-gas. Observing the above three values of  $\alpha$ , one can see the considerable energy dependence of  $\alpha$ , which looks somewhat queer. Leaving out the discussions on the energy dependence to the forthcoming paper, we concentrate ourselves to the point whether the values of  $\alpha$  obtained from the different sources can account for the energy spectrum of gamma-rays from iron irradiated by 14 Mev neutrons.

## § 2. Outline of calculations

Our method of calculations is essentially the same as that in the previous paper I, except the following point. Since we were to do with the integrated yield of gamma rays in I, we took only two steps of evaporations into account and further took the average excitation energy for intermediate nuclei as in (I.8). In the present case such approximations are not permissible but the detailed distribution of excitation energies in each intermediate nucleus has to be taken into account, because we are concerned with the energy

spectrum of gamma rays.

For this purpose we define the branching probability  $P_{ij}(E)$ , which is the probability of emitting particle  $j$  from the intermediate nucleus left after the emission of particle  $i$  by

$$P_{ij}(E) = F_j(E - S_{ij}) / \sum_k F_k(E - S_{ik}), \quad (1)$$

where  $E$  is the excitation energy of the intermediate residual nucleus and  $S_{ij}$  is the separation energy of particle  $j$  from it. Quantity  $F_i$  is proportional to the partial width for disintegration with emission of  $i$  and is defined by (I.5; 6). If a compound nucleus excited by  $E_{ex}$  can evaporate two particles, the probability for having emission of  $i$  then  $j$  is given by

$$J_{ij} / \sum_{kl} J_{kl}. \quad (2)$$

$J_{ij}$  is given, if  $i$  is a particle, by

$$J_{ij} = \frac{2M_i}{\hbar^2} \int_0^{E_{ex} - S_i - S_{ij}} \epsilon \sigma_O^{(i)}(\epsilon) \tau v_R(E_{ex} - S_i - \epsilon) P_{ij}(E_{ex} - S_i - \epsilon) d\epsilon; \quad (3)$$

when  $i$  is a gamma ray,

$$J_{\gamma j} \approx \frac{3}{4} \cdot \frac{e^2}{\hbar c} \cdot \left( \frac{R}{\hbar c} \right)^2 \frac{1}{D_0} \int_0^{E_{ex} - S_j} \epsilon^3 \tau v_R(E_{ex} - \epsilon) P_{\gamma j}(E_{ex} - \epsilon) d\epsilon. \quad (4)$$

Notations adopted here are the same as in I and of usual meaning.

Thus we can obtain the cross section  $\sigma(l; i, j)$  of  $(l; i, j)$  reaction by

$$\sigma(l; i, j) = \sigma_O^{(i)} (J_{ij} / \sum_{k, m} J_{km}). \quad (5)$$

To obtain the gamma ray energy spectrum to be compared with experimental one, we must take into account that the number of gamma quanta excited from nucleus can be one or more before it is left in the ground state. This is performed by iterating the procedure above. The time of iteration, namely the number of emitted gamma quanta, is found to depend strongly upon the temperature or  $u$ . The lower the temperature of the excited nucleus is, the more the number of gamma rays thus emitted becomes. In fact, the residual nucleus after the reemission of a neutron can emit about 3~5 gamma quanta in case (I), while it emits 6~8 gamma quanta in case (III). Since all these gamma rays are integrated, the total cross section obtained from Fig. 3 should be much larger than the inelastic total cross section in a usual sense, namely, 1.4 barns.

A particular attention has to be paid to the low lying excited states of  $^{56}\text{Fe}$ . They are the levels of excitation energies 0.845, 2.66, 2.98 Mev. etc. As the actual level density decreases rapidly near the ground state with decreasing excitation energy contrary to the level density formula we employed, gamma ray transitions to levels lower than 2 Mev, except to the ground state, are cut off in the  $(n; n, \gamma)$  reaction. This cut off does little influence on the high energy portion of the calculated spectrum, but rather on the cascades of gamma rays between low lying levels. This is because the high energy portion is due



mainly to the contribution of the first gamma rays from the residual nucleus to be left at the levels of excitation energies of about several Mev after re-emitting a neutron. Without this cut off, the gamma ray yield should increase about by factor 2 in the low energy portion.

In order to carry out our calculations, we have to consider: (i) the determination of separation energies; (ii) the determination of the cross section for the formation of a compound nucleus; (iii) the determination of level densities; (iv) the evaluation of integrals  $F$  and  $J$ .

Since our calculation refers to the case where 14-Mev neutrons bombard  $^{56}\text{Fe}$  nuclei, the compound nucleus is  $^{57}\text{Fe}$  and its excitation energy is  $14\text{Mev} + S_n$ . The emission of  $\alpha$ ,  $d$ , and  $t$  etc. may be of negligible probability, so that we may take account only of  $n$ ,  $p$  and  $\gamma$ .

(i) *Separation energies*: The observation of  $(n; \gamma)$  reactions gives us 7.6 Mev and 7.3 Mev for  $S_n$  and  $S_{pn}$  respectively<sup>7)</sup>.  $\beta$ -decay data show that the ground state of  $^{56}\text{Mn}$  is lower than that of  $^{56}\text{Fe}$  by 3.5 Mev. From this value we obtain 11.1 Mev for  $S_p$  and 10.8 Mev for  $S_{np}$ . These values satisfy the relation  $S_p + S_{pn} = S_n + S_{np}$ , as it should be.  $S_{nn}$  is obtained from  $(\gamma; n)$  reactions as 11.2 Mev.<sup>7a)</sup>

(ii) *Cross sections for the formation of a compound nucleus*: It is necessary to know the cross section  $\sigma_c$  for neutron and proton in calculating integrals  $J$  and  $F$ . For neutrons the formula of  $\sigma_c$  has been given by Feshbach and Weisskopf.<sup>8)</sup> With this formula we obtain  $\sigma_c$  of 1.4 barns for incident 14-Mev neutron in good agreement with experimental one<sup>9)</sup>. For protons calculated values of  $\sigma_c$  have been listed by Blatt and Weisskopf<sup>10)</sup> for  $Z$  larger than 10. We simply used the values interpolated from the table referred above. The value of  $\sigma_c$  in our case is shown in Fig 1 and Fig 2.

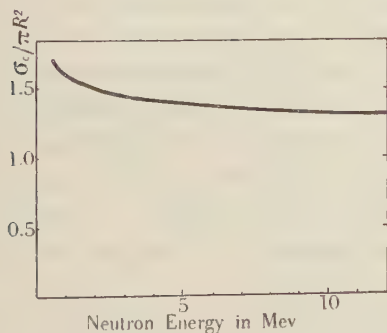


Fig 1. Cross section for the formation of compound nucleus by neutrons, vs. neutron energy for Fe.

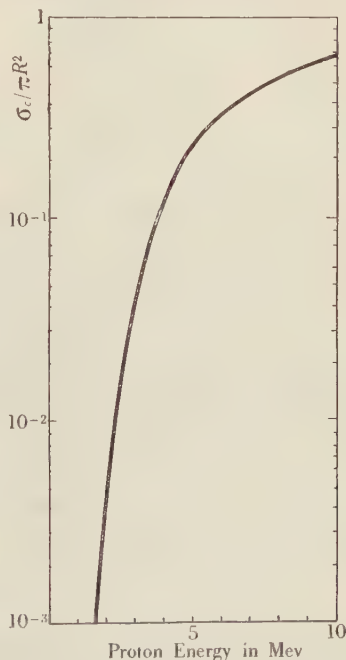


Fig 2. Cross section for the formation of compound nucleus by protons, vs. proton energy for Fe.

(iii) *Level densities*: As mentioned above, we employ the level density  $w(E)$  given by

$$w(E) = c \exp(2\sqrt{aE}) \quad (6)$$

with

$$4c_{\text{even,even}} = 2c_{\text{even,odd}} = c_{\text{odd,odd}}. \quad (7)$$

Here  $a$  is assumed to be constant with a small change in mass number in the course of evaporation. Calculation is performed in three cases of  $a$ , namely (I), (II) and (III).

(iv) *Evaluation of  $F$  and  $J$* : They are evaluated in each step of evaporation by numerical integrations.

### § 3. Results and discussions

Calculated cross sections are shown in Table 1.

Table 1. Cross section in barn

Reaction	(I) Blatt-Weisskopf 1.7 Mev	(II) Gugelot 4.74 Mev	(III) Graves and Rosen 24.5 Mev
$n; n, \gamma$	0.93	1.18	1.24
$n; n, n$	0.40	0.20	$< 10^{-3}$
$n; n, p$	0.001	$< 10^{-3}$	$< 10^{-3}$
$n; p, \gamma$	0.07	0.01	$< 10^{-3}$
$n; p, n$	0.003	$< 10^{-3}$	$< 10^{-3}$
$n; p, p$	$< 10^{-3}$	$< 10^{-3}$	$< 10^{-3}$
$n; \gamma, \gamma$	0.002	0.004	0.05
$n; \gamma, n$	$< 10^{-3}$	0.006	0.11
$n; \gamma, p$	$< 10^{-3}$	$< 10^{-3}$	$< 10^{-3}$

It is remarkable that in case (III)  $\sigma(n; 2n)$  and  $\sigma(n; p)$  are small, whereas  $\sigma(n; \gamma)$  is considerable. This is because the radiation width increases more rapidly than other particle widths with increasing  $a$ . In other words, the increase of  $a$ , that is to correspond to the decrease of temperature, results in the fractional decrease of particle energies greater than that of gamma ray energies. Hence the gamma ray emission predominates far over the particle emission, because of the steeper energy dependence of the former process. The choice of  $a$  can be checked by comparing the calculated  $\sigma(n; p)$  with experiments. Cohen<sup>11)</sup> has obtained the  $(n; p)$  cross section of 18.5 mb from the bombardment of iron by Be- $n$  neutrons. This lies near the case (II). In order to make  $\sigma(n; p)$  the order of the experimental value, the magnitude of  $a$  must lie near Gugelot's  $a$ .

Calculated gamma ray spectrum in each case is shown in Fig 3. In Case (I) the average energy of gamma rays is larger than the observed one. It is seen from this that the small magnitude of  $a$  makes the nuclear temperature too high. In Case (II) and Case (III), the shapes of the gamma-ray spectrum are in rough agreement with the experimental result

but the absolute yield of gamma rays is larger by factor 2. Since the residual energy after the emission of two neutrons is considered to be small, the absolute yield of gamma rays is approximately proportional to the  $(n; n, \gamma)$  cross section. In these cases,  $\sigma(n; n, \gamma)$  may be larger than the actual cross section. This discrepancy perhaps comes from that the formula of  $F_\gamma$  gives the larger magnitude than the actual radiation width<sup>12)</sup>. The modification of the magnitude of  $F_\gamma$  will, however, alter the shape of gamma ray spectra, too.

Thus from the above considerations it may be concluded that the magnitude of  $a$  may be larger at high excitation energies than that deduced for low energies in Case (I). If  $a$  is the magnitude of B-W's at 1 Mev excitation energy and is the order of Graves and Rosen's at approximately 10~14 Mev excitation energy, however, the nuclear temperature has to become lower with increasing excitation energy as pointed out by Cohen<sup>13)</sup>. Such a queer tendency that the temperature increases with decreasing excitation energy can be seen also from the detailed spectrum shapes shown in Fig 3.

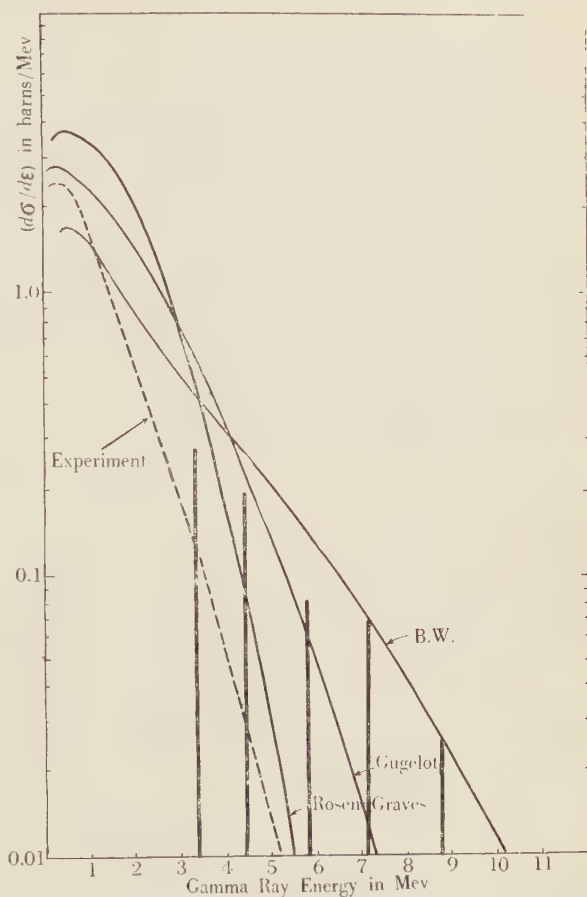


Fig 3. Calculated gamma ray energy spectra from iron excited by 14 Mev neutrons in the following three cases :

- (I) Blatt and Weisskopf,
- (II) Gugelot,
- (III) Graves and Rosen.

Dashed-line shows the experimental spectrum obtained by Scherrer et. al. Vertical rods indicate the discrete spectrum identified by them.

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# On the Renormalization in Tamm-Dancoff Approximation for One-nucleon Problem, II

— Subtraction of Divergences in the Generalized  
Tamm-Dancoff Equations —

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The subtraction of the divergences are carried out in the generalized Tamm-Dancoff equations derived in the previous paper (Part I). The essence of the method is the extension of Fubini's procedure to the arbitrarily higher order approximation. That is, first, we construct the formal solutions which satisfy the equations in question; they contain the infinities and accordingly are quite meaningless. Then, we separate the infinities (including the overlapping divergences) individually at each stage of the construction of these formal solutions. The forms of the nucleon propagation function or the vertex parts which have been made convergent by this method depend on the configurations to which they refer. However, this seems to be inevitable for the present approximation method, i. e. the reduction of the infinite set of coupled integral equations to the finite one.

## § 1. Introduction

In the previous paper (Part I),<sup>1)</sup> we have proposed the generalized Tamm-Dancoff equations. The next problem is how to subtract the divergences from them. Fubini<sup>2)</sup> has already given the answer to this problem in the approximation where the meson number (in our sense) is restricted within two. However, since the generalized T-D equations reduce to the essentially single equation in this order of approximation and the procedure of Fubini rests on this specially simple circumstance, it is not applicable directly to the higher order approximations.

That is, in this order of approximation, the starting equations are eqs. (19), (20) and (21) of Part I with

$$K(\not{p}_1 - \not{k}_3, \not{k}_1, \not{k}_2, \not{k}_3; \not{p}_0, \not{k}_0) = 0.$$

Substituting eq. (I. (21)) into eq. (I. (20)) and performing the subtraction of the self-energy type divergence caused by this substitution, we get the following equations:

$$\begin{aligned} K(\not{p}_1; \not{p}_0, \not{k}_0^\beta) &= g(2\pi)^{-2} \int d\bar{k}_1 S_F(\not{p}_1) \gamma_5 \tau_a K(\not{p}_1 - \not{k}_1; \not{k}_1^\alpha; \not{p}_0, \not{k}_0^\beta), \\ K(\not{p}_1, \not{k}_1^\alpha; \not{p}_0, \not{k}_0^\beta) &= \delta(\not{p}_1 - \not{p}_0) \delta(\not{k}_1 - \not{k}_0) \delta_{\alpha\beta} S_F'(\not{p}_1) \not{A}_F(\not{k}_1) \\ &\quad + g(2\pi)^{-2} S_F'(\not{p}_1) \gamma_5 \tau_a \not{A}_F(\not{k}_1) K(\not{p}_1 + \not{k}_1; \not{p}_0, \not{k}_0^\beta) \end{aligned} \quad (1)$$



$$+ g^2 (2\pi)^{-4} S_F'(\not{p}_1) \not{A}_F(k_1) \gamma_5 \tau_\tau \int d\bar{k}_2 S_F(\not{p}_1 - \bar{k}_2) \gamma_5 \tau_a K(\not{p}_1 + \bar{k}_1 - \bar{k}_2, \not{k}_2^\tau; \not{p}_0, \not{k}_0^B), \quad (2)$$

where  $S_F'(\not{p}_1)$  is the same as that of Fubini (see also § 3). However, the latter equation (2) can be solved (apart from the infinities contained) for  $K(\not{p}_1, \not{k}_1^a; \not{p}_0, \not{k}_0^B)$  without the knowledge of the form of the function  $K(\not{p}_1; \not{p}_0, \not{k}_0^B)$  because of the fact that the conservation of 4-momenta holds here (c. f. the paper of Fubini). But, in the next higher order approximation, our equations become coupled integral equations for  $K(\not{p}_1; \not{p}_0, \not{k}_0)$ ,  $K(\not{p}_1, \not{k}_1; \not{p}_0, \not{k}_0)$  and  $K(\not{p}_1, \not{k}_1, \not{k}_2; \not{p}_0, \not{k}_0)$ , and the simple circumstance such as described above no longer exists for this case, i. e. the equations for  $K(\not{p}_1, \not{k}_1; \not{p}_0, \not{k}_0)$  and  $K(\not{p}_1, \not{k}_1, \not{k}_2; \not{p}_0, \not{k}_0)$  are essentially coupled ones. Accordingly, the method of Fubini is not applicable directly to higher order approximations. Moreover, he has mentioned nothing concerning with the subtraction of the overlapping divergences.

The main purpose of this paper is to show how to extend Fubini's procedure to any higher order approximations. The outline of the method is as follows:

Analogously to the method of Fubini, we find the method for constructing the formal solutions which satisfy the generalized T-D equations in any higher order approximation. These solutions contain the various types of infinities and therefore they are meaningless for themselves. Then we perform the subtraction of those divergences at each stage of constructing the solutions. In this connection, it should be noted that the quantities appearing in our formal solutions generally contain the overlapping divergences in a very complicated form. In this paper, it will also be shown that the convergent parts of them can be consistently defined by the help of the method of Chiba<sup>(3)</sup> and Tanaka and Ito.<sup>(4)</sup>

For clearness, first, we investigate the nature of those formal solutions by the help of Feynman-Dyson diagrams (§ 2). Second, we show the method for constructing those solutions in the analytic form (§ 3), and finally perform the subtraction of the divergences (§ 4). For simplicity, these are all investigated in the approximation where the meson "number" is restricted within three. Section 5 will be devoted to the consideration concerning the case of arbitrarily higher order approximation.<sup>(5)</sup>

## § 2. Consideration with the Feynman-Dyson diagrams

For simplicity, we shall consider the pion-nucleon scattering problem in the approximation where the meson "number" is restricted within three. Because we trace the events along the nucleon line, we shall draw the nucleon line as a straight line from the "initial" stage (below) to the "final" stage (above). Hence the word "meson number at a certain stage" in this article means the number of meson lines which intersect the straight line drawn perpendicularly to the nucleon line at the corresponding stage. As explained in Part I, we keep the order of vertices along the nucleon line as  $(x_0), \tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n(x_1)$ , and omit the factor  $(n!)^{-1}$  in the defining equations of the Feynman kernels (eqs. (7), (8) and (9) in Part 1). Therefore, we do not distinguish between the diagrams which can be transformed to each other by only changing the name of vertices.

Now, suppose that all the diagrams that contribute to the pion-nucleon scattering process are drawn out. It should be possible to classify these diagrams into 3 kinds as follows:

- a) the diagrams that do not possess any intermediate stages of meson number 0 or 1.
- b) the diagrams which neither pass any intermediate stages of meson number 0 nor belong to the class a).
- c) all the diagrams except those belonging to the classes a) or b).

The assembly of all diagrams belonging to the class a) will represent the whole of the processes in which the meson number varies as

$$1 \rightarrow 2 \rightarrow 1$$

$$\text{or} \quad 1 \rightarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow (\text{any times of } 2 \rightarrow 3 \rightarrow 2) \rightarrow 1. \quad (3)$$

In accordance with the notation of § 3, we shall call the whole of these processes as the  $G_0$ -process and the assembly of the diagrams belonging to class a) as the  $G_2$ -diagram. In other words, the  $G_2$ -diagram can be constructed by the following procedure. First, suppose that all the 2-meson scattering diagrams that have no intermediate stage of meson number 0 or 1. We shall name the process corresponding to the assembly of these diagrams as  $R_0$ -process. Next, close the open end of one meson line to the nucleon line at each of the "initial" and "final" stages in those diagrams (dotted line in Fig. 1). Then the assembly of all the diagrams thus constructed is the  $G_2$ -diagram.

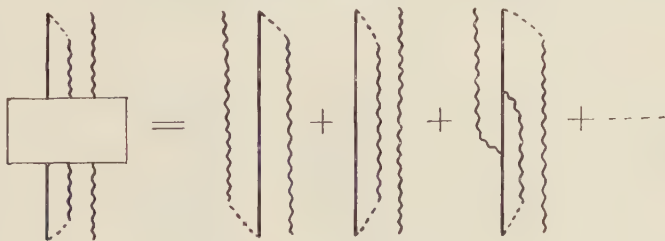


Fig. 1  $G_2$ -diagram

If we had succeeded in solving the equation for the  $R_0$ -process, it would be very easy to construct the analytic expression of the function which represents the  $G_2$ -process, i. e. the process corresponding to the class a).

As regards the diagrams of the class b), we redivide these into the subclasses  $b_1), b_2), \dots, b_n)$ , where the subclass  $b_n)$  consists of all diagrams that possess  $n$  intermediate stages of meson number 1. It is obvious that we can regard the diagrams of subclass  $b_1)$ , as a whole, as consisting of two  $G_2$ -parts connected by one nucleon and one meson lines (cf. Fig. 2). Similarly, we can summarize the diagrams  $b_2)$  into one entity consisting of three  $G_2$ -parts, and the diagrams  $b_n)$  into that of consisting four  $G_2$ -parts, and so on. Accordingly, if we sum up all the diagrams belonging to the classes a) and b), we shall get the whole of the diagrams which represent any times of repetition of the  $G_2$ -process (Fig. 3). We shall call these diagrams belonging to a) or b), as a whole, as  $R_1$ -diagram and the corresponding process as the  $R_1$ -process. In other words, the  $R_1$ -diagram corres-

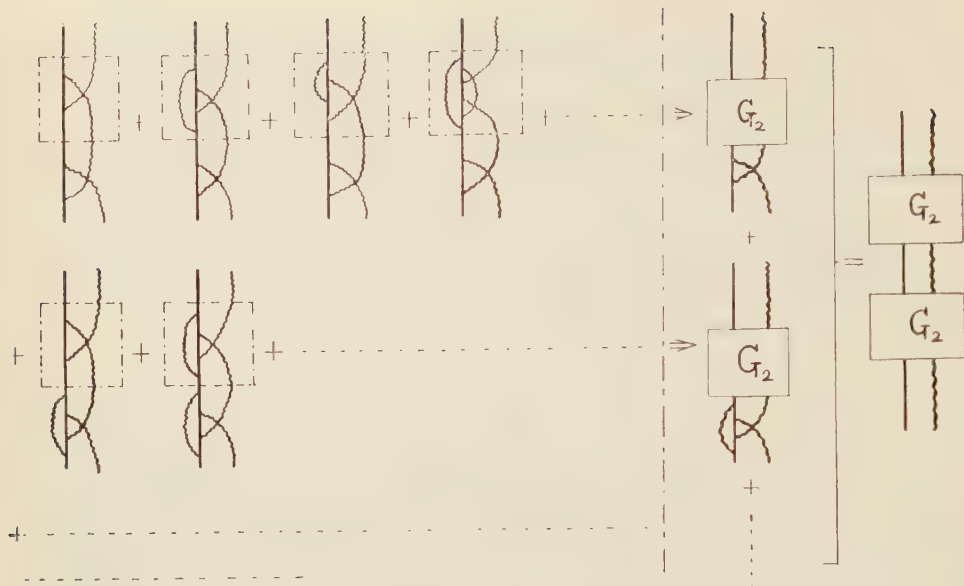


Fig. 2 The diagrams belonging to the class  $b_1$ )

ponds to the solution of an integral equation which has the  $G_2$ -function (the function which represents the  $G_2$ -process) as its kernel, i. e. symbolically

$$R_1 = G_2 + G_2 R_1.$$

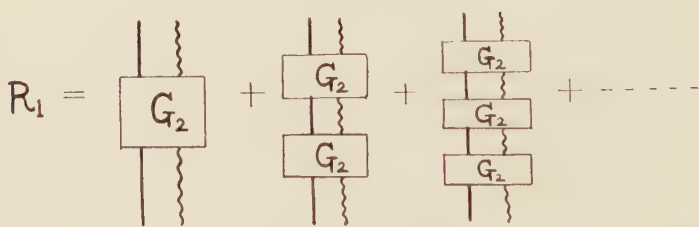


Fig. 3 The  $R_1$ -process

Finally, if we divide each diagram belonging to the class  $c$ ) into two parts at the “last” of the intermediate stages of 0 meson and sum up all of these diagrams analogously to the above cases, it will be found that all the diagrams of class  $c$ ), as a whole, can be constructed by connecting adequately the  $R_1$ -part and the part corresponding to the Feynman kernel  $K(p_0 + k_0; p_0, k_0)$ . The existence of the unknown function  $K(p_0 + k_0; p_0, k_0)$  does not cause any difficulty for solving the equations because of the 4-momentum conservation. (The momentum of a final nucleon in  $K(p; p_0, k_0)$  has always a constant value  $p = p_0 + k_0$ , cf. the work of Fubini.)

Thus, if we had succeeded in solving the  $R_2$ - and  $R_1$ -equations, it would be not difficult to construct the formal solutions of the equations considered here. Moreover, it should be noted that the  $R_2$ -equation is quite divergence-free and  $R_1$ -equation can be made to be also

divergence-free provided only that the subtraction of divergences are carried out in its kernel  $G_0$ .

### § 3. The formal solutions of the generalized $T$ - $D$ equations

In this section, we shall construct the formal solutions of the generalized  $T$ - $D$  equations in the approximation where the meson number is restricted within three. In this order of approximation, the generalized  $T$ - $D$  equations are eqs. (19), (20), (21) and (22') of Part I. Substituting eq. (I.22') into eq. (I.21) and performing the subtraction of the self-energy type divergences caused by that substitution, we get the following equations:

$$K(p_1; p_0, k_0^\varepsilon) = g S_F(p_1) \gamma_5 \tau_\alpha \int dk_1 K(p_1 - k_1, k_1^\alpha; p_0, k_0^\varepsilon), \quad (4)$$

$$\begin{aligned} K(p_1, k_1^\alpha; p_0, k_0^\varepsilon) &= \delta(p_1 - p_0) \delta(k_1^\alpha - k_0^\varepsilon) S_F(p_1) A_F(k_1) \\ &+ g S_F(p_1) \gamma_5 \tau_\alpha A_F(k_1) K(p_1 + k_1; p_0, k_0^\varepsilon) \\ &+ g S_F(p_1) \gamma_5 \tau_\beta \int dk_2 K(p_1 - k_2, k_1^\alpha, k_2^\beta; p_0, k_0^\varepsilon), \end{aligned} \quad (5)$$

and

$$\begin{aligned} K(p_1, k_1^\alpha, k_2^\beta; p_0, k_0^\varepsilon) &= g S_F'(p_1) \gamma_5 \tau_\alpha A_F(k_1) K(p_1 + k_1, k_2^\beta; p_0, k_0^\varepsilon) \\ &+ g S_F'(p_1) \gamma_5 \tau_\beta A_F(k_2) K(p_1 + k_2, k_1^\alpha; p_0, k_0^\varepsilon) \\ &+ g^2 S_F'(p_1) \gamma_5 \tau_\tau \int dk_3 S_F(p_1 - k_3) \gamma_5 \tau_\alpha A_F(k_1) K(p_1 + k_1 - k_3, k_2^\beta, k_3^\tau; p_0, k_0^\varepsilon) \\ &+ g^2 S_F'(p_1) \gamma_5 \tau_\tau \int dk_3 S_F(p_1 - k_3) \gamma_5 \tau_\beta A_F(k_2) K(p_1 + k_2 - k_3, k_1^\alpha, k_3^\tau; p_0, k_0^\varepsilon), \end{aligned} \quad (6)$$

where

$$S_F'(p_1) = [A_0(p_1)]^{-1} S_F(p_1), \quad (7)$$

$$\delta(k_1^\alpha - k_0^\varepsilon) = \delta(k_1 - k_0) \delta_{\alpha\varepsilon}$$

and  $A_0(p_1)$  is the finite part of  $A(p_1)$ ,

$$A(p_1) = 1 - g^2 S_F(p_1) \gamma_5 \tau_\alpha \int dk S_F(p_1 - k) \gamma_5 \tau_\alpha A_F(k). \quad (8)$$

We have rewritten here  $g(2\pi)^{-2}$  of Part I as  $g$ .

The procedure to construct the solutions of these simultaneous integral equations (4), (5) and (6) would be obvious by the considerations of the preceding section. First, one must solve the  $R_2$ -equation:

$$\begin{aligned} R_2(p_1, k_1^\alpha, k_2^\beta; p_0, k_3^\tau, k_4^\delta) &= \delta(p_1 - p_0) S_F'(p_1) A_F(k_1) A_F(k_2) \\ &\times [\delta(k_1^\alpha - k_3^\tau) \delta(k_2^\beta - k_4^\delta) + \delta(k_1^\alpha - k_4^\delta) \delta(k_2^\beta - k_3^\tau)] \end{aligned}$$



$$\begin{aligned}
& + g^2 S_F' (p_1) \gamma_5 \tau_3 \int dk S_F (p_1 - k) \gamma_5 \tau_3 \mathcal{A}_F (k_1) R_2 (p_1 + k_1 - k, k_2^\beta, k^\epsilon; p_0, k_3^\gamma, k_4^\delta) \\
& + g^2 S_F' (p_1) \gamma_5 \tau_3 \int dk S_F (p_1 - k) \gamma_5 \tau_3 \mathcal{A}_F (k_2) R_2 (p_1 + k_2 - k, k_1^\alpha, k^\epsilon; p_0, k_3^\gamma, k_4^\delta).
\end{aligned} \quad (9)$$

Performing the subtraction of the self-energy type divergences caused by the processes corresponding to the diagram of Fig. 4 (e), we have taken  $S_F' (p_1)$  instead of  $S_F (p_1)$  for the "last" nucleon propagators in the right-hand side of eq. (9). This is in accordance to the similar circumstance in eq. (6). The terms of the right-hand side of eq. (9) correspond to the diagrams of Fig. 4 (a), (b), (c) and (d). Especially it should be remembered that the kernel of this integral equation is the same as that of eq. (6).

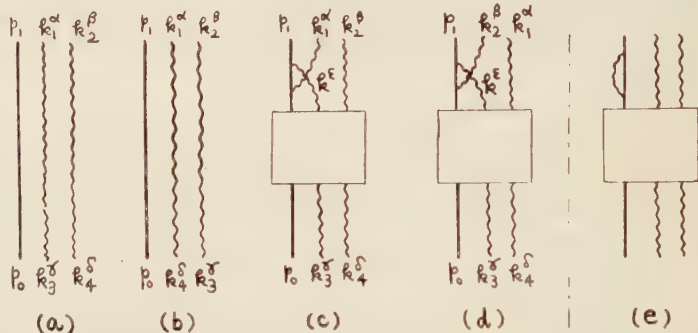


Fig. 4 The correspondence between the diagrams and the terms in the right-hand side of eq. (9).

If we assume that the solution of eq. (9) has been found, our next task is to construct the  $G_2$ -function by enclosing each one of the "initial" and "final" meson lines of  $R_2$  to the nucleon line (Fig. 1).

$$\begin{aligned}
G_2 (p_1, k_1^\alpha; p_0, k_3^\gamma) &= g^2 S_F (p_1) \gamma_5 \tau_3 \int dk_2 \int dk_4 \\
&\times R_2 (p_1 - k_2, k_1^\alpha, k_2^\beta; p_0 - k_4, k_3^\gamma, k_4^\delta) \gamma_5 \tau_3 [\mathcal{A}_F (k_3)]^{-1}.
\end{aligned} \quad (10)$$

Taking into account that this function  $G_2$  is to be used as the kernel of the  $R_1$ -equation in the next stage, the last factor  $[\mathcal{A}_F (k_3)]^{-1}$  in eq. (10) must be inserted to avoid overcounting the meson lines in  $R_1$ . Finally, we have to solve the  $R_1$ -equation (Fig. 3).

$$\begin{aligned}
R_1 (p_1, k_1^\alpha; p_0, k_3^\gamma) &= G_2 (p_1, k_1^\alpha; p_0, k_3^\gamma) \\
&+ \int dp_2 \int dk G_2 (p_1, k_1^\alpha; p_2, k^\beta) R_1 (p_2, k^\beta; p_0, k_3^\gamma).
\end{aligned} \quad (11)$$

It should be noted that the diagrams corresponding to the  $R_1$ -function have not the nucleon and meson lines of their initial stages  $(p_0, k_3^\gamma)$ .

Now, if we had solved the  $R_2$ - and  $R_1$ -equations, the solutions of our starting integral equations (4), (5) and (6) were to be constructed from these functions  $R_2$  and  $R_1$  as considered in the preceding section. In fact, the solutions are given as follows:

$$\begin{aligned}
K (p_1; p_0, k_0^\epsilon) &= [A (p_1) - \Sigma_1 (p_1)]^{-1} \cdot g S_F (p_1) \\
&\times [\gamma_5 \tau_3 \delta (p_1 - p_0 - k_0) S_F (p_0) \mathcal{A}_F (k_0) + A_1 (p_1; p_0, k_0^\epsilon)],
\end{aligned} \quad (12)$$



$$\begin{aligned}
 K(p_1, k_1^\alpha; p_0, k_0^\varepsilon) = & \delta(p_1 - p_0) \delta(k_1^\alpha - k_0^\varepsilon) S_F(p_1) A_F(k_1) \\
 & + R_1(p_1, k_1^\alpha; p_0, k_0^\varepsilon) S_F(p_0) A_F(k_0) + g S_F(p_1) A_F(k_1) \gamma_5 \tau_\alpha K(p_1 + k_1; p_0, k_0^\varepsilon) \\
 & + \int dp A_1^*(p_1, k_1^\alpha; p) K(p; p_0, k_0^\varepsilon), \quad (13)
 \end{aligned}$$

and

$$\begin{aligned}
 K(p_1, k_1^\alpha, k_2^\beta; p_0, k_0^\varepsilon) = & g \int dp_2 dk_3 dk_4 R_2(p_1, k_1^\alpha, k_2^\beta; p_2, k_3^\tau, k_4^\delta) \\
 & \times \gamma_5 \tau_\delta [A_F(k_3)]^{-1} K(p_2 + k_4, k_3^\tau; p_0, k_0^\varepsilon), \quad (14)
 \end{aligned}$$

where

$$A_1(p_1; p_0, k_0^\varepsilon) = g \gamma_5 \tau_\alpha \int dk R_1(p_1 - k, k^\alpha; p_0, k_0^\varepsilon) S_F(p_0) A_F(k_0), \quad (15)$$

$$A_1^*(p_1, k_1^\alpha; p) = g \int dk R_1(p_1, k_1^\alpha; p - k, k^\beta) A_F(k) S_F(p - k) \gamma_5 \tau_\beta, \quad (16)$$

and

$$\Sigma_1(p_1) = g^2 S_F(p_1) \gamma_5 \tau_\alpha \int dp dk_1 dk_2 R_1(p_1 - k_1, k_1^\alpha; p, k_2^\beta) A_F(k_2) S_F(p) \gamma_5 \tau_\beta. \quad (17)$$

That this set of functions is actually the solution can easily be confirmed by substituting this into the starting equations. It is possible to deduce the expression (12) for  $K(p_1; p_0, k_0^\varepsilon)$  from eqs. (4) and (13) by algebraic procedure only, if we take into account that the 4-momentum in  $R_1$  always conserves.

In the purely mathematical language, this method can also be described as follows: First, we assume the form of  $K(p_1, k_1^\alpha k_2^\beta; p_0, k_0^\varepsilon)$  to be given by eq. (14), where  $R_2$  is the function to be determined later. Next, we substitute this  $K(p_1, k_1^\alpha, k_2^\beta; p_0, k_0^\varepsilon)$  into the right-hand side of eq. (6). Then, if we take into account that the first two terms of eq. (6) can be rewritten as

$$\begin{aligned}
 & g S_F'(p_1) [\gamma_5 \tau_\alpha A_F(k_1) K(p_1 + k_1, k_2^\beta; p_0, k_0^\varepsilon) + \gamma_5 \tau_\beta A_F(k_2) K(p_1 + k_2, k_1^\alpha, p_0, k_0^\varepsilon)] \\
 & = g \int dp_2 dk_3 dk_4 \delta(p_1 - p_2) S_F'(p_1) A_F(k_1) A_F(k_2) [\delta(k_1^\alpha - k_3^\tau) \delta(k_2^\beta - k_4^\delta) \\
 & + \delta(k_1^\alpha - k_4^\delta) \delta(k_2^\beta - k_3^\tau)] \gamma_5 \tau_\delta [A_F(k_3)]^{-1} \delta(p_2 + k_4 - p_3) K(p_3, k_3^\tau; p_0, k_0^\varepsilon), \quad (18)
 \end{aligned}$$

it will be found to be sufficient that  $R_2$  is the function which satisfies eq. (9). Finally, substituting  $K(p_1, k_1^\alpha, k_2^\beta; p_0, k_0^\varepsilon)$  of (14) into eq. (5), we get the following equation for  $K(p_1, k_1^\alpha; p_0, k_0^\varepsilon)$ ,

$$\begin{aligned}
 K(p_1, k_1^\alpha; p_0, k_0^\varepsilon) = & \delta(p_1 - p_0) \delta(k_1^\alpha - k_0^\varepsilon) S_F(p_1) A_F(k_1) \\
 & + g S_F(p_1) \gamma_5 \tau_\alpha A_F(k_1) K(p_1 + k_1; p_0, k_0^\varepsilon) \\
 & + \int dp_3 dk_3 G_2(p_1, k_1^\alpha; p_3, k_3^\tau) K(p_3, k_3^\tau; p_0, k_0^\varepsilon), \quad (19)
 \end{aligned}$$

where  $G_2$  is the function given by eq. (10). Eq. (19) differs from the equation studied

by Fubini by only the form of its kernel  $G_2$ , and its solution can be written down according to the Fubini's procedure. Thus we get the equation (13) as the solution of (19). It should need no words as regards the solution of  $K(p_1; p_0, k_0^e)$ .

#### § 4. Subtraction of the divergences

In the preceding section, we have got the formal solutions of our equations. The next task is to subtract the divergences from these solutions according to the program described previously, so as to get significant results. However, the infinities related to self-energies and vertex parts are mixed up in our formalism (especially in  $G_2$ ) in a very complicated manner, and therefore it is necessary to classify these infinities according to their types so as to be able to perform the usual subtraction techniques.

For example, let us consider the infinities in  $G_2$ . The reason why the function  $G_2$  contains infinities in spite of the fact that  $G_2$  is constructed from the quite divergence-free quantity  $R_2$ , lies in the  $k_2$ - and  $k_1$ -integrations in eq. (10). Moreover, that the infinities of various types are contained in  $G_2$  in a mixed form is due to that  $R_2$  itself corresponds to the assembly of the various Feynman-Dyson diagrams. Therefore, to classify these infinities according to their types, it is convenient to make use of the nature of the  $R_2$ -equation.

For convenience, we shall adopt the rule of matrix product in the following. Then, the  $R_2$ -equation (10) can be written as follows,

$$\begin{aligned} R_2(p_1, k_1^a, k_2^b; p_0, k_3^c, k_4^d) = & A(p_1, k_2^b; p_0, k_4^d) J_F(k_1) \delta(k_1^a - k_3^c) \\ & + A(p_1, k_2^b; p_0, k_3^c) J_F(k_1) \delta(k_1^a - k_4^d) \\ & + B(p_1, k_2^b; p_3, k^e) R_2(p_3, k_1^a, k^e; p_0, k_3^c, k_4^d) \\ & + B(p_1, k_1^a; p_3, k^e) R_2(p_3, k_2^b, k^e; p_0, k_3^c, k_4^d), \end{aligned} \quad (20)$$

where

$$A(p_1, k_2^b; p_0, k_3^c) = S_F(p_1) J_F(k_2) \delta(p_1 - p_0) (k_2^b - k_3^c), \quad (21)$$

$$B(p_1, k_2^b; p_3, k^e) = g^2 S_F'(p_1) \gamma_5 \tau_3 S_F(p_1 - k) \gamma_5 \tau_3 J_F(k_2) \delta(p_1 + k_2 - p_3 - k), \quad (22)$$

and  $B(p_1, k_2^b; p_3, k^e) R_2(p_3, k_1^a, k^e; -)$ , for example, means

$$\sum_{\epsilon} \int dp_3 dk B(p_1, k_2^b; p_3, k^e) R_2(p_3, k_1^a, k^e; -).$$

Now, the "renormalization" procedure is usually performed by considering certain parts of the various diagrams as the radiative corrections inserted at the vertices or the propagators of their skeletons. Therefore, also in our case, it would be natural to classify the  $G_2$ -diagrams according to the type of skeleton, i. e. according to whether they are the corrections to the propagator (self-energy type), or the vertex where the meson  $k_1^a$  (or  $k_3^c$ ) is produced (or annihilated), or of other types. At any rate, the history of the meson  $k_1^a$  or  $k_3^c$  would be one of keys for the desired classification of  $G_2$ .

If we, first, pay attention to the vertex where the "final" meson  $k_1^\alpha$  is produced, all of the  $R_2$ -diagrams could be divided into three kinds, that is, (1) the diagrams where the meson  $k_1$  is identical with the meson  $k_3^\tau$ , (2) the diagrams where  $k_1^\alpha$  is identical with  $k_4^\delta$  and (3) others. According to this classification of the  $R_2$ -diagrams, we put

$$R_2(p_1, k_1^\alpha, k_2^\beta; p_0, k_3^\tau, k_4^\delta) = Q(p_1, k_2^\beta; p_0, k_4^\delta) \Delta_F(k_1) \delta(k_1^\alpha - k_3^\tau) \\ + Q'(p_1, k_2^\beta; p_0, k_3^\tau) \Delta_F(k_1) \delta(k_1^\alpha - k_4^\delta) + S(p_1, k_1^\alpha, k_2^\beta; p_0, k_3^\tau, k_4^\delta), \quad (23)$$

where  $S$  is the quantity which contains neither the factor  $\delta(k_1^\alpha - k_3^\tau)$  nor  $\delta(k_1^\alpha - k_4^\delta)$ , and therefore corresponds to the kind (3) of the above classification. Substituting this expression (23) into the  $R_2$ -equation (20) and equating the terms which contain the factor  $\delta(k_1^\alpha - k_3^\tau)$  or  $\delta(k_1^\alpha - k_4^\delta)$  or the remaining terms, respectively, we get the equations for  $Q$  and  $Q'$  or  $S$ . The resulting equations for  $Q$  and  $Q'$  are identical and of the following form,

$$Q(p_1, k_2^\beta; p_0, k_4^\delta) = A(p_1, k_2^\beta; p_0, k_4^\delta) \\ + B(p_1, k_2^\beta; p_3, k^\varepsilon) Q(p_3, k^\varepsilon; p_0, k_4^\delta). \quad (24)$$

This is easily understood by the symmetry of  $R_2$  with respect to  $k_3^\tau$  and  $k_4^\delta$ . From the above equation for  $Q$ , it is obvious that the quantity  $Q$  represents the one-meson scattering process where the meson number varies as  $1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow \dots \rightarrow 1 \rightarrow 2 \rightarrow 1$ . The equation for  $S$  is as follows:

$$S(p_1, k_1^\alpha, k_2^\beta; p_0, k_3^\tau, k_4^\delta) \\ = B(p_1, k_1^\alpha; p_3, k^\varepsilon) Q(p_3, k^\varepsilon; p_0, k_4^\delta) \Delta_F(k_2) \delta(k_2^\beta - k_3^\tau) \\ + B(p_1, k_1^\alpha; p_3, k^\varepsilon) Q(p_3, k^\varepsilon; p_0, k_3^\tau) \Delta_F(k_2) \delta(k_2^\beta - k_4^\delta) \\ + B(p_1, k_2^\beta; p_3, k^\varepsilon) S(p_3, k_1^\alpha, k^\varepsilon; p_0, k_3^\tau, k_4^\delta) \\ + B(p_1, k_1^\alpha; p_3, k^\varepsilon) S(p_3, k_2^\beta, k^\varepsilon; p_0, k_3^\tau, k_4^\delta). \quad (25)$$

Finally, paying attention to the vertex at which the meson  $k_1^\alpha$  is produced and taking into account that the scattering process where the meson number varies as  $1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow 1 \rightarrow \dots \rightarrow 1 \rightarrow 2 \rightarrow 1$  is represented by  $Q$ , we put  $S$  as follows:

$$S(p_1, k_1^\alpha, k_2^\beta; p_0, k_3^\tau, k_4^\delta) \\ = Q(p_1, k_2^\beta; p_3, k^\varepsilon) [S_F'(p_3)]^{-1} [\Delta_F(k)]^{-1} \\ \times B(p_3, k_1^\alpha; p_4, k'^\mu) T(p_4, k'^\mu, k^\varepsilon; p_0, k_3^\tau, k_4^\delta). \quad (26)$$

Taking into account that

$$B(p_1, k_1^\alpha; p_2, k^\varepsilon) Q(p_2, k^\varepsilon; p_4, k''^\nu) \\ = Q(p_1, k_1^\alpha; p_2, k^\varepsilon) [S_F'(p_2)]^{-1} [\Delta_F(k)]^{-1} B(p_2, k^\varepsilon; p_4, k''^\nu) S_F'(p_4) \Delta_F(k''), \quad (27)$$

it follows from eqs. (5) and (26) that

$$\begin{aligned}
& Q(p_1, k_1^\alpha; p_2, k_2^\varepsilon) [S_F'(p_2)]^{-1} [J_F(k)]^{-1} B(p_2, k_2^\varepsilon; p_4, k_4^{\nu\nu}) \\
& \quad \times [T(p_4, k_4^{\nu\nu}; p_0, k_0^\tau, k_4^\delta) - A(p_4, k_4^{\nu\nu}; p_0, k_4^\delta) J_F(k_2) \delta(k_2^\beta - k_3^\tau) \\
& \quad - A(p_4, k_4^{\nu\nu}; p_0, k_0^\tau) J_F(k_2) \delta(k_2^\beta - k_4^\delta) \\
& \quad - B(p_4, k_2^\beta; p_3, k_3^\mu) T(p_3, k_3^\mu, k_4^{\nu\nu}; p_0, k_0^\tau, k_4^\delta) \\
& \quad - B(p_4, k_4^{\nu\nu}; p_3, k_3^\mu) T(p_3, k_3^\mu, k_2^\beta; p_0, k_0^\tau, k_4^\delta)] = 0. \quad (28)
\end{aligned}$$

Therefore, it is found to be sufficient that we take  $T$  as  $R_2$  itself. Thus we have got an alternative equation for  $R_2$  in the following form.

$$\begin{aligned}
R_2(p_1, k_1^\alpha, k_2^\beta; p_0, k_0^\tau, k_4^\delta) &= Q(p_1, k_2^\beta; p_0, k_4^\delta) J_F(k_1) \delta(k_1^\alpha - k_3^\tau) \\
&+ Q(p_1, k_2^\beta; p_0, k_0^\tau) J_F(k_1) \delta(k_1^\alpha - k_4^\delta) + Q(p_1, k_2^\beta; p_3, k_3^\varepsilon) \\
&\times [S_F'(p_3)]^{-1} [J_F(k)]^{-1} B(p_3, k_1^\alpha; p_1, k_1^\mu) R_2(p_4, k_1^\mu, k_2^\varepsilon; p_0, k_0^\tau, k_4^\delta). \quad (29)
\end{aligned}$$

The above derivation of eq. (29) is useful to explain the meaning of such a modification of  $R_2$ -equation. However, there is an alternative method for the derivation of eq. (29), and the latter is more simple and convenient especially for the application to the higher order approximations. That is, if we take into account eq. (2), it will be found that eq. (29) can also be written as follows,

$$\begin{aligned}
R_2(p_1, k_1^\alpha, k_2^\beta; p_0, k_0^\tau, k_4^\delta) &= Q(p_1, k_2^\beta; p_3, k_3^\varepsilon) [S_F'(p_3)]^{-1} [J_F(k)]^{-1} \\
&\times \{R_2(p_3, k_1^\alpha, k_2^\varepsilon; p_0, k_0^\tau, k_4^\delta) \\
&\quad - B(p_3, k_2^\varepsilon; p_2, k_2^\mu) R_2(p_2, k_1^\alpha, k_1^\mu; p_0, k_0^\tau, k_4^\delta)\}. \quad (30)
\end{aligned}$$

Such a relationship as eq. (30) is the one due to the nature of  $Q$ , and generally holds for any function  $F$ . That is, from the fact that the quantity  $Q$  which is the solution of eq. (24) also satisfies the equation

$$\begin{aligned}
Q(p_1, k_2^\beta; p_0, k_4^\delta) &= A(p_1, k_2^\beta; p_0, k_4^\delta) + Q(p_1, k_2^\beta; p_3, k_3^\varepsilon) \\
&\times [S_F'(p_3)]^{-1} [J_F(k)]^{-1} B(p_3, k_2^\varepsilon; p_0, k_4^\delta) S_F'(p_0) J_F(k_4), \quad (31)
\end{aligned}$$

it follows that

$$\begin{aligned}
F(p_1, k_1^\alpha, k_2^\beta, \dots) &= Q(p_1, k_2^\beta; p_3, k_3^\varepsilon) [S_F'(p_3)]^{-1} [J_F(k)]^{-1} \\
&\times \{F(p_3, k_1^\alpha, k_2^\varepsilon, \dots) \\
&\quad - B(p_3, k_2^\varepsilon; p_2, k_2^\mu) F(p_2, k_1^\alpha, k_1^\mu, \dots)\} \quad (32)
\end{aligned}$$

for any function  $F$ . If we notice that the divergences appearing in  $\{dR_2(p_1, k_2, k_1, k_2; \dots)\}$  in the defining equation (10) of  $G_2$  is entirely due to the term  $B(p_1, k_2^\beta; p_3, k_3^\varepsilon) R_2(p_3, k_1^\alpha, k_2^\varepsilon; \dots)$  in the right-hand side of  $R_2$ -equation (20), and that the second term in  $\{ \}$  in eq. (30) corresponds to the former, it would be understood that the modification (29) or (30) of  $R_2$  is useful for the desired classification of  $G_2$ .

Now, the original  $R_2$ -equation (20) can also be written as

$$\begin{aligned}
 R_2(p_1, k_1^\alpha, k_2^\beta; p_0, k_3^\gamma, k_4^\delta) = & A(p_1, k_1^\alpha; p_0, k_4^\delta) \Delta_F(k_3) \delta(k_2^\beta - k_3^\gamma) \\
 & + A(p_1, k_2^\beta; p_0, k_4^\delta) \Delta_F(k_3) \delta(k_1^\alpha - k_3^\gamma) \\
 & + R_2(p_1, k_1^\alpha, k_2^\beta; p_3, k_3^\gamma, k^\varepsilon) [S_F'(p_3)]^{-1} [\Delta_F(k)]^{-1} \\
 & \quad \times B(p_3, k^\varepsilon; p_0, k_4^\delta) S_F'(p_0) \Delta_F(k_4) \\
 & + R_2(p_1, k_1^\alpha, k_2^\beta; p_3, k^\varepsilon, k_4^\delta) [S_F'(p_3)]^{-1} [\Delta_F(k)]^{-1} \\
 & \quad \times B(p_3, k^\varepsilon; p_0, k_3^\gamma) S_F'(p_0) \Delta_F(k_3), \quad (33)
 \end{aligned}$$

by the same reason as that of eq. (27) or (31), i. e.

$$\begin{aligned}
 B(p_1, k_1^\alpha; p, k^\varepsilon) A(p, k^\varepsilon; p_0, k_3^\gamma) \\
 = A(p_1, k_1^\alpha; p, k^\varepsilon) [S_F'(p)]^{-1} [\Delta_F(k)]^{-1} B(p, k^\varepsilon; p_0, k_3^\gamma) S_F'(p_0) \Delta_F(k_3). \quad (34)
 \end{aligned}$$

If we apply a procedure similar to that used in the above derivation of eq. (29) from eq. (20), we can easily derive the following equation for  $R_2$  from eq. (33).

$$\begin{aligned}
 R_2(p_1, k_1^\alpha, k_2^\beta; p_0, k_3^\gamma, k_4^\delta) = & Q(p_1, k_1^\alpha; p_0, k_4^\delta) \Delta_F(k_3) \delta(k_2^\beta - k_3^\gamma) \\
 & + Q(p_1, k_2^\beta; p_0, k_4^\delta) \Delta_F(k_3) \delta(k_1^\alpha - k_3^\gamma) + R_2(p_1, k_1^\alpha, k_2^\beta; p_3, k'^\mu, k^\varepsilon) \\
 & \times [S_F'(p_3)]^{-1} [\Delta_F(k)]^{-1} [\Delta_F(k')]^{-1} B(p_3, k'^\mu; p_4, k_3^\gamma) \Delta_F(k_3) Q(p_4, k^\varepsilon; p_0, k_4^\delta). \quad (35)
 \end{aligned}$$

Substituting eq. (35) into eq. (29), we get

$$\begin{aligned}
 R_2(p_1, k_1^\alpha, k_2^\beta; p_0, k_3^\gamma, k_4^\delta) = & Q(p_1, k_2^\beta; p_0, k_4^\delta) \Delta_F(k_1) \delta(k_1^\alpha - k_3^\gamma) \\
 & + Q(p_1, k_2^\beta; p_3, k_3^\gamma) [S_F'(p_3)]^{-1} Q(p_3, k_1^\alpha; p_0, k_4^\delta) + Q(p_1, k_2^\beta; p_3, k^\varepsilon) \\
 & \times [S_F'(p_3)]^{-1} [\Delta_F(k)]^{-1} B(p_3, k_1^\alpha; p_4, k_3^\gamma) Q(p_4, k^\varepsilon; p_0, k_4^\delta) \Delta_F(k_3) \\
 & + Q(p_1, k_2^\beta; p_3, k^\varepsilon) [S_F'(p_3)]^{-1} [\Delta_F(k)]^{-1} B(p_3, k_1^\alpha; p_4, k'^\mu) \\
 & \times R_2(p_4, k'^\mu, k^\varepsilon; p_5, k''^\nu, k'''^\lambda) [S_F'(p_5)]^{-1} [\Delta_F(k'')]^{-1} [\Delta_F(k''')]^{-1} \\
 & \times B(p_5, k''^\nu; p_6, k_3^\gamma) \Delta_F(k_3) Q(p_6, k'''^\lambda; p_0, k_4^\delta). \quad (36)
 \end{aligned}$$

The important fact is that, in such a form of  $R_2$ -equation as (36), the variables in question,  $k_2$  and  $k_4$ , appear only in the function  $Q$ . The meanings of the terms in the right-hand side of eq. (36) are very obvious. (c. f. Fig. 5)

With the expression (36) for  $R_2$  and the defining equation

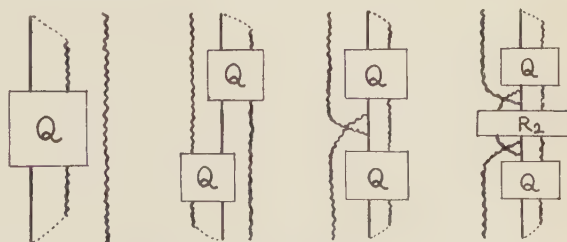


Fig. 5 The correspondence between the terms in eq. (36) and the diagrams. (Dotted lines show the corresponding terms of  $G_3$ )



(10) of  $G_2$ , the classification of the infinities in  $G_2$  can be accomplished as follows;

$$G_2 = G_2^{(a)} + G_2^{(b)} + G_2^{(c)} + G_2^{(d)}, \quad (37)$$

where

$$G_2^{(a)}(p_1, k_1^\alpha; p_0, k_3^\tau) = g^2 S_F(p_1) \Sigma_2(p_1; p_0) \delta(k_1^\alpha - k_3^\tau), \quad (38)$$

$$G_2^{(b)}(p_1, k_1^\alpha; p_0, k_3^\tau) = g^2 S_F(p_1) \int dp_2 A_2(p_1; p_2, k_3^\tau) [S_F'(p_2)]^{-1} \\ \times [J_F(k_2)]^{-1} A_2^*(p_2, k_1^\alpha; p_0), \quad (39)$$

$$G_2^{(c)}(p_1, k_1^\alpha; p_0, k_3^\tau) = g^4 S_F(p_1) J_F(k_1) \int dp_2 dk A_2(p_1; p_2, k^\varepsilon) \\ \times [J_F(k)]^{-1} \gamma_5 \tau_\gamma S_F(p_2 - k_2) \gamma_5 \tau_\alpha A_2^*(p_2 + k_1 - k_2, k^\varepsilon; p_0), \quad (40)$$

and

$$G_2^{(d)}(p_1, k_1^\alpha; p_0, k_3^\tau) = g^6 S_F(p_1) J_F(k_1) \int dp_2 dp_3 dk dk' dk'' dk''' \\ \times A_2(p_1; p_2, k^\varepsilon) [J_F(k)]^{-1} \gamma_5 \tau_\mu S_F(p_2 - k') \gamma_5 \tau_\alpha R_2(p_2 + k_1 - k', k'^\mu, k^\varepsilon; p_3, k''^\nu, k'''^\lambda) \\ \times [J_F(k''')]^{-1} \gamma_5 \tau_\gamma S_F(p_3 - k_3) \gamma_5 \tau_\nu A_2^*(p_3 + k_3 - k'', k'''^\lambda; p_0), \quad (41)$$

with

$$A_2(p_1; p_2, k^\varepsilon) = \gamma_5 \tau_\beta \int dk_2 Q(p_1 - k_2, k_2^\beta; p_2, k^\varepsilon), \quad (42)$$

$$A_2^*(p_1, k^\varepsilon; p_3) = \int dk_4 Q(p_1, k^\varepsilon; p_3 - k_4, k_4^\delta) \gamma_5 \tau_\delta, \quad (43)$$

and

$$\Sigma_2(p_1; p_0) = \gamma_5 \tau_\beta \int dk_2 dk_4 Q(p_1 - k_2, k_2^\beta; p_0 - k_4, k_4^\delta) \gamma_5 \tau_\delta. \quad (44)$$

For the subtraction of the divergences in  $G_2$ , it is obviously sufficient that we perform the subtraction procedure only for the quantities  $A_2$ ,  $A_2^*$  and  $\Sigma_2$ . These contain the divergences of the so-called overlapping type, and the treatments of them are complicated. However, Chiba<sup>3)</sup> and Tanaka-Ito<sup>4)</sup> have given a method for obtaining reasonable convergent parts of such quantities as given by eqs. (42)–(44). For example, let us consider the quantity  $A_2(p_1; p_2, k^\varepsilon)$ . This obeys the following integral equation:

$$A_2(p_1; p_2, k^\varepsilon) = \gamma_5 \tau_\varepsilon S_F'(p_2) J_F(k) \delta(p_1 - p_2 - k) \\ + A_2(p_1; p_4, k'^\mu) [S_F'(p_4)]^{-1} [J_F(k')]^{-1} B(p_4, k'^\mu; p_2, k^\varepsilon) S_F'(p_2) J_F(k). \quad (45)$$

Now, if we construct the finite quantity  $A_{2c}(p_1; p_2, k^\varepsilon)$  by

$$A_{2c}(p_1; p_3, k^E) = A_2^0(p_1; p_3, k^E) + A_2^0(p_1; p_4, k'^\mu) [S_F(p_4)]^{-1} [A_F(k')]^{-1} \\ \times \{B(p_4, k'^\mu; p_3, k'^\nu) - [B(p_4, k'^\mu; p_3, k'^\nu)]_{p_4+k'=p^0}\} Q(p_3, k'^\nu; p_3, k^E), \quad (46)$$

with the solution  $A_2^0$  of the integral equation

$$A_2^0(p_1; p_3, k^E) = \gamma_5 \tau_3 S_F'(p_3) A_F(k) \delta(p_1 - p_3 - k) + A_2^0(p_1; p_4, k'^\mu) \\ \times [S_F'(p_4)]^{-1} [A_F(k')]^{-1} \{B(p_4, k'^\mu; p_3, k^E) - B(p_4, k'^\mu; p_3, 0)\}_{p_4+k'=p^0} \\ \times S_F'(p_3) A_F(k), \quad (47)$$

then this  $A_{2c}$ , according to Tanaka and Ito<sup>(1)</sup>, is just identical with the convergent part of the quantity  $A_2$  derived by applying the procedure of Salam<sup>(6)</sup> to the perturbational expansion (iteration) of eq. (45). In eqs. (46) and (47),  $p^0$  is the 4-momentum which satisfies the relation

$$(p^0)^2 + m^2 = 0. \quad (48)$$

The convergent part of  $A_2^*$  can also be constructed by the similar procedure. The authors quoted above have also shown the possibility of the subtraction equivalent to Salam's method as to the quantity  $\Sigma_2$ . In this case, it is convenient to rewrite eq. (44) as follows: First, consider the  $Q$ -equation which can be written symbolically as (24). If both sides of this equation are multiplied by  $Q(p, k'^E; p_1, k_2^E) [S_F'(p_1)]^{-1} [A_F(k_2)]^{-1}$ , it follows that

$$Q(p, k'^E; p_1, k_2^E) [S_F'(p_1)]^{-1} [A_F(k_2)]^{-1} Q(p_1, k_2^E; p_0, k_4^E) \\ = Q(p, k'^E; p_0, k_4^E) + Q(p, k'^E; p_1, k_2^E) [S_F'(p_1)]^{-1} [A_F(k_2)]^{-1} \\ \times B(p_1, k_2^E; p_3, k^E) Q(p_3, k^E; p_0, k_4^E). \quad (49)$$

Therefore, eq. (44) can be rewritten as follows:

$$\Sigma_2(p_1; p_0) \\ = \int dp dk A_2(p_1; p, k^E) [S_F'(p)]^{-1} [A_F(k)]^{-1} A_2^*(p, k^E; p_0) \\ - \int dp dp' dk dk' A_2(p_1; p, k^E) [S_F'(p)]^{-1} [A_F(k)]^{-1} \\ \times B(p, k^E; p', k'^\mu) A_2^*(p', k'^\mu; p_0). \quad (50)$$

Then, according to Chiba<sup>(3)</sup>, the reasonable convergent part of  $\Sigma_2$  is obtained by applying the usual subtraction technique to the quantity  $\Sigma_2'$  defined as

$$\Sigma_2'(p_1; p_0) = \int dp dk A_{2c}(p_1; p, k^E) [S_F'(p)]^{-1} [A_F(k)]^{-1} A_{2c}^*(p, k^E; p_0) \\ - \int dp dp' dk dk' A_{2c}(p_1; p, k^E) [S_F'(p)]^{-1} [A_F(k)]^{-1} \\ \times B(p, k^E; p', k'^\mu) A_{2c}^*(p', k'^\mu; p_0). \quad (51)$$

Thus, using the procedure of Chiba and Tanaka-Ito, we have got the convergent part of the kernel  $G_2$ . Finally, it should be noted that the effect of the subtraction of the divergence in  $G_2$  is merely to replace eq. (19) by

$$\begin{aligned} K(p_1, k_1^\alpha; p_0, k_0^\epsilon) &= \partial(p_1 - p_0) \partial(k_1^\alpha - k_0^\epsilon) S_F''(p_1) A_F(k_1) \\ &+ g S_F''(p_1) \gamma_5 \tau_\alpha A_F(k_1) K(p_1 + k_1; p_0, k_0^\epsilon) \\ &+ \int dp_3 dk_3 G_{2c}(p_1, k_1^\alpha; p_3, k_3^\tau) K(p_3, k_3^\tau; p_0, k_0^\epsilon), \end{aligned} \quad (52)$$

where

$$G_{2c} = G_{2c}^{(b)} + G_{2c}^{(c)} + G_{2c}^{(d)}, \quad (53)$$

$$S_F''(p_1) = [1 - g^2 S_F(p_1) \Sigma_{2c}(p_1)]^{-1} S_F(p_1), \quad (54)$$

$$(\Sigma_{2c}(p_1; p_0) \equiv \partial(p_1 - p_0) \Sigma_{2c}(p_1))$$

and  $G_{2c}^{(b)}$ ,  $G_{2c}^{(c)}$ ,  $G_{2c}^{(d)}$  are the kernels which are obtained by replacing  $A_2$ ,  $A_2^*$  and the "final"  $S_F(p_1)$  in (39), (40) and (41) by  $A_{2c}$ ,  $A_{2c}^*$  and  $S_F''(p_1)$ , respectively.

As regards the subtraction of divergences after this stage, there occur no more new difficulties and divergences can all be subtracted by a similar or more simple procedure as above. Thus, we have succeeded, at least in principle, in deriving a convergent and significant result in the approximation where the meson "number" is restricted within three.

## § 5. Higher order approximations

In this section, we shall consider briefly the possibility of extending the subtraction procedure investigated so far to higher order approximations.

For example, let us consider the next order of approximation, i. e. the approximation where the meson number is restricted within 4. In this case, eqs. (4) and (5) are kept unchanged. However, we must take

$$\begin{aligned} K(p_1, k_1^\alpha, k_2^\beta; p_0, k_0^\epsilon) &= g S_F(p_1) \gamma_5 \tau_\alpha A_F(k_1) K(p_1 + k_1, k_2^\beta; p_0, k_0^\epsilon) \\ &+ g S_F(p_1) \gamma_5 \tau_\beta A_F(k_2) K(p_1 + k_2, k_1^\alpha; p_0, k_0^\epsilon) \\ &+ g S_F(p_1) \gamma_5 \tau_\tau \int dk_3 K(p_1 - k_3, k_1^\alpha, k_2^\beta, k_3^\tau; p_0, k_0^\epsilon), \end{aligned} \quad (55)$$

and

$$\begin{aligned} K(p_1, k_1^\alpha, k_2^\beta, k_3^\tau; p_0, k_0^\epsilon) &= [g S_F'(p_1) \gamma_5 \tau_\alpha A_F(k_1) K(p_1 + k_1, k_2^\beta, k_3^\tau; p_0, k_0^\epsilon) \\ &+ g^2 S_F'(p_1) \gamma_5 \tau_\mu \int dk_4 S_F(p_1 - k_4) \gamma_5 \tau_\alpha A_F(k_1) \\ &\times K(p_1 + k_1 - k_4, k_2^\beta, k_3^\tau, k_4^\mu; p_0, k_0^\epsilon)] \\ &+ [\text{terms with } k_1^\alpha \text{ and } k_2^\beta \text{ interchanged}] \\ &+ [\text{terms with } k_1^\alpha \text{ and } k_3^\tau \text{ interchanged}], \end{aligned} \quad (56)$$

instead of eq. (6). But, formal solutions of these equations can easily be obtained by a procedure similar to that of § 3. In fact, if we put

$$\begin{aligned} K(p_1, k_1^\alpha, k_2^\beta, k_3^\tau; p_0, k_0^\epsilon) &= \frac{1}{2} g \int d p_2 d k d k' d k'' \\ &\times R_3(p_1, k_1^\alpha, k_2^\beta, k_3^\tau; p_2 - k'', k^\mu, k'^\nu, k''^\lambda) \gamma_5 \tau_\lambda \\ &\times [\mathcal{A}_F(k')]^{-1} [\mathcal{A}_F(k)]^{-1} K(p_2, k^\mu, k'^\nu; p_0, k_0^\epsilon), \end{aligned} \quad (57)$$

it is easily found that the quantity  $R_3$  represents the 3-meson scattering process where the meson number varies as  $3 \rightarrow 4 \rightarrow 3 \rightarrow 4 \rightarrow 3 \rightarrow \dots \rightarrow 3 \rightarrow 4 \rightarrow 3$ .

$$\begin{aligned} R_3(p_1, k_1^\alpha, k_2^\beta, k_3^\tau; p_0, k^\mu, k'^\nu, k''^\lambda) \\ = S_F'(p_1) \mathcal{A}_F(k_1) \mathcal{A}_F(k_2) \mathcal{A}_F(k_3) \delta(p_1 - p_0) \sum \delta(k_1^\alpha - k^\mu) \delta(k_2^\beta - k'^\nu) \delta(k_3^\tau - k''^\lambda) \\ + B(p_1, k_1^\alpha; p_3, k_4^\delta) R_3(p_3, k_2^\beta, k_3^\tau, k_4^\delta; p_0, k^\mu, k'^\nu, k''^\lambda) \\ + B(p_1, k_2^\beta; p_3, k_4^\delta) R_3(p_3, k_1^\alpha, k_3^\tau, k_4^\delta; p_0, k^\mu, k'^\nu, k''^\lambda) \\ + B(p_1, k_3^\tau; p_3, k_4^\delta) R_3(p_3, k_1^\alpha, k_2^\beta, k_4^\delta; p_0, k^\mu, k'^\nu, k''^\lambda), \end{aligned} \quad (58)$$

where the summation in the first term is taken over all the permutations of  $k_1^\alpha$ ,  $k_2^\beta$  and  $k_3^\tau$ . The factor  $1/2$  in the right-hand side of eq. (57) corresponds to the fact that the mesons  $k^\mu$  and  $k'^\nu$  in the intermediate stage can not be distinguished. With this expression (57) for  $K(p_1, k_1, k_2, k_3; p_0, k_0)$ , eq. (55) can be written as

$$\begin{aligned} K(p_1, k_1^\alpha, k_2^\beta; p_0, k_0^\epsilon) &= g S_F(p_1) \gamma_5 \tau_\alpha \mathcal{A}_F(k_1) K(p_1 + k_1, k_2^\beta; p_0, k_0^\epsilon) \\ &+ g S_F(p_1) \gamma_5 \tau_\beta \mathcal{A}_F(k_2) K(p_1 + k_2, k_1^\alpha; p_0, k_0^\epsilon) \\ &+ G_3(p_1, k_1^\alpha, k_2^\beta; p_2, k^\mu, k'^\nu) K(p_2, k^\mu, k'^\nu; p_0, k_0^\epsilon), \end{aligned} \quad (59)$$

where

$$\begin{aligned} G_3(p_1, k_1^\alpha, k_2^\beta; p_2, k^\mu, k'^\nu) &= \frac{1}{2} g^2 S_F(p_1) \gamma_5 \tau_\tau \int d k_3 d k'' \\ &\times R_3(p_1 - k_3, k_1^\alpha, k_2^\beta, k_3^\tau; p_2 - k'', k^\mu, k'^\nu, k''^\lambda) \gamma_5 \tau_\lambda [\mathcal{A}_F(k')]^{-1} [\mathcal{A}_F(k)]^{-1}. \end{aligned} \quad (60)$$

After this stage, the procedure for constructing the formal solutions is quite analogous to that of § 3, because the difference between eqs. (6) and (59) is only the form of their kernels. Thus, it would easily be concluded that we can, at least in principle, construct the formal solutions of the generalized T-D equations in any desired order of approximation only by solving several number of single (non-coupled) integral equations.

As regards the subtraction of the divergences, it would be sufficient only to consider briefly the kernel  $G_3$ . Analogously to the classification of  $G_2$  in § 4, we first rewrite the  $R_3$ -equation by the first method of § 4 as follows:

$$\begin{aligned} R_3(p_1, k_1^\alpha, k_2^\beta, k_3^\tau; p_2, k^\mu, k'^\nu, k''^\lambda) &= R_2(p_1, k_2^\beta, k_3^\tau; p_2, k^\mu, k'^\nu) \\ &\times \mathcal{A}_F(k_1) \delta(k_1^\alpha - k''^\lambda) + R_2(p_1, k_2^\beta, k_3^\tau; p_2, k'^\nu, k''^\lambda) \mathcal{A}_F(k_1) \delta(k_1^\alpha - k^\mu) \end{aligned}$$

$$\begin{aligned}
& + R_2(p_1, k_2^0, k_3^\tau; p_2, k''^\lambda, k^\mu) \mathcal{A}_F(k_1) \delta(k_1^\alpha - k'^\nu) \\
& + \frac{1}{2} R_2(p_1, k_2^0, k_3^\tau; p_3, k_4^0, k_5^\tau) [S_F'(p_3)]^{-1} [\mathcal{A}_F(k_4)]^{-1} [\mathcal{A}_F(k_5)]^{-1} \\
& \times B(p_3, k_1^\alpha; p_4, k_0^0) R_3(p_4, k_4^0, k_5^\tau, k_6^0; p_2, k^\mu, k'^\nu, k''^\lambda), \quad (61)
\end{aligned}$$

where  $R_2$  is the function appearing in § 3, i. e. the solution of eq. (9). This equation can be derived by noticing the vertex at which the meson  $k_1^\alpha$  is produced, the method being quite analogous to that used when we have derived the modified  $R_2$ -equation (29). With this modification (61) of the  $R_2$  equation, we can easily derive the equation for  $R_3$  analogous to eq. (36) for  $R_2$ . By substituting the equation for  $R_2$ , thus derived, into eq. (60), it is easily seen that the problem of defining the convergent part of  $G_3$  can be reduced to that of defining the convergent part of  $\int dk_3 R_2(p_1 - k_3, k_2, k_3; \dots)$  or  $\int dk_4 R_2(\dots; p_2 - k_4, k_1, k_4)$  which can easily be solved by the technique described in § 4.

For the modification of  $R_3$ -equation it is also possible and more simple to apply the second method explained in § 4. Taking into account that the source of the divergences in  $\int dk_3$  in the defining equation (60) of  $G_3$  is the last term in the right-hand side of the  $R_3$ -equation (58), and thus applying the rule (32) with  $k_3^\tau$  instead of  $k_2^0$  to the quantity  $R_3$ , we get the following equation for  $R_3$ :

$$\begin{aligned}
& R_3(p_1, k_1^\alpha, k_2^0, k_3^\tau; p_2, k^\mu, k'^\nu, k''^\lambda) \\
& = [\sum Q(p_1, k_3^\tau; p_2, k^\mu) \delta(k_1^\alpha - k'^\nu) \delta(k_2^0 - k''^\lambda)] \mathcal{A}_F(k_1) \mathcal{A}_F(k_2) \\
& + Q(p_1, k_3^\tau; p_3, k'''^\epsilon) [S_F'(p_3)]^{-1} [\mathcal{A}_F(k''')]^{-1} \\
& \times \{B(p_3, k_1^\alpha; p_4, k_4^0) R_3(p_4, k_2^0, k'''^\epsilon, k_4^0; p_2, k^\mu, k'^\nu, k''^\lambda) \\
& + B(p_3, k_2^0; p_4, k_4^0) R_3(p_4, k_1^\alpha, k'''^\epsilon, k_4^0; p_2, k^\mu, k'^\nu, k''^\lambda)\}, \quad (62)
\end{aligned}$$

where the summation ( $\sum$ ) in the first term is taken over all permutations of  $k^\mu, k'^\nu$  and  $k''^\lambda$ . In this form of  $R_3$ , it would easily be understood that  $\int dk_3 R_3(p_1 - k_3, k_1, k_2, k_3; \dots)$  can be made convergent provided only that  $\int dk_3 Q(p_1 - k_3, k_3; \dots)$  is made convergent. The divergences caused by the integration with respect to  $k''$  in the defining equation (60) of  $G_3$  can also be separated by the similar procedure with the  $R_3$ -equation corresponding to the equation (33) for  $R_2$ .

Thus we see that the convergent contribution of  $G_3$  can, at least in principle, be defined. In the approximation where the meson number is restricted within 4, it is necessary after this stage to construct the solution  $\bar{R}_2$  of the integral equation which has  $G_3$  (made convergent by the above procedure) as its kernel, and moreover to construct the kernel  $\bar{G}_2$  by enclosing one of the meson lines at each of the initial and final stages of  $\bar{R}_2$ . Again, this kernel  $\bar{G}_2$  would contain the overlapping divergences. However, this would also be made convergent by the procedure explained so far. In fact, the source of the overlapping divergences in  $\int dk_2 R_2(p_1 - k_2, k_1, k_2; \dots)$  is only the term (in the  $\bar{R}_2$ -equation) which contains the kernel shown in Fig. 6 (one term of  $G_3$ ), and therefore our method is directly applicable also to this case.



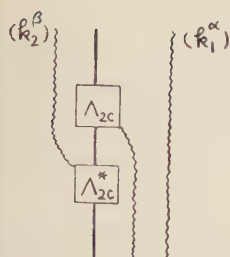


Fig. 6

Thus far, we have considered the case where the meson number is restricted within 4. However, our method is directly applicable to any higher approximation and the significant convergent solution of our formalism can, at least in principle, be derived. The remaining question is only whether our method of defining the convergent contribution can be interpreted as the "renormalization" procedure or not.

## § 6. Discussion and conclusion

In this and the previous papers, we have shown that, by extending the notion of meson number, it is possible to formulate a covariant generalization of the Tamm-Dancoff equations for one-nucleon problem (though in the approximation where the nucleon closed loops are entirely omitted) and that it is also possible, at least in principle, to define the convergent solution of those generalized T-D equations.

Now, it should be noted that, in such an approximation as the T-D method (generalized or not), it is probably impossible to eliminate all divergences only by changing the scales of the mass and the coupling constant as in the case of the perturbational treatment.<sup>7)</sup> Usual renormalization technique is intimately related to the perturbation expansion, and it would be very difficult to apply it directly to other expansion methods, except very special cases. For example, if one wishes to eliminate the divergences, in the present formalism, by the method analogous to that of the perturbational treatment, the renormalized coupling constant would depend on the configuration of that stage at which it appears. This is easily realized by the example shown in Fig. 7. In the approximation where the meson number is restricted within three, there is no correction to the vertex of Fig. 7 (a) and therefore the coupling constant  $g_a$  is unchanged (except the change by the contribution from the self-energy type diagrams.). However, in the same approximation, the process shown in Fig. 7 (c) contributes as the correction to the vertex  $g_b$  shown in Fig. 7 (b), and for the vertex  $g_a$  of Fig. 7 (d) there are more complicated corrections. Thus, after the "renormalization", the modified coupling constants  $g'_a$ ,  $g'_b$  and  $g'_a$  are all different.

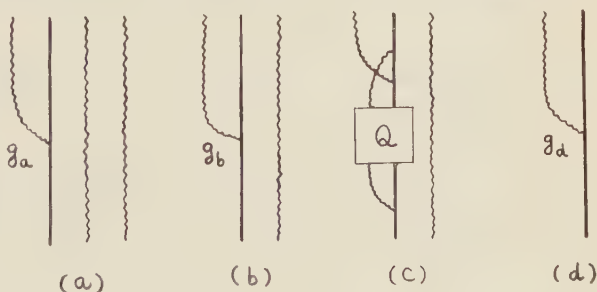


Fig. 7

However, these "renormalized" coupling constants would become identical with each other as we proceed to infinitely higher order approximation, and, in this limit, our subtraction technique would give the correct result. In other words, what we want in such an approximate theory is the approximation in a certain sense to the correct answer of the problem in question. Therefore, supposing that we borrow a certain contribution to  $g_a$  or

$g_b$  from the higher order configurations, we put  $g_a' = g_b' = g_d'$ . The circumstance is quite similar as for the "renormalized" nucleon mass terms. This is a justification for merely omitting the various divergences in our theory. At any rate, the convergent result derived by the procedure of this paper would have the significance as the approximation to the correct result, corresponding to a certain restriction of meson number, and the important one is not  $g'$  but the convergent part of our formal solution.

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# Diagonalization of Hamiltonian and Tamm-Dancoff Equation

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A formalism where Hamiltonian is generally diagonalised is given and the relation of this formalism to the Tamm-Dancoff equation is studied. The Tamm-Dancoff equation in which the energy of the system appearing in the denominator of the equation is eliminated, can be obtained from the solution of a differential equation. In this energy-eliminated Tamm-Dancoff equation, the subtraction of closed loops is possible. Furthermore, some applications are given.

## § 1. Introduction

The recent development of the meson theory has made it clear that the coupling constant between nucleon and meson is not weak enough to be treated by the perturbation method. So several methods other than the simple perturbation have been proposed. One of them is the Tamm-Dancoff<sup>1)</sup> method. This gives comparatively good agreement with the experimental data concerning the pion-nucleon scattering<sup>2)</sup>. This method is also applied to the calculation of nuclear force. Sawada<sup>3)</sup> noticed that we get not usual but imaginary (non-hermitic) potential of Lamb shift if we calculate the potential by the Tamm-Dancoff method. Furthermore, he\*\* noticed that this is due to the fact that the normalisation of the Tamm-Dancoff amplitude is different from that of the original state vector, indicating that the transformation connecting the former with the latter state is not unitary. If we renormalise (this term is used here in a different sense from the usual quantum electrodynamics) the amplitude we can get the usual hermitic potential as expected. We shall show in Chap. 3, that this procedure is equivalent to a unitary transformation. First, in Chap. 2, we shall set up the formalism that will make Hamiltonian diagonal. In Chap. 4, we shall investigate the relation of this method to the adiabatic theorem, and in Chap. 5, we shall treat briefly the relation of the Bethe-Salpeter wave function and the Tamm-Dancoff amplitude. Lastly in Chap. 6, we shall give several applications of the method proposed.

## § 2. Diagonalisation of Hamiltonian

We start from the following eigen-value problem :

$$H\mathcal{F} = E\mathcal{F}. \quad (1)$$

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\*\* The same thing is noticed by Professor Feldman, also. D. Feldman, 4-th Rochester Conference.

Although in the following discussion it does not matter what representation we use, we assume that all operators are those of the time independent representation.

We classify all the eigen-vector of the free Hamiltonian  $H_0$  into two classes  $\{\phi_n\}$  and  $\{\phi_\alpha\}$ ; how to classify is dependent on the nature of the problem. For example, in the problem of the calculation of the nuclear force,  $\{\phi_n\}$  is to be taken as a set of all the eigen-vectors which have two nucleons, no meson and no pair, and  $\{\phi_\alpha\}$  as a set of all the eigen-vectors other than the above mentioned.

Now, let us introduce the projection operator  $\eta$ , which has the following properties:

$$\left. \begin{aligned} \eta \phi_n &= \phi_n \quad (\text{for all } \phi_n \text{ belonging to the first class}), \\ \eta \phi_\alpha &= 0 \quad (\text{for all } \phi_\alpha \text{ belonging to the second class}). \end{aligned} \right\} \quad (2)$$

$\eta$  satisfies the relations:

$$\left. \begin{aligned} \eta^2 &= \eta, \quad (1-\eta)^2 = (1-\eta), \quad \eta(1-\eta) = 0, \\ [H_0, \eta] &= 0. \end{aligned} \right\} \quad (3)$$

$\Psi$  can be expanded by the complete set  $\{\Psi_n\}$  and  $\{\Psi_\alpha\}$ .

$$\Psi = \sum_n a_n \phi_n + \sum_\alpha a_\alpha \phi_\alpha. \quad (4)$$

Let us put

$$\left. \begin{aligned} \Psi_1 &= \eta \Psi = \sum_n a_n \phi_n, \\ \Psi_2 &= (1-\eta) \Psi = \sum_\alpha a_\alpha \phi_\alpha. \end{aligned} \right\} \quad (5)$$

Then the equation (1) is written as

$$\left. \begin{aligned} \eta H \Psi_1 + \eta H \Psi_2 &= E \Psi_1, \\ (1-\eta) H \Psi_1 + (1-\eta) H \Psi_2 &= E \Psi_2, \end{aligned} \right\} \quad (6)$$

or in the matrix notation,

$$\begin{pmatrix} \eta H \eta & \eta H (1-\eta) \\ (1-\eta) H \eta & (1-\eta) H (1-\eta) \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = E \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}. \quad (7)$$

The matrix of the left hand of (7) is hermitic and so it can be diagonalised by a certain unitary transformation\*).

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = U \begin{pmatrix} \Psi'_1 \\ \Psi'_2 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} \Psi'_1 \\ \Psi'_2 \end{pmatrix}, \quad (8)$$

where  $U_{ij}$  are operators which have the following form.

$$\left. \begin{aligned} U_{11} &= \eta U'_{11} \eta, & U_{12} &= \eta U'_{12} (1-\eta) \\ U_{21} &= (1-\eta) U'_{21} \eta, & U_{22} &= (1-\eta) U'_{22} (1-\eta) \end{aligned} \right\}. \quad (8)'$$

\* Similar transformation is proposed by Dr. Tani. See, S. Tani, Prog. Theor. Phys. **12** (1954), 104, and also used by Professor Feldman for the calculation of nuclear force: D. Feldman, Phys. Rev. **93** (1954), 935 A.

From the unitarity of  $U$ ,

$$U^+U=1. \quad (9)$$

$U_{ij}$  can be expressed as

$$\left. \begin{aligned} U_{11} &= (1 + A^+A)^{-1/2} S_1, & U_{22} &= (1 + AA^+)^{-1/2} S_2, \\ U_{21} &= AU_{11}, & U_{12} &= -A^+U_{22}. \end{aligned} \right\} \quad (10)$$

Here  $A$  is an arbitrary operator, which has the following form in the case of (8'):

$$A = (1 - \eta) A' \eta. \quad (11)$$

$S_1$  or  $S_2$  is an arbitrary operator which is unitary in the respective subspace:

$$\left. \begin{aligned} S_1 &= \eta S_1' \eta, & S_2 &= (1 - \eta) S_2' (1 - \eta), \\ S_1^+ S_1 &= \eta, & S_2^+ S_2 &= (1 - \eta). \end{aligned} \right\} \quad (12)$$

In above equations, all operators with prime are arbitrary operators. In order that the matrix of the left hand of (7) is diagonalised, we must have

$$(1 - \eta) (H + [H, A] - AHA) \eta = 0, \quad (13)$$

or

$$(1 - \eta) (1 - A) H (1 + A) \eta = 0. \quad (13)'$$

Then we get

$$\left. \begin{aligned} U_{11}^+ (1 + A^+) H (1 + A) U_{11} \Psi_1' &= E \Psi_1', \\ U_{22}^+ (1 - A) H (1 - A^+) U_{22} \Psi_2' &= E \Psi_2'. \end{aligned} \right\} \quad (14)$$

If we use (13') and notice (8'), (10), (11) and (12), these can be rewritten as follows.

$$\left. \begin{aligned} U_{11}^{-1} H (1 + A) U_{11} \Psi_1' &= E \Psi_1', \\ U_{22}^{-1} H (1 - A^+) U_{22} \Psi_2' &= E \Psi_2'. \end{aligned} \right\} \quad (15)$$

or

$$\left. \begin{aligned} U_{11}^+ (1 + A^+) H (U_{11}^+)^{-1} \Psi_1' &= E \Psi_1', \\ U_{22}^+ (1 - A) H (U_{22}^+)^{-1} \Psi_2' &= E \Psi_2'. \end{aligned} \right\} \quad (15)'$$

Here  $U_{11}^{-1}$  (or  $U_{22}^{-1}$ ) means the inverse operator of  $U_{11}$  (or  $U_{22}$ ) in the respective subspace where it is defined, namely in the matrix notation it is to be interpreted as

$$\begin{pmatrix} U_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

If we define the operator

$$J = 1 + A \quad (16)$$

and  $S_1'$  and  $S_2'$  as the unit operators in (12), we have the following equations from (14), (15) and (15').



$$\left. \begin{aligned} \langle J^+ J \rangle^{-1/2} \langle J^+ H J \rangle \langle J^+ J \rangle^{-1/2} \Psi_1' &= E \Psi_1', \\ \langle J^+ J \rangle^{+1/2} \langle H J \rangle \langle J^+ J \rangle^{-1/2} \Psi_1' &= E \Psi_1', \\ \langle J^+ J \rangle^{-1/2} \langle J^+ H \rangle \langle J^+ J \rangle^{+1/2} \Psi_1' &= E \Psi_1'. \end{aligned} \right\} \quad (17)$$

Here we defined  $\langle Q \rangle$  for an arbitrary operator  $Q$  as

$$\langle Q \rangle = Q_{diagonal} = \eta Q \eta + (1 - \eta) Q (1 - \eta). \quad (18)$$

Of course, it is unnecessary to include in (17) the second term of the right hand of (18) by (11) and (5), but this definition will prove more convenient later.  $J$  defined here corresponds to  $J$  of Sawada et al.<sup>3)</sup> as will be shown in the next chapter. (17) are thus equivalent to theirs. As is clear from (14) or (15),  $\Psi_1'$  and  $\Psi_2'$  are independent of each other. Especially the case  $\Psi_2' = 0$  is interesting, because the Tamm-Dancoff method is an example of this case, as will be shown in Chap. 3. Then, we have from (8)

$$\left. \begin{aligned} \Psi_1 &= U_{11} \Psi_1', \\ \Psi_2 &= U_{21} \Psi_1' = A \Psi_1, \end{aligned} \right\} \quad (19)$$

so that from (16)

$$\Psi = J \Psi_1 \quad (20)$$

and (15) or (17) becomes

$$\langle H J \rangle \Psi_1 = E \Psi_1. \quad (21)$$

We notice that (1) follows from (20), (21) and (13).

Thus, the Hamiltonian is diagonalised into two parts. If we want to diagonalise these further into smaller parts, we have only to repeat the similar procedure. The remaining problem is to solve (13) and obtain  $A$ . This is generally difficult, because (13) is non-linear with respect to  $A$ . A few examples for which (13) is solved are given in Chap. 6.

### § 3. Tamm-Dancoff equation

We divide  $H$  into the free and interaction parts

$$H = H_0 + g H_1. \quad (22)$$

According to Sawada et al.<sup>3)</sup> we get the following equation from the second equation of (6).

$$\Psi_2 = \frac{1}{1 - g \frac{1}{E - H_0} (1 - \eta) H_1} \frac{1}{E - H_0} g (1 - \eta) H_1 \Psi_1. \quad (23)$$

Here we suppose that there are not any incident particles in this space  $\Psi_2$  and solve (23) by the outgoing wave condition, namely  $E$  in the energy denominator has a vanishingly small positive imaginary part. By (23) and the first equation of (6), we have

$$\left[ H_0 + g\eta H_1 \frac{1}{1 - g \frac{1}{E - H_0} (1 - \eta) H_1} \right] \Psi_1 = E \Psi_1. \quad (24)$$

This is the formal Tamm-Dancoff equation.

Now let us define the operator  $A$  which has the form (11) and is independent of  $E$ , by the following equation :

$$\Psi_2 = A \Psi_1, \quad (25)$$

or

$$A \Psi_1 = \frac{1}{1 - g \frac{1}{E - H_0} (1 - \eta) H_1} \frac{1}{E - H_0} g (1 - \eta) H_1 \Psi_1. \quad (26)$$

$A$  can be determined uniquely, because all  $\Psi_1$ 's which satisfy (24) are supposed to make a complete set in the subspace under consideration. We will prove that  $A$  defined here has the same property as  $A$  defined in Chap. 2 ; namely it satisfies (13). For that purpose, let us define  $J$  again by (16). Then, from (1) and (5),

$$\Psi = J \Psi_1 \quad (25)'$$

$$J \Psi_1 = \frac{1}{1 - g \frac{1}{E - H_0} (1 - \eta) H_1} \Psi_1. \quad (26)'$$

Then (24) becomes

$$(H_0 + g \langle HJ \rangle) \Psi_1 = \langle HJ \rangle \Psi_1 = E \Psi_1. \quad (27)$$

(25), (25') and (27) correspond to (19), (20) and (21), respectively. Now we show  $A$  defined here satisfies (13). Since all  $\Psi_1$ 's make a complete set in the subspace, (13) is equivalent to

$$(1 - \eta) (HJ - AHJ) \Psi_1 = 0.$$

From (1), (25') and (27)

$$(1 - \eta) HJ \Psi_1 = HJ \Psi_1 - \langle HJ \rangle \Psi_1 = E (J \Psi_1 - \Psi_1) = EA \Psi_1$$

$$AHJ \Psi_1 = A \langle HJ \rangle \Psi_1 = EA \Psi_1.$$

Therefore

$$(1 - \eta) (HJ - AHJ) \Psi_1 = 0.$$

So, it is known that the formula in Chap. 2 holds good here.

Furthermore, we renormalise  $\Psi_1^*$  :

$$\Psi_1 = (1 + A^+ A)^{-1/2} S_1 \Psi_1' = U_{11} \Psi_1'. \quad (28)$$

Here  $S_1$  and  $U_{11}$  are defined by (12) and (10). (28) corresponds to (19). Then, from (25) and (28),

\* This renormalisation factor is a generalization of  $Z_2$ , that is the renormalisation constant in non-covariant formalism, and has the meaning of the probability of the initial state. See, G. F. Chew, Phys. Rev. **94** (1954), 1748. T. D. Lee, Phys. Rev. **95** (1954), 1329.

$$(\Psi\Psi) = (\Psi_1, (1 + A^+A)\Psi_1) = (\Psi_1', \Psi_1').$$

So, it follows that  $\Psi_1'$  is the renormalised amplitude, and from (25) and (28),

$$\Psi = \begin{pmatrix} \Psi_1' \\ \Psi_2' \end{pmatrix} = \begin{pmatrix} U_{11}\Psi_1' \\ AU_{11}\Psi_1' \end{pmatrix} = \begin{pmatrix} U_{11} & -A^+U_{22} \\ AU_{11} & U_{22} \end{pmatrix} \begin{pmatrix} \Psi_1' \\ 0 \end{pmatrix} = U \begin{pmatrix} \Psi_1' \\ 0 \end{pmatrix}.$$

Although the above equation holds good whatever  $\Psi_1'$  is, it is put equal to the operator defined by (10) and (12) so that  $U$  may be unitary and equivalent to one defined in Chap. 2. When we compare this result with (8), we have  $\Psi_2' \equiv 0$ . This is due to the fact that in Chap. 2 we made only the unitary transformation for the Hamiltonian, while here we used the relation between  $\Psi_1$  and  $\Psi_2$  by considering the boundary condition that there are not any incident particles in the subspace  $\Psi_2$ . In fact, when (25) holds, we get from (8)

$$\Psi_2' = U_{12}^+\Psi_1 + U_{22}^+\Psi_2 = U_{22}^+(-A\Psi_1 + \Psi_2) = 0.$$

If we put  $S_1$  to unit in (28) then (27) becomes

$$\langle J^+J \rangle^{+1/2} \langle HJ \rangle \langle J^+J \rangle^{-1/2} \Psi_1' = E\Psi_1'. \quad (17)$$

This is the second equation of (17). The effective Tamm-Dancoff Hamiltonian which is the term of the left hand of (27) is not hermitic, generally. This is due to the fact that the transformation which connects  $\Psi$  with  $\Psi_1$  is not unitary. But the operators in the left hand of (17) are hermitic, because the transformation connecting  $\Psi$  with  $\Psi_1'$  is unitary as has been shown above. So the renormalisation of the Tamm-Dancoff amplitude is necessary. The examples which show this are given in Chap. 6.

It is to be stressed that  $A$  satisfying (13) is not necessarily equivalent to the corresponding  $A$  in the Tamm-Dancoff method. For that  $A$  must satisfy (26).

#### § 4. Time independent formulation and adiabatic theorem

First, let us notice that from (11) and (3),

$$(1 - \eta)H_0\eta = AH_0A = 0.$$

Thus, (13) can be rewritten as

$$(1 - \eta)(gH_1 + [H_0, A] + g[H_1, A] - gAH_1A)\eta = 0, \quad (29)$$

or

$$(1 - \eta)(gH_1\eta + [H_0, \eta] - g\eta[H_1, \eta])\eta = 0. \quad (29)'$$

Now, introducing the interaction representation for an arbitrary time independent operator  $Q$ ,

$$Q(t) = \exp(iH_0t)Q\exp(-iH_0t) \quad (30)$$

and noticing (3), (11) and (29'), we get

$$i\frac{d}{dt}J(t) = (1 - \eta)[gH_1(t)J(t) - gJ(t)\langle H_1(t)J(t) \rangle]\eta. \quad (31)$$

Furthermore, we consider the next equation :

$$i \frac{d}{dt} V(t) = [gH_1(t) V(t) - gV(t) \langle H_1(t) V(t) \rangle], \quad (32)$$

which was introduced by H. Tanaka<sup>4)</sup> for the first time.

From (32),  $V(t)$  has the solution of the following form :

$$V(t) = 1 + (\text{non-diagonal}) = 1 + \eta V(t) (1 - \eta) + (1 - \eta) V(t) \eta. \quad (33)$$

Then, the solution of (31) is given by

$$\left. \begin{aligned} J(t) &= 1 + (1 - \eta) V(t) \eta, \\ A(t) &= (1 - \eta) V(t) \eta \end{aligned} \right\} \quad (34)$$

as is ascertained by the following relations in combination with (33) and (34):

$$\left. \begin{aligned} J(t) \eta &= V(t) \eta, \\ (1 - \eta) J(t) &= (1 - \eta) V(t). \end{aligned} \right\} \quad (35)$$

So, defining  $V$  as

$$V = V(0), \quad (36)$$

we obtain

$$\left. \begin{aligned} J &= 1 + (1 - \eta) V \eta, \\ A &= (1 - \eta) V \eta. \end{aligned} \right\} \quad (34)'$$

We can thus find  $A$  by solving the differential equation (32). The initial condition may well be arbitrary, and one choice of them corresponds to the Tamm-Dancoff method. To obtain it, we replace the coupling constant  $g$  by

$$g \exp(\epsilon t) \quad (\epsilon > 0).$$

Then from (26), we have

$$A(-\infty) = 0 \quad (36)$$

or

$$\left. \begin{aligned} J(-\infty) &= 1, \\ V(-\infty) &= 1. \end{aligned} \right\} \quad (38)$$

With the initial condition (38),  $V(t)$  is uniquely determined by (32). This solution is the one corresponding to the Tamm-Dancoff equation. To give more rigorous proof, it is only necessary that  $A$  corresponding to this solution satisfies (26). Of course, we take the case  $\mathcal{V}'_0 \equiv 0$  in Chap. 2. First, we notice that all  $J$ 's in the equations of Chap. 2 can be replaced by  $V$ 's as a consequence of (35). Then, from (1) and (20),

$$(E - H_0 - gH_1) (V\mathcal{V}_1) = 0.$$

Let us notice from (21) or (27)

$$g(1-\eta)II_1I^*\Psi_1 = gII_1I^*\Psi_1 - g\langle II_1I^* \rangle \Psi_1 = gH_1I^*\Psi_1 - (E-H_0)\Psi_1$$

and we have

$$(E-H_0-g(1-\eta)H_1I^*)\Psi_1 = (E-H_0)\Psi_1.$$

Let  $\Phi$  be an arbitrary vector which satisfies

$$(E-H_0)\Phi = 0.$$

Then, we have

$$I\Psi_1 = I^*\Psi_1 = \frac{1}{1 - \frac{1}{E-H_0}g(1-\eta)H_1}(\Psi_1 + \Phi).$$

So,

$$\begin{aligned} (V-1)\Psi_1 &= A\Psi_1 = \frac{1}{1 - \frac{1}{E-H_0}g(1-\eta)H_1} \frac{1}{E-H_0}g(1-\eta)H_1\Psi_1 \\ &+ \frac{1}{1 - \frac{1}{E-H_0}g(1-\eta)H_1}\Phi. \end{aligned}$$

From the construction of  $I^*$  (see (32) and (38) and notice that  $g$  is replaced by  $g \exp \epsilon t$ )  $(V-1)\Psi_1$  contains only the outgoing wave, so that  $\Phi$  must be zero. Then this gives (26). Thus, it is proved that the choice of  $V$  which is the solution of (32) under the initial condition (38) corresponds to the Tamm-Dancoff method.

Tanaka<sup>4)</sup> proved the following relation holds in the limit when  $\epsilon$  tends to zero:

$$HV = V\langle HI^* \rangle. \quad (39)$$

So, (1) holds good, considering (20) and (21), when  $I$  there is replaced by  $I^*$ . But it was not clear in his formalism that  $\Psi_1$  is the Tamm-Dancoff amplitude and the formalism has the close connection with the Tamm-Dancoff method.

Furthermore, let us define  $U(t)$  according to him, as follows:

$$\left. \begin{aligned} U(t) &= V(t)U_1(t), \\ I^*(t) &= I(\exp(-i \int_{-\infty}^t dt g \langle II_1(t) I^*(t) \rangle)). \end{aligned} \right\} \quad (40)$$

Then  $U(t)$  satisfies

$$\left. \begin{aligned} idU(t)/dt &= gH_1(t)U(t), \\ U(-\infty) &= 1. \end{aligned} \right\} \quad (41)$$

So,  $I^*(t)$  is the usual transformation function which transforms the Heisenberg operators to the interaction representation.



From (33) and (40),

$$\left. \begin{aligned} U_1(t) &= \langle U(t) \rangle \\ U(t) &= V(t) \langle U(t) \rangle \end{aligned} \right\} \quad (42)$$

or

$$V(t) = U(t) \langle U(t) \rangle^{-1}. \quad (42)'$$

If we restrict the projection operator  $\eta$  to that for a pure state  $\Phi_0$  which is an eigenstate of  $H_0$ , then  $\mathcal{P}_1$  is a numerical multiple of  $\Phi_0$ .

Then, as shown by Tanaka<sup>4)</sup>, we get

$$\mathcal{P} = V \mathcal{P}_1 = c V \Phi_0 = c' U(0) \Phi_0. \quad (43)$$

This is the adiabatic theorem obtained by Ferretti<sup>5)</sup> and Gell-Mann and Low.<sup>6)</sup> (25') is the generalisation of (43), when  $J$  is replaced by  $V$ . It is worth-while to note that  $U_1$  represents the wave matrix for the Hamiltonian (27) as known from the form (40) of  $U_1$ . So, if we consider the scattering problem,  $\mathcal{P}_1$  is obtained from a free state  $\Phi$ :

$$\mathcal{P}_1 = U_1(0) \Phi$$

and

$$\mathcal{P} = V \mathcal{P}_1 = V(0) U_1(0) \Phi = U(0) \Phi.$$

This is equivalent to (43) except the numerical factor. The difference is due to the fact that  $\Phi$  is not normalised to unity.

The result of Cini and Fubini<sup>7)</sup> is obtained by neglecting the second term of the right hand of (32) and solving it under a proper boundary condition.

Lastly, let us consider the problem of the nuclear force up to the fourth order in the coupling constant  $g$ . The projection operator  $\eta$  is that which selects only two nucleon, no pair and no meson states, and we assume that the interaction Hamiltonian is linear with respect to the meson creation and annihilation operator. Then the energy-eliminated Tamm-Dancoff equation is obtained by solving (32) and using (21),

$$\begin{aligned} (E - E_n) a_n = & -g^2 \frac{\langle (n|H_1|\alpha) (\alpha|H_1|m) \rangle}{E_\alpha - E_m} a_m - g^4 \frac{\langle (n|H_1|\alpha) (\alpha|H_1|\beta) (\beta|H_1|\gamma) (\gamma|H_1|m) \rangle}{(E_\alpha - E_m) (E_\beta - E_m) (E_\gamma - E_m)} a_m \\ & + g^4 \frac{\langle (n|H_1|\alpha) (\alpha|H_1|l) \rangle \langle (l|H_1|\beta) (\beta|H_1|m) \rangle}{(E_\alpha - E_m) (E_\alpha - E_l) (E_\beta - E_m)} a_m. \end{aligned} \quad (44)$$

This, also is applicable to the one nucleon problem. Here the summation symbols are dropped for brevity and latin indices mean the states in which only two nucleons exist and Greek indices the other states, and the energy denominators have a vanishingly small negative imaginary part. (44) is equivalent to the usual Tamm-Dancoff equation in which  $E$  is eliminated, as is shown below. The Tamm-Dancoff equation up to the fourth order is given by

$$(E - E_n) a_n = -g^2 \frac{\langle (n|H_1|\alpha) (\alpha|H_1|m) \rangle}{E_\alpha - E} a_m - g^4 \frac{\langle (n|H_1|\alpha) (\alpha|H_1|\beta) (\beta|H_1|\gamma) (\gamma|H_1|m) \rangle}{(E_\alpha - E) (E_\beta - E) (E_\gamma - E)} a_m. \quad (45)$$

\* It is worthy to notice that this derivation given here has not to do with the perturbation method.

From (45),

$$(E - E_n) a_n = O(g^2) a_n.$$

So,  $E$  in the denominator of the second term of (45) can be replaced by  $E_m$  in this approximation, and then this term gives the second term of (44). The first term of (45) is rewritten as follows:

$$-g^2 \frac{\langle n | H_1 | \alpha \rangle \langle \alpha | H_1 | m \rangle}{E_\alpha - E_m} a_m - g^2 \frac{\langle n | H_1 | \alpha \rangle \langle \alpha | H_1 | m \rangle}{(E_\alpha - E)(E_\alpha - E_m)} (E - E_m) a_m.$$

The second term of the above equation is iterated by using (45) and then the energy  $E$  is replaced by  $E_m$ , and we get (44).

Thus, it is known that the approximation of the usual Tamm-Dancoff method is different from the approximation made in the energy-eliminated Tamm-Dancoff equation (27). For example, the second order usual Tamm-Dancoff equation involves a part of the higher order terms of (27) in the coupling constant. One merit of the energy-eliminated Tamm-Dancoff equation is that it makes possible to subtract the disconnected loops in contrast to the usual Tamm-Dancoff equation, since the energy denominator there does not contain  $E$  any more. In fact, we can see it from (44) in the second order of  $g$ . In the fourth order, the sum of all the graphs of type (a) in Fig. 1 is cancelled by the sum of type (b) which are the third term of (44) and arise from the elimination of  $E$  in the denominator of the second order graph. So, remaining closed loops are of the type (c) and are absorbed in  $E$  in the left hand of (44).

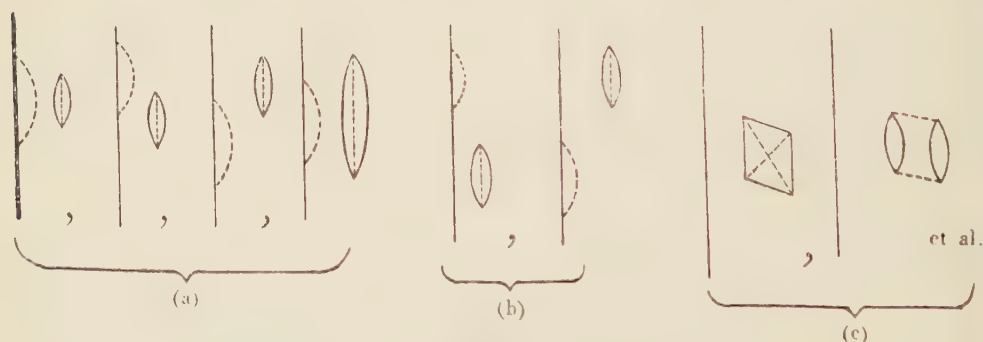


Fig. 1.

Generally, this is proven as follows: From (27) and (42'), the interaction part in the energy-eliminated Tamm-Dancoff equation is given by

$$g \langle H_1 U'(0) \rangle \langle U'(0) \rangle^{-1}.$$

Since the closed loops which are contained in  $U$  cancel\* each other, we have only to consider the loops which are produced by contraction of  $H_1$  and  $U$ . Then the corresponding part is

$$\langle H_1 U \rangle = \langle H_1 U \rangle_{\text{closed loop}} \times \langle U \rangle + \dots$$

\* Here, we use the method of the Feynman diagram instead of the diagram of Tamm-Dancoff method. This is possible, because  $U$  is the usual transformation function.

and so

$$g\langle H_1 U \rangle \langle U \rangle^{-1} = [\text{Connected Part}] + \langle H_1 U \rangle_{\text{closed loop}}.$$

Therefore the remaining disconnected loops are only of the type (c) in Fig. 1, and they can be subtracted.

Next, let us consider the energy-eliminated renormalised Tamm-Dancoff equation up to the fourth order. If we put the solution of (32) in the following form :

$$I' = 1 + gV_1 + g^2 I'_2 + g^3 I'_3 + \dots.$$

Then, (17) is given by

$$\begin{aligned} & \{H_0 + g^2/2 \cdot (\langle H_1 V_1 \rangle + \langle V_1^+ H_1 \rangle) + g^4/2 \cdot (\langle H_1 V_3 \rangle + \langle V_3^+ H_1 \rangle) \\ & + g^4/8 \cdot [\langle V_1^+ V_1 \rangle, \langle H_1 V_1 \rangle - \langle V_1^+ H_1 \rangle]\} \Psi_1' = E \Psi_1'. \end{aligned} \quad (46)$$

Here the higher order than the fourth are omitted. If we put

$$\begin{aligned} H_2 &= 1/2 \cdot (\langle H_1 I'_1 \rangle + \langle I_1'^+ H_1 \rangle), \\ H_4 &= 1/2 \cdot (\langle H_1 I'_3 \rangle + \langle I_3'^+ H_1 \rangle) + 1/8 \cdot [\langle I_1'^+ I'_1 \rangle, \langle H_1 V_1 \rangle - \langle V_1^+ H_1 \rangle], \end{aligned} \quad (46)'$$

we obtain

$$\begin{aligned} \langle n | H_2 | m \rangle &= \frac{1}{2} \langle (n | H_1 | \alpha) (\alpha | H_1 | m) \left[ \frac{1}{E_m + i\epsilon - E_\alpha} + \frac{1}{E_n - i\epsilon - E_\alpha} \right], \\ \langle n | H_4 | m \rangle &= -1/2 \cdot \langle (n | H_1 | \alpha) (\alpha | H_1 | \beta) (\beta | H_1 | \gamma) (\gamma | H_1 | m) \rangle \\ &\quad \times \left[ \frac{1}{(E_\alpha - E_m - i\epsilon) (E_\beta - E_m - i\epsilon) (E_\gamma - E_m - i\epsilon)} \right. \\ &\quad \left. + \frac{1}{(E_\alpha - E_n + i\epsilon) (E_\beta - E_n + i\epsilon) (E_\gamma - E_n + i\epsilon)} \right] \\ &\quad + 1/2 \cdot \langle (n | H_1 | \alpha) (\alpha | H_1 | l) \rangle \langle (l | H_1 | \beta) (\beta | H_1 | m) \rangle \\ &\quad \times \left[ \frac{1}{(E_\alpha - E_m - i\epsilon) (E_\alpha - E_l - i\epsilon) (E_\beta - E_m - i\epsilon)} \right. \\ &\quad \left. + \frac{1}{(E_\alpha - E_n + i\epsilon) (E_\beta - E_l + i\epsilon) (E_\beta - E_n + i\epsilon)} \right. \\ &\quad \left. + \frac{1}{4} \frac{2E_l - (E_n + E_m)}{(E_n - E_\alpha - i\epsilon) (E_\alpha - E_l - i\epsilon) (E_\beta - E_m - i\epsilon) (E_l - E_\beta - i\epsilon)} \right]. \end{aligned} \quad (47)$$

It is easy to calculate the nuclear force from these equations. (47) can be used for the problem of the meson-nucleon scattering. In that case  $H_2$  of (46') gives of the effective Hamiltonian of the second order, and this is equivalent to the result obtained by Sawada<sup>8)</sup>. If we neglect the nucleon recoil, we can solve exactly the problem of the meson-nucleon scattering, as shown by him<sup>8)</sup>. The reason is that the matrix element of  $H_2$  has the energy denominators such as the initial or final minus intermediate and is different from the energy denominator of the Tamm-Dancoff equation, as is seen from (47).

### § 5. Bethe-Salpeter wave function and Tamm-Dancoff amplitude

Usually, the relation between Bethe-Salpeter wave function<sup>(6)</sup> and the Tamm-Dancoff amplitude is obtained by using the Bethe-Salpeter equation<sup>(7)</sup>. Here, we seek the relation by using not the equation but the definition of the wave function directly. For the two body problem, the wave function<sup>(6)</sup> is given by

$$\chi(x, y) = (\mathcal{F}_0, T(\phi_1(x)\phi_2(y))\mathcal{F}). \quad (48)$$

Here,  $T$  represents Wick's chronological ordering operator<sup>(10)</sup>, and  $\mathcal{F}_0$  and  $\mathcal{F}$  are true vacuum and two body states, respectively.  $\phi$  is the Heisenberg operator defined at  $t=0$ . In the system of the center of gravity (which simply means the state where the total momentum is null)  $\chi(x, y)$  must have the form

$$\chi(x, y) = f(x-y) \exp\{-iE(x_0 + y_0/2)\}. \quad (49)$$

Here  $E$  represents the total energy of the system minus the energy of true vacuum. Hereafter, we put the energy of vacuum equal to zero by subtracting this energy from the original Hamiltonian. Thus,  $E$  represents just the total energy of the system and the interaction Hamiltonian is given by

$$H' = gH_1 - E_{\text{vacuum}}. \quad (50)$$

By (25') and (35),

$$\mathcal{F} = I\mathcal{F}_1 = I'\mathcal{F}_1,$$

where  $\mathcal{F}_1$  is the Tamm-Dancoff amplitude

$$\mathcal{F}_1 = \int a(\mathbf{K}) \phi(\mathbf{K}, -\mathbf{K}) d^3\mathbf{K}. \quad (51)$$

$\phi(\mathbf{K}, -\mathbf{K})$  is the eigen-vector of  $H_0$  that each particle has the momentum  $\mathbf{K}$  and  $-\mathbf{K}$  respectively. (48) becomes

$$\chi(x, y) = (\mathcal{F}_0, T(\phi_1(x)\phi_2(y))I'(0)\mathcal{F}_1). \quad (48)'$$

by From (51) and (48'), we can represent  $\chi$  by a series expansion in  $g$  in terms of  $a(\mathbf{K})$ . Here we consider only up to the second order in  $g$ . From (32), within this order, we have

$$V(0) = U(0) + i \int_{-\infty}^0 dt \langle H'(t) V(t) \rangle + O(g^3).$$

So, omitting the numerical factor,

$$\begin{aligned} \chi(x, y) = & (\phi_0, T(U(+\infty, -\infty)\phi_1(x)\phi_2(y))\mathcal{F}_1) \\ & + (\phi_0, T(\phi_1(x)\phi_2(y))i \int_{-\infty}^0 dt \langle H'(t) I'(t) \rangle \mathcal{F}_1) + O(g^3). \end{aligned}$$

Here  $\phi_1(x)$  is the operator of the interaction representation defined in  $t=0$ , and  $\phi_0$  is the free vacuum state and  $U(+\infty, -\infty)$  is the usual  $S$ -matrix and the method used to obtain the above equation is due to Gell-Mann and Low.<sup>(6)</sup>

$$\begin{aligned} \chi(x, y) = & (\Phi_0, T(\psi_1(x) \psi_2(y)) [1 + i \int_{-\infty}^0 dt \langle H'(t) V(t) \rangle] \Psi_1) \\ & - \frac{1}{2} \left( \Phi_0, \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' T(\psi_1(x) \psi_2(y) H'(t) H'(t') \Psi_1) + O(g^3) \right). \end{aligned} \quad (52)$$

The second term can be calculated by the usual Feynman graphical method. First, let us consider the contribution from the first term;

$$\begin{aligned} \chi^{(1)}(x, y) = & \frac{1}{(2\pi)^3} \int d\mathbf{K} u^{(1)}(\mathbf{K}) u^{(2)}(-\mathbf{K}) \exp[i\mathbf{K}(\mathbf{X} - \mathbf{Y}) - iE_{\mathbf{K}}(x_0 + y_0)] \\ & \times [\alpha(\mathbf{K}) + \int d\mathbf{K}' (\mathbf{K}, -\mathbf{K}) | i \int_{-\infty}^0 dt \langle H'(t) V(t) \rangle | \mathbf{K}', -\mathbf{K}' \rangle \alpha(\mathbf{K}')]. \end{aligned} \quad (53)$$

Here

$$(\alpha \mathbf{K} + \beta m) u(\mathbf{K}) = E_{\mathbf{K}} u(\mathbf{K}).$$

In order that  $\chi^{(1)}(x, y)$  has the form (49), it is necessary to replace  $E_{\mathbf{K}}$  by  $E/2$  in the exponential of (53). In the lowest order  $g^0$ , to do so is right, because  $\alpha(\mathbf{K})$  satisfies, by (45),

$$(E - 2E_{\mathbf{K}} + O(g^2)) \alpha(\mathbf{K}) = 0. \quad (54)$$

To replace  $E_{\mathbf{K}}$  by  $E/2$  is legitimate up to the second order in  $g$ . This is shown as follows. Within the second order in  $g$ , we notice that the following equation holds:

$$\begin{aligned} \exp(-iH_0 t) [1 + i \int_{-\infty}^0 dt \langle H'(t) V(t) \rangle] \\ = [1 + i \int_{-\infty}^0 dt \langle H'(t) V(t) \rangle] \exp(-i(H_0 + \langle H'V \rangle)t). \end{aligned}$$

This results also from the fact that  $U_1$  of (40) is the wave matrix of the Hamiltonian  $\langle H'V \rangle$ . So, operating this equation on  $\Psi_1$  and observing (27) and (35), we get

$$\begin{aligned} \exp(-iE_{\mathbf{K}}(x_0 + y_0)) [\alpha(\mathbf{K}) + \int d\mathbf{K}' (\mathbf{K}, -\mathbf{K}) | i \int_{-\infty}^0 dt \langle H'(t) V(t) \rangle | \mathbf{K}', -\mathbf{K}' \rangle \alpha(\mathbf{K}')] \\ = \exp\left(-i \frac{E}{2} (x_0 + y_0)\right) [\alpha(\mathbf{K}) + \int d\mathbf{K}' (\mathbf{K}, -\mathbf{K}) | i \int_{-\infty}^0 dt \langle H'(t) V(t) \rangle | \mathbf{K}', -\mathbf{K}' \rangle \alpha(\mathbf{K}')]. \end{aligned}$$

In order to show the correspondence to Levy's results<sup>9)</sup>, we expand  $f$  of (49)

$$f(x - y) = \frac{1}{(2\pi)^3} \int d^3\mathbf{K} dk_0 \exp[i\mathbf{K}(\mathbf{X} - \mathbf{Y}) - ik_0(x_0 - y_0)] \sum_{i,j} u_i^{(1)}(\mathbf{K}) u_j^{(2)}(-\mathbf{K}) A_{ij}(\mathbf{K}, k_0). \quad (55)$$

The meaning of suffices in  $u$  is found in the reference (9). The contribution from  $\chi^{(1)}$  is given by

$$A^{(1)}(\mathbf{K}, k_0) = \delta(k_0) [\alpha(\mathbf{K}) + \int d\mathbf{K}' (\mathbf{K}, -\mathbf{K}) | i \int_{-\infty}^0 dt \langle H'(t) V(t) \rangle | \mathbf{K}', -\mathbf{K}' \rangle \alpha(\mathbf{K}')] ]$$



or within this order,

$$A^{(1)}(\mathbf{K}, k_0) = \delta(k_0) \left[ \alpha(\mathbf{K}) - g^2 \int d\mathbf{K}' d\alpha \frac{(\mathbf{K}, -\mathbf{K} | H_1 | \alpha) (\alpha | H_1 | \mathbf{K}', -\mathbf{K}')}{(2E_{\mathbf{K}} - 2E_{\mathbf{K}'})(E_{\alpha} - 2E_{\mathbf{K}'})} \alpha(\mathbf{K}') \right],$$

By (54), we can replace  $2E_{\mathbf{K}'}$  by  $E$  in the second order.

$$A^{(1)}(\mathbf{K}, k_0) = \delta(k_0) \left[ \alpha(\mathbf{K}) - \frac{g^2}{E - 2E_{\mathbf{K}}} \int d\mathbf{K}' d\alpha \frac{(\mathbf{K}, -\mathbf{K} | H_1 | \alpha) (\alpha | H_1 | \mathbf{K}', -\mathbf{K}')}{E - E_{\alpha}} \alpha(\mathbf{K}') \right]. \quad (56)$$

By the counter part of (50), the closed loop graph is cancelled as was seen in the preceding chapter, and so the right hand of (56) has no term corresponding to this graph. From (45), we know that (56) is zero in the case of the bound state.

The second term of (52) is calculated by the usual Feynman-Dyson graphical method. Here, we omit the self-energy graph for brevity and the closed loop graph need not be taken into consideration, because the counter term of (50) cancels it.

$$A_{ij}^{(2)}(\mathbf{K}, k_0) = \frac{-ig^2}{(2\pi)^4} \frac{1}{[A_i(\mathbf{K}) + k_0][A_j(\mathbf{K}) - k_0]} \sum_{k,l} \int \frac{d^3\mathbf{P}}{\omega_{\mathbf{P}}^2 - k_0^2} \\ \times \Gamma_{ik}^{(1)}(\mathbf{K}, \mathbf{K} + \mathbf{P}) \Gamma_{jl}^{(2)}(-\mathbf{K}, -\mathbf{K} - \mathbf{P}) A_{kl}(\mathbf{K} + \mathbf{P}), \quad (57)$$

where

$$\Gamma_{ij}(\mathbf{K}, \mathbf{K}') = (u_i^*(\mathbf{K}), Ou_j(\mathbf{K}')),$$

$$A_i(\mathbf{K}) = E/2 - E_i(\mathbf{K}).$$

In the case of the ps-ps meson theory,  $()$  is  $i\gamma_1\gamma_3$ . We used (54) and replaced  $E(\mathbf{K} + \mathbf{P})$  in the denominator by  $E/2$ . So

$$A(\mathbf{K}, k_0) = A^{(1)}(\mathbf{K}, k_0) + A^{(2)}(\mathbf{K}, k_0); \quad (58)$$

$A^{(1)}$  is zero for the bound state. This, also results from the Bethe-Salpeter equation. The equation is given by<sup>9)</sup>

$$A_{ij}(\mathbf{K}, k) = - \frac{ig^2}{(2\pi)^4} \frac{1}{[A_i(\mathbf{K}) + k_0][A_j(\mathbf{K}) - k_0]} \sum_{k,l} \int \frac{d^3\mathbf{P}}{\omega_{\mathbf{P}}^2 - k_0^2} \\ \times \Gamma_{ik}^{(1)}(\mathbf{K}, \mathbf{K} + \mathbf{P}) \Gamma_{jl}^{(2)}(-\mathbf{K}, -\mathbf{K} - \mathbf{P}) A_{kl}(\mathbf{K} + \mathbf{P}, k_0 + k_0). \quad (59)$$

Inserting (58) in (59), and observing that  $A^{(2)}$  in the left hand of (59) is cancelled by the right hand term which appears from the insertion of  $A^{(1)}$ . We get within the second order the following equation:

$$A^{(1)} \equiv 0. \quad (60)$$

This is the Tamm-Dancoff equation for the bound state.

For the scattering problem, by (56),

$$A^{(1)} \equiv \delta(k_0) \delta(\mathbf{K} - \mathbf{K}_0) \quad (60)'$$

so that in the right hand of (59), the term  $\delta(k_0)\delta(\mathbf{K}-\mathbf{K}_0)$  must be added. In the case of bound state, by (60) and (58), we have

$$A(\mathbf{K}, k_0) = A^{(2)}(\mathbf{K}, k_0) \quad (61)$$

which can be obtained by inserting  $A^{(1)}$  in  $A$  of the right hand of (59). Thus these procedures are self-consistent. (57) and (61) are different from the equation (35) of Lévy, who inserted  $-2I(\mathbf{K})a(\mathbf{K})[I^2(\mathbf{K})-k_0^2]^{-1}$  for  $A$  in the right hand of (59).

## § 6. Examples

### (Ex. 1) Lévy-Klein Method

Let us consider the second order nuclear force as an example that shows the necessity of the renormalisation of the Tamm-Dancoff amplitude. In the non-relativistic approximation of ps-meson theory, the Tamm-Dancoff equation is given by (45) as follows :

$$\left(E - \frac{\mathbf{P}^2}{M}\right)a(\mathbf{P}) = \frac{1}{(2\pi)^3} \left(\frac{g}{2M}\right)^2 \int \frac{\sigma_1(\mathbf{P}-\mathbf{P}')\sigma_2(\mathbf{P}-\mathbf{P}')(\tau_1\tau_2)}{\omega(\mathbf{P}-\mathbf{P}') \left[E - \frac{\mathbf{P}^2}{2M} - \frac{\mathbf{P}'^2}{2M} - \omega(\mathbf{P}-\mathbf{P}')\right]} a(\mathbf{P}') d^3\mathbf{P}' \quad (62)$$

Here we take the center of mass system. Let us expand the denominator of (62) as follows :

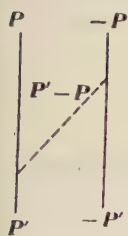


Fig. 2

$$\begin{aligned} & \frac{1}{E - \frac{\mathbf{P}^2}{2M} - \frac{\mathbf{P}'^2}{2M} - \omega(\mathbf{P}-\mathbf{P}')} \\ &= -\frac{1}{\omega(\mathbf{P}-\mathbf{P}')} \left[ 1 + \frac{E - \frac{\mathbf{P}^2}{2M} - \frac{\mathbf{P}'^2}{2M}}{\omega(\mathbf{P}-\mathbf{P}')} + \dots \right] \end{aligned}$$

In the Lévy-Klein<sup>(1)</sup> procedure to obtain the nuclear force, one makes use of the approximation

$$E - \frac{\mathbf{P}^2}{2M} - \frac{\mathbf{P}'^2}{2M} \approx E - \frac{\mathbf{P}'^2}{2M} \quad (63)$$

and iterate (62) for eliminating  $E$  from the right hand of (62). Transforming to the coordinate space, we have

$$\left(E + \frac{\Delta}{M}\right)a(\mathbf{X}) = [V_1(\mathbf{X}) + V_2(\mathbf{X})V_1(\mathbf{X})]a(\mathbf{X}) \quad (64)$$

where

$$\left. \begin{aligned} V_1(\mathbf{X}) &= -\frac{1}{(2\pi)^3} \left(\frac{g}{2M}\right)^2 (\tau_1\tau_2) \int (\sigma_1\mathbf{P})(\sigma_2\mathbf{P}) \frac{1}{\omega^2(\mathbf{P})} \exp(i\mathbf{P}\mathbf{X}) d^3\mathbf{P}, \\ V_2(\mathbf{X}) &= -\frac{1}{(2\pi)^3} \left(\frac{g}{2M}\right)^2 (\tau_1\tau_2) \int (\sigma_1\mathbf{P})(\sigma_2\mathbf{P}) \frac{1}{\omega^3(\mathbf{P})} \exp(i\mathbf{P}\mathbf{X}) d^3\mathbf{P}. \end{aligned} \right\} \quad (65)$$

$V_1$  is the usual second order potential and  $V_2V_1$  is the difference of the potential of

Taketani, Machida and Onuma from that of Brueckner-Watson.<sup>12)</sup> But the approximation (63) has little meaning. In fact, if we do not make use of (63) and eliminate  $\bar{E}$  from the right hand of (62) by the iteration, we get, within the approximation that the higher order terms than  $(\mu/M)^2$  are neglected,

$$(E + \Delta/M) a(\mathbf{X}) = [V_1(\mathbf{X}) + V_2(\mathbf{X}) V_1(\mathbf{X}) + 1/2 M \cdot \Delta V_2(\mathbf{X}) + 1/M \cdot \partial V_2 / \partial \mathbf{X} \cdot \partial / \partial \mathbf{X}] a(\mathbf{X}). \quad (66)$$

Thus we have two extra potentials not contained in (64). Especially, the last term of (66) is a velocity-dependent non-hermitic potential. The reason why such a non-hermitic potential appeared is due to the fact

$$(\Psi, \Psi) \neq \int a^*(\mathbf{X}) a(\mathbf{X}) d^3\mathbf{X}.$$

Now let us put

$$a'(\mathbf{X}) = [1 - V_2(\mathbf{X}) + \text{self-energy term}]^{1/2} a(\mathbf{X}).$$

$a'(\mathbf{X})$  satisfies, within the approximation considered,

$$(\Psi, \Psi) = \int a'^*(\mathbf{X}) a'(\mathbf{X}) d^3\mathbf{X}.$$

Then (66) becomes

$$\begin{aligned} (E + \Delta/M) a'(\mathbf{X}) &= (V_1(\mathbf{X}) + 1/2 \cdot V_2(\mathbf{X}) V_1(\mathbf{X}) + 1/2 \cdot V_1(\mathbf{X}) V_2(\mathbf{X})) a'(\mathbf{X}) \\ &= (V_1(\mathbf{X}) + V_2(\mathbf{X}) V_1(\mathbf{X})) a'(\mathbf{X}). \end{aligned}$$

This agrees with (64). Thus the Lévy-Klein procedure is justified. The result obtained here can be more easily obtained from (47).

If we don't expand the denominator of the first equation of (17) in the power series in  $g$ , we have a potential which is different from the one by perturbation and has the character that is intermediate between the one obtained by Taketani, Machida and Onuma, and the one by Brueckner and Watson. This is being studied by Sato and Sawada.<sup>13)</sup>

#### (Ex. 2) *Pauli Approximation for Dirac Equation*

Let us consider the Dirac equation as the second example.

$$(\alpha \mathbf{P} + \beta m - eV) \psi = E \psi. \quad (67)$$

When we eliminate the small component, we have

$$(1/2m \cdot \mathbf{P}^2 + m - eV + e/4m^2 \cdot (\boldsymbol{\sigma}, \mathbf{E} \times \mathbf{P}) + e/4m^2 \cdot \text{div } \mathbf{E} - e/4m^2 i \cdot (\mathbf{E} \cdot \mathbf{P})) \psi_+ = E \psi_+, \quad (68)$$

where

$$\mathbf{E} = -\text{grad } V$$

and the higher order terms than  $1/m^2$  are neglected. The last term of the left hand of (63) is non-hermitic. The reason is the fact that the transformation connecting  $\psi$  with

$\psi_+$  is not unitary, as is clear from

$$(\psi\psi) = (\psi_+\psi_+) + (\psi_-\psi_-) \neq (\psi_+\psi_+).$$

If we put

$$\psi_+ = (1 - \mathbf{P}^2/8m^2)\varphi \quad (69)$$

then,

$$(\psi\psi) = (\varphi\varphi)$$

and we have for (68)

$$\left( \frac{1}{2m} \mathbf{P}^2 + m - eV + \frac{e}{4m^2} (\boldsymbol{\sigma}, \mathbf{E} \times \mathbf{P}) + \frac{e}{8m^2} \text{div } \mathbf{E} \right) \varphi = E\varphi. \quad (70)$$

The non-hermitic term thus disappears. This result is identical to that obtained by the Foldy-Wouthouysen transformation<sup>14)</sup>.\*

(Ex. 3) Dirac Equation as Example of Chap. 2

First, take the free Dirac equation.

$$(\alpha\mathbf{P} + \beta m)\psi = E\psi. \quad (71)$$

We take  $\eta$  as the projection operator which picks up the large component only. From (11), we put

$$A = \begin{pmatrix} 0 & 0 \\ (\boldsymbol{\sigma}\mathbf{P}) & 0 \end{pmatrix} f(p^2). \quad (72)$$

Then (13) or (13') is satisfied if the following relation holds:

$$1 - 2mf - p^2 f^2 = 0$$

or

$$f \equiv (-m \pm \sqrt{p^2 + m^2})/p^2. \quad (72)$$

If we take the upper sign, and equate  $S_1$  and  $S_2$  of (12) to unity, we get

$$U = \sqrt{(\sqrt{p^2 + m^2} + m)/2\sqrt{p^2 + m^2} - \beta(\alpha\mathbf{P})/p} \cdot \sqrt{(\sqrt{p^2 + m^2} - m)/2\sqrt{p^2 + m^2}}.$$

This is equivalent to the Foldy-Wouthouysen transformation.<sup>14)</sup>

Next, we consider the case that there is a potential:

$$(\alpha\mathbf{P} + \beta m - eV)\psi = E\psi.$$

(13) is satisfied, up to the order  $1/m^2$ , by

$$A = \begin{pmatrix} 0 & 0 \\ \frac{1}{2m}(\boldsymbol{\sigma}\mathbf{P}) + \frac{i}{4m^2}(\boldsymbol{\sigma}\mathbf{E}) & 0 \end{pmatrix}. \quad (74)$$

Then the effective Hamiltonian is obtained by (17) and is identical with (70).

\* Recently, Case applied a similar transformation in the case of  $\beta$ -formalism. In this case, the situation is almost the same as stated here. See, K. M. Case, Phys. Rev. **95** (1954), 1323.

(Ex. 4) *Breit Equation*

$$H = \alpha_1(\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1) + \alpha_2(\mathbf{P}_2 + \epsilon_2 \mathbf{A}_2) + \beta_1 m_1 + \beta_2 m_2 - V - \frac{\epsilon_1 \epsilon_2}{2r} \left[ (\alpha_1 \alpha_2) + \frac{(\alpha_1 \mathbf{r})(\alpha_2 \mathbf{r})}{r^2} \right], \quad (75)$$

where

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_{12} \quad r = |\mathbf{r}|.$$

Now, we take  $\eta$  as the operator which picks up the large-large component. Then from (11),  $A$  has the following form:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ A_3 & 0 & 0 & 0 \end{pmatrix}. \quad (76)$$

The matrix means the Kronecker's product of two matrices. Then up to the second order of  $1/m_1$  and  $1/m_2$ ,

$$\left. \begin{aligned} A_1 &= \frac{1}{2m_2} \sigma_2(\mathbf{P}_2 + \epsilon_2 \mathbf{A}_2) + \frac{i}{4m_2^2} (\sigma_2 \mathbf{E}_2) + \frac{1}{4m_1 m_2} S \sigma_1(\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1) \\ &\quad + \frac{1}{4m_2(m_1 + m_2)} \sigma_1(\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1) S, \\ A_2 &= \frac{1}{2m_1} \sigma_1(\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1) + \frac{i}{4m_1^2} (\sigma_1 \mathbf{E}_1) + \frac{1}{4m_1 m_2} S \sigma_2(\mathbf{P}_2 + \epsilon_2 \mathbf{A}_2) \\ &\quad + \frac{1}{4m_1(m_1 + m_2)} \sigma_2(\mathbf{P}_2 + \epsilon_2 \mathbf{A}_2) S, \\ A_3 &= \frac{1}{2(m_1 + m_2)} S + \frac{1}{4m_1 m_2} \sigma_1(\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1) \sigma_2(\mathbf{P}_2 + \epsilon_2 \mathbf{A}_2), \end{aligned} \right\} \quad (77)$$

where

$$\left. \begin{aligned} \mathbf{E} &= -\text{grad } V, \\ S &= -\epsilon_1 \epsilon_2 / 2r \cdot (\sigma_1 \sigma_2) - \epsilon_1 \epsilon_2 / 2r^3 \cdot (\sigma_1 \mathbf{r})(\sigma_2 \mathbf{r}). \end{aligned} \right\} \quad (78)$$

From (17), the effective Hamiltonian is given by

$$\begin{aligned} H &= (m_1 + m_2 - V + S^2/2(m_1 + m_2)) + [1/2m_1 \cdot \{\sigma(\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1)\}^2 \\ &\quad + i/8m_1^2 \cdot [\sigma_1(\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1), \sigma_1 \mathbf{E}_1] + (1 \leftrightarrow 2)] \\ &\quad + 1/4m_1 m_2 \cdot [\sigma_1(\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1) \sigma_2(\mathbf{P}_2 + \epsilon_2 \mathbf{A}_2) S + \sigma_1(\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1) S \sigma_2(\mathbf{P}_2 + \epsilon_2 \mathbf{A}_2) \\ &\quad + \text{hermitic conjugate term}]. \end{aligned} \quad (79)$$

This is identical with the result obtained by the renormalised Pauli approximation. If we rewrite (17) explicitly, we have



$$\begin{aligned}
 H = m_1 + m_2 - V + & \left[ \frac{1}{2m_1} \mathbf{P}_1^2 + \frac{\epsilon_1}{m_1} (\mathbf{A}_1 \mathbf{P}_1) + \frac{\epsilon_1}{2m_1} (\boldsymbol{\sigma}_1 \mathbf{H}_1) + \frac{1}{2m_1} \epsilon_1^2 \mathbf{A}_1^2 \right. \\
 & + \frac{1}{8m_1^2} \operatorname{div} \mathbf{E}_1 + \frac{1}{4m_1^2} (\boldsymbol{\sigma}_1, \mathbf{E}_1 \times (\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1)) \\
 & + \frac{\epsilon_1 \epsilon_2}{2m_1 m_2} \frac{1}{r^3} (\boldsymbol{\sigma}_1, \mathbf{r}_{12} \times (\mathbf{P}_2 + \epsilon_2 \mathbf{A}_2)) + (1 \leftrightarrow 2) \Big] \\
 & - \frac{\epsilon_1 \epsilon_2}{2m_1 m_2} \frac{1}{r^3} \sum_{i,j} (\mathbf{r}_i \mathbf{r}_j + \delta_{ij} r^2) (\mathbf{P}_1 + \epsilon_1 \mathbf{A}_1)_i (\mathbf{P}_2 + \epsilon_2 \mathbf{A}_2)_j \\
 & + \frac{\epsilon_1 \epsilon_2}{4m_1 m_2} \frac{1}{r^3} \left[ (\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2) - \frac{3}{r^2} (\boldsymbol{\sigma}_1 \mathbf{r}) (\boldsymbol{\sigma}_2 \mathbf{r}) \right] \\
 & + \frac{\epsilon_1^2 \epsilon_2^2}{4(m_1 + m_2)} \left[ \frac{3 - 2(\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2)}{r^2} + \frac{(\boldsymbol{\sigma}_1 \mathbf{r}) (\boldsymbol{\sigma}_2 \mathbf{r})}{r^4} \right]. \quad (80)
 \end{aligned}$$

The most parts agree with the results of Bethe and Breit.<sup>15)</sup> The only difference is that the non-hermitic terms there are replaced by the Darwin terms. In this method, it is not necessary to distinguish the case  $m_1 = m_2$  from the case  $m_1 \neq m_2$ , as is the case in the method of Chraplyvy.<sup>16)</sup>

(Ex. 5) *Neutral Scalar Meson*

$$H = a^* a + g(a + a^*), \quad [a, a^*] = 1. \quad (81)$$

We take  $\eta$  as the operator which picks up the no meson state:

$$a\eta = 0, \quad \eta a^* = 0.$$

Now, we put

$$A = f(a^*)\eta. \quad (82)$$

According to (11), however,

$$f(0) = 0 \quad (83)$$

and noticing that

$$\eta(af)\eta = \langle df/da^* \rangle_0 = \text{const.} = c. \quad (84)$$

Then (13) or (29) is satisfied by

$$(a^* + g)df/da^* + g(a^* - c)(1 + f) = 0. \quad (85)$$

The solution of (85) is given by

$$f = \exp[-ga^* + g(g + c) \log(g + a^*/g)] - 1. \quad (86)$$

This satisfies (83) and (84), letting  $c$  undetermined. Unless  $c$  is equal to  $-g$ ,  $g=0$  is a singular point of  $f$ , and it is not clear what this means. If we put  $c$  to  $-g$ , we get

$$f = \exp(-ga^*) - 1. \quad (87)$$

If we take  $S_1 = \eta$  and choose  $S_2$  properly in (12), we can get

$$U = \exp g(a - a^*).$$

This is the usual transformation which diagonalise (76) most simply.

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**Configuration Mixing and Magnetic Moments of Odd Nuclei\***

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The deviations of magnetic moments of odd nuclei from the Schmidt lines are calculated by taking into account the effect of departure from the single-particle model in nuclear states caused by configuration mixing. The calculations are based upon the simple perturbation theory and the values of the deviations are estimated by adopting the two-body interaction strengths and integrals which are determined from the empirical data on pairing energies. The general trend as well as the fluctuation of the deviations of magnetic moments from the Schmidt lines are accounted for fairly well.

**§ 1. Introduction**

It is well known that many properties of nuclei can be interpreted quite successfully, although qualitatively, by the shell model with strong spin-orbit interaction.<sup>1),2)</sup> There are, however, accumulating evidences for the mixed configurations in nuclear states. The empirically-found smooth variations of the first excited states of even-even nuclei and beta-transitions with anomalous  $ft$  values have been discussed on the basis of configurational mixing in somewhat qualitative manner.<sup>3),4)</sup> The absence of the symmetry between particles and holes in a subshell has also been counted as an evidence for the mixed configuration.<sup>5)</sup>

On the other hand, it is often convenient in the analysis of nuclear properties based upon the shell model to take into account the so-called pairing energy which is derived as a consequence of the short-range attractive forces between nucleons.<sup>1)</sup> It is, however, expected that the different configurations should inevitably become intermingled with those predicted by the shell model to a certain extent under such short-range attractive forces.<sup>5)</sup> The short-range force may usually give rise to much more configuration interaction in comparison with the long-range force, as can readily be seen.

Therefore, it seems probable that some properties of nuclei can deviate from those given by the single particle shell model (SPM) without the configuration mixing. Here, we consider the effect of the configuration mixing upon the magnetic moments of odd nuclei. Blin-Stoyle attempted the application of the configuration mixing to the magnetic moments of spin 1/2 nuclei under special assumptions,<sup>6)</sup> while we treat those of odd nuclei in more general case.

The observed values of magnetic moments of odd-neutron and odd-proton nuclei are divided into two groups corresponding to the Schmidt lines of the SPM, with a very few

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exceptions. Therefore, it can be taken for granted that the configurations of the ground states of odd nuclei can be described by the SPM fairly accurately. The small amplitudes of different configurations which are mixed in, then, will cause the deviations of magnetic moments from the Schmidt limits. If this holds, most of the deviation will be proportional to the coefficients of mixing of those configurations and we may neglect the effects quadratic in their coefficients of mixing, on the assumption that all the coefficients of these configurations are small. Thus it is sufficient to exploit only such configurations that have non-vanishing matrix elements of the magnetic moment operator with the configuration predicted by the SPM. The configurations which contribute to the deviation from the Schmidt lines by their mixing, differ in a single orbit from that of SPM, since the magnetic moment operator is of the single-particle type, and the different orbits must have the same orbital angular momentum since the operator is composed of spin and orbital angular momenta. The coefficients of mixing of these configurations can be calculated from the short-range attractive force which was used in estimating the pairing energy in the shell model. We assume, for simplicity, that the radial dependence of the interaction is of  $\delta$ -function type as the limit of short-range.

## § 2. Deviation of magnetic moments from the Schmidt lines

The magnetic moments of odd-proton and odd-neutron nuclei with  $j^p$  configurations ( $p$ =odd) and with spin  $j$  are, as is well known, equal to the single particle values (the Schmidt limits) :

$$\mu_{sp} = j \{ g_l \pm (g_s - g_l) / (2l + 1) \}, \quad j = l \pm 1/2 \quad (1)$$

where  $g_s = 5.585$  and  $-3.826$  and  $g_l = 1$  and  $0$  (n. m.) for proton and neutron, respectively. There are, however, several configurations which can be mixed in the configuration predicted by the shell model and give rise to the deviation of magnetic moments from the Schmidt lines. In such nuclei, there exist besides several numbers of orbits which are occupied by even numbers of nucleons so as to make the total angular momentum zero in each orbit, some of these orbits being not necessarily closed. First of all, we estimate the effects of excitation of the even group caused by the interaction with odd numbers of nucleons in the outermost orbit.

We consider the type of configuration mixing in which a single nucleon in one orbit of the even group is coupled to another orbit, and we represent the zeroth-order state of the nucleus schematically as

$$\Psi(j_1^{n_1}(0)j_2^{n_2}(0)j^p(j)jm),$$

where the last  $j$  and  $m$  denote the total angular momentum and its  $z$ -component of this system, and nucleons in  $j_1 = l_1 + 1/2$  and  $j_2 = l_1 - 1/2$  are like nucleons and  $n_1$  and  $n_2$  are even numbers ( $0 \leq n_2 < 2j_2 + 1$ ). Usually,  $n_2 = 0$  if  $n_1 < 2j_1 + 1$  (unfilled) and  $n_1 = 2j_1 + 1$  (closed) if  $n_2 > 0$ . A sequence of excited states

$$\Psi([j_1^{n_1-1}(j_1)j_2^{n_2+1}(j_2)](J)j^p(j)jm)$$



can be mixed with the above state, where  $J$  is restricted by the relation  $|j_1 - j_2| \leq J \leq j_1 + j_2$ . However, it is easy to see that only the state with  $J=1$  contributes to the magnetic moment linearly in the coefficient of mixing because of the vector property of the magnetic moment operator. Hence, the mixed configuration which is interested in, is expressed as

$$\Psi(j_1^{n_1}(0)j_2^{n_2}(0)j^p(j)jm) + \alpha\Psi([j_1^{n_1-1}(j_1)j_2^{n_2+1}(j_2)](1)j^p(j)jm), \quad (2)$$

where  $\alpha$  is the coefficient of mixing to be determined in the following, the magnitude of which is usually found to be of the order 0.1. We can neglect the reduction of the amplitude of the first term in (2), assuming that all the coefficients of mixing of the configurations, including those not contributing to the magnetic moment, are small, and we neglect also all the contributions to the magnetic moment which are quadratic in the coefficients of mixing.

The expectation value of the magnetic moment operator for the state represented by the first term in (2) with  $m=j$  gives, of course, the Schmidt limit (1), while the main correction caused by the addition of the second term is obtained by the cross term between the first and second terms in (2) with  $m=j$ . Thus the deviation of magnetic moment from the Schmidt limits is given by

$$\delta\mu = 2\alpha(j_1^{n_1}(0)j_2^{n_2}(0)j^p(j)jj|\sum\mu_z|[j_1^{n_1-1}(j_1)j_2^{n_2+1}(j_2)](1)j^p(j)jj). \quad (3)$$

This expression can easily be reduced by the standard method for the problem involving angular momenta<sup>7),8)</sup> to a simple formula:

$$\delta\mu = 2\alpha[n_1(2j_2+1-n_2)/(2j_2+1)]^{1/2}[j_1l_1/3(j+1)(2l_1+1)]^{1/2}(g_s-g_l), \quad (4)$$

where  $g_s - g_l = 4.585$  or  $-3.826$  (n. m.) according to whether the exciting nucleon is proton or neutron. In the derivation of (4) from (3), we used the following values of the coefficients of fractional parentage<sup>9)</sup> for the simple cases in  $j-j$  coupling.

$$(j^{n-1}(j)j0||j^n0) = 1, \quad (5a)$$

and

$$(j^n(0)jj^*||j^{n+1}j) = [(2j+1-n)/(n+1)(2j+1)]^{1/2}. \quad (5b)$$

The phases of these coefficients do not affect the final results, since the same coefficients will appear in the expression of  $\alpha$ .

On the other hand, the coefficient of mixing  $\alpha$  can be estimated by the perturbation theory. If we put the zeroth-order energy difference between the second and the first configurations in (2)  $\Delta E$ , it is given by

$$\alpha = -(j_1^{n_1}(0)j_2^{n_2}(0)j^p(j)jm|\sum_{i < k} V_{ik} |[j_1^{n_1-1}(j_1)j_2^{n_2+1}(j_2)](1)j^p(j)jm)/\Delta E, \quad (6)$$

where  $V_{ik}$  is the interaction between the  $i$ -th and  $k$ -th nucleons. The calculation of (6) is tedious for the general interaction, but the results become much simpler if we employ the short-range interaction of  $\delta$ -function type:

$$V_{ik} = [V_s(1 - \sigma_i \cdot \sigma_k)/4 + V_t(3 + \sigma_i \cdot \sigma_k)/4] \delta(r_i - r_k) \delta(\cos \omega - 1)/r_i r_k, \quad (7)$$



Table I. Deviation of magnetic moments of nuclei whose state is  $\Psi(j_1^{n_1}(0)j_2^{n_2}(0)j^p(j)jm)$ , by the mixing of  $\Psi([j_1^{n_1-1}(j_1)j_2^{n_2+1}(j_2)](1)j^p(j)jm)$  where  $j_1=l_1+1/2$  and  $j_2=l_1-1/2$

Nucleus	$j$	$\delta\mu/[n_1(2j_2+1-n_2)/(2j_2+1)]$	Contribution from even numbers of
odd proton	$l+1/2$	$-\frac{(l+2)l_1}{(2l+3)(2l_1+1)} \times \left\{ \frac{(-V_s I)/\Delta E \times 4.585}{1/2 \cdot (V_s - V_t) I/\Delta E \times 3.826} \right\}$	protons neutrons
	$l-1/2$	$-\frac{(l-1)l_1}{(2l+1)(2l_1+1)} \times \left\{ \frac{(-V_s I)/\Delta E \times 4.585}{1/2 \cdot (V_s - V_t) I/\Delta E \times 3.826} \right\}$	protons neutrons
odd neutron	$l+1/2$	$\frac{(l+2)l_1}{(2l+3)(2l_1+1)} \times \left\{ \frac{1/2 \cdot (V_s - V_t) I/\Delta E \times 4.585}{(-V_s I)/\Delta E \times 3.826} \right\}$	protons neutrons
	$l-1/2$	$-\frac{(l-1)l_1}{(2l+1)(2l_1+1)} \times \left\{ \frac{1/2 \cdot (V_s - V_t) I/\Delta E \times 4.585}{(-V_s I)/\Delta E \times 3.826} \right\}$	protons neutrons

where  $\omega$  is the angle between the radii vectors of the two nucleons and  $V_s$  and  $V_t$  are the interaction strengths in the singlet and triplet states of the two nucleons. When (7) is employed as the internucleonic interaction,  $\alpha$  is given by (cf. Appendix A)

$$\alpha = -[n_1(2j_2+1-n_2)/(2j_2+1)]^{1/2} \cdot [3l_1/j(j+1)(2l_1+1)]^{1/2} \times [1 + (-1)^{j-l-1/2}(2j+1)/2] \cdot 1/4 \times \left\{ \frac{(-V_s I)/\Delta E}{1/2 \cdot (V_t - V_s) I/\Delta E} \right\}. \quad (8)$$

The values in the curly bracket in (8) must be chosen in such a way that the even numbers of nucleons in the orbits  $j_1$  and  $j_2$  are like or unlike nucleons with those in the outermost orbit  $j$ . And  $I$  is the simplest integral reduced from the generalized Slater integrals in virtue of the  $\delta$ -function interaction

$$I = \frac{1}{2} \int_0^\infty R_{j_1}^2(r) R_{j_1}(r) R_{j_2}(r) r^2 dr. \quad (9)$$

Inserting the coefficient of mixing  $\alpha$  given by (8) into (4), we get the deviations of magnetic moments from the Schmidt lines due to the mixing of the configuration represented by the second term in (2). The results for the deviation  $\delta\mu$  for each two groups ( $j=l \pm 1/2$ ) of the odd-proton and odd-neutron nuclei are given in Table I.

There are also other modes of configuration mixing in odd nuclei. If the outermost orbit of the odd-nucleons is  $j=l-1/2$ , the like nucleons in the orbit  $j_1=l+1/2$  can be excited into the outermost orbit. This mode of excitation is merely the case in which  $j_2$  coincides with  $j$  in the above-mentioned type of excitation. However, the state is characterized in somewhat different way as

$$\Psi(j_1^n(0)j^p(j)jm) + \sum_J \beta_J \Psi(j_1^{n-1}(j_1)j^{p+1}(J)jm), \quad (j=l-1/2) \quad (10)$$

where  $\beta_J$  are the coefficients of mixing and the states  $j^{p+1}(J)$  represent those of seniority-two<sup>(8,9)</sup> and  $J=2, 4, \dots, 2j-1$ . The deviations of the magnetic moments of the nuclei with  $j^p$  configuration and with spin  $j(j=l-1/2)$  by such configuration mixing represented

as (10), is obtained in a formula similar to those of the corresponding cases in Table I (c. f. Appendix B),

$$\delta\mu = n \frac{2j-p}{2j-1} \cdot \frac{(l-1)l}{(2l+1)^2} (g_s - g_l) (-V_s I) / \langle \Delta E \rangle, \quad (j=l-1/2) \quad (11)$$

where  $n$  usually equals  $2j_1+1$  (i. e., closed) and  $\langle \Delta E \rangle$  is the average value of the zeroth-order energy differences between each of the second and the first configuration in (10). We also used in the derivation of (11), the formulae for the coefficients of fractional parentages

$$(j^p(j)fJ|\{j^{p+1}J) = [2(2j-p)/(p+1)(2j-1)]^{1/2}, \quad (J=2, 4, \dots, 2j-1) \quad (12)$$

where the phase factors of these coefficients do not give any influence on the results (11). When the outermost orbit of the odd-nucleon is  $j=l-1/2$ , the deviation of magnetic moment from the Schmidt limit is given by the sum of the values which can be obtained by both Table I and eq. (11).

If the outermost orbit of the odd-nucleon is  $j=l+1/2$  and the number of nucleons (say  $p$ ) in this orbit is larger than unity, there are also other modes of mixing of configurations. A single nucleon in this orbit can be excited into the orbit  $j_1=l-1/2$  of the like nucleon. Therefore, the mixed configuration can be represented as

$$\Psi(j^p j m) + \sum_J \gamma_J \Psi(j^{p-1}(J) j_1 j m), \quad (j=l+1/2, p > 1) \quad (13)$$

where  $\gamma_J$  are the coefficients of mixing and the states  $j^{p-1}(J)$  are also seniority-two. The result for the additional deviation of magnetic moment of the nucleus with  $j^p$  configuration and spin  $j$  ( $j=l+1/2, p > 1$ ) by the configuration mixing given by (13), is obtained as

$$\delta\mu = -(p-1) \frac{(l+1)(l+2)}{(2l+1)(2l+3)} (-V_s I) / \langle \Delta E \rangle, \quad (j=l+1/2, p > 1) \quad (14)$$

where  $\langle \Delta E \rangle$  stands for the similar meaning as before. This formula also resembles those of the corresponding cases in Table I. It must be noted that the integrals  $I$  in eqs. (11) and (14) contain relevant wave functions to each case.

### § 3. Qualitative discussion on the theoretical expression of $\delta\mu$

First of all, we note that the signs of theoretical values of  $\delta\mu$  given by Table I eqs. (11) and (14) are in accord with those of the empirical deviations of magnetic moments from the Schmidt lines. The interaction between nucleons can be taken to be attractive in view of the pairing energy and it seems a plausible assumption that the attractive force in the triplet state is stronger than that in the singlet state. Therefore, the interaction strengths  $V_s$  and  $V_t$  are negative and  $|V_t| > |V_s|$ . Hence, by taking into account the values of  $g_s - g_l$  for proton or neutron, it is easily seen from Table I and eqs. (11) and (14) that the theoretical values of the deviation of odd-proton or odd-neutron nuclei have proper signs for each case to fit the empirical data, i. e.,  $\delta\mu < 0$  for odd-proton nuclei with  $j=l+1/2$  and for odd-neutron nuclei with  $j=l-1/2$ , and  $\delta\mu > 0$  for odd-proton nuclei

with  $j=l-1/2$  and for odd-neutron nuclei with  $j=l+1/2$ . Thus, we can account for the empirical fact that the observed values of magnetic moments lie between the Schmidt lines with a very few exceptions among very light nuclei.

Respecting this empirical fact, the individual particle model<sup>(9),(11)</sup> does not provide satisfactory assurance, since the calculated values based upon it do not necessarily lie between the Schmidt lines. It seems necessary for the validity of this model to prove that the calculated values of magnetic moments of the ground states always lie between the Schmidt lines. On the other hand, the deformed core model gives the outward deviation from the Schmidt lines for nuclei with large  $j=l-1/2$  values in both weak and strong coupling limits.<sup>(12),(13)</sup> This contradicts with the empirical data and it is not likely that the desirable tendency will be obtained in the intermediate coupling without resorting to configuration mixing.

For both odd-proton and odd-neutron nuclei with  $p_{1/2}$  as the outermost orbit, the deviation of the magnetic moments given by Table I and eq. (11) vanishes. The observed values of magnetic moments for these nuclei deviate in a less degree than those with other outermost orbits. This trend is more conspicuous in odd-proton ( $1/2-$ ) nuclei than in odd-neutron ( $1/2-$ ) nuclei. Small deviations of magnetic moments of these nuclei might be accounted for by the second order effects which are neglected throughout this paper.

If we can further assume the ratio of the interaction strengths in the triplet and singlet states of the two nucleons to be 1.5 or so in accordance with the experimental data of two-nucleon system at low energy, we get at once from Table I that the contributions to the deviations are smaller by one-fourth than those from the like nucleons under the same condition. Hence, if we could completely neglect the contributions from the unlike nucleons, we should have the deviations of magnetic moments proportional to the corresponding value of  $g_s - g_l$  to the odd-proton and odd-neutron nuclei from Table I and eqs. (11) and (14). Thus, the ratio of the deviations of odd-proton ( $Z$ ) nuclei to the odd-neutron ( $N'$ ) nuclei,  $Z=N'$ , would be given by

$$|\delta\mu_p|/|\delta\mu_{N'}| \approx 4.585/3.826 = 1.20.$$

This approximate rule was pointed out by de-Shalit<sup>(14)</sup> as the evidence against the quenching theory of nuclear magnetic moments.<sup>(15),(16),(17)</sup> Some of the observed ratios of this quantity have the values near 1.20, but there are large fluctuations around these values. It shows that the complete neglect of the contributions from the unlike nucleon is not a good approximation.

## § 4. Comparison with the experimental data

### (a) Determination of the parameters

We shall compare the theoretical values of the deviations of magnetic moments of odd nuclei from the single particle Schmidt limits with the experimental values. To do this, we must fix the values of parameters appearing in Table I and eqs. (11) and (14). First of all, we assume that the interactions between nucleons are attractive and the attractive



force in the triplet state is stronger than that in the singlet state of the two nucleons, in the ratio

$$|I_t'| = 1.5 |I_s'|$$

as was assumed in the preceding section. The integral  $I$  which was used in various meaning in each case of Table I and eqs. (11) and (14), depends upon the quantum numbers of the wave functions involved in it. However, we shall ignore the dependence upon these quantum numbers to avoid complexities, and take into account only its dependence upon the mass numbers of nuclei, though, in some cases, the former dependence will be referred to in detailed analysis of the experimental data. The values of  $I$  which are averaged in this way, can be estimated in terms of the experimental data on pairing energy of equivalent nucleons. It is carried out in an analogous way to the earlier estimation by M. G. Mayer<sup>1)</sup> and gives similar values

$$V_s I = -25/A \text{ Mev},$$

where  $A$  is the mass number of the nucleons. It agrees with the value adopted by Pryce in the analysis of energy levels of fairly heavy nuclei ( $A \sim 200$ ).<sup>18)</sup> The dependence upon the mass number reflects the assumption that the interaction strength between two nucleons is constant with respect to the mass number of nuclei.

The other quantity, the values of which must be estimated, is the energy difference between the unperturbed ground and excited configurations. For example, in the case of Table I,  $\Delta E$  consists of two parts: one is the energy difference of single-particle levels  $j_1$  and  $j_2$ , and the other is approximately the difference of the pairing energy of  $j_1^2$  and the energy of  $j_1 j_2$  configuration. It is not easy to calculate the latter energy differences in each case, but it is likely they have the tendency to increase  $\Delta E$ . Hence, we adopt as  $\Delta E$  somewhat enlarged values of energy differences of single-particle levels  $j_1$  and  $j_2$ . These orbits have the same orbital angular momentum  $l_1$  as mentioned before so that the energy difference corresponds to the doublet splitting due to the spin-orbit interaction. We put this to be nearly proportional to  $(2l_1 + 1)$  and  $A^{-2/3}$ , and the proportionality constant is determined so as to fit  $\Delta E = 5 \text{ Mev}$  for  ${}_8\text{O}_{17}^{17}$ ,  $l_1 = 2$ .<sup>19), 20)</sup> The values which are employed for the actual calculations of the deviations of magnetic moments are fixed as is shown in Table II irrespective of the mass number  $A$ . Those values adopted may seem to be too large in view of the empirical data on isomeric levels, but it is probable that those levels are not the single-particle levels but appear as the results of many body effects and our

Table II. Energy difference  $\Delta E$  adopted by our calculations (Mev)

orbit	$\Delta E$	orbit	$\Delta E$
$1d_{5/2} - 1d_{3/2}$	5	$2d_{5/2} - 2d_{3/2}$	3/2
$1f_{7/2} - 1f_{5/2}$	3	$1h_{11/2} - 1h_{9/2}$	2
$2p_{3/2} - 2p_{1/2}$	3/2	$2f_{7/2} - 2f_{5/2}$	2
$1g_{9/2} - 1g_{7/2}$	5/2	$1i_{13/2} - 1i_{11/2}$	2

values of  $\Delta I$  include partly the differences between pairing energy in  $j_1^2$  and the energy of  $j_1 j_2$  configuration. The average values of energy difference  $\langle \Delta I \rangle$  are also assumed to be the same as  $\Delta E$  in Table II, for the sake of simplicity.

### (b) Analysis of the magnetic moments of odd nuclei

Adopting the parameter values determined as above, we attempt a somewhat detailed analysis of the magnetic moments of odd nuclei on the basis of our theoretical consideration. Since our assumptions may not hold well for very light nuclei, the analysis will not be carried out for nuclei up to  $A=16$ . Furthermore, our calculations concern only the odd nuclei in normal coupling, so that we can say nothing about nuclei such as  $\text{Na}^{23}$ ,  $\text{Mn}^{55}$  etc. in abnormal coupling.<sup>21)</sup>

The empirical data of the magnetic moments are based upon Klinkenberg<sup>22)</sup> and some data which appeared after that compilation are added following Murakawa and Kamei.<sup>23)</sup> The nucleon configurations are determined following the table of Klinkenberg unless otherwise stated. Some attempts are also made of the determination of the configurations of odd nucleons while the configurations of even nucleons in odd-nuclei do not seriously affect the results of the calculated values as was noted in the preceding section.

The tables for the comparison between the calculated and experimental values are divided according to spin and parity of the odd-nuclei. In each table, the proton- and neutron-configurations show only those which affect the calculated values of magnetic moments including the deviations caused by the configuration mixing. For the sake of simplicity,  $j_1^n$  configurations ( $j_1=l_1+1/2$ ) are dropped in these entries of configurations when the  $j_2=l_1-1/2$  orbits of the like nucleon are further occupied.

#### (i) $(1/2+)$ nuclei

Table III. Magnetic moments of  $(1/2+)$  nuclei

odd-proton nuclei ( $\mu_{sp}=2.79$ )				
nucleus	<i>P</i> -configuration	<i>N</i> -configuration	$\mu_{\text{cal}}$	$\mu_{\text{exp}}$
$9\text{F}^{19}$	$s_{1/2}$	$(d_{5/2})^2$	2.66	2.63
$15\text{P}^{31}$	$(d_{5/2})^6 s_{1/2}$	$(d_{5/2})^6$	1.36	1.13
$81\text{Ti}^{203}$	$(h_{11/2})^{12} s_{1/2}$	$(i_{13/2})^{12} (f_{3/2})^4$	1.44	1.61
$81\text{Ti}^{205}$	$(h_{11/2})^{12} s_{1/2}$	$(i_{13/2})^{14} (f_{3/2})^4$	1.45	1.63
odd-neutron nuclei ( $\mu_{sp}=-1.91$ )				
$14\text{Si}^{29}$	$(d_{5/2})^6$	$(d_{5/2})^6 s_{1/2}$	-0.54	-0.56
$48\text{Cd}^{111}$	$(g_{9/2})^8$	$(g_{7/2})^6 (d_{5/2})^6 s_{1/2}$	-0.49	-0.59
$48\text{Cd}^{113}$	$(g_{9/2})^8$	$(d_{5/2})^6 s_{1/2}$	-0.77	-0.62
$50\text{Sn}^{115}$	$(g_{9/2})^{10}$	$(d_{5/2})^6 s_{1/2}$	-0.73	-0.92
$50\text{Sn}^{117}$	$(g_{9/2})^{10}$	$(d_{5/2})^6 (h_{11/2})^2 s_{1/2}$	-0.50	-1.00
	"	$(d_{5/2})^2 s_{1/2}$	-1.26	



$_{50}\text{Sn}^{119}$	$(g_{9/2})^{10}$	$(d_{5/2})^6$ $(h_{11/2})^4$ $s_{1/2}$	-0.28	-1.05
	"	$(d_{3/2})^2$ $(h_{11/2})^2$ $s_{1/2}$	-0.95	
	"	$s_{1/2}$	-1.62	
$_{52}\text{Te}^{123}$	$(g_{7/2})^2$	$(d_{5/2})^6$ $(h_{11/2})^6$ $s_{1/2}$	-0.04	-0.74
	"	$(d_{5/2})^2$ $(h_{11/2})^4$ $s_{1/2}$	-0.82	
	"	$(h_{11/2})^2$ $s_{1/2}$	-1.39	
$_{52}\text{Te}^{125}$	$(g_{7/2})^2$	$(d_{5/2})^6$ $(h_{11/2})^8$ $s_{1/2}$	-0.03	-0.89
	"	$(d_{3/2})^2$ $(h_{11/2})^6$ $s_{1/2}$	-0.60	
	"	$(h_{11/2})^4$ $s_{1/2}$	-1.24	
$_{54}\text{Xe}^{129}$	$(g_{7/2})^4$	$(d_{5/2})^6$ $(h_{11/2})^{10}$ $s_{1/2}$	+0.14	-0.78
	"	$(d_{3/2})^2$ $(h_{11/2})^8$ $s_{1/2}$	-0.46	
	"	$(h_{11/2})^6$ $s_{1/2}$	-1.10	

The calculated values for odd-proton nuclei show good agreement with the experimental values. It is shown that the deviation for  $\text{F}^{19}$  can be explained by  $(d_{5/2})^2$  neutron-configuration and it does not seem necessary from this viewpoint to change it to  $s^2$  configuration. For  $\text{Ti}^{203}$ , we adopt the neutron-configuration as  $(i_{13/2})^{12}$   $(p_{3/2})^4$  instead of  $(i_{13/2})^{14}$   $(p_{3/2})^2$ , while this assignment may be supported by the positive quadrupole moment of  $\text{Hg}^{201}$  since the positive value suggests its neutron-configuration as  $(i_{13/2})^{12}$   $(p_{3/2})^3$ .

Except for  $\text{Si}^{29}$ , there occurs serious problem concerning the order of neutron levels  $1/2_{11/2}$  and  $2/2_{3/2}$ . The choice of the zeroth-order configurations has large effects on the results as can be seen from the table. For example, the experimental value of  $\text{Sn}^{117}$  lies almost at the middle of the two calculated values based upon two different zeroth-order configurations. This suggests that the zeroth-order configurations in these nuclei may be essentially mixed configurations. The similar situation will often appear in other groups of nuclei of this analysis.

The values obtained by the deformed core model taking into account the configuration mixing, show fairly good agreements for these nuclei.<sup>(12), (24)</sup> However, the agreement becomes worse for higher spin nuclei, especially for  $\text{Bi}^{209}$ .

## (ii) $(1/2-)$ nuclei

Table IV. Magnetic moments of  $(1/2-)$  nuclei

odd-proton nuclei ( $\mu_{sp} = -0.26$ )		odd-neutron nuclei ( $\mu_{sp} = +0.64$ )	
nucleus	$\mu_{\text{exp}}$	nucleus	$\mu_{\text{exp}}$
$_{7}\text{N}^{15}$	-0.28	$_{6}\text{C}^{13}$	0.70
$_{39}\text{Y}^{89}$	-0.14	$_{34}\text{Se}^{77}$	0.53
$_{45}\text{Rh}^{103}$	-0.10	$_{70}\text{Yb}^{171}$	0.5
$_{47}\text{Ag}^{107}$	-0.11	$_{78}\text{Pt}^{195}$	0.61
$_{47}\text{Ag}^{109}$	-0.13	$_{80}\text{Hg}^{199}$	0.50
		$_{82}\text{Pb}^{207}$	0.59

(iii)  $(3/2-)$  nucleiTable V. Magnetic moments of  $(3/2-)$  nuclei

odd-proton nuclei ( $\mu_{sp}=3.79$ )				
nucleus	$P$ -configuration	$N$ -configuration	$\mu_{cal}$	$\mu_{exp}$
$^{63}_{29}\text{Cu}$	$(f_{7/2})^8 p_{3/2}$	$(p_{3/2})^4 (f_{5/2})^2$	2.17	2.23
$^{65}_{29}\text{Cu}$	$(f_{7/2})^8 p_{3/2}$	$(p_{3/2})^4 (f_{5/2})^4$	2.30	2.38
$^{69}_{31}\text{Ga}$	$(f_{7/2})^8 (p_{3/2})^3$	$(p_{3/2})^4$	1.58	2.02
$^{71}_{31}\text{Ga}$	$(f_{7/2})^8 (p_{3/2})^3$	—	1.82	2.56
$^{75}_{33}\text{As}$	$(f_{5/2})^2 (p_{3/2})^3$	$(g_{9/2})^2$	2.21	1.44
	$(g_{9/2})^2 (p_{3/2})^3$	"	1.62	
$^{79}_{35}\text{Br}$	$(f_{5/2})^4 (p_{3/2})^3$	$(g_{9/2})^4$	2.55	2.11
	$(g_{9/2})^2 (f_{5/2})^2 (p_{3/2})^3$	"	2.06	
$^{81}_{35}\text{Br}$	$(f_{5/2})^4 (p_{3/2})^3$	$(g_{9/2})^6$	2.52	2.27
	$(g_{9/2})^2 (f_{5/2})^2 (p_{3/2})^3$	"	1.91	
$^{87}_{37}\text{Rb}$	$(p_{3/2})^3$	$(g_{9/2})^8$	2.79	2.75
odd-neutron nuclei ( $\mu_{sp}=-1.91$ )				
$^{53}_{24}\text{Cr}$	$(f_{7/2})^4$	$(f_{7/2})^8 p_{3/2}$	-0.49	-0.47
$^{57}_{26}\text{Fe}$	$(f_{7/2})^6$	$(f_{7/2})^8 (p_{3/2})^3$	+0.4	$\sim 0$
$^{61}_{28}\text{Ni}$	$(f_{7/2})^8$	$(f_{5/2})^2 (p_{3/2})^3$	-0.03	$\sim 0$
$^{201}_{80}\text{Hg}$	$(h_{11/2})^{12}$	$(i_{13/2})^{12} (p_{3/2})^3$	-0.51	-0.56

If we adopt  $(p_{3/2})^3$  proton-configuration for Ga isotopes, the magnitudes of the calculated deviations are larger than the observed values, though the relative values of deviations are given correctly; on the contrary, the magnitudes become smaller than the observed values if we adopt  $(f_{5/2})^2 p_{3/2}$  configuration. The observed values lie almost in the middle of these two alternatives. However, positive values of quadrupole moments of Ga isotopes may suggest that the  $(p_{3/2})^3$  configuration is predominant in these nuclei.

The magnetic moment of  $\text{As}^{75}$  lies in the so-called forbidden zone while we can barely obtain the close value by employing  $(f_{9/2})^2 (p_{3/2})^3$  configuration. The configuration is not supported by other empirical evidences but the calculated values of Br isotopes based on the similar configuration provides fairly good agreement.

The neutron-configuration of  $\text{Hg}^{201}$  are assumed as  $(i_{13/2})^{12} (p_{3/2})^3$  in view of its positive quadrupole moment.<sup>21)</sup>

We cannot interpret by our theoretical considerations such a large deviation of  $\text{Os}^{189}$  that gives rise to displacement of the moment from the lower branch to almost the upper branch of the Schmidt lines. Even if we can employ the neutron configuration  $(f_{5/2})^0 (i_{13/2})^{10} (p_{3/2})^3$  instead of  $(f_{5/2})^6 (i_{13/2})^1 (p_{3/2})^3$ , we get only  $\delta\mu=2.11$  in place of its observed value  $\delta\mu_{exp}=2.61$  n. m.

(iv)  $(3/2+)$  nuclei

Table VI. Magnetic moments of  $(3/2+)$  nuclei

odd-proton nuclei ( $\mu_{sp}=0.12$ )					
nucleus	<i>P</i> -configuration	<i>N</i> -configuration	$\mu_{cal}$	$\mu_{cal}^*$	$\mu_{exp}$
$^{17}\text{Cl}^{35}$	$d_{3/2}$	$(d_{3/2})^2$	0.46	0.65	0.82
$^{17}\text{Cl}^{37}$	$d_{3/2}$	—	0.42	0.57	0.68
$^{19}\text{K}^{39}$	$(d_{3/2})^3$	—	0.12	0.12	0.39
$^{19}\text{K}^{41}$	$(d_{3/2})^3$	$(f_{7/2})^2$	0.15	0.15	0.22
$^{77}\text{Ir}^{191}$	$(h_{11/2})^{10} (d_{3/2})^3$	$(i_{13/2})^8$	0.44	0.29	0.17
$^{77}\text{Ir}^{193}$	$(h_{11/2})^{10} (d_{3/2})^3$	$(i_{13/2})^8$	0.44	0.29	0.17
$^{79}\text{Au}^{197}$	$(h_{11/2})^{12} (d_{3/2})^3$	$(i_{13/2})^{12}$	0.43	0.28	0.14
odd-neutron nuclei ( $\mu_{sp}=1.15$ )					
$^{16}\text{S}^{33}$	$(d_{5/2})^6$	$d_{3/2}$	0.79	0.61	0.64
$^{16}\text{S}^{35}$	$(d_{5/2})^6$	$(d_{3/2})^3$	1.07	1.03	1.00
$^{54}\text{Xe}^{131}$	$(g_{7/2})^4$	$(h_{11/2})^{12} d_{3/2}$	0.48	0.70	0.70
$^{56}\text{Ba}^{135}$	$(g_{7/2})^6$	$(h_{11/2})^{12} (d_{3/2})^3$	0.74	0.94	0.83
$^{56}\text{Ba}^{137}$	$(g_{7/2})^6$	$(h_{11/2})^{12} (d_{3/2})^3$	0.77	0.95	0.94

Although the calculated values of S isotopes show very good agreement with the experimental values, the magnetic moments of K isotopes cannot be clearly explained by our calculation. Therefore, we are forced to expect the interpretation of these relatively small deviations for the second-order effects which are ignored throughout this work.

For Ir isotopes, we adopt proton-configuration  $(h_{11/2})^{10} (d_{3/2})^3$  in view of their positive quadrupole moments. The smallness of their deviation of magnetic moments can also be accounted for by this configuration since the large contribution from the excitation of  $d_{5/2}$  proton becomes vanishing. This configuration assignment is consistent with that of  $\text{Au}^{197}$  which has similar magnetic and quadrupole moments.

We shall attempt to investigate the effects of the dependence of the integral *I* upon the quantum numbers of wave functions involved in it. Since the configurations of these nuclei are rather definitely determined, this group is suitable for the present purpose. If it is possible to make use of the oscillator wave functions, we have the following values of ratios of the integrals approximately:

$$I(1d, 1d) : I(2d, 2d) : I(2d, 1g) : I(2d, 1f) = 31 : 20 : 9 : 8.$$

Adopting  $I(2d, 2d)$  as the standard of magnitude and inserting these values of ratios, we obtain  $\mu_{cal}^*$  in the above table. These modifications give good agreement essentially for odd-neutron nuclei.

(v)  $(5/2+)$  nuclei

Table VII. Magnetic moment of  $(5/2+)$  nuclei

odd-proton nuclei ( $\mu_{sp}=4.79$ )				
nucleus	$P$ -configuration	$N$ -configuration	$\mu_{cal}$	$\mu_{exp}$
$^{13}\text{Al}^{27}$	$(d_{5/2})^5$	$(d_{5/2})^6$	3.38	3.64
$^{51}\text{Sb}^{121}$	$(g_{9/2})^{10} d_{5/2}$	$(d_{5/2})^6 (h_{1/2})^6$	3.49	3.36
$^{53}\text{I}^{127}$	$(g_{7/2})^2 d_{5/2}$	$(d_{5/2})^6 (h_{1/2})^{10}$	3.69	2.81
	$(g_{9/2})^{10} (d_{5/2})^3$	"	3.04	
$^{55}\text{Cs}^{131}$	$(g_{7/2})^4 d_{5/2}$	$(d_{5/2})^6 (h_{11/2})^{12}$	3.89	3.48
	$(g_{7/2})^2 (d_{5/2})^3$	"	3.27	
$^{59}\text{Pr}^{141}$	$d_{5/2}$	$(h_{11/2})^{12}$	4.53	4.0
	$(g_{7/2})^6 (d_{5/2})^3$	"	3.95	
$^{63}\text{Eu}^{151}$	$(d_{5/2})^5$	$(h_{11/2})^{12} (f_{7/2})^6$	3.7	3.6
$^{75}\text{Re}^{185}$	$(h_{11/2})^{12} (d_{5/2})^5$	$(i_{13/2})^4$	3.19	3.17
$^{75}\text{Re}^{187}$	$(h_{11/2})^{12} (d_{5/2})^5$	$(i_{13/2})^6$	3.17	3.20
odd-neutron nuclei ( $\mu_{sp}=-1.91$ )				
$^8\text{O}^{17}$	—	$d_{5/2}$	-1.91	-1.89
$^{12}\text{Mg}^{25}$	$(d_{5/2})^4$	$(d_{5/2})^5$	-0.65	-0.86
$^{40}\text{Zr}^{91}$	—	$(g_{9/2})^{10} d_{5/2}$	-0.8	-1.3
$^{42}\text{Mo}^{95}$	$(g_{9/2})^2$	$(g_{9/2})^{10} (d_{5/2})^3$	-0.35	-0.91
	"	$(g_{7/2})^2 d_{5/2}$	-1.08	
$^{42}\text{Mo}^{97}$	$(g_{9/2})^2$	$(g_{9/2})^{10} (d_{5/2})^5$	+0.05	-0.93
	"	$(g_{7/2})^2 (d_{5/2})^3$	-0.65	
$^{46}\text{Pd}^{105}$	$(g_{9/2})^6$	$(g_{7/2})^4 (d_{5/2})^5$	-0.45	-0.57
$^{48}\text{Cd}^{111}$	$(d_{5/2})^2 (g_{9/2})^6$	$(g_{7/2})^4 (d_{5/2})^5$	-0.7	-0.7

The problem concerning the order of  $d_{5/2}$  and  $f_{7/2}$  orbit occurs for several nuclei in this group. For  $\text{I}^{127}$ , the negative quadrupole moment suggests  $(d_{5/2})^1$  configuration while better agreement is obtained for the magnetic moments by  $(d_{5/2})^3$  configuration. On the other hand, very small value of quadrupole moment of  $\text{Pr}^{141}$  suggests  $(d_{5/2})^3$  configuration which gives better agreement for magnetic moment. However, it seems natural to consider that the zeroth-order configurations for nuclei in this group are essentially mixed configurations and more thorough investigation of their magnetic and quadrupole moments is needed.

The calculated value for  $\text{Eu}^{151}$  shows good agreement with the experimental value, while, for  $\text{Eu}^{153}$ , the calculated value based upon the similar configuration is almost the same as for  $\text{Eu}^{151}$ . This contradicts seriously with the experiment, so that the configuration assignment by Moszkowski and Townes<sup>(21)</sup> might be necessary.

For  $\text{O}^{17}$ , the deviation vanishes in our approximation. This corresponds to the very small deviation of the observed value from the Schmidt limit.

(vi)  $(5/2-)$  nuclei

Table VIII. Magnetic moments of  $(5/2-)$  nuclei

odd-proton nuclei ( $\mu_{sp}=0.86$ )				
nucleus	<i>P</i> -configuration	<i>N</i> -configuration	$\mu_{cal}$	$\mu_{exp}$
$_{37}\text{Rb}^{85}$	$(f_{3/2})^4 (f_{5/2})^5$	$(g_{9/2})^8$	1.32	1.35
odd-neutron nuclei ( $\mu_{sp}=1.37$ )				
$_{30}\text{Zn}^{67}$	$(f_{7/2})^8 (p_{3/2})^2$	$(p_{3/2})^4 (f_{5/2})^5$	0.81	0.88

Except  $\text{Yb}^{173}$ , the calculated values are in good agreement with the experimental values. Because  $\text{Yb}^{173}$  has a large quadrupole moment, its proton-configuration was assigned as  $(f_{7/2})^5_{5/2} (f_{5/2})^6_0$  by Moszkowski and Townes. However, there remains the question whether this configuration is more favored in energetical consideration than any other configurations.

(vii)  $(7/2-)$  nucleiTable IX. Magnetic moments of  $(7/2-)$  nuclei

odd-proton nuclei ( $\mu_{sp}=5.79$ )				
nucleus	<i>P</i> -configuration	<i>N</i> -configuration	$\mu_{cal}$	$\mu_{exp}$
$_{21}\text{Sc}^{45}$	$f_{7/2}$	$(f_{7/2})^4$	5.62	4.75
	$(d_{3/2})^2 (f_{7/2})^3$	"	4.74	
$_{23}\text{V}^{51}$	$(f_{7/2})^3$	$(f_{7/2})^6$	5.02	5.15
$_{27}\text{Co}^{57}$	$(f_{7/2})^7$	$(p_{3/2})^2 (f_{7/2})^6$	4.1	4.6
$_{27}\text{Co}^{59}$	$(f_{7/2})^7$	$(p_{3/2})^4 (f_{7/2})^6$	4.10	4.65
odd-neutron nuclei ( $\mu_{sp}=-1.91$ )				
$_{20}\text{Ca}^{43}$	—	$(f_{7/2})^3$	-1.44	-1.31
$_{22}\text{Ti}^{49}$	$(f_{7/2})^2$	$(f_{7/2})^7$	-0.58	-1.10
$_{60}\text{Nd}^{143}$	$(d_{5/2})^2$	$(h_{11/2})^{12} f_{7/2}$	-0.84	-1.0
$_{60}\text{Nd}^{145}$	$(d_{5/2})^2$	$(h_{9/2})^2 f_{7/2}$	-1.05	-0.65
	"	$(h_{13/2})^{12} (f_{7/2})^3$	-0.64	
$_{62}\text{Sm}^{147}$	$(d_{5/2})^4$	$(h_{9/2})^2 f_{7/2}$	-1.01	-0.76
	"	$(h_{11/2})^{12} (f_{7/2})^3$	-0.60	
$_{62}\text{Sm}^{149}$	$(d_{5/2})^4$	$(h_{9/2})^4 f_{7/2}$	-1.21	-0.64
	"	$(h_{9/2})^2 (f_{7/2})^3$	-0.81	

By the usual assignment of proton-configuration  $(f_{7/2})^1$  of  $\text{Sc}^{45}$ , we have too small deviation for its magnetic moment. This contradicts with the observed value. If we could employ  $(d_{3/2})^{-2} (f_{7/2})^3$  configuration in stead of  $(f_{7/2})^1$ , the calculated value would show very good agreement with the observed value. We wonder whether such assignment is



energetically plausible and the so-called stabilization effect can make this configuration favorable.<sup>23)</sup>

The calculated value of deviations for Co isotopes are too large. Similar situation holds for Ti<sup>49</sup> in the odd-neutron group. It might be a general trend that somewhat larger values of deviations are obtained for such nuclei that many (odd) nucleons occupy the outermost orbit, for instance (7/2)<sup>7</sup> or (9/2)<sup>9</sup>. This might be revealed by more satisfactory estimation of the average value of energy difference  $\langle \Delta E \rangle$  than ours.

For Nd<sup>146</sup> and Sm isotopes, there is a competition between  $f_{7/2}$  and  $h_{9/2}$  neutron levels. Better agreements are obtained by the configuration assignment  $(f_{7/2})^3$  in place of  $(f_{7/2})^1$ .

(viii) (7/2+) nuclei

Table X. Magnetic moments of (7/2+) nuclei

odd-proton nuclei ( $\mu_N = 1.72$ )				
nucleus	$P$ -configuration	$N$ -configuration	$\mu_{\text{cal}}$	$\mu_{\text{exp}}$
<sup>51</sup> Sb <sup>123</sup>	$g_{7/2}$	$(d_{5/2})^6 (h_{11/2})^6$	2.46	2.55
<sup>53</sup> I <sup>129</sup>	$(g_{7/2})^3$	$(d_{5/2})^6 (h_{11/2})^{12}$	2.34	2.62
	$(d_{5/2})^2 g_{7/2}$	"	2.84	
	"	$(h_{11/2})^{10} (d_{3/2})^2$	2.69	
<sup>55</sup> Cs <sup>133</sup>	$(g_{7/2})^5$	$(h_{11/2})^{12} (d_{3/2})^2$	2.10	2.58
	$(d_{5/2})^2 (g_{7/2})^3$	"	2.42	
	$(d_{5/2})^4 g_{7/2}$	"	2.75	
<sup>55</sup> Cs <sup>135</sup>	$(g_{7/2})^5$	$(h_{11/2})^{12}$	2.05	2.72
	$(d_{5/2})^2 (g_{7/2})^3$	"	2.37	
	$(d_{5/2})^4 g_{7/2}$	"	2.68	
<sup>55</sup> Cs <sup>137</sup>	$(g_{7/2})^5$	$(h_{11/2})^{12}$	2.04	2.84
	$(d_{5/2})^2 (g_{7/2})^3$	"	2.36	
	$(d_{5/2})^4 g_{7/2}$	"	2.67	
<sup>57</sup> La <sup>139</sup>	$(g_{7/2})^7$	$(h_{11/2})^{12}$	1.88	2.78
	$(d_{5/2})^2 (g_{7/2})^5$	"	2.19	
<sup>71</sup> Lu <sup>175</sup>	$(d_{5/2})^6 (h_{11/2})^8 (g_{7/2})^7$	$(f_{5/2})^4$	2.4	2.9
<sup>73</sup> Ta <sup>181</sup>	$(d_{5/2})^6 (h_{11/2})^{10} (g_{7/2})^7$	$(i_{13/2})^2$	2.6	2.1
	$(h_{11/2})^8 (d_{5/2})^2 (g_{6/2})^7$	"	2.3	
	$(h_{11/2})^6 (g_{7/2})^7$	"	2.0	

For these nuclei, the competition between  $f_{7/2}$  and  $d_{5/2}$  proton-levels occurs. The calculated value shows good agreement with the observed value for Sb<sup>123</sup> where the competition does not matter. For I<sup>129</sup> the  $(d_{5/2})^2 g_{7/2}$  configuration gives better agreement with the experiment than  $(f_{7/2})^3$  configuration gives, but it seems likely that the zeroth-order configuration of this nucleus is essentially mixed. For remaining nuclei of this group, we cannot obtain good agreement between calculated and experimental values. In view of very large values of quadrupole moment of La<sup>139</sup> Lu<sup>175</sup> and Ta<sup>181</sup>, we may anticipate some relation between the deviation of magnetic moment and the magnitude of quadrupole moment as was pointed out by other authors.<sup>15), 25)</sup>

(ix)  $(9/2-)$  nucleiTable XI. Magnetic moments of  $(9/2+)$  nuclei

odd-proton nuclei ( $\mu_{sp}=6.79$ )				
nucleus	<i>P</i> -configuration	<i>N</i> -configuration	$\mu_{cal}$	$\mu_{exp}$
$^{93}\text{Nb}$	$g_{9/2}$	$(g_{7/2})^2$	6.60	6.16
	$(f_{3/2})^4 (g_{9/2})^3$	"	5.71	
$^{99}\text{Tc}$	$(g_{9/2})^3$	$(g_{7/2})^6$	6.45	5.68
	$(f_{3/2})^4 (g_{9/2})^5$	"	5.61	
$^{113}\text{In}$	$(g_{9/2})^9$	$(d_{5/2})^6$	5.62	5.49
$^{115}\text{In}$	"	$(h_{11/2})^2 (d_{5/2})^6$	5.59	5.50
odd-neutron nuclei ( $\mu_{sn}=-1.91$ )				
$^{83}\text{Kr}$	$(f_{5/2})^4 (p_{3/2})^4$	$(g_{9/2})^7$	-0.83	-0.97
$^{87}\text{Sr}$	$(f_{3/2})^4$	$(g_{9/2})^9$	-0.68	-1.09

In view of the competition between  $g_{9/2}$  and  $f_{11/2}$  proton-levels which is well known in the isomeric transitions, we calculate the magnetic moments of  $\text{Nb}^{93}$  and  $\text{Tc}^{99}$  based upon two alternatives. The observed value for  $\text{Nb}^{93}$  is just the average of the values calculated for two configurations. For  $\text{Tc}^{99}$ , the calculated value based upon  $(f_{9/2})^5$  configuration shows better agreement with the experimental value.

The larger calculated deviation for  $\text{Kr}^{83}$  and  $\text{Sr}^{87}$  than their observed values may be due to the crude estimate of  $\langle J/\rangle$  as was mentioned for Co isotopes and  $\text{Ti}^{49}$  in  $(7/2-)$  nuclei. However, the situation is somewhat complex, since the calculated values of In isotopes show fairly good agreement with the experimental values in spite of their  $(f_{9/2})^9$  configurations. It is desirable to investigate more thoroughly the problem of competition between levels and the estimate of  $\langle \Delta E \rangle$ .

(x)  $(9/2-)$  nucleiTable XII. Magnetic moment of  $(9/2-)$  nuclei

odd-proton nuclei ( $\mu_{sp}=2.62$ )				
nucleus	<i>P</i> -configuration	<i>N</i> -configuration	$\mu_{cal}$	$\mu_{exp}$
$^{209}\text{Bi}$	$(h_{11/2})^{12} h_{9/2}$	$(i_{13/2})^{14}$	3.43	4.08

The calculated deviation of the magnetic moment of  $\text{Bi}^{209}$  is somewhat small, but it will become large enough to fit the observed value if we put the doublet splittings  $\Delta E(h)$  and  $\Delta E(i)$  to be about 1.5 Mev.

#### § 4. Conclusion

The coefficients of admixture  $\alpha$  were estimated in a course of evaluation of magnetic moments and their values were found to be of the order of 0.1. Both the reduction of the amplitude of the unperturbed configuration and the second-order effect on the magnetic moments can usually be negligible. However, there can be so many mixing configurations which do not contribute to the magnetic moments linearly in the coefficients of mixing that they might give rise to additional deviation of the moment to those which we have discussed. Therefore, it is desirable to take into account the second-order effects for the more detailed analysis of empirical data. The small deviations of  $(1/2-)$  nuclei and the small differences of deviations of some isotopes will be interpreted as the second-order effects. For the differences of isotopes with the same spin, the small variation of mixing in the zeroth-order configuration may also play an important role.

There are some nuclei for which the magnetic moments lie at the middle of two values calculated for two different zeroth-order configurations. The fact may be taken as a evidence of the essential configurational mixing discussed by other authors.<sup>3),4)</sup>

In order to improve the accuracy of the calculated values, there is another problem which have not been fully worked out. It is the detailed estimation of the energy differences of the zeroth-order configurations which have been treated very simply. This question concerns the analysis of empirical data accumulated by the study of beta and isomeric transitions, and it is very important in the nuclear spectroscopy in the medium and heavy nuclei.

It is a very simple but remarkable result of this work that even the large deviation of magnetic moments from the Schmidt lines can be accounted for by the small admixture of excited configurations. In spite of rather simplified assumptions, the calculated values based upon our theoretical consideration show fairly good agreement with the experimental values of magnetic moments of odd-nuclei. Furthermore, the determination of the parameters in the formulae has been very crude, but the results represent sufficiently the actual variations in the empirical data.

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#### Appendix A. Derivation of eq. (8)

The interaction given by (7) can be written in another form

$$V_{ij} = [V_0 + V_1(\sigma_i \cdot \sigma_j)] \delta(r_i - r_j) \delta(\cos \omega - 1) / r_i r_j \quad (\text{A1})$$

where  $I_0' = (I_s' + 3I_t')/4$  and  $I_1' = (I_t' - I_s')/4$ . Expanding the angular part of the interaction into spherical harmonics, we obtain

$$V_{ij} = \sum_{\kappa k r} (-)^{\kappa+k-r} V_{\kappa} v_{\kappa}(r_i, r_j) (t_i^{(\kappa k; r)} \cdot t_j^{(\kappa k; r)}), \quad (\text{A2})$$

when

$$\mathbf{t}_q^{(nk; k)} = \mathbf{C}_q^{(k)} = [4\pi / (2k+1)]^{1/2} \Theta(kq) \Phi(q), \quad (\text{A3a})$$

$$\mathbf{t}_p^{(lk; r)} = [\boldsymbol{\sigma}^{(l)} \times \mathbf{C}^{(k)}]_p^{(r)}, \quad (r=k, k \pm 1) \quad (\text{A3b})$$

and

$$v_k(r_i, r_j) = (2k+1)/2 \cdot \delta(r_i - r_j) / r_i r_j.$$

On the other hand, it is easy to see that

$$\begin{aligned} & (j_1^{n_1}(0) j_2^{n_2}(0) j^p(j) j^m | \sum_{i < j} V_{ij} | j_1^{n_1-1}(j_1) j_2^{n_2+1}(j_2) ) (1) j^p(j) j^m) \\ &= [n_1(n_2+1)]^{1/2} (j_1^{n_1} 0 \{ | j_1^{n_1-1}(j_1) j_1 0 \} (j_1 j_1(0) j^p(j) j^m | \sum V_{ij} | j_1 j_2(1) j^p(j) j^m) \\ & \quad \cdot (j_2^{n_2}(0) j_2 j_2 | \{ j_2^{n_2+1} j_2 \}) \end{aligned} \quad (\text{A5})$$

where  $(j_1^{n_1} 0 \{ | j_1^{n_1-1}(j_1) j_1 0 \})$  and  $(j_2^{n_2}(0) j_2 j_2 | \{ j_2^{n_2+1} j_2 \})$  are the coefficients of fractional parentages, the values of which are given by (5a) and (5b), respectively. The matrix element of the right-hand side in (A5) is easy to calculate if the exchange between nucleons in orbits  $j_1 j_2$  and  $j$  can be ignored. By (A2) and RII (38),

$$\begin{aligned} & (j_1 j_1(0) j^p(j) j^m | \sum V_{ij} | j_1 j_2(1) j^p(j) j^m) \\ &= \sum_{\kappa k r} V_{\kappa} (j_1 j_1 0 \| \mathbf{t}^{(\kappa k; r)} \| j_1 j_2 1) (j^p j \| \sum_i \mathbf{t}_i^{(\kappa k; r)} \| j^p j) \\ & \quad \cdot [(2j+1)(2r+1)]^{-1/2} (2k+1) I \delta(r, 1), \end{aligned} \quad (\text{A6})$$

where  $I$  is the integral defined in (9), and the use is made of the fact  $\kappa + k + r = \text{even}$  due to the non-vanishing of  $(j \| \mathbf{t}^{(\kappa k; r)} \| j)$ .<sup>26)</sup> Since  $k$  must also be even by the similar argument,  $\kappa$  must be odd, i. e.,  $\kappa = 1$ . Therefore, making use of RII (44b) and analogous equation to RIII (69a), we get

$$\begin{aligned} & (j_1 j_1(0) j^p(j) j^m | \sum V_{ij} | j_1 j_2(1) j^p(j) j^m) \\ &= [3(2j+1)(2j_1+1)]^{-1/2} \sum_k (2k+1) (j_1 \| \mathbf{t}^{(1k; 1)} \| j_2) (j \| \mathbf{t}^{(1k; 1)} \| j) V_1 I. \end{aligned} \quad (\text{A7})$$

The summation over  $k$  can be carried out by means of similar method as was employed by de-Shalit<sup>27)</sup> and (A7) is given as

$$[3I_1 / (2I_1 + 1) j(j+1)]^{1/2} \{1 + (-1)^{1/2+I-j} (2j+1)/2\} / 2 \cdot V_1 I. \quad (\text{A8})$$

Collecting these equations, we obtain the matrix element (A5) when the exchange of nucleons between  $j_1$ ,  $j_2$  and  $j$ -orbits is ignored. This corresponds to the case where the nucleons in  $j_1$  and  $j_2$  are not identical with ones in  $j$ , in eq. (8). It becomes necessary to take into account the exchange when the nucleons in  $j_1$  and  $j_2$  are identical with ones in  $j$  and the derivation is somewhat complicated. However, the procedure in RII sec. 5 with slight modification makes the calculations easy, and we obtain another result in eq. (8).

## Appendix B. Derivation of eqs. (11) and (14)

The deviation of magnetic moment from the Schmidt limits due to the configurational mixing (10) is given by

$$\delta\mu = \sum_j 2\beta_j(j_1^n(0) j^p(j) j j | \sum \mu_z | j_1^{n-1}(j_1) j^{p+1}(J) j j) \quad (\text{A9})$$

where  $j=l-1/2$  and  $j_1=l+1/2$ . This can be reduced by RIII (28), RII (44a, b), (5) and (12) to

$$\delta\mu = [n(2j-p)/(2j-1)]^{1/2} \cdot [2j(l+1)/3(j+1)(2l+1)]^{1/2} (g_8 - g_l) \cdot \sum_j 2\beta_j(j_1 j(1) j j | j_1, j j(J) j), \quad (\text{A10})$$

where the last factor represents the transformation matrix between two coupling schemes of three angular momenta  $j_1$ ,  $j$  and  $j$  to form resultant  $j$ . The admixture  $\beta$ 's are determined from the two-body interaction of  $\delta$ -function type by means of perturbation theory. The matrix elements

$$(j_1^n(0) j^p(j) j m | \sum V_{ij} | j_1^{n-1}(j_1) j^{p+1}(J) j m) \quad (\text{A11})$$

must be expressed in terms of  $(j_1 j J' M' | I_{12} | j_1 j J' M')$  with complicated coefficients, where  $J'$  are the allowed angular momenta of  $j^2$  and  $j_1 j$ -configurations. For the  $\delta$ -function interaction (7), the matrix elements are given by

$$(j_1 j J' M' | V_{12} | j j J' M') = [(2j_1+1)(2j+1)^3]^{1/2} (j_1 j \frac{1}{2} - \frac{1}{2} | j_1 j J' 0) \cdot (j j \frac{1}{2} - \frac{1}{2} | j j J' 0) V_s I / [2(2J+1)] \quad (\text{A12})$$

for arbitrary  $j_1$  and  $j$  where, of course, proper parity consideration must be paid. Therefore, (A11) will include only the combination of the interaction strengths  $I_s = I_0 - 3I_1$ . Thus, it is necessary to calculate only the coefficient of either  $I_0$  or  $I_1$  of (A11). It is easy to obtain the coefficient of  $I_1$  since the similar relation as RIII (69a) can be applicable. The result of this procedure is given by

$$(j_1^n(0) j^p(j) j m | \sum V_{ij} | j_1^{n-1}(j) j^{p+1}(J) j m) = (-1)^{j_1+j} [2n(2j-p)/(2j-1)]^{1/2} \cdot (2j+1) I_s I / 2 \cdot (j_1 j \frac{1}{2} - \frac{1}{2} | j_1 j J 0) (j j \frac{1}{2} - \frac{1}{2} | j j J 0) / (2J+1)^{1/2}. \quad (\text{A13})$$

If, further, the average value of  $J/I$  over  $J$  is employed, the summation with respect to  $J$  in (A10) can be carried out,  $J$  being the allowed angular momenta of  $j^2$  and  $j_1 j$ -configurations.

$$\begin{aligned} & \sum_j 2(2J+1)^{-1/2} (j_1 j(1) j j | j_1, j j(J) j) (j_1 j \frac{1}{2} - \frac{1}{2} | j_1 j J 0) (j j \frac{1}{2} - \frac{1}{2} | j j J 0) \\ & = - (l-1)/2 \cdot [3/2 j(j+1)(2j+1)(2l+1)]^{1/2}. \end{aligned} \quad (\text{A14})$$

Thus, collecting these formulae, we get eq. (11).

The derivation of eq. (14) is almost similar as that of eq. (11).



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## Renormalization of Two-electron Green-function

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The difference between the ordinary method of renormalization of Dyson and that of the present authors developed in the previous paper is emphasized. In order to clarify this difference, the renormalization of the two-electron Green-function is investigated. The proof of the finiteness of the renormalized Green-function seems to be given in quite a brief way. It is the essential point in this proof that there is no ambiguity in the order of integration appearing in course of calculation of Green-functions.

## Introduction

In our previous paper<sup>1)</sup> the renormalization of one-body Green-function<sup>2)</sup> was investigated by using a method slightly different from the usual one.<sup>3)4)5)6)</sup> That is, in place of taking into account the so-called “*b*-divergence”, the transformation

$$G = Z_1 G', \quad \mathfrak{G} = \mathfrak{G}', \quad \Gamma^\mu = 1/Z_1 \cdot \Gamma'^\mu, \quad e = 1/\sqrt{Z_1} \cdot e' \quad (a)$$

was introduced, where the primed and unprimed quantities stand for renormalized and unrenormalized ones, respectively. In this transformation, the last one, the renormalization of the elementary charge, is characteristic of our theory and  $Z_1$  appearing in this charge renormalization plays the part for  $Z_1$  arising from the “*b*-divergences”.



Fig. 1.

In order to make clear the situation, let us consider, as an example, a graph of the type of electron self-energy. In the ordinary Dyson-Feynman method, the expression corresponding to this graph contains in general overlapping integrals, and since sub-integrals contained in this expression do not uniformly converge, the result that will be obtained depends essentially on the order of integration. To integrate the multiple integrals symmetrically with respect to both vertex parts *a* and *b* gives rise to the “*b*-divergence”. (Fig. 1) Thus the expression  $\Sigma^*$  corresponding to the electron self-energy can be written as

$$\Sigma^*(k) = Z_1^{-1} \{ A + B(i\gamma^\mu k_\mu + m) + C(k) \}, \quad (b)$$

where *A* and *B* are divergent constants and *C*(*k*) is the finite part.

On the other hand, in Schwinger's theory of Green-functions  $\Sigma^*$  has the expression

$$\Sigma^*(k) = ie_1^2 \gamma^\mu \int G(k-l) \mathfrak{G}_{\mu\nu}(l) \Gamma^\nu(k-l, k) dl \equiv ie_1^2 \gamma^\mu G(\mathfrak{G}_{\mu\nu} \Gamma^\nu), \quad (c)$$

where the vertex parts  $\alpha$  and  $b$  are treated in quite an asymmetrical way. In this expression,  $G$ ,  $\mathfrak{G}$  and  $I'$ , of course, contain in themselves much complicated multiple integrals. From the view-point of the theory of Green-functions, however, these integrations should have been performed before the last integration with respect to  $l$  is done. Thus the order of integration is automatically determined and there occur no overlapping integrals in this theory. From these situations, it comes out to be unnecessary to take into account the "b-divergences." Accordingly, in order to eliminate all the divergences (except the divergences of the perturbation expansion) within the frame work of the theory of Green-functions, it seems indispensable to introduce the renormalization of charge constant

$$c_1 = 1/\sqrt{Z_1} \cdot c_1',$$

instead of taking account of "b-divergence". In fact (c) is rewritten as

$$\Sigma^* = i 1/Z_1 \cdot e_1'^2 \gamma^\mu G' \mathfrak{G}'_{\mu\nu} I'^\nu = 1/Z_1 \cdot \Sigma^*'. \quad (d)$$

Under the assumption that  $G'$ ,  $\mathfrak{G}'$  and  $I''$  were already proved to be finite, the integral

$$\Sigma'^* = i c_1'^2 \gamma^\mu \int G'(k-l) \mathfrak{G}'_{\mu\nu}(l) I'^\nu(k-l, l) dl$$

has the nature similar to the integral

$$i c_1'^2 \gamma^\mu \int S_F(k-l) D_F(l) \delta_{\mu\nu} \gamma^\nu dl,$$

and the result after integration is

$$\Sigma'^*(k) = A + B(i k_\mu \gamma^\mu + m_1) + C(k).$$

From the argument stated up to now, one can easily conjecture that the renormalization of  $e_1$ , plays the part for the effect of "b-divergence".

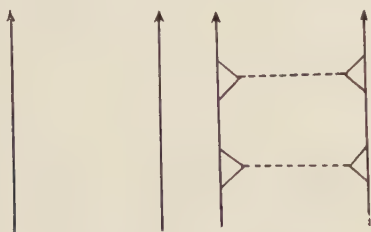
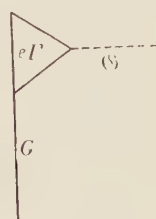
Fig. II<sub>1</sub>Fig. II<sub>2</sub>

Fig. III

To clarify in more detail the difference between the ordinary method of renormalization and ours, let us consider another example, i.e. the two-electron problem. (Fig. II<sub>1</sub>, II<sub>2</sub>) In the Dyson theory, any matrix element of  $S$  including all the radiation effects is given by replacing respectively  $G$ ,  $\mathfrak{G}$  and  $I'$  with all  $S_F$ 's,  $D_F$ 's and  $\gamma$ 's appearing in the matrix element corresponding to the irreducible Dyson-graph. For example, the matrix element corresponding to Fig. II<sub>2</sub> is given by

$$e_1^4 G \Gamma^\mu G \Gamma^\nu G \mathfrak{G}_{\mu\rho} \mathfrak{G}_{\nu\sigma} G \Gamma^\rho G \Gamma^\sigma G,$$

which after the renormalization turns out to be

$$Z_1^{-2} \{ e_1^4 G' I'^{\mu} G' I'^{\nu} \mathfrak{G}'_{\mu\nu} \mathfrak{G}'_{\nu\sigma} G' I'^{\rho} G' I'^{\sigma} G' \}.$$

In general, any matrix element corresponding to a graph having  $n$  open electron polygons comes to have  $Z_1^n$  as a factor after the renormalization has been carried out, because the expression  $e_1 I'^{\mu} G \mathfrak{G}_{\mu\nu}$  corresponding to Fig. III is invariant under the transformation of renormalization.

On the other hand, in Schwinger's theory all the  $\gamma$ 's appearing in the irreducible graph are not always replaced with  $I'$ 's, but some of the vertex parts are treated in quite an asymmetrical way. The matrix elements corresponding to Fig. II<sub>1</sub> and Fig. II<sub>2</sub> have such expressions as

$$Z_1^{-2} G' G' \text{ and } e_1^4 / Z_1^2 \cdot Z_1^{-2} G' I'^{\mu} G' I'^{\nu} G' \mathfrak{G}_{\mu\rho}' \mathfrak{G}'_{\nu\sigma} G' I'^{\rho} G' I'^{\sigma} G',$$

respectively, after the renormalization. From this fact one can easily see that the part of the two electron Green-function corresponding to the disconnected Dyson-graph and that corresponding to connected graphs can not be renormalized in a unified way, but rather should be treated separately.

In the present paper it will be shown how to renormalize many-electron Green-functions by dealing with the simplest case, i.e. the two-electron Green-function<sup>5)</sup>. In our proof of the finiteness of Green-functions, it will be the essential point that there is no ambiguity in the order of integration.

## § 1. Preliminary considerations

The Green-function of two-electron problem is given by

$$G^{II}(x_1 x_2; x_1' x_2') = G(x_1 x_1') G(x_2 x_2') - G(x_1 x_2') G(x_2 x_1') + \int G(x_1 1) G(x_2 2) K(1 2; x_1' x_2') d1 d2,$$

where

$$\begin{aligned} K(y_1 y_2; x_1' x_2') &= \int I(y_1 y_2; 1 2) G^{II}(12; x_1' x_2') d1 d2 \\ &= -ie_1^{-2} \gamma^{\mu} \int I^{\nu}(y_2 1; 2) \mathfrak{G}_{\mu\nu}(y, 2) G^{II}(y_1 1; x_1' x_2') d1 d2 \quad (1 \cdot 2) \\ &\quad - ie_1^{-2} \gamma^{\mu} \int G(y, 1) \partial/\partial y_{\mu} J_{\mu}(y_1) \cdot K(1 y_2; x_1' x_2') d1. \end{aligned}$$

In these equations  $G$ ,  $\mathfrak{G}$ , and  $I'$  are the unrenormalized functions in one-electron problem whose definitions were given in § 4 of U. S. I.<sup>1)</sup>.  $I$  is the proper interaction kernel in two-electron problem.<sup>2)b)</sup>

By substituting (1.1) into (1.2), it reads as

$$\begin{aligned} K(x_1 x_2; x_1' x_2') &= -ie_1^{-2} \gamma^{\mu} \int I^{\nu}(x_2 1; 2) \mathfrak{G}_{\mu\nu}(x_1 2) \{ G(x_1 x_1') G(1 x_2') \\ &\quad - G(x_1 x_2') G(1 x_1') \} d1 d2 \end{aligned}$$

$$\begin{aligned}
 & -ie_1^2 \gamma^\mu \int \Gamma^\nu(x_2 1; 2) \mathfrak{G}_{\mu\nu}(x_1 2) G(x_1 3) G(14) \\
 & K(34; x_1' x_2') d1 d2 d3 d4 \\
 & -ie_1^2 \gamma^\mu \int G(x_1 1) \delta/\epsilon_1 \delta J^\mu(x_1) \cdot K(1x_2; x_1' x_2') d1. \quad (1.3)
 \end{aligned}$$

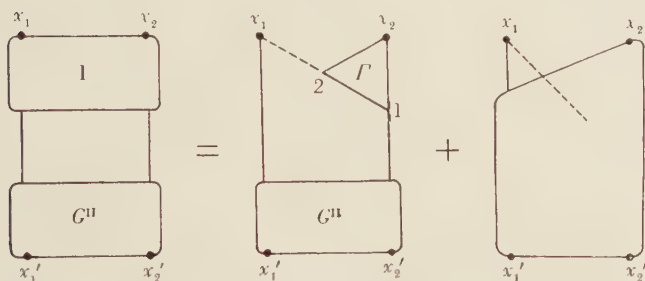


Fig. 1.1 Graphical representation of (1.2)

By using the iteration method, (1.1) is easily solved in a form as

$$G^{11} = \sum_{n=0}^{\infty} (G \cdot G I)^n G \times G,$$

where

$$G \times G = G(1 1') G(2 2') - G(1 2') G(2 1').$$

The graphical representation is given in Fig. 1.2.

Now it will happen that this graph is divided by broken lines, as shown in Fig. 1.3, into several pieces with actual electron- and photon-lines in both ends. However, the case where electron pairs are created (c.f. Fig. 1.5) will be excluded from our consideration if the incident energy is not enough to create actual pairs, and the case where

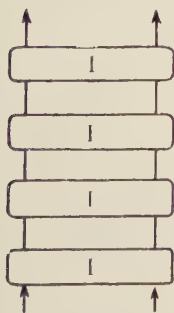


Fig. 1.2

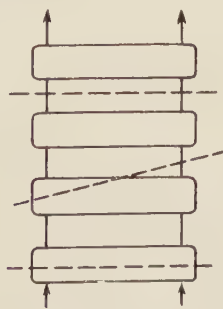


Fig. 1.3

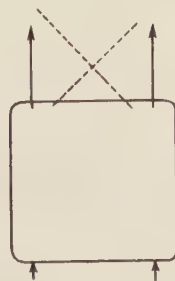


Fig. 1.4

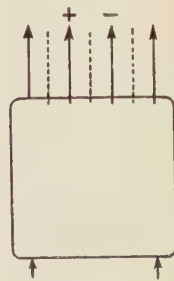


Fig. 1.5

actual photon-lines appear (c.f. Fig. 1.4) can be left out of our arguments by virtue of the fact that the quantity corresponding to the graph Fig. 1.4 can be obtained by differentia-



ting undivisible  $G''$  with respect to  $\int_{\mu}$ , provided that undivisible  $G''$  is finite. Thus  $G''$  is given by a sum of integrals whose integrands consist of several factors corresponding to undivisible pieces of graphs with two actual electron-lines and actual photon-lines in both ends. Since these integrations should be carried out over the limited domain where the energy and momentum of each connection-line satisfy the condition for actual particles,  $G''$  is finite if undivisible  $G''$  is finite. In order to restrict our problem only to undivisible graphs, or in other words, to the case where no displaced poles appear in course of calculation of  $G''$ , let us take  $\epsilon$  and  $i\lambda$ , the well-known imaginary parts of mass of electrons and photons, sufficiently large compared with the incident energies and momenta.

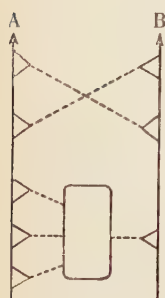


Fig. 1-6

For the sake of convenience, we define a term "bridge". From the structure of Eq. (1.2) we can express  $G''$  as a functional of  $G$ ,  $\mathfrak{G}$ ,  $\gamma$  and  $\Gamma$ . Thus  $G''$  is a sum of contributions corresponding to Feynman-Dyson graphs as shown in Fig. 1.6, where full and dotted lines correspond to  $G$ 's and  $\mathfrak{G}$ 's respectively. Now one can easily see that any graph of the type of Fig. 1.6 can be divided into right and left parts by cutting only photon-lines which connect both parts. Let us choose a special way of division where the number of cut lines is minimum among the possible ways of cutting. Furthermore, if there are several ways having minimal number of cut photon-lines, we take a particular way where the boundary between both parts of the graph is

nearest to the open electron-polygon  $B$ . In this special case of division, the part containing  $A$ -open-polygon is called the "left-graph", the other the "right-graph", and finally the cut photon-lines are called the "bridges".<sup>5)</sup>

## § 2. Renormalization of $G''$

In this section we shall prove the finiteness of  $G''$ . In order to do this, let us perform the following transformation:

$$G = aG' \quad (a = Z_1), \quad e_1 = 1/\sqrt{a} \cdot e'_1, \quad J = \sqrt{a} J', \quad \mathfrak{G} = \mathfrak{G}', \quad \Gamma = 1/a \cdot \Gamma', \quad A = 1/a \cdot A', \text{ etc.}, \quad (2.1)$$

in the same way as U.S.I., and

$$K = bK', \quad (2.2)$$

where  $b$  is a constant to be determined later.

Now let us postulate that  $G'''$  satisfies an equation of the same form as that of  $G''$  and define it as follows:

$$G''' = G' \times G' + G' G' K', \quad (2.3)$$

then we can easily see that  $G'''$  will be finite if  $K'$  be finite, because all the primed quantities of one-body problem were proved to be finite in U.S.I.. Therefore, we shall show the finiteness of  $K'$ .

By using (2.1) and (2.2), (1.3) becomes

$$K' = -\frac{i}{b} e_1'^2 \gamma^\mu \Gamma'^\nu \mathfrak{G}'_{\mu\nu} G' \times G' - i e_1'^2 \gamma^\mu \Gamma'^\nu \mathfrak{G}'_{\mu\nu} G' G' K' - i e_1'^2 \gamma^\mu G' \frac{\partial}{e_1' \partial f_{\mu}'} K'. \quad (2.4)$$

Here, we expand  $K'$  as

$$K' = \sum_{n=1}^{\infty} K_n', \quad (2.5)$$

where  $K_n'$  is all the contributions from the  $n$ -bridge-graphs.

Substituting (2.5) into (2.4), we have the equations;

$$K_1' = -\frac{i}{b} e_1'^2 \gamma^\mu \Gamma'^\nu \mathfrak{G}'_{\mu\nu} G' \times G' - i e_1'^2 \gamma^\mu G' \frac{\partial^L}{e_1' \partial f_{\mu}'} K_1' \quad (2.6)_1$$

and

$$K_{n+1}' = -i e_1'^2 \gamma^\mu \Gamma'^\nu \mathfrak{G}'_{\mu\nu} G' G' K_n' - i e_1'^2 \gamma^\mu G' \frac{\partial^R}{e_1' \partial f_{\mu}'} K_n' \quad (2.6)_2$$

$$- i e_1'^2 \gamma^\mu G' \frac{\partial^L}{e_1' \partial f_{\mu}'} K_{n+1}' \quad \text{for } n=1, 2, \dots,$$

where  $\partial^R/\partial f'$  stands for the functional differentiation of the "right-graph" only and  $\partial^L/\partial f'$  that of the remaining part, so that  $\partial^R/\partial f'$  adds a bridge to this graph, but  $\partial^L/\partial f'$  does not.

In the first place, we show that (3.5) has the following solution

$$K_1' = -i e_1'^2 \Gamma'^\mu \Gamma'^\nu \mathfrak{G}'_{\mu\nu} G' \times G'. \quad (2.7)$$

Proof: Substituting with (2.7) into the second term of the right hand side in (2.6)<sub>1</sub>, we have

$$\begin{aligned} & -i e_1'^2 \gamma^\mu G' (x_1 \ 1) \frac{\partial^L}{e_1' \partial f_{\mu}'} K_1' (1 \ x_2; x_1' \ x_2') \\ &= -i e_1'^2 \gamma^\mu G' (x_1 \ 1) \frac{\partial^L}{e_1' \partial f_{\mu}'} [(-i e_1'^2) \Gamma'^\rho (12; 3) \mathfrak{G}'_{\rho\sigma} (34) \Gamma'^\sigma (x_1' \ 5; 4) \\ & \quad \times G' (2 \ x_2) G' (5 \ x_2')] + [\text{the same term in which } x_2 \text{ is exchanged for } x_2'] \\ &= -i e_1'^2 \gamma^\mu G' (x_1 \ 1) \frac{\partial}{e_1' \partial f_{\mu}'} [ (+i e_1'^2) \frac{\partial G^{-1}(12)}{e_1' \partial f_{\sigma}'} G' (2 \ x_2) ] \\ & \quad \times \Gamma'^\sigma (x_1' \ 5; 4) G' (5 \ x_2') + ['' ] \\ &= e_1'^4 \gamma^\mu \left\{ \frac{\partial}{e_1' \partial f_{\mu}'} \left( G' (x_1 \ 1) \frac{\partial G^{-1}(12)}{e_1' \partial f_{\sigma}'} G' (2 \ x_2) \right) - \frac{\partial G' (x, 1)}{e_1' \partial f_{\mu}'} \frac{\partial G^{-1}(12)}{e_1' \partial f_{\sigma}'} \right. \\ & \quad \left. \times G' (2 \ x_2) \right\} \Gamma'^\sigma (x_1' \ 5; 4) G' (5 \ x_2') + ['' ] \\ &= \left\{ -e_1'^4 \gamma^\mu \frac{\partial}{e_1' \partial f_{\mu}'} \frac{\partial G' (x_1 x_2)}{e_1' \partial f_{\sigma}'} + i e_1'^2 \Sigma^{*'} (x_1 3) G' (31) \frac{\partial G^{-1}(12)}{e_1' \partial f_{\sigma}'} G' (2 \ x_2) \right\} \\ & \quad \times \Gamma'^\sigma (x_1' \ 5; 4) G' (5 \ x_2') + ['' ] \end{aligned}$$

$$\begin{aligned}
&= \left[ i e_1'^2 \frac{\partial}{e_1' \partial f_{\sigma}'(4)} \{ \Sigma^{*'}(x_1, 2) G'(2x_2) \} - i e_1' \Sigma^{*'}(x_1, 2) \frac{\partial G(2x_2)}{e_1' \partial f_{\sigma}'(4)} \right] \\
&\quad \times I''^{\sigma}(x_1', 5; 4) G'(5x_2') + [ \text{''} ] \\
&= i e_1'^2 \frac{\partial \Sigma^{*'}(x_1, 2)}{e_1' \partial f_{\sigma}'(4)} G'(2x_2) I''^{\sigma}(x_1', 5; 4) G'(5x_2') + [ \text{''} ] \\
&= -i e_1'^2 A'_{\rho}(x_1, 2; 3) \mathfrak{G}'_{\rho\sigma}(34) G'(2x_2) I''^{\sigma}(x_1', 5; 4) G'(5x_2') + [ \text{''} ].
\end{aligned}$$

Therefore, the right-hand side of (2.6), is

$$R.H.S. = -i e_1'^2 \{ 1/b \cdot \gamma^{\mu} + A'^{\mu} \} I''^{\nu} \mathfrak{G}'_{\mu\nu} G' \times G'.$$

Putting  $b^{-1} = a$ , we get

$$R.H.S. = -i e_1'^2 I''^{\mu} I''^{\nu} \mathfrak{G}'_{\mu\nu} G' \times G' = K_1', \quad \text{Q.E.D.}$$

Since  $I'$ ,  $\mathfrak{G}'$  and  $G'$  are finite, (2.7) is evidently finite.

In the second place, we consider (2.6)<sub>2</sub>. Here we expand  $K_n'$  in powers of  $e_1'$ ;

$$K_n' = \sum_{m=0}^{\infty} e_1'^{2n+m} K'_{n,m}. \quad (2.8)$$

Substituting (2.8) into (2.6)<sub>2</sub> and comparing both sides of (2.6)<sub>2</sub>, we get the following equations:

$$K_{n+1,m} = -i (\gamma^{\mu} I''^{\nu} \mathfrak{G}'_{\mu\nu} G' G' K_n')_m - i \gamma^{\mu} (G' \frac{\partial^L}{e_1' \partial f_{\mu}'} K_n')_m \quad \text{for } m=0, 1, \quad (2.9)$$

and

$$\begin{aligned}
K_{n+1,m} = & -i (\gamma^{\mu} I''^{\nu} \mathfrak{G}'_{\mu\nu} G' G' K_n')_m - i \gamma^{\mu} (G' \frac{\partial^L}{e_1' \partial f_{\mu}'} K_n')_m - i \gamma^{\mu} (G' \frac{\partial^L}{e_1' \partial f_{\mu}'} K'_{n+1})_{m-2} \\
& \text{for } m \geq 2. \quad (2.10)
\end{aligned}$$

Now we shall prove the finiteness of  $K'$  by means of the mathematical induction with assumption of the convergence of series in coupling constant.

Let us assume

$$K'_{n,m} \text{ are finite for } m=0 \text{ and } 1, \quad (2.11)$$

then in the right-hand side of (2.9), the first term is finite on account of this assumption, and the second term is also finite, because its graph is given by such one as Fig. 2.1, thus  $K'_{n+1,m}$  for  $m=0$  and 1 are finite. Furthermore let us assume

$$\begin{aligned}
&K'_{n,k} \quad \text{for } m \geq k \geq 0 \\
&\text{and} \\
&K'_{n+1,k} \quad \text{for } m-2 \geq k \geq 0 \text{ are finite.}
\end{aligned} \quad (2.12)$$

In the right-hand side of (2.10), the first and second terms are finite by virtue of (2.12)

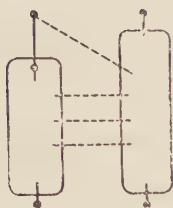


Fig. 2.1

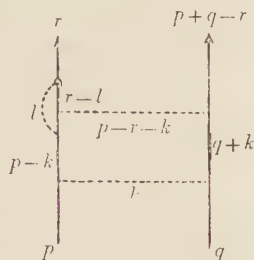


Fig. 2.2



Fig. 2.3

from the same reason as that in the above argument. In dealing with the third term, however, some attention should be paid, because this term contains such graphs as Fig. 2.2 and 2.3 which might cause new divergences. Thus we should investigate only the cases Fig. 2.2 and 2.3, since these are only graphs having possibilities to cause divergences.

As already emphasized in the introduction, the order of momentum integration in (2.10) is determined beforehand from the view-point of the theory of Green-function. As an example, let us consider the integral corresponding to Fig. 3.2. This is written as

$$\int \mathfrak{G}'(l) G'(r-l) F(p, q, r, l) dl, \quad (2.13)$$

where  $F(p, q, r, l) = \int \mathfrak{G}'(k) G'(p-k) G'(p-k-l) \mathfrak{G}'(p-r-k) dk$ .

By using Feynman's method of parameter integration, it can be proved that  $F$  is  $O(l^{-2+\alpha})$  ( $1 \gg \alpha > 0$ ) for sufficiently large  $l$ . (c.f. Appendix A). Thus it is easily seen that (2.13) converges. Similar consideration is also possible in the case of Fig. 2.3. In this way we have finished the proof of the finiteness of  $\Lambda'$ , since  $\Lambda'_1$  is already got in a form being free from divergences.

### Acknowledgements

The authors would like to express their gratitude to Dr. Nakai for his valuable discussions.

### Appendix A

We shall prove that  $F(pqr)$  in (2.13) depends on  $l$  in a form of  $l^{-2+\alpha}$  ( $1 \gg \alpha > 0$ ) for sufficiently large  $l$ . As a first step, let us evaluate an integral corresponding to a graph which is given by replacing  $G'$ ,  $\mathfrak{G}'$  and  $\Gamma'$  in Fig. 2.2 with  $S_F$ ,  $D_F$  and  $\gamma$  respectively. This integral is written as

$$f(pqr) = \text{const.} \int \frac{i(p-k-l)\gamma + m}{(p-k-l)^2 + m^2} \cdot \frac{i(p-k)\gamma + m}{(p-k)^2 + m^2} \cdot \frac{i(q-k)\gamma + m}{(q-k)^2 + m^2} \cdot \frac{1}{k^2 + \kappa^2} \\ \times \frac{1}{(p-r-k)^2 + \kappa^2} d^4k. \quad (1)$$

Using the well-known method of parameter integration, we can transform (1) into a linear combination of the following expressions ;

$$\int_0^1 \left\{ \frac{(1; P_\sigma; P_\sigma P_\tau; -P_\lambda P_\sigma P_\tau)}{(-P^2 + D)^3} - \frac{1}{4} \frac{(0; 0; \partial_{\sigma\tau}; 2(\partial_{\sigma\lambda} P_\tau + \partial_{\tau\lambda} P_\sigma + \partial_{\sigma\tau} P_\lambda))}{(-P^2 + D)^2} \right\} \times u^3 z^2 y du dx dy dz, \quad (2)$$

where

$$P_\mu = -(1-u)l_\mu + p_\mu + uA_\mu$$

$$J = (1-u)l^2 - 2(1-u)l \cdot p + B$$

$$-P^2 + J = (1-u)ul^2 + 2(1-u)ul \cdot A + C$$

$$A_\mu = (q-p)_\mu z - q_\mu y z - (p-r)_\mu x y z$$

$$B = p^2 + m^2 + (q^2 - p^2)zu + (\lambda^2 - m^2 - q^2)yzu + (p-r)^2 uxyz$$

$$C = -(uA + p)^2 + B,$$

after carrying out the momentum integration. Now it must be remarked that  $l_0$  can be replaced with  $i l_0$  since contributions from displaced poles can be left from our consideration. Dividing the domain of  $u$ -integration as

$$I_1 + I_2 + I_3 + I_4 + I_5 = \int_0^{l^{-2-\alpha}} + \int_{l^{-2-\alpha}}^{l^{-2+\alpha}} + \int_{l^{-2+\alpha}}^{1-l^{-2+\alpha}} + \int_{1-l^{-2-\alpha}}^{1-l^{-2-\alpha}} + \int_{1-l^{-2-\alpha}}^1,$$

we can get the following  $l$ -dependences approximately

$$I_1 \sim \int \frac{x^2 y}{C^m} dx dy dz l^n (l^{-2-\alpha})^3 (l^{-2-\alpha}) \sim O(l^{-8+n-4\alpha})$$

$$I_2 \sim \int \frac{x^2 y}{(1+C)^m} dx dy dz l^n (l^{-2})^3 (l^{-2+\alpha}) \sim O(l^{-8+n+\alpha})$$

$$I_3 \sim \int \frac{(1-u)^n l^n}{(u(1-u)l^2)^m} u^3 du \sim O\left(\left[(1-u)^{-m+n+1}\right]_{u=l^{-2-\alpha}}^{u=1-l^{-2+\alpha}} \times l^{n-2m}\right) \sim O(l^{-2+\alpha-n-\alpha n-\alpha m})$$

$$I_4 \sim \int \frac{x^2 y}{(1+C)^m} dx dy dz (l^{-2})^n (l^n) (l^{-2+\alpha}) \sim O(l^{-2+\alpha-n})$$

$$I_5 \sim \int \frac{x^2 y}{C^m} dx dy dz (l^{-2+\alpha})^n (l^n) (l^{-2-\alpha}) \sim O(l^{-2-\alpha-n+n\alpha}),$$

where  $m=3$ ,  $n=0,1,2,3$  or  $m=2$  and  $n=0,1$ . Thus we get

$$f(l) \sim O(l^{-2+\alpha}).$$

To get the conclusion in § 2, it is also necessary to assume that  $G'$ ,  $\mathcal{G}'$  and  $I''$  have the following functional forms with respect to  $k$ ;



$$G'(k) \sim \varphi_1(k) \frac{ik\gamma + b}{k^2 + a^2}$$

$$\mathfrak{G}'(k) \sim \varphi_2(k) \frac{1}{k^2 + c^2}$$

$$I''_\mu(k, l) \sim \varphi_3(k) \gamma_\mu,$$

where  $\varphi_i$ 's are  $O(k^0 l^0)$  for large  $k$  and  $l$ , and are assumed to be well-behaved functions<sup>7)</sup> (c.f. Appendix B). (For example, it is one of the sufficient conditions for  $\varphi_i$ 's that  $\varphi_i$ 's have no poles at least in the upper (or lower) half-plane of  $k_0$ .) This assumption seems to be reasonable because it can be considered that the procedure of renormalization does not change radically the nature of quantities if they are renormalizable.

## Appendix B

In this appendix we shall show that  $G'(k)$ ,  $\mathfrak{G}'(k)$  and  $I''(k, l)$  are respectively  $O(k^{-1})$ ,  $O(k^{-2})$  and  $O(k^0, l^0)$  for sufficiently large  $k$  and  $l$ .

Let us consider an integral of the following form;

$$I(p_1, p_2) = \int F_r(k) G_s(p_1 - k) H_t(p_2 - k) K_u(p_1 \pm p_2 - k) d^{4n}k,$$

where  $F_r(k)$  is a product of  $\mathfrak{G}'$  and  $G'$  depending only on  $k_1, k_2, \dots, k_n$  and is  $O(k^{-r})$  for large  $k$ .  $G$ ,  $H$  and  $K$  are assumed to have also the nature similar to that of  $F$ . For example, we can easily see that  $I$  represents respectively

- i) the irreducible electron-self-energy part when  $r+s=3$ ,  $n=1$  and  $H_t=K_u=1$ ,
- ii) the irreducible photon-self-energy part when  $r+s=2$ ,  $n=1$  and  $H_t=K_u=1$ , and
- iii) the irreducible vertex part when  $r+s+t+u=4n$ .

Now we shall restrict our consideration only to  $p_1$ -dependence of  $I$ , because  $p_2$ -dependence can be similarly treated. After the transformation of all the energy variables from real to imaginary ones, let us introduce a system of  $4n$ -dimensional polar coordinates in the Euclidian  $k$ -space. Then  $I$  can be written as

$$I = \int F_r G_s H_t K_u R^{4n+1} dR d\Omega,$$

where  $\Omega$  represents angular variables. Divide the domain of  $R$ -integration as

$$I_1 + I_2 + I_3 + I_4 + I_5 = \int_0^{\max(\lambda, m, p_2)} + \int_{\max(\lambda, m, p_2)}^{p_1^{(1-\alpha)}} + \int_{p_1^{(1-\alpha)}}^{p_1^{(1+\alpha)}} + \int_{p_1^{(1+\alpha)}}^{\infty}$$

where  $\alpha$  is a positive constant being possible to be chosen as arbitrarily small. Then the  $p_1$ -dependence of each integral is approximately given as

$$I_1 \sim p_1^{-s-u} \times \{ [R^{4n}]_0^{R=\max(\lambda, m, p_2)} \} \times \int (FH)_{R=0} d\Omega \sim (p_1^{-s-u})$$

$$I_2 \sim p_1^{-s-u} \times (\max(\lambda, m, p_2))^{4n} \times \int (FH)_{R \sim \lambda, m, p_2} d\Omega \sim O(p_1^{-s-u})$$

$$\begin{aligned}
I_3 &\sim p_1^{-s-u} \times \{ [R^{4n-r-t}]_{R=\max(\lambda, m, p_2)}^{(1-\alpha)} \} \times \int (FHR^{r+t})_{R=\infty} d\Omega \sim O(p^{-s-u+(4n-r-t)(1-\alpha)}) \\
I_4 &\sim p_1^{-r-s-t-u+4n-1} \times p_1^{1+\alpha} \times \int (FGHKKR^{r+s+t+u})_{R=p_1 \sim \infty} d\Omega \sim O(p_1^{4n-r-s-t-u+\alpha}) \\
I_5 &\sim \{ [R^{4n-r-s-t-u}]_{R=p_1}^{(1+\alpha)} \} \times \int (FGHKKR^{r+s+t+u})_{R=\infty} d\Omega \sim O(p_1^{(4n-r-s-t-u)(1+\alpha)}).
\end{aligned}$$

Here two points must be noted. One is the finiteness of  $\Omega$ -integration for a fixed  $R$ . The other is that the contribution from the upper limit in  $I_5$  can be neglected because it should be cancelled out by the contribution from the renormalization-term. Since the infinitesimal constant  $\alpha$  does not play role in the discussion of divergences of any integral, we can get the following conclusion

$$I \sim O(p^{4n-r-s-t-u}).$$

This is just the one we wish to prove.

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# On the Binding Energy and Properties of Lower Energy Levels of $\text{Li}^6$

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By modifying the usual shell model treatment, so as to take into account the correlation of the position coordinates of the constituent particles, the binding energy and other properties of the lower energy levels of  $\text{Li}^6$  are calculated. Using two body central and tensor interactions, the parameters of which are adjusted to the low energy two body data, fairly good agreement with experiment is obtained for the binding energy. However, the splitting between the second excited and the ground states is far too small, because of the large cancellation of the tensor S-D interference terms, and this is reflected in the too large value of the magnetic moment of the ground state. The quadrupole moment is very small in agreement with experiment, although it might be considered as an incidental agreement.

## § 1. Introduction

The primary purpose of this paper is to investigate whether two particle interactions, which are consistent with two body problems, are sufficient to give the binding energy of  $\text{Li}^6$  and properties of its lower levels, such as level order and spacings of these levels, and the magnetic moment and the quadrupole moment of the ground state, etc. However, since a great many papers have been published on the same subjects, it would be instructive first to give a brief summary of these works.

Feenberg and Wigner<sup>(1)</sup> have given general expressions for central force matrix elements in the L-S coupling approximation for all states which occur in the  $1p$  shell nuclei, by assuming only Majorana and Heisenberg type exchange forces. Feenberg and Phillips<sup>(2)</sup> have generalized their calculation so as to include also Wigner and Bartlett type central forces. The results given in these two papers are similar, and it is found that, if the central force with dominant Majorana force is used and if the depth parameters are normalized in such a way as to give correct binding energies of  $\text{He}^4$  and  $\text{O}^{16}$ , then the binding energies of other  $1p$  shell nuclei are also obtained in fair agreement with experiments. It is also found that the mass difference of  $\text{Li}^6$  and  $\text{He}^6$ , which is equivalent to the level spacing of the second excited and the ground states of  $\text{Li}^6$ , is correctly given. However, if we make use of depth parameters consistent with two body problems, the binding energies

are considerably smaller than the experimental values.

Kurath<sup>3)</sup> has performed similar calculations for the  $1p$  shell nuclei in the  $j-j$  coupling shell model. His result is quite similar in its essence to the L-S coupling result. However, there are some cases, in which  $j-j$  coupling gives correct results, while L-S coupling does not, and vice versa. Especially, for the ground state spin of  $\text{Li}^6$ ,  $j-j$  coupling approximation gives  $J=3$ , while L-S coupling approximation gives  $J=1$  in agreement with experiment. As regards the ground state of  $\text{B}^{10}$ , theoretical values are the same as in the case of  $\text{Li}^6$ , while the experimental values give  $J=3$ ; so the  $j-j$  coupling approximation is better in this case.

In order to overcome these inconsistencies, Zeldes<sup>4</sup> has performed calculations by the intermediate coupling approximation, and showed that both ground state spins of  $\text{Li}^6$  and  $\text{B}^{10}$  could be explained by using a common value for the parameter  $a/K$ , the ratio of the strength of the spin-orbit interaction and central interactions.

In an elaborate review article, Inglis<sup>5)</sup> has given a more thorough discussion about the level structures of all  $1p$  shell nuclei in the intermediate coupling approximation, and arrived at a conclusion similar to Zeldes. However, it should be noted that in the case of  $\text{Li}^6$  the value of  $a/K$  had to be taken smaller than the average value in other nuclei, consistent with the fact that L-S coupling approximation is more appropriate than  $j-j$  coupling one, in this case. The necessity of the intermediate coupling approximation which, in the  $j-j$  coupling terminology, is equivalent to the configuration interaction between  $1p_{3/2}$  and  $1p_{1/2}$  shells has already been pointed out by Horie and Yoshida<sup>6)</sup> and applied to explain magnetic and quadrupole moments of several nuclei, including  $\text{Li}^6$ . By actually solving secular equations, similar calculations have recently been carried out by Tauber and Wu<sup>7)</sup> and by Schulten.<sup>8)</sup>

A powerful method in constructing the wave functions for a system consisting of many particles in equivalent orbits and the evaluation of various matrix elements has been investigated by Racah<sup>9)</sup>, and applied extensively to atomic spectroscopy. This work has been extended to nuclear shell structure by Jahn<sup>10)</sup> and Jahn and Wieringen<sup>11)</sup> in the L-S coupling approximation, and by Flowers and Edmonds<sup>12)</sup> in the  $j-j$  coupling approximation.

The result of Jahn and Wieringen<sup>11)</sup> has been used by Elliot<sup>13)</sup>, and the matrix elements of non-central forces, i.e., the tensor force and the two-particle spin-orbit force, in the  $1p$  shell configuration were calculated, and applied especially to the investigation of the properties of lower energy levels of  $\text{Li}^6$ ,  $\text{Li}^7$  and  $\text{B}^{10}$ . It was possible to fit the several data of these nuclei if the condition imposed by the value of the magnetic moment of  $\text{Li}^6$  is relaxed, by taking into account the contribution of the exchange moment. However, so long as the depth parameters of the interaction of particles are taken from the two body data, both the binding energy and the splitting between lower energy levels are considerably smaller than the experimental values, and Elliot thus concluded the importance of configuration interactions. Similar calculation have been done by Komoda and Sasaki<sup>14)</sup> and by Ishidzu and Obi<sup>15)</sup>.

Feingold<sup>16)</sup> has made an interesting calculation, taking into account the configuration interaction by some specific procedure and using only tensor forces as non-central interactions.



His result for the splitting of the  $^1S_0$  and  $^3S_1$  states of  $\text{Li}^6$  was in much better agreement with experiment than those obtained by the second order perturbation theoretic calculation. However, the splitting he has obtained is almost half the experimental value and also quadrupole moment is several times larger than the experimental value.

Recent discoveries of new technique of integration have made it possible to calculate easily the binding energies of light nuclei. Indeed, Pease and Feshbach<sup>17)</sup> and Irving<sup>18)</sup> have calculated the binding energies of  $\text{H}^3$  and  $\text{He}^4$ , respectively, and they found that the phenomenological potentials consistent with the two body data, i.e., the sum of central and tensor forces were sufficient to account for the experimental binding energies. It would be interesting, therefore, to investigate, whether the same conclusion could be reached for heavier nuclei than  $\text{He}^4$  or not. To our regret,  $\text{He}^5$  and  $\text{Li}^5$  which come next to  $\text{He}^4$  have no bound states, and so in order to investigate the nature of their level structures, we must treat the scattering problem which needs tedious numerical calculations.<sup>19)</sup> In this paper, therefore, we would make calculations of the binding energy and other properties of  $\text{Li}^6$ .

A method which has been used by one of the present authors (T.T.) for the calculation of neutron- $\text{He}^4$  scattering<sup>19)</sup> is extended in this paper so as to treat the six body problem. In this calculation, we consider the wave functions to be constructed as vectors which constitute the basis of symmetric group, and so they are specified by appropriate partition numbers. Such methods making use of the irreducible representations of the symmetric group to construct the wave functions have been used by several authors. Villars<sup>20)</sup> applied it to the calculation of the magnetic moment of  $\text{H}^3$ , Verde<sup>21)</sup> to the calculation of the neutron-deuteron scattering, Trainor<sup>22)</sup> to the calculation of the level structure of  $\text{He}^4$  and Gamba et al. to the general discussion of the selection rules of nuclear reactions.

Our method which will be explained in § 2 may be considered as an extension of these works. However, for the actual construction of wave functions the technique developed by Jahn and Wieringen<sup>11)</sup> is most powerful, and used extensively in our paper. As for the spatial part of our wave functions, we make some deviation from the shell model individual particle treatment<sup>11)</sup>, and the actual form used in our calculation is very similar to that used by Irving<sup>18)</sup>. This form of wave function is very convenient for applying to bound state problems, because it permits one to perform any integration analytically, so long as we assume Yukawa or exponential radial dependence for the two body potentials.

In § 3 through § 6, we calculate matrix elements of central, tensor and Coulomb interactions, kinetic energy, magnetic and quadrupole moments. Numerical results are shown in § 7, and some discussion of the results are given in § 8.

## § 2. Construction of the wave functions

In constructing our wave functions, we start from the stand point of L-S coupling scheme. However, as is mentioned in the introduction, we do not follow the conventional shell model treatment, and so the terminology "L-S coupling scheme", has somewhat



different meaning from the usual one. It is used here only to mean that the charge-spin functions of the six particles span some space of the irreducible representations of the unitary group  $U_4$  in four dimensions, while the spatial wave functions span that of the irreducible representations which are adjoint to the former.

The structure of the functions which span an irreducible space of the representation of a unitary group can be specified by a partition  $[\lambda]$ , as was found by Young. As is shown by Weyl<sup>(24)</sup> this partition also specifies the irreducible representation of the symmetric group on the elements which constitute the tensor space, to which the unitary transformation is operated. On the other hand, Yamanouchi<sup>(25)</sup> found a very convenient method for obtaining standard orthogonal representations of symmetric group, and his method was extensively used by Jahn and Wieringen<sup>(11)</sup> for the calculation of the coefficients of fractional parentage. This method will also be used in this paper.

In the usual shell model, the lowest configuration for  $Li^6$  is believed to be  $(1s)^4(1p)^2$ . The symmetry character of the spatial wave functions constructed from this configuration can be specified by two kinds of partitions, i.e.,  $[42]$  and  $[411]$ . When these irreducible representations are decomposed into the irreducible representations of the three dimensional rotation group, they contain S and D states, and P state, respectively. We will limit our calculation to these two partitions. Then, as is easily seen from the Table 3 of reference 11, this means that we are limiting our calculation to the following states;  $^{13}S$ ,  $^{13}D$ ,  $^{31}S$ ,  $^{31}D$ ,  $^{11}P$  and  $^{33}P$ , where the first and the second superscripts mean the charge and the spin multiplicities, respectively.

The degrees of the irreducible representations of the symmetric group for  $n$  elements corresponding to the partition  $[\lambda] = [\lambda_1, \lambda_2, \dots, \lambda_m, \dots, \lambda_k]$  can be calculated by using the formulae

$$n_{[\lambda]} = n! D(h_1, h_2, \dots, h_m, \dots, h_k) / h_1! h_2! \dots h_m! \dots h_k!, \quad (1)$$

$$h_m = \lambda_m + k - m \quad (m=1, 2, \dots, k), \quad D(h_1, h_2, \dots, h_k) = \prod_{i < j}^k (h_i - h_j),$$

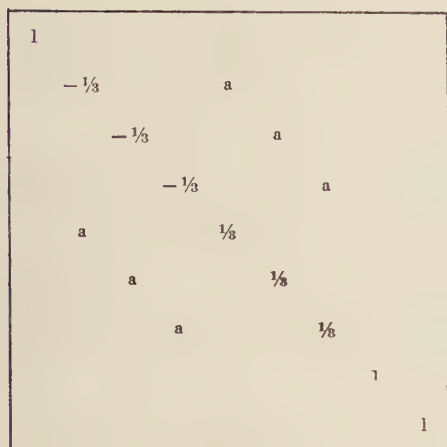
given by Weyl<sup>(24)</sup>, and in our case this gives nine and ten degrees for  $[42]$  and  $[411]$ , respectively.

Because our calculation in the following are very similar for these two partitions, we will show the details of the calculation for  $[42]$ , and for  $[411]$  we show only the results. If we write the nine spatial functions which span the representation space  $[42]$  as  $\phi_1, \phi_2, \dots, \phi_9$  and the corresponding charge-spin functions which span the representation space  $[\widetilde{42}]$ , which is adjoint to  $[42]$  as  $\psi_1, \psi_2, \dots, \psi_9$ , then the totally antisymmetrized wave function is clearly given by

$$\Psi = 1/\sqrt{9N} \sum_{i=1}^9 \phi_i \psi_i, \quad (2)$$

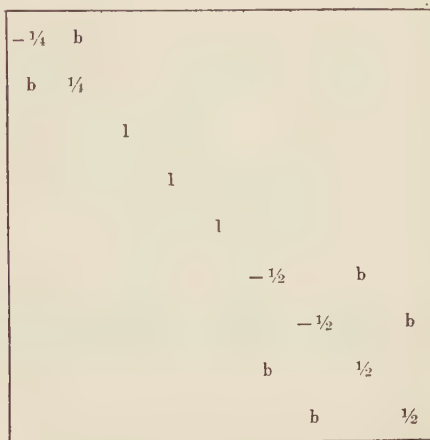
where  $1/\sqrt{N}$  is the normalization factor. Although these normalization factors can differ from state to state, it is shown in App. B that they are all equal, and we will not discriminate them in the following.

$P_{56}$



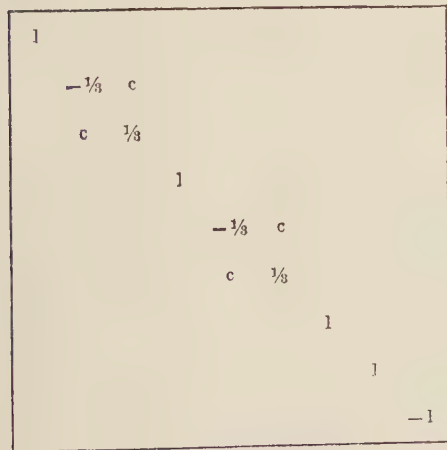
$$a = \sqrt{8/3}$$

$P_{45}$



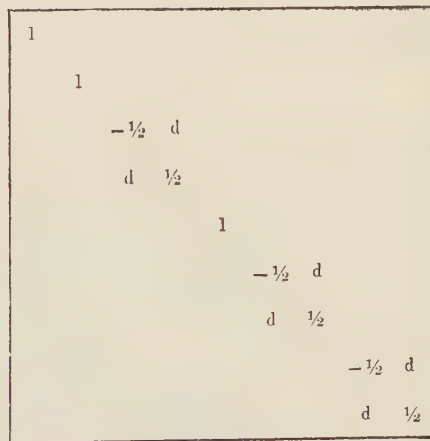
$$b = \sqrt{3}/2$$

$P_{34}$



$$c = \sqrt{8/3}$$

$P_{23}$



$$d = \sqrt{3}/2$$

$P_{12}$



Table I.  
Standard orthogonal Yamanoichi representation  
for partition [42].

To obtain the explicit form of the functions  $\phi_i$ 's and  $\psi_i$ 's, we specify them, as mentioned above, as functions of appropriate Yamanouchi symbols. As the meaning of the Yamanouchi symbols and the method of constructing corresponding orthogonal irreducible representations have been quoted and summarized by Jahn and Wieringen<sup>11</sup>, we will not dare to repeat them. The explicit form of the representation for partition [42] are shown in Table I. Clearly, corresponding standard representations for  $\widetilde{[42]}$  are obtained from those of the Table I, merely by changing the sign of all the elements of the matrices. However, as we are using same set of the Yamanouchi symbols, as those used by Jahn and Wieringen, our  $\psi_i$ 's are nothing but their charge-spin functions  $I'(r_1, r_2, \dots, r_6)$ 's, and so they can immediately be written down, by using the coefficients of fractional parentage, prepared in Table 3 of their paper. Explicitly, they are given as follow,

$$\begin{aligned}\psi_1(\widetilde{221111}) &= [(\widetilde{1111}), 11, 22, 13], \\ \psi_2(\widetilde{212111}) &= -1/\sqrt{5}[(\widetilde{2111}), 13, 22, 13] - 1/\sqrt{5}[(\widetilde{2111}), 31, 22, 13] \\ &\quad - \sqrt{3/5}[(\widetilde{2111}), 33, 22, 13], \quad (3a)\end{aligned}$$

$$\begin{aligned}\psi_3(\widetilde{122111}) &= -1/\sqrt{10}[(\widetilde{2111}), 13, 22, 13] + 1/\sqrt{10}[(\widetilde{2111}), 31, 22, 13] \\ &\quad - 1/\sqrt{5}[(\widetilde{2111}), 13, 24, 13] + \sqrt{3/5}[(\widetilde{2111}), 33, 24, 13],\end{aligned}$$

$$\begin{aligned}\psi_4(\widetilde{112211}) &= 1/\sqrt{20}[(\widetilde{2111}), 11, 22, 13] - \sqrt{3/20}[(\widetilde{2111}), 33, 22, 13] \\ &\quad + \sqrt{3/10}[(\widetilde{2111}), 33, 24, 13] + 1/\sqrt{2}[(\widetilde{2111}), 15, 24, 13],\end{aligned}$$

$\psi_5$  and  $\psi_6$  are obtained from  $\psi_2$  by replacing  $(\widetilde{2111})$  by  $(\widetilde{1211})$  and  $(\widetilde{1121})$ , respectively,  $\psi_6$  and  $\psi_7$  are obtained from  $\psi_3$  by replacing  $(\widetilde{2111})$  by  $(\widetilde{1211})$  and  $(\widetilde{1121})$ , respectively, and  $\psi_8$  is obtained from  $\psi_4$  by replacing  $(\widetilde{2111})$  by  $(\widetilde{2121})$ . (3b)

In these equation, e.g.,  $|(\widetilde{2111}), 13, 22, 13|$  means a charge-spin function for six particles, which is constructed from a charge singlet, spin triplet function of the first four particles, characterized by the Yamanouchi symbol  $(\widetilde{2111})$ , by adding the charge and spin functions of the fifth particle, vectorially, so as to make up a charge and spin doublet state, and then adding the charge and spin functions of the sixth particle, finally making up a charge singlet and spin triplet state. The meaning of other functions may be read off similarly.

So far the construction of our wave functions has no difference from the usual shell model treatment. Here we will make an essential deviation from it, in connection with the construction of the spatial parts of the functions.

In the usual shell model treatment, the spatial parts of the wave functions, corresponding to the configuration  $(1s)^4(1p)^2$  can be thought of as linear combination of the functions such as

$$f(r_1)f(r_2)g(r_3)g(r_4)g(r_5)g(r_6) \sum_{m_1 m_2} (1m_1 1m_2 | LM) Y_{1m_1}(r_1) Y_{1m_2}(r_2), \quad (4)$$

where  $Y_{1m_1}(r_1)$  etc. are spherical harmonics of order one, while  $f$ 's and  $g$ 's are radial wave functions of the particles in the  $1p$  and  $1s$  orbits, respectively. It is usual to assume that  $f$ 's and  $g$ 's to be the zero node oscillator functions, and so to use (4) in the form

$$r_1 r_2 \exp[-\mu \sum_{i=1}^6 r_i^2] \sum_{m_1 m_2} (1m_1 1m_2 | LM) Y_{1m_1}(r_1) Y_{1m_2}(r_2), \quad (5)$$

apart from a constant factor. In this treatment, all the coordinates of the constituent particles are measured from an assumed center of mass, which is different from the correct center of mass  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2 + \dots + \mathbf{r}_6)/6$ .

Although, such a treatment may be permissible for heavier nuclei, its validity is very doubtful for lighter nuclei such as  $\text{Li}^6$ . Therefore we will first modify the expression (5), by rewriting  $r_1 Y_{1m_1}(r_1)$  as  $Y_{1m_1}(\mathbf{r}_1)$ , i.e., as a solid harmonics\*, where the vector  $\mathbf{r}_1$  is to be measured from the center of mass  $\mathbf{R}$ , defined above.

The exponential factor in (5), which is frequently used in the shell model calculation to make the evaluation of the matrix elements easy, will not be appropriate, especially in light nuclei, because of its Gaussian dependence at large distance. The radial dependence should be exponential, so long as we consider a negative energy solution of the Schrödinger equation with short range potential. Therefore, the second modification of (5) we make, is to replace the exponential factor of (5) as follow,

$$\exp[-\mu \sum_{i=1}^6 r_i^2] \longrightarrow \exp[-\lambda \sqrt{\sum_{i < j=1}^6 r_{ij}^2}]. \quad (6)$$

This sort of replacement is suggested from the work of Irving<sup>18)</sup>, on the calculation of the binding energy of  $\text{He}^4$ , and is stressed by him to be very appropriate, both from the theoretical point of view, and for easier computation. This replacement has also an advantage of taking into consideration the correlation of the positions of the constituent particles.

Our final task in this section is to write down explicit forms of  $\phi_i$ 's. According to the preceding discussion, they are linear combinations of  $L_{ij}$ 's where  $L_{ij}$  is defined by

$$L_{ij} = \sum_{m_i m_j} (1m_i 1m_j | LM) Y_{1m_i}(\mathbf{r}_i) Y_{1m_j}(\mathbf{r}_j) \exp[-\lambda \sqrt{\sum_{i < j=1}^6 r_{ij}^2}]. \quad (7)$$

The Yamanouchi symbol for  $\phi_1$  suggests, that it may be generated from  $L_{36}$  by operating to it a Young's symmetrization operator  $P_N$ , where, by writing a position exchange operator between particles  $i$  and  $j$  as  $P_{ij}^{(r)}$

$$N = (1 - P_{15}^{(r)}) (1 - P_{26}^{(r)}), \quad (8a)$$

and

$$P = [\text{sum of all possible permutations among members from 1 to 4}] \times (1 + P_{56}^{(r)}). \quad (8b)$$

Taking into account the fact that, e.g.,  $L_{36}$  is totally symmetric among the particles from

\* Throughout in this paper, symbols  $Y_{lm}()$ , whose arguments are written by Italic and Gothic letters, mean surface and solid harmonics, respectively.

first to fourth particles, and symmetric between fifth and sixth particles, the result can be obtained in a rather simple form

$$\begin{aligned}\phi_1(221111) = & 1/\sqrt{240} [12L_{50} + 2(L_{12} + L_{13} + L_{14} + L_{23} + L_{24} + L_{34}) \\ & - 3(L_{15} + L_{25} + L_{35} + L_{45} + L_{16} + L_{26} + L_{36} + L_{46})].\end{aligned}\quad (9)$$

The other  $\phi_i$ 's may be obtained in a similar manner. However, as the results are not mutually orthogonal if obtained in this way, it is better to use the standard orthogonal representations, given in Table I.

For example, if we operate  $P_{45}^{(r)}$  on  $\phi_1$ , then we get

$$\begin{aligned}P_{45}^{(r)}\phi_1 = & -1/4 \cdot \phi_1 + (\sqrt{15}/4) \cdot 1/12 [2(L_{12} + L_{13} + L_{23}) - 2(L_{14} + L_{24} + L_{34}) \\ & + (L_{15} + L_{25} + L_{35}) - 3(L_{16} + L_{26} + L_{36}) - 3L_{45} + 9L_{46}].\end{aligned}\quad (10)$$

Comparison with the standard representation for  $P_4$ , in Table I shows that it is given by

$$\begin{aligned}\phi_2(212111) = & 1/12 [2(L_{12} + L_{13} + L_{23}) - 2(L_{14} + L_{24} + L_{34}) + (L_{15} + L_{25} + L_{35}) \\ & - 3(L_{16} + L_{26} + L_{36}) - 3L_{45} + 9L_{46}].\end{aligned}\quad (11)$$

Similarly  $\phi_i$  is obtained by operating  $P_{34}^{(r)}$  on  $\phi_2$ , and comparing the result with Table I, and so on. We summarize the results in the following.

$$\begin{aligned}\phi_3(211211) = & 1/\sqrt{72} [2L_{12} - (L_{13} + L_{23}) + (L_{14} + L_{24}) - 2L_{34} + (L_{15} + L_{25}) - 2L_{35} \\ & - 3(L_{16} + L_{26}) + 6L_{36}], \\ \phi_4(211121) = & 1/\sqrt{24} [(L_{13} + L_{14} + L_{15}) - (L_{23} + L_{24} + L_{25}) - 3(L_{16} - L_{26})], \\ \phi_5(122111) = & 1/\sqrt{18} [(L_{12} + L_{13} + L_{23}) - (L_{14} + L_{24} + L_{34}) - (L_{15} + L_{25} + L_{35}) + 3L_{45}], \\ \phi_6(121211) = & 1/6 [2L_{12} - (L_{13} + L_{23}) + (L_{14} + L_{24}) - 2L_{34} + 4L_{35} - 2(L_{15} + L_{25})], \\ \phi_7(121121) = & 1/\sqrt{12} [(L_{13} + L_{14} - 2L_{15}) - (L_{23} + L_{24} - 2L_{25})], \\ \phi_8(112211) = & 1/\sqrt{12} [2L_{12} - (L_{13} + L_{23} + L_{14} + L_{24}) + 2L_{34}], \\ \phi_9(112121) = & 1/2 [(L_{13} - L_{14}) - (L_{23} - L_{24})].\end{aligned}\quad (12)$$

The results for the partition  $[411]$ , namely  $^{11}\text{P}$  and  $^{31}\text{P}$  states, as well as for  $^{31}\text{S}$  and  $^{31}\text{D}$  states are given in App. A.

### § 3. Evaluation of the matrix element for central force potentials

We write the central interaction potentials as

$$V_c = \sum_{i < j=1}^6 V_{ij}^{(c)}, \quad (13)$$

where

$$V_{ij}^{(c)} = (w + mP_{ij}^{(r)} + bP_{ij}^{(s)} + hP_{ij}^{(z)})f_c(r_{ij}). \quad (14)$$



The expectation value  $E_c$  of  $V_c$  is given by using the wave function (2), as

$$E_c = (\Psi | V_c | \Psi) = (\Psi | \sum_{i < j=1}^6 V_{ij}^{(c)} | \Psi). \quad (15)$$

As is clear from its construction,  $\Psi$  is totally antisymmetric between exchange of any pair of the particles, and  $V_c$  is totally symmetric, so that the whole expression of (15) is totally symmetric. Therefore it can be written as

$$\begin{aligned} E_c &= 15 (\Psi | V_{56}^{(c)} | \Psi) \\ &= 15/9N (\sum_{i=1}^9 \phi_i \psi_i | V_{56}^{(c)} | \sum_{j=1}^9 \phi_j \psi_j). \end{aligned} \quad (16)$$

For the evaluation of (16), it is easier first to perform the sum over charge-spin variables, and the result can conveniently be expressed in a matrix form as

$$E_c = 15/9N (\phi_1^*, \phi_2^*, \dots, \phi_9^*) f_c(r_{56}) (\tau v \mathbf{1} + m \bar{P}_{56}^{(r)} + b \bar{P}_{56}^{(o)} + h \bar{P}_{56}^{(\tau)}) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_9 \end{pmatrix}, \quad (17)$$

in which we take  $f_c(r_{56}) = V_c \exp[-r_{56}/r_c]/(r_{56}/r_c)$ . Here, of course  $\mathbf{1}$ ,  $\bar{P}_{56}^{(r)}$ ,  $\bar{P}_{56}^{(o)}$  and  $\bar{P}_{56}^{(\tau)}$  are square matrices with nine rows and columns, and are given explicitly in Table II. How they are obtained will be explained as follows.

We first rewrite (16) in the form

$$\begin{aligned} E_c &= 15/9N (\sum_{i=1}^9 \phi_i \psi_i | f_c(r_{56}) (\tau v + m P_{56}^{(r)} + b P_{56}^{(o)} + h P_{56}^{(\tau)}) | \sum_{j=1}^9 \phi_j \psi_j) \\ &= 15/9N \sum_{i=1}^9 \sum_{j=1}^9 (\phi_i | f_c(r_{56}) | \phi_j) \\ &\quad \times [\tau v (\psi_i | 1 | \psi_j) + m (\psi_i | P_{56}^{(r)} | \psi_j) + b (\psi_i | P_{56}^{(o)} | \psi_j) + h (\psi_i | P_{56}^{(\tau)} | \psi_j)], \end{aligned} \quad (18)$$

then it is easy to see that the matrix elements of the matrices in Table II are nothing but the elements in the square brackets of (18), and so their meaning is clear. For Wigner force, from the ortho-normality of  $\psi_i$ 's, it is clear that the corresponding matrix is a unit matrix. Next the  $\bar{P}_{56}^{(r)}$  in Table II is just the same as the  $P_{56}$  in Table I. The reason is as follows. If  $P_{56}^{(r)}$  is operated on a totally antisymmetrized function, it can be replaced by  $-P_{56}^{(o)} P_{56}^{(\tau)}$ . However in our formalism  $P_{56}^{(o)} P_{56}^{(\tau)}$  is the adjoint representation of  $P_{56}^{(r)}$ , i.e., its matrix elements are just minus those of  $P_{56}^{(r)}$ . This is the reason why  $(\psi_i | P_{56}^{(r)} | \psi_j)$  coincides with the matrix elements of  $P_{56}^{(r)}$  in Table I.

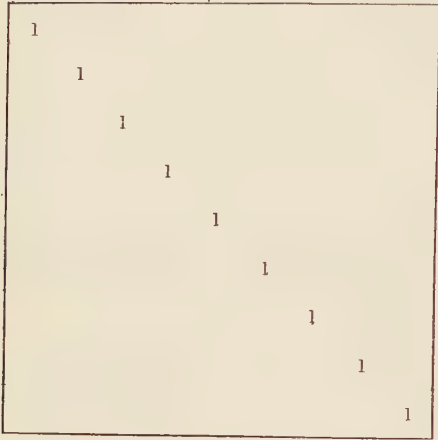
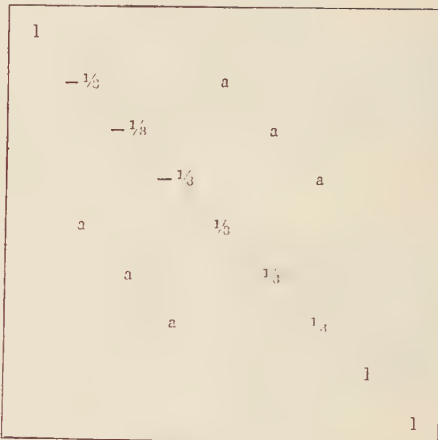
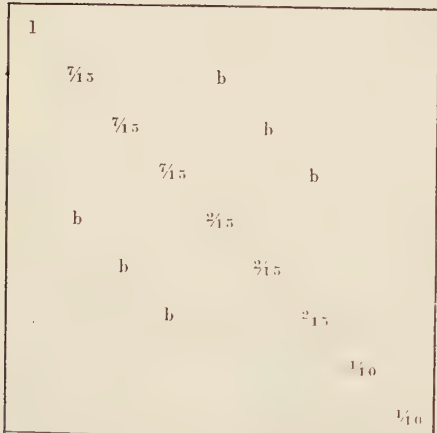
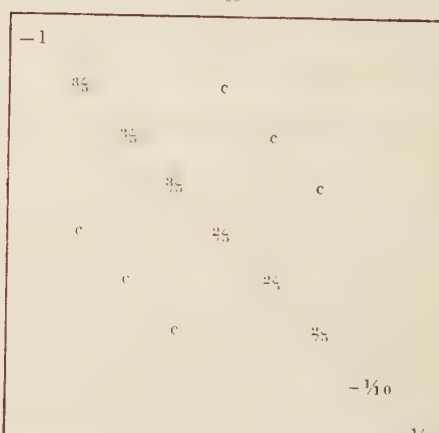
For the calculation of  $(\psi_i | P_{56}^{(o)} | \psi_j)$ , we use the explicit form of (3) for  $\psi_i$ 's. If we write a term in the righthand side symbolically as

$$[(\widetilde{r_4 r_3 r_2 r_1}), T_1 S_1, T_2 S_2, T_3 S_3], \quad (19)$$

and  $S_i = 2l_i + 1$ , then this expression is proportional to  $|l_1 l_2 l_3\rangle$ , where

$$|l_1 l_2 l_3\rangle = \sum_{m_1 m_2 m_3 m'_1 m'_2 m'_3} (l_1 m_1 \frac{1}{2} m_3 | l_2 m_2) (l_2 m_2 \frac{1}{2} m'_3 | l_3 m_3) \varphi_{l_1 m_1}^{(1,2,3,4)} \chi_{m_3}^{(5)} \chi_{m'_3}^{(6)}. \quad (20)$$

Table II.  
Charge-spin matrices for central forces for [42].

$I$	$\overline{P}_{50}^{(r)}$
	 <p style="text-align: center;"><math>a = \sqrt{8}/3</math></p>
$\overline{P}_{56}^{(o)}$	$\overline{P}_{50}^{(s)}$
 <p style="text-align: center;"><math>b = -\sqrt{2}/3</math></p>	 <p style="text-align: center;"><math>c = -\sqrt{2}/5</math></p>

Here  $\varphi_{l_1 m_1}^{(1,2,3,4)}$  is a spin function constructed from the spin functions of the first through fourth particles, while  $\chi_{m_5}^{(5)} \chi_{m_6}^{(6)}$  are spin functions of the fifth and sixth particles, respectively. The operation of  $P_{50}^{(o)}$  on (20) results in

$$P_{50}^{(o)} |l_1 l_2 l_3\rangle = \sum_k (-1)^{l_1 + l_2 + l_3 + k} [(2k+1)(2l_2+1)]^{1/2} W(l_1 \frac{1}{2} \frac{1}{2} l_3; l_2 k) |l_1 k l_3\rangle, \quad (21)$$

where  $W(abcd; ef)$  is the Racah coefficient.<sup>9)</sup>

Therefore, the results of operation of  $P_{50}^{(o)}$  on (19) is written down, as

$$P_{50}^{(o)} [(\widetilde{r_4 r_3 r_2 r_1}), T_1 S_1, T_2 S_2, T_3 S_3]$$

$$= \sum_k (-)^{l_1+l_2+l_3+k} [(2k+1)(2l_2+1)]^{1/2} W(l_1 \frac{1}{2} \frac{1}{2} l_3; l_2 k) \\ \times [(\widetilde{r_4 r_3 r_2 r_1}), T_1 S_1, T_2 2k+1, T_3 S_3]. \quad (22)$$

From this formula the matrix elements of (18) for the Bartlett force are easily derived using the fact that the charge-spin functions for first to fourth particles are orthogonal, so long as they have different value of  $S_i$ ,  $T_i$ , and of Yamanouchi symbol  $(\widetilde{r_4 r_3 r_2 r_1})$ . Moreover, it should be noted that the charge states constructed adding fifth and sixth particles to those of preceding four particles must be the same in order to obtain non-vanishing matrix elements. The calculation for the Heisenberg force is similar.

As we have obtained the expression (17), the remainder of our calculation is a matter of matrix multiplication. The calculation is elementary but rather lengthy, because each  $\phi_i$  itself contains many terms. However, the symmetry property of  $L_{ij}$ , mentioned just after (8), limits the number of the independent matrix elements only to twelve, and abbreviating  $(L_{ij}|f_e(r_{56})|L_{i'j'})$  as  $V_e(ij|i'j')$ , the result is given by

$$E_e = V_e/N [\tau v \{ (56|56) + 6(12|12) + (12|34) - 6(12|13) + 8(15|15) - 6(15|25) \\ - 2(15|16) + 4(15|26) + 2(12|56) - 4(15|56) - 12(12|15) + 8(12|35) \} \\ + m \{ (56|56) + 6(12|12) + (12|34) - 6(12|13) - 2(15|15) + 4(15|25) \\ + 8(15|16) - 6(15|26) + 2(12|56) - 4(15|56) + 12(12|15) + 8(12|35) \} \\ + b \{ (56|56) + (12|34) + 4(15|15) - 2(15|25) - 4(15|16) + 6(15|26) \\ + 2(12|56) - 4(15|56) - 4(12|35) \} \\ + h \{ - (56|56) - (12|34) + 4(15|15) - 6(15|25) - 4(15|16) \\ + 2(15|26) - 2(12|56) + 4(15|56) + 4(12|35) \} ]. \quad (23)$$

The evaluation of the matrix elements needs some long calculation. It is illustrated in App. B, and the results are summarized in Table B-I. By using this Table, it is seen that all the matrix elements can be expressed by four independent integrals  $I_1$ ,  $I_2$ ,  $I_3$ , and  $I_4$ , defined by (B-11), of App. B, and we can rewrite (23) as follows. (Since from here on, the results are different for  $^{13}\text{S}$  and  $^{13}\text{D}$ , we will write them separately.)

$$E_e(^{13}\text{S}) = V_e/N [\tau v \{ 46/3 I_1 - 11/2 I_2 + 7/3 I_3 + 3/16 I_4 \} \\ + m \{ 46/3 I_1 - 11/2 I_2 - 13/3 I_3 + 3/16 I_4 \} \\ + b \{ 10/3 I_1 - 5/2 I_2 + 5/3 I_3 + 3/16 I_4 \} \\ + h \{ -10/3 I_1 + 5/2 I_2 + 11/3 I_3 - 3/16 I_4 \} ], \quad (24)$$

$$E_e(^{13}\text{D}) = V_e/N [\tau v \{ 92/15 I_1 + 5 I_2 + 10/3 I_3 + 3/40 I_4 \} \\ + m \{ 92/15 I_1 + 5 I_2 - 10/3 I_3 + 3/40 I_4 \} \\ + b \{ 4/3 I_1 - I_2 + 8/3 I_3 + 3/40 I_4 \} \\ + h \{ -4/3 I_1 + I_2 + 8/3 I_3 - 3/40 I_4 \} ]. \quad (25)$$

In App. B (B-11), values for  $I_1/N$ , etc. are given. Therefore we obtain the final results for  $E_c$ ,

$$E_c(^{13}\text{S}) = V_c(5/96) (2\sqrt{3}\lambda r_c)^{19} (3\cdot 5\cdots 17/2\cdot 4\cdots 14) \\ \cdot [\omega(164/15A + 784/45B + 224/15C) + m(164/15A - 1456/45B + 224/15C) \\ + b(4/3A + 112/9B + 224/15C) + h(-4/3A + 1232/45B - 224/15C)], \quad (26)$$

$$E_c(^{13}\text{D}) = V_c(5/96) (2\sqrt{3}\lambda r_c)^{19} (3\cdot 5\cdots 17/2\cdot 4\cdots 14) \\ \cdot [\omega(152/15A + 224/9B + 448/75C) + m(152/15A - 224/9B + 448/75C) \\ + h(-8/15A + 896/45B - 448/75C) + b(+8/15A + 896/45B + 448/75C)], \quad (27)$$

where

$$A = \int_0^1 (1-y^2)^7 y(y+c)^{-18} dy, \quad B = \int_0^1 (1-y^2)^6 y^3(y+c)^{-18} dy, \\ C = \int_0^1 (1-y^2)^5 y^5(y+c)^{-18} dy, \quad (28)$$

with

$$c = 2\sqrt{3}\lambda r_c. \quad (29)$$

Although it is possible to integrate  $A$ ,  $B$ , and  $C$  analytically, it is very tedious calculation, and we find it better to evaluate them numerically, and so leave them in the integral form of (28).

The results for  $^{31}\text{S}$  and  $^{31}\text{D}$  states are obtained from above formulae by exchanging  $b$  and  $h$ , as is easily seen from (A-5) of App. A.

Similar calculations for  $^{11}\text{P}$  and  $^{33}\text{P}$  states, which belong to partition [411], are performed, and the following results are obtained.

$$E_c(^{11}\text{P}) = E_c(^{33}\text{P}) \\ = V_c/N [\omega \{ (56|56) + 8(15|15) - 6(15|25) + 4(15|56) - 2(15|16) \\ + 6(12|12) - 6(12|13) - 12(12|15) \} \\ + m \{ - (56|56) - 2(15|15) - 4(15|56) + 8(15|16) - 6(15|26) \\ + 6(12|12) - 6(12|13) - 12(12|16) \} \\ + (b+h) \{ - (56|56) + 4(15|15) - 6(15|25) - 4(15|56) \\ - 4(15|16) + 6(15|26) \} ] \quad (30) \\ = (V_c/N) [\omega(12I_2 + 4I_3) + m(12I_2 - 4I_3) + (b+h)(144/25I_2 - 22/25I_3)] \\ = V_c(5/96) (2\sqrt{3}\lambda r_c)^{19} (3\cdot 5\cdots 17)/(2\cdot 4\cdots 14) \\ \cdot [\omega(48/5A + 448/15B) + m(48/5A - 448/15B) \\ + (b+h)(576/125A - 2464/375B)].$$

### § 4. Tensor and Coulomb forces

In this section we calculate the matrix elements of the tensor and the Coulomb interactions. We assume the tensor interaction potential to be given by

$$V_t = \sum_{i < j=1}^6 V_{ij}^{(t)}, \quad (31)$$

where, neglecting any exchange character other than Wigner

$$V_{ij}^{(t)} = S_{ij} f_t(r_{ij}), \quad (32)$$

and

$$S_{ij} = [3(\boldsymbol{\sigma}_i \cdot \mathbf{r}_{ij})(\boldsymbol{\sigma}_j \cdot \mathbf{r}_{ij})/r_{ij}^2] - (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) \quad (33)$$

is the usual tensor operator. The matrix element for (31) is given, as in (16) (again for partition [42]), by

$$E_t = (15/9N) \left( \sum_{i=1}^9 \phi_i \psi_i | V_{56}^{(t)} | \sum_{i=1}^9 \phi_i \psi_i \right). \quad (34)$$

In evaluating (34), it is again convenient first to perform summation over charge-spin variables, and express it in a matrix form, such as in (17). In this case, as (31) does not contain any operator for charge variables, the ortho-normality for charge functions are evident. Therefore it is only necessary to calculate matrix elements concerning ordinary spin states.

It is convenient to rewrite (33) as

$$S_{ij} = \sqrt{6 \cdot 4\pi/5} \sum_{\mu m \cdot m'} (-)^{\mu} Y_{2-\mu}(r_{ij}) (1m1m' | 2\mu) \sigma_m^{(i)} \sigma_{m'}^{(j)}, \quad (35)$$

where  $Y_{2-\mu}(r_{ij})$  is a surface harmonics of order two. Then using the notation defined in (20), the matrix element now in concern is written down as follows,

$$\begin{aligned} (l_1' l_2' | S_{56} | l_1 l_2 l_3') &= \sqrt{6 \cdot 4\pi/5} \left( \frac{1}{2} \| \boldsymbol{\sigma} \| \frac{1}{2} \right)^2 (1/3) \sum_{\mu} (-)^{\mu} Y_{2-\mu}(\tau v) \\ &\cdot \sum_{\mu} (-)^{1+m_s+m_s'+m_5+m_6} (l_1 m_1 \frac{1}{2} m_s | l_2 m_2) (l_2 m_2 \frac{1}{2} m_s' | l_3 m_3) (l_1 m_1 \frac{1}{2} m_s' | l_2' m_2') \\ &\cdot (l_2' m_2' \frac{1}{2} m_s''' | l_3' m_3') (1 m_5 1 m_6 | 2\mu) \left( \frac{1}{2} - m_s \frac{1}{2} m_s''' | 1 - m_5 \right) \left( \frac{1}{2} - m_s' \frac{1}{2} m_s''' | 1 - m_6 \right), \end{aligned} \quad (36)$$

where  $(1/2 \| \boldsymbol{\sigma} \| 1/2) = \sqrt{6}$ . The second summation in (36) is easily evaluated by using Racah's formula<sup>9)</sup>, and formulae given, e.g., by Jahn and Hope<sup>25)</sup>, and (36) now becomes

$$\begin{aligned} (l_1' l_2' | S_{56} | l_1 l_2 l_3') &= 2 \sqrt{6 \cdot 4\pi/5} \sum_{\mu} (-)^{\mu} Y_{2-\mu}(\tau v) (l_3 - m_3 l_3' m_3' | 2 - \mu) \\ &\cdot (-)^{l_2+l_2'+l_3+1+m_3} \sqrt{(2l_2+1)(2l_2'+1)} W(l_1 l_2' \frac{1}{2} 1; \frac{1}{2} l_2) \chi(l_2 \frac{1}{2} l_2' \frac{1}{2}; l_3 l_3', 11, 2), \end{aligned} \quad (37)$$

where  $\chi$  is the function defined in reference 25, and in our case it is

$$\begin{aligned} \chi(l_2 \frac{1}{2} l_2' \frac{1}{2}; l_3 l_3', 11, 2) &= 3[(2l_3+1)(2l_3'+1)]^{1/2} (-)^{l_2+l_3'} \sum_x (2x+1) \\ &\times W(l_2 l_2' 21; 1x) W(\frac{1}{2} \frac{1}{2} x l_2'; 1 l_3') W(l_3 l_3' l_2 x; 2 \frac{1}{2}). \end{aligned} \quad (38)$$



Using this result and the explicit form of charge-spin wave functions given in (3), we obtain, for  $E_t$  of (34), the following expression,

$$E_t = \frac{15}{9N} \sqrt{\frac{6\pi}{5}} (\phi_1^*, \phi_2^*, \dots, \phi_9^*) \cdot T \cdot F \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_9 \end{pmatrix}, \quad (39)$$

where

$$\begin{aligned} T = & \sum_{\mu} (-)^M (S - M_s, S' M_s' | 2 - \mu) \\ & \times (S M_s L M_l | J M) (S' M_s' L' M_l' | J M) \\ & \times f_t(\tau\psi) Y_{2-\mu}(\tau\psi). \end{aligned} \quad (40)$$

In  $F$  we have replaced  $l_3$  and  $l'_3$  by more appropriate notations  $S$  and  $S'$ . The nine row and nine column matrix  $T$  is given in Table III.

Now we write the matrix element of  $F$ , between states  $L_{ij}$  and  $L_{i'j'}$  defined in (7) as  $(ij|F|i'j')$ . Then the result of matrix multiplication of (39) is obtained in the following short form

$$\begin{aligned} E_t = (1/N) \sqrt{6\pi/5} [ & -4(56|F|56) - 8(15|F|25) - 8(15|F|26) - 4(12|F|56) \\ & - 4(56|F|12) + 8(15|F|56) + 8(56|F|15) ], \end{aligned} \quad (41)$$

It should be noted that there are only seven terms, while in (23) there have been twelve terms. This comes from the fact that  $F$  contains a factor  $Y_{2-\mu}(\tau\psi)$ , and so, to obtain non-vanishing value for  $(ij|F|i'j')$  after integration over  $d\Omega_w$ , at least two of the indices  $i, j, i'$  and  $j'$  must be 5 or 6, because only vectors  $\mathbf{r}_5$  and  $\mathbf{r}_6$  contain the vector  $\mathbf{w}$ . The reason that  $(15|F|15)$  and  $(15|F|16)$  do not appear in (41) is that their coefficients themselves vanish, incidentally. On the other hand, e.g.,  $(15|F|56)$  and  $(56|F|15)$  contribute differently to D-S or S-D interference terms, and so they are kept separated.

The explicit form of the terms which appear in (41), e.g., of  $(15|F|25)$  is

$$\begin{aligned} (15|F|25) = & \int (-)^M (1m_1 1m_2 | L M_l) (1m'_2 1m'_1 | L' M_l') (S - M_s, S' M_s' | 2 - \mu) \\ & \cdot (S - M_s L M_l | J M) (S' M_s' L' M_l' | J M) \int Y_{1m_1}^*(\mathbf{r}_1) Y_{1m_2}^*(\mathbf{r}_5) \\ & \cdot \exp[-2\lambda r] \exp[-\tau\psi/r_t] (\tau\psi/r_t) Y_{2-\mu}(\tau\psi) Y_{1m_2'}(\mathbf{r}_2) Y_{1m_5'}(\mathbf{r}_5) du dv dw/ds dt, \end{aligned} \quad (42)$$

in which we have taken

$$f_t(r_{56}) = V_t \exp[-r_{56}/r_t]/(r_{56}/r_t). \quad (43)$$

$T =$

				-4
$\frac{1}{2}15$			a	
	$\frac{1}{2}15$			a
		$\frac{1}{2}15$		a
a			$\frac{1}{2}15$	
	a			$\frac{3}{2}15$
		a		$\frac{3}{2}15$
				-25
				-25

$a = 4\sqrt{2}/15$

Table III

Charge-spin matrix for tensor force for [42].

The integration over angle variables and summation over magnetic quantum numbers are similarly performed as in the case of central forces (see App. B), and we show only the results,

$$\begin{aligned}
 (15|F|25) &= V_t(-)^{L+J}(1/6) \sqrt{(5/6\pi)(2L+1)(2L'+1)} W(1L'12; 1L) \\
 &\quad \times W(SLS'L'; J2) I_3', \\
 (15|F|26) &= -(15|F|25), \\
 [(15|F|56) + (56|F|15)] &= V_t[(-)^{1+J}(1/6 \sqrt{2\pi(2J+1)} (\partial_{L,0} \partial_{L',2} + \partial_{L,2} \partial_{L',0}) \\
 &\quad + (1/3) \sqrt{(5/6\pi)(2L+1)(2L'+1)} W(1L'12; 1L) W(SLS'L'; J2) \\
 &\quad \cdot \{(-)^{L+J+1} + (-)^{L'+L+J}\}] I_3', \quad (44) \\
 [(12|F|56) + (56|F|12)] &= V_t(1/6 \sqrt{2\pi(2J+1)} (-)^{J+1} (\partial_{L,0} \partial_{L',2} + \partial_{L,2} \partial_{L',0}) I_3', \\
 (56|F|56) &= V_t[\{(2/3) \sqrt{(5/6\pi)(2L+1)(2L'+1)} W(1L'12; 1L) W(SLS'L'; J) \\
 &\quad \cdot ((-)^{L+J+1} + (-)^{L'+L+J}) - (1/3 \sqrt{2\pi(2J+1)}) (-)^{J+1} \\
 &\quad \cdot (\partial_{L,0} \partial_{L',2} + \partial_{L,2} \partial_{L',0})\} I_3' + (9/16) \sqrt{5/2\pi} (-)^{S+S'+L+J} (1010|L0) (1010|L'0) \\
 &\quad \cdot (L0L'0|20) W(LL'SS'; 2J) I_4'].
 \end{aligned}$$

In these formulae,  $I_3'$  and  $I_4'$  can be obtained from  $I_3$  and  $I_4$  which are defined in (B-11) of App. B, respectively, by changing  $r_e$  by  $r_t$ .

Substituting (44) in (41), we obtain

$$\begin{aligned}
 E_t &= (V_t/N) [2 \sqrt{3/5(2J+1)} (\partial_{L,0} \partial_{L',2} + \partial_{L,2} \partial_{L',0}) (-)^{J+1} I_3' \\
 &\quad - (9 \sqrt{3}/4) (-)^{S+S'+L+J} (1010|0L) (1010|L'0) \\
 &\quad \cdot (L0L'0|20) W(LL'SS'; 2J) I_4']. \quad (45)
 \end{aligned}$$

Using the expression of  $I_3'/N$  and  $I_4'/N$ , given by (B-21) of App. B, we finally get the following results

$$\begin{aligned}
 E_t(^{13}S_1; ^{13}D_1) &= V_t(7/9 \sqrt{5}) (3 \cdot 5 \cdots 17/2 \cdot 4 \cdots 14) (2 \sqrt{3} \lambda r_t)^{19} (B' - 4C'), \\
 E_t(^{13}D_1; ^{13}D_1) &= -V_t(28/45) (3 \cdot 5 \cdots 17/2 \cdot 4 \cdots 14) (2 \sqrt{3} \lambda r_t)^{19} C', \quad (46a) \\
 E_t(^{13}D_2; ^{13}D_2) &= -(W(1212; 22)/W(1212; 12)) E_t(^{13}D_1; ^{13}D_1) = -E_t(^{13}D_1; ^{13}D_1), \\
 E_t(^{13}D_3; ^{13}D_3) &= (W(1212; 32)/W(1212; 12)) E_t(^{13}D_1; ^{13}D_1) \\
 &= (2/7) E_t(^{13}D_1; ^{13}D_1),
 \end{aligned}$$

where  $B'$  and  $C'$  are obtained from  $B$  and  $C$ , which are defined in (B-22) of App. B, by changing  $r_e$  by  $r_t$ .

Similar calculations are performed for  $^{33}\text{P}_{0,1,2}$  states, and the following results are obtained,

$$\begin{aligned} E_t(^{33}\text{P}_0; ^{33}\text{P}_0) &= -V_t(14/81)(3 \cdot 5 \cdots 17)/2 \cdot 4 \cdots 14)(2\sqrt{3}\lambda r_t)^{19}B', \\ E_t(^{33}\text{P}_1; ^{33}\text{P}_1) &= -(W(1111; 12)/W(1111; 02))E_t(^{33}\text{P}_0; ^{33}\text{P}_0) \\ &= -(1/2)E_t(^{33}\text{P}_0; ^{33}\text{P}_0), \\ E_t(^{33}\text{P}_2; ^{33}\text{P}_2) &= (W(1111; 22)/W(1111; 02))E_t(^{33}\text{P}_0; ^{33}\text{P}_0) = (1/10)E_t(^{33}\text{P}_0; ^{33}\text{P}_0). \end{aligned} \quad (46b)$$

It would be appropriate, here, to evaluate the matrix elements of the Coulomb interactions, because the calculations are very similar to that of the tensor interactions.

The expression corresponding to (34) is, again for the partition [42],

$$E_{\text{coul}} = (15/9N) \left( \sum_{i=1}^9 \phi_i \phi_i | H_C^{(46)} | \sum_{i=1}^9 \phi_i \phi_i \right), \quad (47)$$

where

$$H_C^{(56)} = (1/4)(1 - \tau_z^{(5)})(1 - \tau_z^{(6)})(e^2/\tau v). \quad (48)$$

The operator (48) is now free from ordinary spin operator, but contain  $\tau$ -spin operators. Therefore, for performing sum over charge-spin variables, it is convenient to define a function  $|t_1 t_2 t_3\rangle$ , which are composed of  $\tau$ -spin wave functions of the constituent particles, and has the similar meaning as the  $|t_1 t_2 t_3\rangle$ , defined in (22). Then the matrix element, which has similar meaning as (36), is

$$\begin{aligned} (t_1' t_2' t_3' | H_C^{(56)} | t_1 t_2 t_3) &= (e^2/4\tau v) \sum (t_1 m_1 \frac{1}{2} m_3 | t_2 m_2) (t_2 m_2 \frac{1}{2} m_3 | t_3 m_3) \\ &\cdot (t_1' m_1' \frac{1}{2} m_3' | t_2' m_2') (t_2' m_2' \frac{1}{2} m_3' | t_3' m_3') \chi_{m_5}^{(5)} \chi_{m_6}^{(6)} * (1 - \tau_z^{(5)} - \tau_z^{(6)} + \tau_z^{(5)} \tau_z^{(6)}) \chi_{m_5'}^{(5)} \chi_{m_6'}^{(6)} \\ &= (e^2/4\tau v) [\partial_{t_2 t_2'} \partial_{t_3 t_3'} \delta_{m_2 m_2'} \delta_{m_3 m_3'} + (-)^{1+t_3+t_3'} \sqrt{6(t_3'+1)} W(t_2' t_3' \frac{1}{2} 1; \frac{1}{2} t_3') \\ &\quad \cdot W(t_3' m_3' 10 | t_3 m_3) \delta_{t_2 t_2'} \delta_{m_5 m_5'} \\ &\quad + (-)^{1+t_2+t_2'} \sqrt{6(2t_2'+1)(2t_2'+1)(2t_3'+1)} W(t_1' t_2' \frac{1}{2} 1; \frac{1}{2} t_2) W(\frac{1}{2} t_3' t_2 1; t_2' t_3) \\ &\quad \cdot (t_3' m_3' 10 | t_3 m_3) \delta_{m_6 m_6'} \\ &\quad + (-)^{1+t_3+t_3'} \sqrt{6(2t_2'+1)(2t_2'+1)} W(t_1' t_2' \frac{1}{2} 1; \frac{1}{2} t_2) \sum_{k,q} W(t_2' 1 \frac{1}{2} t_3; t_2 k) \\ &\quad \cdot W(\frac{1}{2} 1 t_2' k; \frac{1}{2} t_3') (kq 10 | t_3 m_3) (kq 10 | t_3' m_3')]. \end{aligned} \quad (49)$$

In this formula, only the first and the fourth terms contribute to  $T=0$  states, while all four terms contribute for  $T=1$  states.

Using (3), and the expressions of the charge-spin functions for other states, given in App. A, we can obtain matrices which correspond to those of Table II, and we show them in Table IV.

Using Table IV, the calculation of the matrix multiplication is straightforward, and abbreviating  $(L_{ij} | 1/\tau v | L_{ij})/N$  by  $(ij || i'j')$ , we obtain the following results.

Table IV  
Charge-spin matrices for Coulomb force.

$^{13}S, ^{13}D, ^{31}S, ^{31}D \ (T_z=0)$		$^{31}S, ^{31}D \ (T_z=1)$	
<div style="border: 1px solid black; padding: 10px; width: fit-content; margin: 10px auto;"> <math display="block">  \begin{array}{ccccccc}  2 &amp; &amp; &amp; &amp; &amp; &amp; \\  &amp; ^1\frac{1}{2}\frac{1}{2} &amp; &amp; a &amp; &amp; &amp; \\  &amp; &amp; ^1\frac{1}{2}\frac{1}{2} &amp; &amp; a &amp; &amp; \\  &amp; &amp; &amp; ^1\frac{1}{2}\frac{1}{2} &amp; &amp; a &amp; \\  a &amp; &amp; &amp; &amp; ^1\frac{1}{2}\frac{1}{2} &amp; &amp; \\  &amp; a &amp; &amp; &amp; &amp; ^1\frac{1}{2}\frac{1}{2} &amp; \\  &amp; &amp; a &amp; &amp; &amp; &amp; ^1\frac{1}{2}\frac{1}{2} \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{7}{5} \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{7}{5}  \end{array}  </math> </div> <p style="text-align: center;"><math>a=2\sqrt{2}/15</math></p>		<div style="border: 1px solid black; padding: 10px; width: fit-content; margin: 10px auto;"> <math display="block">  \begin{array}{ccccccc}  0 &amp; &amp; &amp; &amp; &amp; &amp; \\  &amp; 0 &amp; &amp; &amp; &amp; &amp; \\  &amp; &amp; 0 &amp; &amp; &amp; &amp; \\  &amp; &amp; &amp; 1\frac{2}{5} &amp; &amp; &amp; \\  &amp; &amp; &amp; &amp; 1\frac{2}{5} &amp; &amp; \\  &amp; &amp; &amp; &amp; &amp; 1\frac{2}{5} &amp; \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{6}{5} \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{6}{5}  \end{array}  </math> </div>	
$^{11}P, ^{33}P \ (T_z=0)$		$^{33}P \ (T_z=1)$	
<div style="border: 1px solid black; padding: 10px; width: fit-content; margin: 10px auto;"> <math display="block">  \begin{array}{ccccccc}  2 &amp; &amp; &amp; &amp; &amp; &amp; \\  &amp; ^1\frac{1}{2}\frac{1}{2} &amp; &amp; b &amp; &amp; &amp; \\  &amp; &amp; ^1\frac{1}{2}\frac{1}{2} &amp; &amp; b &amp; &amp; \\  &amp; &amp; &amp; ^1\frac{1}{2}\frac{1}{2} &amp; &amp; b &amp; \\  b &amp; &amp; &amp; &amp; ^1\frac{1}{2}\frac{1}{2} &amp; &amp; \\  &amp; b &amp; &amp; &amp; &amp; ^1\frac{1}{2}\frac{1}{2} &amp; \\  &amp; &amp; b &amp; &amp; &amp; &amp; ^1\frac{1}{2}\frac{1}{2} \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{1}{3} \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{1}{3} \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{1}{3}  \end{array}  </math> </div> <p style="text-align: center;"><math>b=2\sqrt{6}/15</math></p>		<div style="border: 1px solid black; padding: 10px; width: fit-content; margin: 10px auto;"> <math display="block">  \begin{array}{ccccccc}  0 &amp; &amp; &amp; &amp; &amp; &amp; \\  &amp; 0 &amp; &amp; &amp; &amp; &amp; \\  &amp; &amp; 0 &amp; &amp; &amp; &amp; \\  &amp; &amp; &amp; 2\frac{0}{5} &amp; &amp; &amp; \\  &amp; &amp; &amp; &amp; 2\frac{0}{5} &amp; &amp; \\  &amp; &amp; &amp; &amp; &amp; 2\frac{0}{5} &amp; \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{1}{3} \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{1}{4} \\  &amp; &amp; &amp; &amp; &amp; &amp; \frac{1}{4}  \end{array}  </math> </div>	

$$\begin{aligned}
 E_{\text{coul}}(^{13}S) &= E_{\text{coul}}(^{13}D) = E_{\text{coul}}(^{31}S)_{T_z=0} = E_{\text{coul}}(^{31}D)_{T_z=0} \\
 &= (e^2/4) [2(56\|56) + 8(12\|12) + 2(12\|34) - 8(12\|13) + 8(15\|15) - 4(15\|25) \\
 &\quad + 4(15\|26) + 4(12\|56) - 8(15\|56) - 16(12\|15) + 8(12\|35)], \quad (50a) \\
 E_{\text{coul}}(^{31}S)_{T_z=1} &= E_{\text{coul}}(^{31}D)_{T_z=1} = (e^2/4) \cdot 8 \cdot [(12\|12) - (12\|13) + (15\|15) \\
 &\quad - (15\|25) - 2(12\|15) + 2(12\|35)].
 \end{aligned}$$

For states belonging to partition  $[411]$ , the results are

$$E_{\text{coul}}(^{11}P) = E_{\text{coul}}(^{33}P)_{T_z=0} = (e^2/4) [2(56\|56) + 8(15\|15) - 4(15\|25)]$$

$$+ 8(15||56) - 4(15||26) + 8(12||12) - 8(12||13) - 16(12||15)], \quad (50b)$$

$$E_{\text{coul}}(^{31}\text{P})_{T_z=1} = (e^2/4) [8(15||15) - 8(15||25) + 8(12||12) - 8(12||13) - 16(12||15)].$$

As is easily seen from the definition,  $(ij||i'j')$  is obtained from  $(ij|i'j')$  defined in § 3, by replacing  $\exp[-\tau w/r_e]/(\tau w/r_e)$  in the integrand for the spatial integration by  $1/\tau w$ . However, in the present case, all four kinds of integrals  $I_1$ ,  $I_2$ ,  $I_3$  and  $I_4$ , which appeared in § 3, can be expressed by only one integral  $H$ , by the following relations,

$$I_1 \rightarrow H, \quad I_2 \rightarrow (4/5)H, \quad I_3 \rightarrow (16/15)H, \quad I_4 \rightarrow (512/135)H \quad (51)$$

where

$$H = (4\pi)^{-2} \int \tau^4 \exp[-2\lambda Z] (1/\tau w) d\tau. \quad (52)$$

The integration of (52) is easily performed, and we obtain

$$H/N = (5\sqrt{3}\lambda/768) (3 \cdot 5 \cdots 17/2 \cdot 4 \cdots 14). \quad (53)$$

Because of these discussions, it is clear that each matrix element in (50) can be expressed as a multiple of  $H/N$ , and we show them in Table V, for S, D and P states, separately.

Table V  
Spatial matrix elements for Coulomb forces.

$$(L_{ij}|r_{56}^{-1}|L_{ij'}) (135||I) = (ij||i'j') (135N/H)$$

	S	D	P
(12  12)	116	104	96
(12  13)	-4	-16	-24
(12  34)	20	8	0
(12  15)	-4	-16	-24
(12  35)	20	8	0
(15  15)	96	84	76
(15  25)	0	-12	-20
(15  16)	16	4	-4
(15  26)	16	4	-4
(56  56)	80	70.4	64
(12  56)	8	8	*
(15  56)	-16	-16	16

$$H = (4\pi)^{-2} \int \tau^4 \exp[-2\lambda Z] d\tau; \quad H/N = (5\sqrt{3}\lambda/3 \times 16^2) (3 \cdot 5 \cdots 17/2 \cdot 4 \cdots 14).$$

\*: This term does not appear in P state.

Combining (50) and the figures of Table V, we get the following results,

$$E_{\text{coul}}(^{13}\text{S}) = E_{\text{coul}}(^{31}\text{S})_{T_z=0} = 88/5G,$$

$$E_{\text{coul}}(^{13}\text{D}) = E_{\text{coul}}(^{31}\text{D})_{T_z=0} = 432/25G,$$



$$E_{\text{coul}}(^{11}\text{P}) = E_{\text{coul}}(^{31}\text{P})_{T_z=0} = 256/15G, \quad (54)$$

$$E_{\text{coul}}(^{31}\text{S})_{T_z=1} = E_{\text{coul}}(^{31}\text{D})_{T_z=1} = E_{\text{coul}}(^{33}\text{P})_{T_z=1} = 704/45G,$$

with

$$G = (e^2/4) (5 \sqrt{3} \lambda / 768) (3 \cdot 5 \cdot \dots \cdot 17 / 2 \cdot 4 \cdot \dots \cdot 14).$$

## § 5. Kinetic energy

The kinetic energy operator is given by

$$H_k = (\hbar^2/2M) \cdot \sum_{i=1}^6 A_i, \quad (55)$$

where  $M$  is the nucleon mass. As this operator is free from charge-spin operators, its expectation value, in the states belonging to the partition [42], is reduced into a simple form, as the following equation shows,

$$\begin{aligned} E_k &= \langle \Psi | H_k | \Psi \rangle \\ &= 1/9N \left( \sum_{i=1}^9 \phi_i \phi_i | H_k | \sum_{i=1}^9 \phi_i \phi_i \right) \\ &= 1/9N \sum_{i=1}^9 (\phi_i | H_k | \phi_i) = (1/9N) (\hbar^2/2M) \sum_{j=1}^9 (\phi_j | \sum_{i=1}^6 A_i | \phi_j). \end{aligned} \quad (56)$$

By using the symmetry properties of  $L_{ij}$ 's, (56) can easily be rewritten as

$$E_k = (\hbar^2/2M) [ (L_{56} | \sum_{i=1}^6 J_i | L_{56}) - 2 (L_{16} | \sum_{i=1}^6 J_i | L_{56}) + (L_{12} | \sum_{i=1}^6 J_i | L_{56}) ]. \quad (57a)$$

Similarly for partition [411]

$$E_k = (\hbar^2/2M) [ (L_{56} | \sum_{i=1}^6 J_i | L_{56}) - 2 (L_{16} | \sum_{i=1}^6 J_i | L_{56}) ]. \quad (57b)$$

In these equations, e.g.,  $(L_{56} | \sum_{i=1}^6 J_i | L_{56})$  has the following explicit form

$$\begin{aligned} (L_{56} | \sum_{i=1}^6 J_i | L_{56}) &= (1/N) \sum (1m_5 1m_6 | LM_i) (1m_5' 1m_6' | LM_i) \\ &\cdot \int Y_{1m_5}^*(\mathbf{r}_5) Y_{1m_6}^*(\mathbf{r}_6) \exp[-\lambda Z] \left( \sum_{i=1}^6 J_i \right) \exp[-\lambda Z] Y_{1m_5'}(\mathbf{r}_5) Y_{1m_6'}(\mathbf{r}_6) d\mathbf{u} d\mathbf{v} d\mathbf{w} d\mathbf{s} d\mathbf{t}. \end{aligned} \quad (58)$$

The evaluation of this matrix element is illustrated in App. C. We write here, only the results,

$$E_k(^{13}\text{S}) = E_k(^{31}\text{S}) = E_k(^{13}\text{D}) = E_k(^{31}\text{D}) = E_k(^{11}\text{P}) = E_k(^{33}\text{P}) = (\hbar^2/2M) 6\lambda^2. \quad (59)$$

## § 6. Quadrupole and magnetic moments

In the first part of this section, we evaluate the matrix elements for quadrupole moment. As only the quadrupole moment of the ground state is measured, and, so long

as we consider only the tensor forces for non-central interactions, the ground state may be considered as a mixture of  $^{13}\text{S}_1$  and  $^{13}\text{D}_1$  states, we evaluate the matrix elements only for these states. Moreover it is clear that the S state can have no quadrupole moment, so it is only necessary to calculate the D-D matrix element and the S-D interference term.

The quadrupole moment operator is given by

$$H_Q = \sum_{i=1}^6 H_Q^{(i)} = \sum_{i=1}^6 (1/2) (1 - \tau_z) (3z_i^2 - r_i^2) = \sqrt{4\pi/5} \sum_{i=1}^6 (1 - \tau_z^{(i)}) Y_{20}(\mathbf{r}_i). \quad (60)$$

Then its matrix element  $Q$  is written as

$$Q = \langle \Psi | H_Q | \Psi \rangle = 1/9 \cdot V \left( \sum_{i=1}^9 \phi_i \phi_i' | H_Q | \sum_{j=1}^9 \phi_j \phi_j' \right) = 2/3 \cdot V \left( \sum_{i=1}^9 \phi_i \phi_i' | H_Q^{(6)} | \sum_{j=1}^9 \phi_j \phi_j' \right). \quad (61)$$

For the evaluation of (61) it is convenient, as in the preceding sections, first to perform sum over charge-spin variables. However, as  $H_Q^{(6)}$  is an one-particle operator, the summation is much simpler than in the case of two-particle operators, and we need not use  $\phi_i$ 's in the form of (3), but in the following more simplified form,

$$\begin{aligned} \phi_1(\widetilde{221111}) &= [\widetilde{(21111)}, 22, 13], & \phi_2(\widetilde{212111}) &= [\widetilde{(12111)}, 22, 13], \\ \phi_5(\widetilde{122111}) &= 1/\sqrt{5} [\widetilde{(22111)}, 22, 13] - 2/\sqrt{5} [\widetilde{(22111)}, 24, 13], \\ \phi_8(\widetilde{112211}) &= 1/\sqrt{5} [\widetilde{(12211)}, 22, 13] - 2/\sqrt{5} [\widetilde{(12211)}, 24, 13]. \end{aligned} \quad (62)$$

The meaning of these expressions is similar to those of (3). Other  $\phi_i$ 's can be expressed similarly.

Now we define a  $\tau$ -spin eigen-function  $|t_2 t_3\rangle$ , which should have similar meaning as the  $|t_1 t_2 t_3\rangle$ , defined below of (48), and calculate the following matrix element

$$\begin{aligned} \langle t_2 t_3 | H_Q^{(6)} | t_2' t_3' \rangle &= \langle t_2 t_3 | \sqrt{4\pi/5} Y_{20}(\mathbf{r}_6) (1 - \tau_z^{(6)}) | t_2' t_3' \rangle \\ &= \sqrt{4\pi/5} Y_{20}(\mathbf{r}_6) \sum_{mm' t_2} \langle t_2 m_{2\frac{1}{2}} m | t_3 m_3 \rangle \langle t_2 m_{2\frac{1}{2}} m' | t_3' m_3' \rangle \chi_m^{(6)*} (1 - \tau_z^{(6)}) \chi_m^{(6)}, \end{aligned} \quad (63)$$

where  $\chi_m^{(6)}$  means the  $\tau$ -spin eigen-function of the sixth particle. The summation in (63) can easily be done, and replacing  $t_3, t_3', m_3$  and  $m_3'$  by  $T, T', M_T$  and  $M_T'$ , respectively, we obtain

$$\begin{aligned} \langle t_2 t_3 | H_Q^{(6)} | t_2' t_3' \rangle &= \sqrt{4\pi/5} Y_{20}(\mathbf{r}_6) [\partial_{TT'} + (-)^{T+T'+1} \sqrt{6(2T'+1)} W(t_2 T' \frac{1}{2} 1; \frac{1}{2} T) \\ &\quad \cdot (T' M_T' 10 | T M_T)]. \end{aligned} \quad (64)$$

As we consider only the ground state of  $\text{Li}^6$ , and this state has  $T = T' = 0$ , it is clear that only the first term of (64) contributes to our calculation. Then the matrix corresponding to Table II is now a unit matrix, multiplied by  $\sqrt{4\pi/5} Y_{20}(\mathbf{r}_6)$ , and the matrix multiplication is very simple. Writing  $(I_{ij} | Q^{(6)} | I_{i'j'})/N$  as  $(ij | Q | i'j')$ . We obtain the following expression for  $Q$ ,

$$Q = 2 [2(12|Q|12) - 3(12|Q|13) + (12|Q|34) - (12|Q|16) - (16|Q|12) + (12|Q|36) + (36|Q|12) + (16|Q|16) - (16|Q|26)]. \quad (65)$$

Here, e.g.,  $(12|Q|12)$  has the explicit form

$$(12|Q|12) = \sqrt{4\pi/5} \sum (1m_1 1m_2 | LM_L) (1m_1' 1m_2' | L' M_L') (LM_L SM_s | JJ) \cdot (L' M_L' SM_s | JJ) \cdot \int Y_{1m_1}^*(\mathbf{r}_1) Y_{1m_2}^*(\mathbf{r}_2) Y_{20}(\mathbf{r}_1) Y_{m_1'}(\mathbf{r}_1) Y_{m_2'}(\mathbf{r}_2) \exp[-2\lambda Z] d\mathbf{u} d\mathbf{v} d\mathbf{w} d\mathbf{s}. \quad (66)$$

The evaluation of this matrix element is shown in App. D, and the result after integration over angle variables and sum over magnetic quantum numbers is given in the following form,

$$(12|Q|12) = 1/V \int [(4/729)t^2 Q_2 + (1/81) \{1 + 2(-)^L + (-)^{L+L'}\} \delta_{LL'} t^4 s^2 Q_1 + (1/81) \{-1 + 2(-)^L - (-)^{L+L'}\} \delta_{LL'} t^4 Q_1 + (1/81) \{\delta_{L',0} \delta_{L,2} \delta_{J,1} + \delta_{L',2} \delta_{L,0} \delta_{J,1}\} (s^2 t^4 - u^2 t^4) Q_3] \exp[-2\lambda Z] d\tau, \quad (67)$$

where  $Q_1$ ,  $Q_2$  and  $Q_3$  are designed in (D-1) and (D-2), together with other  $Q_i$ 's, which are necessary for our calculation.

After performing radial integrals and substituting the values of  $Q_i$ 's, (67) becomes

$$(12|Q|12) \begin{cases} = -25\sqrt{5}/(3888\lambda^2) & \text{for } L=0, L'=2, \text{ or for } L=2, L'=0, \\ = -7/(648\lambda^2) & \text{for } L=L'=2. \end{cases} \quad (68)$$

Table VI  
Spatial matrix elements for quadrupole moments.  
 $(L_{ij}|Q|L_{ij'})$  (6075  $N/K$ )

	SD	DD			
			$(L_{16} Q L_{12})$	$50\sqrt{5}$	77
$(L_{12} Q L_{12})$	$-10\sqrt{5}$	-28	$(L_{12} Q L_{36})$	$-46\sqrt{5}$	-28
$(L_{12} Q L_{13})$	$2\sqrt{5}$	-7	$(L_{36} Q L_{12})$	$-10\sqrt{5}$	-28
$(L_{12} Q L_{34})$	$14\sqrt{5}$	-14	$(L_{16} Q L_{16})$	$-310\sqrt{5}$	-490
$(L_{12} Q L_{16})$	$14\sqrt{5}$	77	$(L_{16} Q L_{26})$	$-10\sqrt{5}$	35

$$K = (4\pi)^{-2} \int t^6 \exp[-2\lambda Z] ; K/N = 75/(32\lambda^2).$$

Other matrix elements in (65) can be calculated similarly, and the results are tabulated in Table VI. From this table and (65) we obtained the following very simplified final results,

$$Q = \begin{cases} -\sqrt{5}/(3\lambda^2), & \text{for } L=0, L'=2, \text{ or for } L=2, L'=0, \\ 49/(81\lambda^2), & \text{for } L=L'=2. \end{cases} \quad (69)$$

The second part of this section concerns with the evaluation of the magnetic moment. It is, however, very fortunate that we are considering a self-conjugate nucleus, i.e., a nucleus in which the number of protons and neutrons are equal, because in such a nucleus no tedious calculation is needed, but the  $g$ -factor can be obtained, as is shown by Sachs<sup>27)</sup>, only if we know the amplitudes of the substates, in the L-S coupling version, mixed in the concerning state, and is given by the following formula. (This is the eq. (8) of reference 27.)

$$g = 1/4(1 + g_n + g_p) + 1/4(1 - g_n - g_p) \sum_{L,S} |C_{L,S}|^2 \{ L(L+1) - S(S+1) \} / J(J+1). \quad (70)$$

As we are considering only S and D states, if we denote their amplitudes by  $\alpha$  and  $\beta$ , respectively, i.e., if we take  $C_{01} = \alpha$ ,  $C_{21} = \beta$ , with  $\alpha^2 + \beta^2 = 1$ , (70) becomes

$$\begin{aligned} g &= 1/4(1 + g_n + g_p) + 1/4(1 - g_n - g_p)(-\alpha^2 + 2\beta^2) \\ &= 0.87990 - 0.56985 \beta^2. \end{aligned} \quad (71)$$

## § 7. Numerical calculations and comparison with experiments

The binding energy of  $\text{Li}^6$  can be calculated by using (26), (27), (46a) and (50a). As for the parameters for two body potentials, we first take two kinds of them which are used by Irving<sup>18)</sup> in Tables I and IV of his paper. These values are reproduced in our Tables VII and VIII.

So long as we assume tensor force, exclusively, as non-central forces, the ground state can be considered as a mixture of  $^{13}\text{S}$  and  $^{13}\text{D}$  states, and to obtain the binding energy we must solve a secular equation. As it is tedious to minimize the solution of this secular equation against  $\lambda$ , the reciprocal of the nuclear radius, we first determine the value of  $\lambda$  so as to minimize the energy of the pure  $^{13}\text{S}$  state and using same value of  $\lambda$  also for  $^{13}\text{D}$  state, solve the secular equation. (In actual cases, it is found that the  $\lambda$  determined so as to minimize the energy of pure  $^{13}\text{S}$  state also minimizes that of pure  $^{13}\text{D}$  state.)

The binding energy, thus obtained for Irving's two kinds of parameters, are 12.1 Mev and 42.6 Mev, respectively, which are too small and too large, compared with experimental value 31.9 Mev.

To seek for more appropriate parameters, we referred to the work of Feshbach and Schwinger<sup>28)</sup>. In Table V of their paper, there are given sixty sets of parameters which fit to deuteron data. However, we find that, among these sets, those with  $\gamma$  ( $\gamma = (M\epsilon / \hbar)^{1/2} r_c$ ,  $\epsilon$  being the binding energy of deuteron) other than 0.275 cannot give reasonable binding energy of  $\text{Li}^6$ . (The Irving's set, in our Table IX, is one of these sets with  $\gamma = 0.275$ ).

The most appropriate set is given in Table IX, together with the energy matrix elements used in the secular equation. This set gives a value 28.7 Mev for the binding energy of  $\text{Li}^6$ , which is fairly close to the experimental value. From this value it is also possible to calculate the splitting between the second excited,  $^{31}\text{S}_0$ , state and the ground



state. We obtain a value 0.16 Mev, which is too small compared with experimental value, 3.58 Mev.<sup>29)</sup> On this point some discussion will be given in the next section.

The secular equation also gives the amplitude  $\beta$  of  $^{13}\text{D}_1$  state in the ground state, which is defined in § 6. We obtain, for two body parameters of Table IX,  $\beta^2=0.01426$  and this gives for the magnetic moment of the ground state a value 0.872 n. m., which is much larger than the experimental value, 0.821 n. m.. The quadrupole moment can also be calculated and it is evaluated to be 0.77 mb. Experimentally only the absolute value of this quantity is known, and it is less than 1 mb. Therefore our theoretical result may be said to be consistent with experiments.

Table VII  
Two body parameters Set I.

$r_o=1.184 \times 10^{-13}$ cm
$r_t=1.67 \times 10^{-13}$ cm
$V_o=46.1$ Mev
$V_t/V_o=0.54$
$w=1.004$
$b=-0.004$

Table VIII  
Two body parameters Set II.

$r_o=1.184 \times 10^{-13}$ cm
$r_t=2.121 \times 10^{-13}$ cm
$V_o=54.59$ Mev
$V_t/V_o=0.2305$
$w=0.926$
$b=0.074$

Table IX  
Two body parameters Set III, and energy matrices used in the secular equation.

$r_o=1.184 \times 10^{-13}$ cm	$E(^{13}\text{S})=-28.6$ Mev
$r_t=2.12 \times 10^{-13}$ cm	$E(^{13}\text{D})=-15.9$ Mev
$V_o=49.3$ Mev	$E(^{13}\text{S}, ^{13}\text{D})=1.33$ Mev
$V_t/V_o=0.300$	Binding Energy of $\text{Li}^6=28.7$ Mev
$w=1$	
$b=0$	

## § 8. Discussions

As is shown in the preceding section, we have succeeded in obtaining binding energy of  $\text{Li}^6$  in close agreement with experiment, assuming only two body interactions which are consistent with low energy two body problems. Although our calculation may seem to be drastically different from the ordinary shell model calculations, as is seen from the very complicated calculations of matrix elements compared with the case of shell model treatment, the main difference is the correlation of the position coordinates of the constituent particles taken into account, i.e., there is no essential difference as to the transformation properties of the wave functions. It is gratifying that such a slight modification could improve the situation fairly well.

The magnitudes of the matrix elements for the tensor interactions are also increased



compared with shell model calculation. For example  $E_t(^{13}\text{D}, ^{13}\text{D})$ , which is defined in (46a) is 5.9 Mev, while the corresponding shell model result, as shown, e.g., by Feingold<sup>16)</sup>, is 0.58 Mev.

It is very unfortunate, however, that the S-D interference term is very small, i.e., the same order of magnitude with the shell model result. Its explicit form is, as shown in (46a),

$$E_t(^{13}\text{S}, ^{13}\text{D}) = V_t(7/9\sqrt{5})(3.5\cdots 17/2.4\cdots 14)(2\sqrt{3}\lambda r_t)^{10}(B' - 4C'), \quad (72)$$

and its numerical value, with our parameters in Table IX, is

$$\begin{aligned} E_t(^{13}\text{S}, ^{13}\text{D}) &= -2.52(2\sqrt{3}\lambda r_t)^{10}(B' - 4C') = -2.52(4.71 - 5.24)\text{Mev} \\ &= (-11.86 + 13.19)\text{Mev} = 1.33\text{ Mev}. \end{aligned} \quad (73)$$

showing cancellation, very large compared with the value of each term. Similar cancellation also occurs in the shell model case; however, it is not so drastic as in our case, as is seen from the following numerical relation, which is again given by Feingold<sup>16)</sup>,

$$\begin{aligned} E_t(^{13}\text{S}, ^{13}\text{D}) &= -2/\sqrt{5}(M - N) = -2/\sqrt{5}(-3.0 + 3.8)\text{Mev}, \\ &= (3.40 - 2.68)\text{Mev} = 0.72\text{ Mev}; \end{aligned} \quad (74)$$

We have investigated whether tensor interactions with exchange character other than Wigner could change such a situation. It is found, however, that all exchange characters give the same result, if the nuclear parameters are modified so as to fit the deuteron data.

Our calculation does not either give reasonable result for the level ordering among  $^{13}\text{D}_{1,2,3}$  states. Although the amount of splitting due to tensor force is sufficiently large, as is mentioned above, the  $J=2$  state comes lowest, while experimentally  $J=3$  state comes lowest<sup>20)</sup>. This would perhaps have resulted from the fact that our wave function has essentially the same transformation property as the usual shell model wave function, and, indeed, our level ordering coincides with the result of Ishidzu and Obi<sup>15)</sup>.

As for the P states, they are found to lie high enough, where no experimental data are known, and we will not discuss them in detail.

The calculated values of the quadrupole moment is in fairly good agreement with experiments. This has resulted from the small admixture of  $^{13}\text{D}_1$  state to the ground state, and so is a consequence of the small S-D interference term. Therefore, so long as we have not succeeded in obtaining good result for the splitting between the second excited and the ground states, we would have to conclude that our good agreement is only an incidental one. This is also seen from too large value of the magnetic moment.

Recently Lyons and Feingold<sup>30)</sup> reported that they had extended Feingold's former work<sup>16)</sup>, and succeeded in obtaining reasonable value for the splitting between the  $^{13}\text{S}_1$  and the  $^{31}\text{S}_0$  states, as well as the correct level ordering and splitting among  $^{13}\text{D}_{1,2,3}$  states. As is illustrated in reference 16, in the comparison of their result with Dancoff's<sup>31)</sup> one of the second order perturbational calculation, for the doublet splitting in  $\text{He}^5$ , the former can give a new matrix element of the tensor interaction which has larger magnitude than the latter, with opposite sign. Similar situation might have led the  $J=3$  level to come to

the lowest among the  $^{13}\text{D}$  states.

Although our method gives reasonable result concerning the central interactions, the results become unreasonable as soon as the effect of the tensor interaction concerns. Comparing this situation with Feingold's one, we may conclude that the transformation properties of interaction concerns. Comparing our wave function is too simple, although in it the effect of the higher configurations, is considered to some extent, if interpreted in the terminology of the shell model, by taking into account the correlation of the position coordinates of the constituent particles.

It is not reported in the note of Lyons and Feingold<sup>(30)</sup> whether or not they have calculated the binding energy of  $\text{Li}^6$ . If it had become clear that it were impossible to obtain the binding energy, by their method, it would be permitted to consider that our method and Feingold's one are mutually complementary. Such investigations will be done in near future.

### Acknowledgement

The authors should like to express their sincere thanks to Dr. H. Horie for his stimulating discussions, and to Profs. S. Tomonaga, T. Yamanouchi and N. Fukuda for their kind interest.

### Appendix A

The standard Young-Yamanouchi orthogonal representation corresponding to the partition [411] is given by Table A-I.

The space wave functions are

$$\begin{aligned}
 \phi_1(321111) &= 1/\sqrt{24} [4L_{36} + (L_{15} + L_{25} + L_{35} + L_{45}) - (L_{16} + L_{26} + L_{36} + L_{46})], \\
 \phi_2(312111) &= 1/\sqrt{360} [15L_{46} + 4(L_{14} + L_{24} + L_{34}) - 5(L_{16} + L_{26} + L_{36}) \\
 &\quad + (L_{15} + L_{25} + L_{35}) - 3L_{45}], \\
 \phi_3(311211) &= 1/\sqrt{180} [10L_{36} + 3(L_{13} + L_{23}) - 2(L_{34} + L_{35}) - 5(L_{16} + L_{26}) \\
 &\quad + (L_{15} + L_{25}) + (L_{14} + L_{24})], \\
 \phi_4(311121) &= 1/\sqrt{60} [5(-L_{16} + L_{26}) + 2L_{12} + (L_{15} - L_{25}) + (L_{13} + L_{14} - L_{23} - L_{24})], \\
 \phi_5(132111) &= 1/\sqrt{15} [3L_{45} + (L_{14} + L_{24} + L_{34}) - (L_{15} + L_{25} + L_{35})], \\
 \phi_6(131211) &= 1/\sqrt{120} [8L_{35} + 3(L_{13} + L_{23}) - 2L_{34} - 4(L_{15} + L_{25}) + (L_{14} + L_{24})], \\
 \phi_7(131121) &= 1/\sqrt{40} [4(-L_{15} + L_{25}) + 2L_{12} + (L_{13} + L_{14} - L_{23} - L_{24})], \\
 \phi_8(113211) &= 1/\sqrt{8} [2L_{34} + (L_{13} + L_{23}) - (L_{14} + L_{24})], \\
 \phi_9(113121) &= 1/\sqrt{24} [3(L_{24} - L_{14}) + 2L_{12} + (L_{13} - L_{23})], \\
 \phi_{10}(111321) &= 1/\sqrt{3} [-L_{13} + L_{23} + L_{12}].
 \end{aligned} \tag{A-1}$$

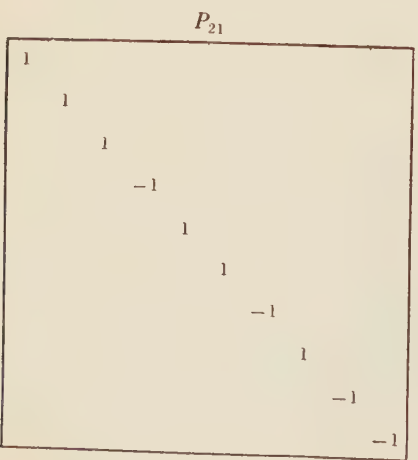
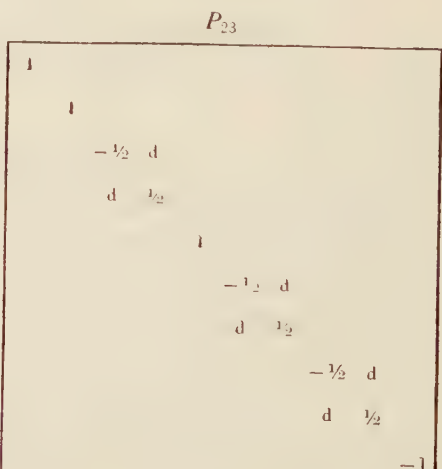
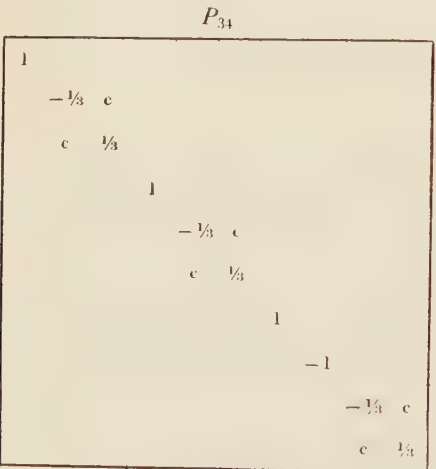
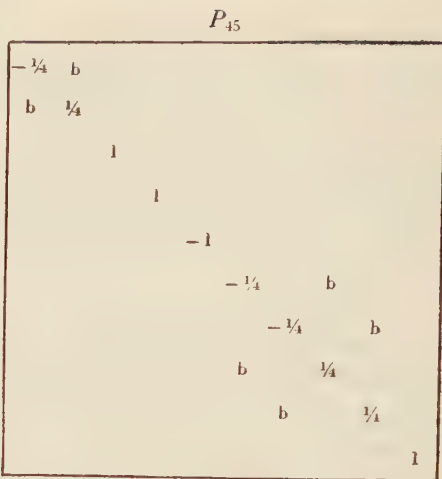
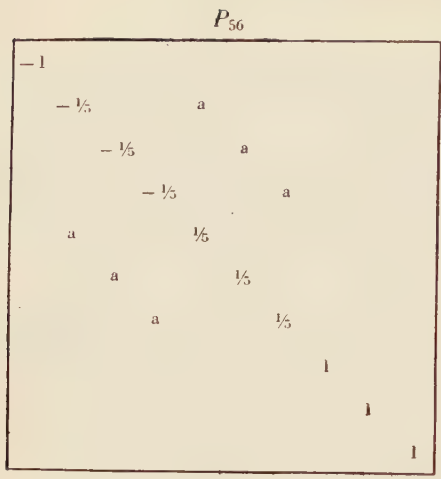


Table A-I.  
Standard orthogonal Yamanouchi representation  
for partition [411].

It should be noted that  $L_{ij}$ , in this case, changes sign when  $i$  and  $j$  are interchanged.

The charge-spin wave functions must be given separately for  $^{11}\text{P}$ , and  $^{33}\text{P}$  states. For  $^{11}\text{P}$  state they are

$$\psi_1(\widetilde{321111}) = [(\widetilde{1111}), 11, 22, 11],$$

$$\begin{aligned} \psi_2(\widetilde{312111}) = & -1/\sqrt{5} [(\widetilde{2111}), 13, 22, 11] - 1/\sqrt{5} [(\widetilde{2111}), 31, 22, 11] \\ & - \sqrt{3/5} [(\widetilde{2111}), 33, 22, 11], \end{aligned}$$

$$\begin{aligned} \psi_5(\widetilde{132111}) = & -\sqrt{3/10} [(\widetilde{2111}), 13, 22, 11] - \sqrt{3/10} [(\widetilde{2111}), 31, 22, 11] \\ & + \sqrt{2/5} [(\widetilde{2111}), 33, 22, 11], \end{aligned}$$

$$\psi_8(\widetilde{113211}) = 1/\sqrt{2} [(\widetilde{3211}), 13, 22, 11] - 1/\sqrt{2} [(\widetilde{3211}), 31, 22, 11]. \quad (\text{A-2})$$

$\psi_3$  and  $\psi_4$  are obtained from  $\psi_2$  by replacing  $(\widetilde{2111})$  by  $(\widetilde{1211})$  and  $(\widetilde{1121})$ , respectively.

$\psi_6$  and  $\psi_7$  are obtained from  $\psi_5$  by replacing  $(\widetilde{2111})$  by  $(\widetilde{1211})$  and  $(\widetilde{1121})$ , respectively.

$\psi_9$  and  $\psi_{10}$  are obtained from  $\psi_8$  by replacing  $(\widetilde{3211})$  by  $(\widetilde{3121})$  and  $(\widetilde{1321})$ , respectively. (A-3)

From  $^{33}\text{P}$  state they are

$$\psi_1(\widetilde{321111}) = [(\widetilde{1111}), 11, 22, 33],$$

$$\begin{aligned} \psi_2(\widetilde{312111}) = & -1/\sqrt{5} [(\widetilde{2111}), 13, 22, 33] - 1/\sqrt{5} [(\widetilde{2111}), 31, 22, 33] \\ & - \sqrt{3/5} [(\widetilde{2111}), 33, 22, 33], \end{aligned}$$

$$\begin{aligned} \psi_5(\widetilde{132111}) = & 1/(3\sqrt{30}) [(\widetilde{2111}), 13, 22, 33] + 1/(3\sqrt{30}) [(\widetilde{2111}), 31, 22, 33] \\ & - 2/9\sqrt{10} [(\widetilde{2111}), 33, 22, 33] + \sqrt{15}/9 [(\widetilde{2111}), 13, 24, 33] \\ & + \sqrt{5}/9 [(\widetilde{2111}), 33, 24, 33] + \sqrt{15}/9 [(\widetilde{2111}), 31, 42, 33] \\ & + \sqrt{5}/9 [(\widetilde{2111}), 33, 42, 33] + \sqrt{40}/9 [(\widetilde{2111}), 33, 44, 33], \end{aligned}$$

$$\begin{aligned} \psi_8(\widetilde{113211}) = & -1/(9\sqrt{2}) [(\widetilde{3211}), 13, 22, 33] + 1/9\sqrt{2} [(\widetilde{3211}), 31, 22, 33] \\ & + 1/9 [(\widetilde{3211}), 13, 24, 33] - \sqrt{10}/9 [(\widetilde{3211}), 35, 24, 33] \\ & - 1/9 [(\widetilde{3211}), 31, 42, 33] + 1/3 [(\widetilde{3211}), 33, 42, 33] \\ & + \sqrt{10}/9 [(\widetilde{3211}), 53, 42, 33] + \sqrt{20}/9 [(\widetilde{3211}), 35, 44, 33] \\ & - \sqrt{20}/9 [(\widetilde{3211}), 53, 44, 33] - 1/3 [(\widetilde{3211}), 33, 24, 33]. \quad (\text{A-4}) \end{aligned}$$

$\psi_2; \psi_4; \psi_6; \psi_7$  and  $\psi_9; \psi_{10}$  are obtained from  $\psi_2, \psi_5$  and  $\psi_8$ , respectively, as in (A-3).

The matrices  $\overline{P}_{56}^{(\sigma)}$  and  $\overline{P}_{56}^{(\tau)}$ , which correspond to Table II in the text, are given below;

<p>for <math>^{11}P, \overline{P}_{56}^{(\sigma)} = \overline{P}_{56}^{(\tau)}</math></p> <div style="border: 1px solid black; padding: 10px; margin: 10px auto; width: 350px; height: 200px; position: relative;"> <span style="position: absolute; top: 5px; left: 5px;">-1</span> <span style="position: absolute; top: 15%; left: 15%;">3/5</span> <span style="position: absolute; top: 15%; left: 30%;">a</span> <span style="position: absolute; top: 25%; left: 20%;">3/5</span> <span style="position: absolute; top: 25%; left: 35%;">a</span> <span style="position: absolute; top: 35%; left: 30%;">3/5</span> <span style="position: absolute; top: 35%; left: 40%;">a</span> <span style="position: absolute; top: 45%; left: 10%;">a</span> <span style="position: absolute; top: 45%; left: 25%;">2/5</span> <span style="position: absolute; top: 55%; left: 15%;">a</span> <span style="position: absolute; top: 55%; left: 30%;">3/5</span> <span style="position: absolute; top: 65%; left: 25%;">a</span> <span style="position: absolute; top: 65%; left: 35%;">2/5</span> <span style="position: absolute; top: 75%; left: 40%;">0</span> <span style="position: absolute; top: 85%; left: 45%;">0</span> <span style="position: absolute; top: 95%; left: 48%;">0</span> </div> <p style="text-align: center;"><math>a = -\sqrt{6}/5</math></p>	<p>for <math>P, \overline{P}_{56}^{(\sigma)} = \overline{P}_{56}^{(\tau)}</math></p> <div style="border: 1px solid black; padding: 10px; margin: 10px auto; width: 350px; height: 200px; position: relative;"> <span style="position: absolute; top: 5px; left: 5px;">1</span> <span style="position: absolute; top: 15%; left: 15%;">7/15</span> <span style="position: absolute; top: 15%; left: 30%;">b</span> <span style="position: absolute; top: 25%; left: 20%;">7/15</span> <span style="position: absolute; top: 25%; left: 35%;">b</span> <span style="position: absolute; top: 35%; left: 30%;">7/15</span> <span style="position: absolute; top: 35%; left: 40%;">b</span> <span style="position: absolute; top: 45%; left: 10%;">b</span> <span style="position: absolute; top: 45%; left: 25%;">1 4/15</span> <span style="position: absolute; top: 55%; left: 15%;">b</span> <span style="position: absolute; top: 55%; left: 30%;">1 4/15</span> <span style="position: absolute; top: 65%; left: 25%;">b</span> <span style="position: absolute; top: 65%; left: 35%;">1 4/15</span> <span style="position: absolute; top: 75%; left: 40%;">0</span> <span style="position: absolute; top: 85%; left: 45%;">0</span> <span style="position: absolute; top: 95%; left: 48%;">0</span> </div> <p style="text-align: center;"><math>b = -7\sqrt{6}/45</math></p>
---	---

The charge-spin wave functions for  $^{31}S$  and  $^{31}D$  states, which belong to partition [42], are obtained from (3) in the text, by interchanging all the charge and spin variables in it. Therefore we obtain the following relations.

$$\begin{aligned}
 \overline{P}_{56}^{(\sigma)} \text{ for } ^{31}S &= \overline{P}_{56}^{(\tau)} \text{ for } ^{13}S, \\
 \overline{P}_{56}^{(\tau)} \text{ for } ^{31}S &= \overline{P}_{56}^{(\sigma)} \text{ for } ^{13}S.
 \end{aligned}
 \tag{A-5}$$

## Appendix B

In this appendix we illustrate the evaluation of the spatial integral in the central force matrix elements, which appeared in § 2, in the text. For that purpose, we first take the relative coordinates as are shown in Fig. B-I. Then defining the position vectors of the center of mass by  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4 + \mathbf{r}_5 + \mathbf{r}_6)/6 = 0$ , the position vectors of the constituent six particles measured from this center of mass are given as follow;

$$\begin{aligned}
 \mathbf{r}_1 &= \mathbf{t}/3 - \mathbf{s}/2 - \mathbf{u}/2, & \mathbf{r}_2 &= \mathbf{t}/3 - \mathbf{s}/2 + \mathbf{u}/2, \\
 \mathbf{r}_3 &= \mathbf{t}/3 + \mathbf{s}/2 - \mathbf{v}/2, & \mathbf{r}_4 &= \mathbf{t}/3 + \mathbf{s}/2 + \mathbf{v}/2, \\
 \mathbf{r}_5 &= -2\mathbf{t}/3 - \mathbf{w}/2, & \mathbf{r}_6 &= -2\mathbf{t}/3 + \mathbf{w}/2.
 \end{aligned}
 \tag{B-1}$$

Correspondingly, we obtain

$$\sum_{i < j=1}^6 r_{ij}^2 = 8t^2 + 6s^2 + 3(u^2 + v^2 + w^2),
 \tag{B-2}$$

which we denote as  $Z^2$ . The volume element of integration is transformed into

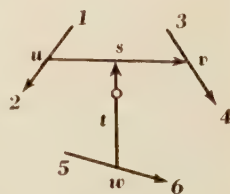


Fig. B-I.



$$d\mathbf{r}_1 d\mathbf{r}_2 \cdots d\mathbf{r}_6 = J_1 d\mathbf{R} du dv dw ds dt, \quad (\text{B-3})$$

where  $J_1$  is the constant factor of the Jacobian.

In this coordinate system, we can write  $(L_{12}|f_c(r_{56})|L_{12})$  as

$$\begin{aligned} (L_{12}|f_c(r_{56})|L_{12})/N &= (V_c J_1/N) \sum (1m_1 1m_2 |LM_i) (1m_1' 1m_2' |L'M_i') \\ &\cdot \int Y_{1m_1}^*(\mathbf{r}_1) Y_{1m_2}^*(\mathbf{r}_2) Y_{1m_1'}(\mathbf{r}_1) Y_{1m_2'}(\mathbf{r}_2) \exp[-2\lambda Z] \exp[-w/r_c] \\ &\cdot (w/r_c) du dv dw \cdots dt. \end{aligned} \quad (\text{B-4})$$

Here we assumed that the central interaction has the Yukawa radial dependence. In this equation, we replace  $\mathbf{r}_1$  and  $\mathbf{r}_2$  by the expression of (B-1), then it becomes

$$\begin{aligned} (L_{12}|f_c(r_{56})|L_{12})/N &= (V_c J_1/N) \sum (1m_1 1m_2 |LM_i) (1m_1' 1m_2' |L'M_i') \\ &\cdot \int \exp[-2\lambda Z] \exp[-w/r_c] (w/r_c) \\ &\cdot \{ (1/3) Y_{1m_1}^*(\mathbf{t}) - (1/2) Y_{1m_1}^*(\mathbf{s}) - (1/2) Y_{1m_1}^*(\mathbf{u}) \} \\ &\cdot \{ (1/3) Y_{1m_2}^*(\mathbf{t}) - (1/2) Y_{1m_2}^*(\mathbf{s}) + (1/2) Y_{1m_2}^*(\mathbf{u}) \} \\ &\cdot \{ (1/3) Y_{1m_1'}(\mathbf{t}) - (1/2) Y_{1m_1'}(\mathbf{s}) - (1/2) Y_{1m_1'}(\mathbf{u}) \} \\ &\cdot \{ (1/3) Y_{1m_2'}(\mathbf{t}) - (1/2) Y_{1m_2'}(\mathbf{s}) + (1/2) Y_{1m_2'}(\mathbf{u}) \} \\ &\cdot du dv \cdots dt. \end{aligned} \quad (\text{B-5})$$

From this form, the merit of our use of the solid harmonics  $Y_{1m_1}(\mathbf{r}_1)$ , instead of the surface harmonics  $Y_{1m_1}(r)$  will be clear, because it permits us to perform any desired coordinate transformations.

(B-5) can be developed as a linear combination of terms, each containing in the integrand four solid harmonics of order one. Abbreviating, e.g., the term which contain  $Y_{1m_1}^*(\mathbf{t}) Y_{1m_1'}(\mathbf{s}) Y_{1m_2}^*(\mathbf{t}) Y_{1m_2'}(\mathbf{s})$  as  $tsst$ , the results of integration over angle variables and summation over magnetic quantum numbers, are written down as

$$\begin{aligned} tttt &= (t^4/(4\pi)^2) (-)^L 3 \sum_{L_\alpha} (L_\alpha 010|10)^2 (2L_\alpha + 1) W(1111; LL_\alpha) \delta_{LL'} \delta_{M_L M_{L'}}, \\ tss &= ssst = (t^2 s^2/(4\pi)^2) \delta_{LL'} \delta_{M_L M_{L'}}, \\ tsst &= stst = (t^2 s^2/(4\pi)^2) 3 \delta_{L,0} \delta_{L',0}, \\ tssst &= ststt = (t^2 s^2/(4\pi)^2) (-)^L \delta_{LL'} \delta_{M_L M_{L'}}. \end{aligned} \quad (\text{B-6})$$

Omitting the Kronecker-delta symbol, except for  $\delta_{L,0} \delta_{L',0}$ , (B-5) now becomes

$$\begin{aligned} (L_{12}|f_c(r_{56})|L_{12})/N &= (V_c J_1/(4\pi)N) \int [3 \sum_{L_\alpha} (L_\alpha 010|10)^2 (2L_\alpha + 1) W(1111; LL_\alpha) \\ &\cdot \{ (1/81) t^4 + (1/16) (s^4 + u^4) \} + \{ (1/18) (t^2 s^2 + t^2 u^2) + (1/8) s^2 u^2 \} \\ &+ \{ (1/18) (t^2 s^2 - t^2 u^2) - (1/8) s^2 u^2 \} \{ (-)^L + 3 \delta_{L,0} \delta_{L',0} \}] du dv \cdots dt. \end{aligned} \quad (\text{B-7})$$

For performing remaining integration, we generalize Irving's method<sup>17)</sup> and make the following transformation of the integral variables,

$$\begin{aligned} w &= R \cos \theta, & v &= R \sin \theta_1 \cos \theta_2, & u &= R \sin \theta_1 \sin \theta_2 \cos \theta_3, \\ s &= \sqrt{1/2} R \sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_4, & t &= \sqrt{3/8} R \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4, \end{aligned} \quad (\text{B-8})$$

then

$$dudvkwdsdt = J_2 R^4 \sin^3 \theta_1 \sin^2 \theta_2 \sin \theta_3 dR d\theta_1 d\theta_2 d\theta_3 d\theta_4, \quad (\text{B-9})$$

where  $J_2$  is the constant factor of the Jacobian.

Now using the notation, defined in (B-11) below, we finally obtain

$$\begin{aligned} (L_{12} | f_c(r_{56}) | L_{12}) / N &= (V_c / N) [46/27 I_1 - 19/18 I_2] && \text{for S-state,} \\ &= (V_c / N) (8/9) I_2 && \text{for P-state,} \\ &= (V_c / N) [92/135 I_1 + 1/9 I_2] && \text{for D-state.} \end{aligned} \quad (\text{B-10})$$

Other matrix elements are calculated similarly, and we find that they are all expressed by only four independent integrals  $I_1$ ,  $I_2$ ,  $I_3$  and  $I_4$ . The results are summarized in Table B-I, and the explicit form of  $I_i$ 's are given below;

Table B-I  
Spatial matrix elements for central forces.

$$(L_{4j} | f(r_{56}) | L_{4j'}) / I^*_{cc} = (ij | i'j')$$

	S	D	P
(12 12)	$46/27 I_1 - 19/18 I_2$	$92/135 I_1 + 1/9 I_2$	$8/9 I_2$
(12 13)	$-8/27 I_1 + 1/3 I_2$	$-16/135 I_1$	$-2/9 I_2$
(12 34)	$10/27 I_1 - 5/18 I_2$	$20/135 I_1 - 1/9 I_2$	0
(12 15)	$-2/27 I_1 + 1/18 I_2$	$-4/135 I_1 - 1/9 I_2$	$-2/9 I_2$
(12 35)	$-2/27 I_1 + 5/18 I_2$	$-4/135 I_1 + 1/9 I_2$	0
(15 15)	$4/27 I_1 + 1/3 I_2 + 5/18 I_3$	$8/135 I_1 + 1/3 I_2 + 5/18 I_3$	$1/3 I_2 + 5/18 I_3$
(15 25)	$4/27 I_1 - 1/9 I_2 - 1/18 I_3$	$8/135 I_1 - 1/9 I_2 - 1/18 I_3$	$-1/9 I_2 - 1/18 I_3$
(15 16)	$4/27 I_1 + 1/3 I_2 - 5/18 I_3$	$8/135 I_1 + 1/3 I_2 - 5/18 I_3$	$1/3 I_2 - 5/18 I_3$
(15 26)	$4/27 I_1 - 1/9 I_2 + 1/18 I_3$	$8/135 I_1 - 1/9 I_2 + 1/18 I_3$	$-1/9 I_2 + 1/18 I_3$
(56 56)	$16/27 I_1 - 2/3 I_3 + 3/16 I_4$	$32/135 I_1 + 3/40 I_4$	$4/9 I_3$
(12 56)	$4/27 I_1 - 1/3 I_2 + 1/6 I_3$	$8/135 I_1$	does not appear
(15 56)	$-8/27 I_1 + 1/6 I_3$	$-16/135 I_1$	$1/9 I_3$

$$\begin{aligned} I_1 &= (4\pi)^{-3} \int r^4 \exp[-2\lambda Z] \exp[-w/r_c] / (w/r_c) d\tau, \\ I_2 &= (4\pi)^{-3} \int r^2 s^2 \exp[-2\lambda Z] \exp[-w/r_c] / (w/r_c) d\tau, \\ I_3 &= (4\pi)^{-3} \int r^2 w^2 \exp[-2\lambda Z] \exp[-w/r_c] / (w/r_c) d\tau, \\ I_4 &= (4\pi)^{-3} \int w^4 \exp[-2\lambda Z] \exp[-w/r_c] / (w/r_c) d\tau, \end{aligned} \quad (\text{B-11})$$

with

$$d\tau = J_2 (4\pi)^5 u^2 \tau^2 v^2 s^2 t^2 du dv d\tau ds dt. \quad (B-12)$$

In expressing the further results of the integration of (B-11), it is convenient to do so in the form  $I_i/N$ , because  $I_i$ 's always appear in this form, and also because  $I_i$  and  $N$  have rather complicated common factors. Therefore it will be appropriate, here, to explain the calculation of  $N$ .

$N$  is defined, e.g., for S and D states, by

$$N_{S \text{ or } D} = 1/\sqrt{9} \left( \sum_i \phi_i \phi_i \middle| \sum_j \phi_j \phi_j \right) = 1/\sqrt{9} \sum_i (\phi_i | \phi_i), \quad (B-13)$$

and can easily be shown to be reduced to the following simple formula

$$N_{S \text{ or } D} = (L_{12} | L_{12}) - 2(L_{12} | L_{13}) + (L_{12} | L_{34}), \quad (B-14)$$

where, e.g.,

$$(L_{12} | L_{13}) = \sum_{m_1 m_2 m_1' m_2'} (1m_1 1m_2 | LM) (1m_1' 1m_2' | LM) \cdot \int Y_{1m_1}^*(\mathbf{r}_1) Y_{1m_2}^*(\mathbf{r}_2) \exp[-2\lambda Z] Y_{1m_1'}(\mathbf{r}_1) Y_{1m_2'}(\mathbf{r}_2) du dv \dots dt. \quad (B-15)$$

Defining two independent integrals  ${}_1I$  and  ${}_2I$  by

$${}_1I = (4\pi)^{-2} \int t^4 \exp[-2\lambda Z] d\tau, \quad {}_2I = (4\pi)^{-2} \int t^2 s^2 \exp[-2\lambda Z] d\tau, \quad (B-16)$$

(B-14) becomes

$$N_S = 8/3 {}_1I - 2{}_2I, \quad N_D = 16/15 {}_1I. \quad (B-17)$$

In a similar way we obtain

$$N_P = (L_{12} | L_{12}) - 2(L_{12} | L_{13}) = 4/3 {}_2I. \quad (B-18)$$

However, actual calculation shows that  ${}_2I = 4/5 {}_1I$ , and we can easily see that

$$N_S = N_P = N_D = N, \quad (B-19)$$

where

$$N = (16/15) {}_1I = (16/15) (4\pi)^3 (3/8)^3 (\sqrt{3}/8) 18! (2\sqrt{3}\lambda)^{-19} (\pi/2)^2 J_1 J_2 / 7 \cdot 9 \cdot \dots \cdot 17. \quad (B-20)$$

The relation (B-19) is the reason, why we have not discriminated the normalization factors for any states, throughout in this paper.

From (B-11) and (B-20) we obtain the following expressions

$$\begin{aligned} I_1/N &= \{(5/96) (2\sqrt{3}\lambda r_c)^{19} 3 \cdot 5 \cdot \dots \cdot 17 / 2 \cdot 4 \cdot \dots \cdot 14\} A, & I_2 &= 4/5 I_1, \\ I_3/N &= \{(5/96) (2\sqrt{3}\lambda r_c)^{19} 3 \cdot 5 \cdot \dots \cdot 17 / 2 \cdot 4 \cdot \dots \cdot 14\} (112/15) B, \\ I_4/N &= \{(5/96) (2\sqrt{3}\lambda r_c)^{19} 3 \cdot 5 \cdot \dots \cdot 17 / 2 \cdot 4 \cdot \dots \cdot 14\} (3584/45) C, \end{aligned} \quad (B-21)$$

where

$$A = \int_0^1 (1-y^2)^7 y^7 dy, \quad B = \int_0^1 (1-y^2)^6 y^3 dy, \quad C = \int_0^1 (1-y^2)^5 y^5 dy, \quad (\text{B-22})$$

with

$$\ell = 2\sqrt{3}\lambda r_c. \quad (\text{B-23})$$

### Appendix C

In this appendix we show the explicit evaluation of (58) in the text. Using the coordinate system given by (B-1), and further replacing  $\mathbf{s}$  and  $\mathbf{t}$  by

$$\mathbf{s}' = \sqrt{2} \mathbf{s} \quad \text{and} \quad \mathbf{t}' = \sqrt{8/3} \mathbf{t}, \quad (\text{C-1})$$

we have

$$\sum_{i=1}^6 \mathcal{A}_i = 2(\mathcal{A}_s + \mathcal{A}_t + \mathcal{A}_u + \mathcal{A}_v + \mathcal{A}_w), \quad (\text{C-2})$$

and

$$Z^2 = 3Z'^2 = 3(t'^2 + s'^2 + u^2 + v^2 + w^2). \quad (\text{C-3})$$

Expressing  $\mathcal{A}_u$  as a scalar product by

$$\mathcal{A}_u = \sum_m (-)^m \mathcal{P}_m^{(u)} \mathcal{P}_{-m}^{(u)}, \quad (\text{C-4})$$

and using the relation

$$\Gamma_m^{(u)} f(u) Y_{lm'}(u) = \sum_{L_\alpha M_\alpha} (-)^1 (L_\alpha 0 1 0 | L 0) (l m' 1 m | L_\alpha 1 L_\alpha) Y_{L_\alpha M_\alpha}^{*}(u) D^{L_\alpha} f_l(u), \quad (\text{C-5})$$

with

$$D^{[l+1]} = (d/du) - (l/u) \quad \text{and} \quad D^{[l-1]} = (d/du) + (l+1/u). \quad (\text{C-6})$$

we can perform the differentiations.

If we write, e.g.,

$$\sum (1m_3 1m_6 | LM_l) (1m_3' 1m_6' | LM_l) Y_{1m_3}^{*}(\mathbf{w}) Y_{1m_6}^{*}(\mathbf{w}) \mathcal{A}_u Y_{1m_3'}(\mathbf{w}) Y_{1m_6'}(\mathbf{w}) \exp[-\sqrt{3}\lambda Z'], \quad (\text{C-7})$$

as  $(\hat{u}\hat{v}|\mathcal{J}_u|\hat{u}\hat{v})$  and perform differentiation, as in (C-5), and integration over angle variables and summation over magnetic quantum numbers, we see that the seven independent terms which are necessary in expressing the kinetic energy matrix elements are given in the following formulae.

$$\begin{aligned} & (\hat{u}\hat{v}|\mathcal{J}_u|\hat{u}\hat{v}) \\ &= (w^4/(4\pi)^2) (-)^1 3 \sum_{L_\alpha} (L_\alpha 0 1 0 | 1 0)^2 (2L_\alpha + 1) W(1111; LL_\alpha) \\ & \quad (\partial/\partial u + 2/u) (\partial/\partial u) \exp[-\lambda\sqrt{3}Z'] \delta_{LL'}, \end{aligned}$$





$$Y_{20}(\mathbf{r}_i) = \sqrt{10\pi/3} \sum_{mm'} (1m1m'|20) Y_{1m}(\mathbf{r}_i) Y_{1m'}(\mathbf{r}_i), \quad (\text{D-1})$$

then there appears quite many terms, which contain six solid harmonics of order one in the integrand.

A typical term, which we abbreviate as  $sstttt$  is

$$\begin{aligned} sstttt &= Y_{1m_1}^*(\mathbf{s}) Y_{1m_2}^*(\mathbf{t}) Y_{1m_7}(\mathbf{t}) Y_{1m_8}(\mathbf{t}) Y_{1m_1'}(\mathbf{s}) Y_{1m_2'}(\mathbf{t}) \sqrt{4\pi/5} \sqrt{10\pi/3} \\ &\quad \cdot (1m_7 1m_8 | 20) (1m_1 1m_2 | LM_L) (LM_L SM_S | JM) (1m_1' 1m_2' | L'M_L') \\ &\quad \cdot (L'M_L' SM_S | JM) \\ &= Y_{1m_1}^*(\mathbf{s}) Y_{1m_1'}(\mathbf{s}) Y_{1m_2}^*(\mathbf{t}) Y_{1m_2'}(\mathbf{t}) Y_{1m_7}(\mathbf{t}) Y_{1m_8}(\mathbf{t}) \cdot (\dots\dots\dots) \\ &= (s^2 t^4 / (4\pi)^2) (-)^{-L-M+S+1} \sqrt{(6/5) (2L+1) (2L'+1) (2J+1) (2J'+1)} \\ &\quad \cdot W(1L' 12; 1L) W(LJL'J'; S2) V(JJ'2, -MM'0) \delta_{LL'} = (-)^{-L} t^4 s^2 Q_1, \end{aligned} \quad (\text{D-2})$$

By this formula a symbol  $Q_1$  is defined.

Other independent terms, which appear in the calculation of matrix elements of (65) in the text, are evaluated similarly and summarized in the following formulae, in which symbols  $Q_2, Q_3, \dots, Q_9$  are defined. It should be noted that the order of appearance of the letters  $t, s$  and  $w$ , in these formulae have essential meaning, as is seen, e.g., from (D-2).

$$\begin{aligned} tssst &= (-)^{-L} t^4 s^2 Q_1, \\ tssst &= t^4 s^2 Q_1, \\ stsst &= (-)^{-L-L'} Q_1, \\ ttttt &= (t^6 / (4\pi)^2) (9 / \sqrt{5}) (-)^{L-M'+S} \sum_{L_\alpha L_\beta} (1010 | L_\alpha 0) (L_\alpha 010 | L_\beta 0) (L_\beta 010 | 20) \\ &\quad \cdot \sqrt{(2L_\alpha+1) (2L_\beta+1) (2L+1) (2L'+1) (2J+1) (2J'+1)} W(1L L_\alpha L_\beta; 1L) \\ &\quad \cdot W(L_\beta 21 L'; 1L) W(LJL'J'; S2) V(JJ'2, -MM'0) \delta_{LL'} = t^6 Q_2, \\ tstst &= (t^4 s^2 / (4\pi)^2) (-)^{1-J'+M} (3 \sqrt{2} (2J+1) / 5) V(JJ'2, -MM'0) \delta_{L'0} \delta_{J,1} \delta_{L,2} \\ &\quad = t^4 s^2 Q_3 \delta_{L',0} \delta_{J,1} \delta_{L,2}, \\ ststst &= t^4 s^2 Q_3 \delta_{L,0} \delta_{J,1} \delta_{L',2}, \\ tttwtw &= tttwtwt = (t^4 w^2 / (4\pi)^2) 3 \sqrt{5/2} \sum_{L_\alpha L_\beta} (1010 | L_\alpha 0) (L_\alpha 010 | L_\beta 0) \\ &\quad \cdot (L_\beta 010 | 00) (-)^{L_\alpha-1+S-M'} \\ &\quad \cdot \sqrt{(2L_\alpha+1) (2L_\beta+1) (2L+1) (2L'+1) (2J+1) (2J'+1)} \\ &\quad \cdot W(111 L_\beta; L L_\alpha) W(1L' L_\beta 2; 1L) W(LJ' L'J; S2) V(JJ'2, -MM'0) \delta_{LL'} \\ &= t^4 w^2 Q_4, \\ tttwtw &= tttwtwt = (t^4 w^2 / (4\pi)^2) 3 \sqrt{5/2} \sum (1010 | L_\alpha 0) (L_\alpha 010 | L_\beta 0) \end{aligned}$$

$$\begin{aligned}
 & \cdot (L_{\beta}010|00) (-)^{L_{\beta}+S-M} \\
 & \cdot \sqrt{(2L_{\alpha}+1)(2L_{\beta}+1)(2L+1)(2L'+1)(2J+1)(2J'+1)} \\
 & \cdot W(111L_{\beta}; L' L_{\alpha}) W(1LL_{\beta}2; 1L') W(LJL'J'; S2) V(JJ'2, -MM'0) \delta_{LL'} \\
 & = t^4 \tau^2 Q_5,
 \end{aligned}$$

$$\begin{aligned}
 sst\tau w\tau w &= sst\tau w\tau w = ss\tau w\tau w = ss\tau w\tau w \\
 &= (s^2 t^2 \tau^2 w^2 / (4\pi)^2) (-)^{1+S-M'} \sqrt{(5/6)(2L+1)(2L'+1)(2J+1)(2J'+1)} \\
 & \cdot W(1L1L'; 12) W(LJL'J'; S2) V(JJ'2, -MM'0) \delta_{LL'} = s^2 t^2 \tau^2 w^2 Q_6,
 \end{aligned}$$

$$\begin{aligned}
 \tau w t t t w &= \tau w t t t w = (t^4 \tau w^2 / (4\pi)^2) 3 \sqrt{5/2} \sum_{L_{\alpha} L_{\beta}} (1010|L_{\alpha}0) \\
 & \cdot (L_{\alpha}010|L_{\beta}0) (L_{\beta}010|00) (-)^{L_{\beta}+L'+S-M} \\
 & \cdot \sqrt{(2L_{\beta}+1)(2L'+1)(2J+1)(2J'+1)} \\
 & \cdot W(1L'L_{\beta}L; 12) W(LJL'J'; S2) V(JJ'2, -MM'0) \delta_{LL'} = t^4 \tau w^2 Q_7,
 \end{aligned}$$

$$\begin{aligned}
 \tau w t t t t w &= \tau w t t t t w \\
 &= (t^4 \tau w / (4\pi)^2) 3 \sqrt{5/2} \sum_{L_{\alpha} L_{\beta}} (1010|L_{\alpha}0) (L_{\alpha}010|L_{\beta}0) (L_{\beta}010|00) \\
 & \cdot \sqrt{(2L_{\alpha}+1)(2L_{\beta}+1)(2L+1)(2L'+1)(2J+1)(2J'+1)} (-)^{1+L+S-M} \\
 & \cdot W(1L'L_{\alpha}L_{\beta}; 11) W(1L1L'; 12) W(LJL'J' S2) V(JJ'2; -MM'0) \delta_{LL'} \\
 & = t^4 \tau w^2 Q_8,
 \end{aligned}$$

$$tsswwt = tsswwt = (-)^{-L} s^2 t^2 \tau w^2 Q_9,$$

$$\begin{aligned}
 stswwt &= stswwt \\
 &= (s^2 t^2 \tau w^2 / (4\pi)^2) \sqrt{(2J'+1)/2} V(JJ'2, -MM'0) \\
 & \cdot (-)^{-J+M'+1} \delta_{L,0} \delta_{L',2} = s^2 t^2 \tau w^2 Q_9 \delta_{L,0} \delta_{L',2}.
 \end{aligned} \tag{D-3}$$

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## Note added in proof

1) After this work has been submitted to publication, the authors knew that a similar calculation had already be performed by a Chinese author, as early as in 1950, although his calculation had been limited to the central force only: S. N. King, *Chinese Jour. Phys.* **7** (1950), 445.

2) Recently similar calculation has also be done by J. Irving and D. S. Schonland (to be published). We wish to thank them for their courtesy in sending us a pre-publication copy of their paper.

## Letters to the Editor

### A Generalization of the Foldy Transformation in Ps-Ps Meson Theory

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Two of the present authors (S.T. and K.Y.) studied<sup>1)</sup> the limitation of the solution to the one-body problem in the whole scheme of quantized fields, when a fermion is interacting with boson fields. If one makes use of the results obtained by them, it is possible to compute the correction terms for the one-body solution coming from the effects of vacuum fermions, although it is difficult and yet no progress has been made in the direct attack of the many-body problem.

In this note we will report on the one-body solution for a fermion interacting with the neutral pseudoscalar meson field by ps-coupling, in which a generalization of the Foldy transformation<sup>2)</sup> hitherto used is considered.

The physical meaning of the Foldy transformation will be revealed by the following considerations. The effective mass of a physical nucleon becomes heavier by taking into account the effect of the attached meson field. In the approximation which deals with the meson field variable as a classical quantity, the quantum mechanical effects of the meson field are disregarded and the Hamiltonian for the system under consideration is given by

$$H_N = M\rho_3 + \rho_1(\sigma \mathbf{P}) + g\rho_2\phi(X), \quad (1)$$

where  $X$  is the coordinate of the nucleon.

If we disregard the recoil effects, too, the above Hamiltonian can be made diagonal by means of the Foldy transformation, and the transformed Hamiltonian is given by

$$U_F^{-1}H_N U_F = \sqrt{M^2 + g^2\phi(X)^2} \cdot \rho_3 + \rho_1 U_F^{-1}(\sigma \mathbf{P}) U_F, \quad (2)$$

with

$$U_F = \exp\left[\frac{i}{2} \tan^{-1}\left(\frac{g}{M} \rho_1 \phi(X)\right)\right],$$

where the second term in (2) is responsible for the recoil effects. The original mass  $M$  becomes the effective mass  $M^*$  by including the attached meson fields, i.e.

$$M \rightarrow M^* = \sqrt{M^2 + g^2\phi(X)^2}. \quad (3)$$

Incidentally, the inverse Dyson transformation, which was first considered by Sawada and Takagi<sup>3)</sup> and has been applied to the computation of the effective Hamiltonian by Drell and Henley,<sup>4)</sup> is a good approximation to the Foldy transformation, when the coupling constant is small, as it has been discussed by Akiba and Sawada.<sup>5)</sup>

On the other hand, if we do not wish to make any approximation we must proceed with the total Hamiltonian, in which the energy operator for the meson field is included, namely,

$$H_T = H_N + H_M \quad (4)$$

where  $H_M$  is the Hamiltonian for the free meson field,

$$H_M = \frac{1}{2} \int dV \left\{ \pi(x)^2 + (\nabla \phi(x))^2 + \mu^2 \phi(x)^2 \right\}. \quad (5)$$

If we perform the Foldy transformation which is valid in the sense mentioned above, the total Hamiltonian becomes

$$\begin{aligned} U_F^{-1}H_T U_F &= M^* \rho_3 + \rho_1(\sigma \mathbf{P}) \\ &+ g \left\{ \frac{M}{4M^{*2}} (\sigma \nabla \phi) + (\sigma \nabla \phi) \frac{M}{4M^{*2}} \right\} \\ &+ H_M + g \left\{ \frac{M}{4M^{*2}} \rho_1 \vec{\pi} + \rho_1 \vec{\pi} \frac{M}{4M^{*2}} \right\} \\ &+ g^2 \int \left\{ \frac{M}{M^{*2}} \delta(X' - X) \right\}^2 dX'. \end{aligned} \quad (6)$$

In this well known result, it is noticeable that the time-component of the effective pv-coupling is born out of the transformed meson Hamiltonian.

Now, it is desirable to obtain the representation of the total Hamiltonian, from which a greater part of the odd term (associated with the transition of the nucleon from a positive energy state into a negative energy state and vice versa) is eliminated. In such a representation, we can safely adopt the non-relativistic approximation for the nucleon and investigate the meson-nucleon interaction in the low-energy region, although the four-dimensional invariance of the transformed Hamiltonian is lost.\* It is sensible for us to take into account the quantum mechanical effects of the meson field but to neglect the recoil effects, so that we may go beyond the limit of approximation which is set when the Foldy transformation is derived. Indeed, the main odd term which is still existing after the Foldy transformation, namely the time-component of the effective  $\rho$ -coupling, comes from the meson Hamiltonian; on the other hand, the recoil effects yield the space-components of the effective  $\rho$ -coupling, which are mainly even terms. Hence it is essential to take the meson Hamiltonian into account, when we try to reduce the odd term still smaller.

If we neglect the recoil effects, we should try to make the following Hamiltonian diagonal by means of a suitable transformation

$$H = H_0 + H' \quad (7)$$

where  $H_0$  is the sum of the nucleon mass and the energy of the meson field

$$H_0 = M\rho_3 + H_M$$

which is diagonal and even (i.e., free from Dirac matrices  $\rho_1$  or  $\rho_2$ ), while  $H'$  is the interaction energy and is nondiagonal and odd,

$$H' = g\rho_2\phi(X).$$

We assume that it is convenient to make the 'Ansatz' of half-arctangent type, and will try to find the transformation function including the next correction term to the Foldy transformation, which is required by addition of  $H_M$ . Then we put the transformation function into the form

$$U = \exp((i/2) \cdot \tan^{-1} G) \quad (8)$$

where  $G$  should be given by the solution of the equation<sup>6)</sup>

$$[H_0, iG] = -2H' - 1/2 \cdot \{ \{ H' G \} G \} \\ + 1/2 \cdot [ [ H', (1+G^2)^{1/2} ], (1+G^2)^{1/2} ]. \quad (9)$$

(curled bracket means to take the anti-commutator)

Eq. (9) can be solved exactly, irrespective of the magnitude of the coupling constant  $g$ , up to the first order in the expansion into powers of  $(\bar{\omega}/M)$  where  $\bar{\omega}$  is the average frequency of mesons which is relevant to our consideration; in other words, we can find out the solution to eq. (9) in a closed form, if we confine ourselves to take those terms only in which  $\pi(\pi=\dot{\phi})$  appears at most once. The result is written down as

$$G = (g/M)\rho_1\phi(X) - (g/2M)\rho_2[(\pi(X)/M) \\ + \sum_{n=1}^{\infty} (g/M)^{2n} \{ \{ \{ \pi(X)/M \cdot \phi \} \phi \} \cdots \phi \} \}]. \quad (10)$$

*2n anti-brackets*

In the present stage of the development of computational techniques we can carry out the ordering procedure only for those operators, which can be expressed as functions of a linear form of creation-annihilation operators, i.e., a linear form of  $\phi$  and  $\pi$ . In view of this circumstance, it is sensible to put the operator  $G$  into the form

$$G = \lambda\rho_1\phi(X) - \mu\rho_2\pi(X) \quad (11)$$

instead of (10), and to search for the most favorable value of  $\lambda$  and  $\mu$ . If it is allowed to approximate the effects due to pairs of  $\phi$  by multiplying by the vacuum expectation value of these pairs of  $\phi$ , then the replacement of eq. (10) by eq. (11) may be justified.  $\lambda$  and  $\mu$  should be chosen in such a way that the odd terms in the transformed Hamiltonian become ineffective as possible.

It is possible that in the case of charge-symmetrical interaction the term which is proportional to  $(\tau \cdot [\phi \times \pi])$  in the transformed Hamiltonian has a larger effective coupling parameter; this fact is favourable in the discussion of S-wave interaction of nucleon-meson system.

Detailed calculations and the examination of the results will be published in our forthcoming paper.

\*) The failure of the four-dimensional invariance cannot be avoided. So long as we resort to the Hamiltonian formalism and try to solve a dynamical problem, we must introduce a surface normal to the three-dimensional surface which we adopt in the four-dimensional formulation.

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## Notes on the Unstable Particles

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Among the various decay processes of the unstable particles which have recently been observed, there seem to exist universally certain interactions which have peculiar regularity. As one of such interactions we have taken out weak Boson-Fermion interaction (W B-F I) in the preceding letter,<sup>1)</sup> while we may expect weak Fermi interaction also to stand the above criterion. In order to test the possible universality of these interactions, we have studied, in this note, their predictions on the possible competition between decay and nuclear capture of unstable particles with negative charge, when they stop in dense materials.

First, we shall study the competition due to W B-F I, by which a fermion can interact with spin 0 boson such as  $\pi$ -meson having strong coupling with nucleon. Therefore a fermion with negative charge, when brought into nuclear Coulomb field, is able to be absorbed by nucleus through the mediation of the boson ( $\pi$ -meson) field (see, Fig. 1). Such a study is also done assuming interactions in the case where unstable particles are known to decay into  $\pi$ -mesons,

but its precise form of interaction is not well established as in the case of  $\tau$ -meson (see, Fig. 2). Secondly, we take account of Fermi interaction by which an unstable fermion bound near nucleus is able to decay into three lighter ones or is absorbed by nucleus (see, Fig. 3) as is familiar to us in the case of  $\mu$ -meson. We summarize our analysis as follows:

(I) *Interactions* We take following four weak couplings which cause the decay of unstable particles.

(1) W B-F I

$$H' = g \bar{\psi}_a \gamma_5' \gamma_\mu \psi_b \partial_\mu \phi_\pi + \text{c.c.}^*$$

where  $\gamma_5' = 1$  or  $\gamma_5$  to make the interaction Lorentz invariant.

(2) Assumed type of interaction among three bosons.

$$H' = g \phi_a \phi_b \phi_\pi + \text{c.c.}$$

(3) Interaction of  $\tau \rightarrow 3\pi$  decay.

$$H' = g \phi_\tau \phi_\pi \phi_\pi \phi_\pi + \text{c.c.}$$

(4) Weak Fermi interaction.<sup>2)</sup>

$$H' = g (\bar{\psi}_a O \psi_b) (\bar{\psi}_c O \psi_d) + \text{c.c.}$$

The capture process due to the interactions (1), (2) and (3) needs another interaction, that is, strong interaction between  $\pi$ -meson and nucleon. We assume for it the pseudo-scalar type with coupling constant  $G$ .

(II) *Numerical results.* Calculated results on some possible reactions are presented in tables. In Table 1, the ratio  $\rho$  of capture to decay is given for unstable particles with assumed masses in the case where these particles come into  $K$ -orbit of nuclear Coulomb field. Effective nuclear charge  $Z_{eff}$ <sup>3)</sup> is replaced by that in extremely heavy ( $R \rightarrow \infty$ ) nuclei, except  $\mu$ -meson case. This procedure is not so far from reality in the case of heavy nuclei such as

\* Numerical value of this  $g$  is to be referred to A.<sup>1)</sup>

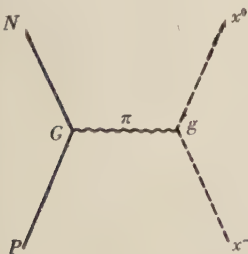


Fig. 1

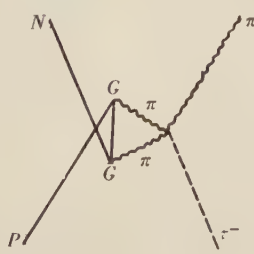


Fig. 2

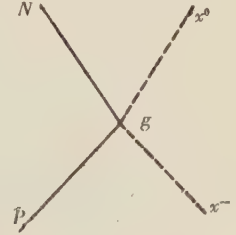


Fig. 3

Table 1.

Particle	mass ( $m_e$ )	decay scheme	capture scheme	$\rho (G^2 \cdot Z/A)$	interaction	recoil energy to neutron E (Mev)
$\Lambda^-$	2380	$\Lambda^- \rightarrow N + \pi^-$	$\Lambda^- + P \rightarrow N + N$	$1.2 \times 10^{-1}$	(1) $\gamma_5' = \gamma_5$	137
$\Lambda^-$	"	"	"	$2.2 \times 10^{-2}$	(1) $\gamma_5' = 1$	"
$\Sigma^-$	1400	$\Sigma^- \rightarrow \pi^- + \nu$	$\Sigma^- + P \rightarrow N + \nu$	$1.7 \times 10^{-2}$	(1)	154
$\Sigma^-$	1200	"	"	$2.1 \times 10^{-2}$	(1)	120
$\Sigma^-$	1400	$\Sigma^- \rightarrow \pi^- + \pi^0$	$\Sigma^- + P \rightarrow N + \pi_0$	$1.2 \times 10^{-2}$	(2)	148
$\Sigma^-$	1200	"	"	$1.3 \times 10^{-2}$	(2)	114
$\tau^-$	970	$\tau^- \rightarrow 3\pi$	$\tau^- + P \rightarrow N + \pi^0$	$1.3 \times 10^2 \cdot G^2$	(3)	99
$\mu^-$	207		$\mu^- + P \rightarrow N + \nu$	$0.3 \cdot G^2 \cdot (Z_{eff})^4$	(1)	8

Table 2.\*

particle	mass ( $m_e$ )	decay scheme $a \rightarrow b + c + d$	decay probability ( $\eta^2/\text{sec}$ )	capture scheme $a + c \rightarrow b + d$	capture probability ( $\eta^2/\text{sec}$ ) $\times (Z/A)$
$\Sigma^-$	1400	$\Sigma^- \rightarrow \mu + \nu + \nu$	$(3.3 + 2.9K) \times 10^8$	$\Sigma^- + P \rightarrow N + \nu$	$1.2 \times 10^{10} \{ (1 + K_1) + 0.51K_2 + 0.86K_3 \}$
$\Sigma^-$	"	"	"	$\Sigma^- + P \rightarrow \Lambda^0 + \nu$	$0.8 \times 10^{10} \{ (1 + K_1) + 0.38K_2 + 0.93K_3 \}$
$\Sigma^-$	1200	"	$(1.2 + 1.5K) \times 10^8$	$\Sigma^- + P \rightarrow N + \nu$	$0.82 \times 10^{10} \{ (1 + K_1) + 0.46K_2 + 0.89K_3 \}$
$\Sigma^-$	"	"	"	$\Sigma^- + P \rightarrow \Lambda^0 + \nu$	$0.62 \times 10^{10} \{ (1 + K_1) + 0.32K_2 + 0.95K_3 \}$
$\Lambda^-$	2380	$\Lambda^- \rightarrow N + e + \nu$	$(3.1 + 2.9K) \times 10^7$	$\Lambda^- + P \rightarrow N + N$	$2.8 \times 10^9 \{ (1 + K_1) + 0.14K_2 + 0.99K_3 \}$
$\mu^-$	207	$\mu^- \rightarrow e + \nu + \nu$	$2.8 \times 10^4$	$\mu^- \rightarrow e + P \rightarrow N + N$	$5.9 \cdot 10^{-10} \cdot \eta^2 \cdot (Z_{eff})^4 \cdot \{ (1 + K_1) + 0.1K_2 + 0.9K_3 \}$

Ag and Br, and is used also for Table 2. Although our results in the first table somewhat lose clearness due to the appearance of strong-coupling constant  $G$  which does not always make the perturbation treatment correct, the capture probabilities other than in the case of  $\tau$ -meson seem to be much smaller than that of decay, even in heavy nuclei, as far as  $G^2 \ll 10^2$ . With this coupling constant  $\mu$ -meson capture\* is found not to fit the experiment if we use only interaction (1). The large value  $\rho$  in the case of  $\tau$ -meson is mainly caused by the small decay probability which is due to small  $Q$ -value in decay process.

In the Table 2, we see the calculated results on Fermi interaction. In this table, equating coupling constant  $g$  in capture to that in decay, we find the capture probability of  $\mu$ -meson is somewhat larger than that experimentally given, even if we take account of the reduction factor  $\sim 1/4^{(3)}$  due to Pauli principle effective to final neutrons. If we take the coupling constants separately in decay and in capture to gain the coincidence with the experiment in  $\mu$ -mesic case, then we find, in heavy nuclei, the capture probabilities of other heavy unstable particles with

negative charge are of the same order of magnitude as those of decay.

Now, we shall turn our attention to the coupling constants of the so-called weak interactions (1) - (4). These coupling constants are widely different from each other in their numerical values and dimensions. However, if we introduce  $\gamma_0$  having dimension of length as another universal constant besides  $\hbar$  and  $c$ , then we can bring the above coupling constants into the same dimension by the multiplication of the appropriate power of these three constants and we find these coupling constants almost coincide with each other in order of magnitude if we use  $\gamma_0 = 10^{-13} \text{cm}$ . Full account will be published elsewhere in this journal.

We wish to express our hearty thanks to the group of Nagoya and Kanazawa University for their helpful discussions.

\*  $K_0 = g_1^2 + 4g_2^2 + 6g_3^2 + 4g_4^2 + g_5^2 = \eta^2 \cdot (10^{-49} \text{erg} \cdot \text{cm}^3)^2$ ,

$K = (g_1^2 - 2g_2^2 + 2g_4^2 - g_5^2) / K_0$ ,

$K_1 = (2g_1g_2 - 6g_2g_3 + 6g_3g_4 - 2g_4g_5) / K_0$ ,

$$K_2 = (2g_2^2 + 4g_3^2 + 2g_4^2 - 2g_1g_3 - 4g_2g_4 - 2g_3g_5 + 2g_1g_5 - 6g_2g_3 + 6g_4g_5 - 2g_3g_5) / K_0,$$

$$K_3 = (g_1^2 - 2g_2^2 + 2g_4^2 - g_5^2 + 2g_1g_2 + 6g_2g_3 + 6g_3g_4 + 2g_4g_5) / K_0.$$

- 1) S. Ogawa, H. Okonogi and S. Oneda, Prog. Theor. Phys. **11** (1954), 330, cited as A. K. Iwata, S. Ogawa, H. Okonogi, B. Sakita and S. Oneda, *ibid* to be published
- 2) L. Michel, Proc. Phys. Soc. **83** (1950), 514 should be referred to. Signs of  $g_2$  and  $g_5$  in our Table 2 are inversed to those given by him.
- 3) J. A. Wheeler, Rev. Mod. Phys. **21** (1949), 133.

## Zero-energy Mesonic Processes and Renormalization in D.T.G.'s Formalism

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Recently, Deser, Thirring and Goldberger<sup>1)</sup> (D. T.G.) have proposed a new method of charge renormalization and treated the meson problems in low energy limit with zero external meson mass. Their proposal, however, does not seem to be so significant as compared with other definitions of mesic charge. We shall here treat the same problem with consistent use of Dyson's prescriptions.

The modified propagation function of the nucleon moving in the constant external meson field  $a_t$  (symmetrical pseudoscalar) can be written as follows (in the notations similar to D. T. G.'s):

$$\begin{aligned} S'^{-1}(p, a) = & i\gamma p \gamma_5 (p^2, a^2) + m f(p^2, a^2) + i a \tau \gamma_5 \\ & \times d(p^2, a^2) + (i\gamma p + m) i a \tau \gamma_5 (i\gamma p + m) a(p^2, a^2) \\ & + \{i\gamma p + m, i a \tau \gamma_5\} h(p^2, a^2). \end{aligned} \quad (1)$$

This form of propagation function is different from D.T.G.'s, but is required for the correct definition of  $Z_1$ . Also, this form is of the expansion in powers of  $(i\gamma p + m)$  in place of  $(i\gamma p + m + i a \tau \gamma_5)$ , because the renormalization constants should be determined as the quantities to be renormalized for the particle which is perfectly free. Further, we assume that, in the functions  $g(p^2, a^2)$ ,  $f(p^2, a^2)$ ,  $d(p^2, a^2)$ ,  $a(p^2, a^2)$  and  $h(p^2, a^2)$ , the mass renormalization is already performed for the self-energy insertions, but not for the final self-energy divergence.

The self-mass  $\Sigma_0$  is given by

$$\Sigma_0 = m[f(-m^2, 0) - g(-m^2, 0)], \quad (2)$$

and the renormalization constants  $Z_1$  and  $Z_2$  are determined by the requirements

$$\begin{aligned} \bar{\psi}(p) \lim_{\alpha \rightarrow 0} 1/i \cdot \partial / \partial a_k S'^{-1}(p, a) \psi(p) \\ = Z_1^{-1} \bar{\psi}(p) \tau_k \gamma_5 \psi(p), \end{aligned} \quad (3)$$

$$\begin{aligned} \bar{\psi}(p) \lim_{\alpha \rightarrow 0} 1/i \cdot \partial / \partial p_\mu S'^{-1}(p, a) \psi(p) \\ = Z_2^{-1} \bar{\psi}(p) \gamma_\mu \psi(p), \end{aligned} \quad (4)$$

where  $\bar{\psi}(p)$  and  $\psi(p)$  are Dirac spinors satisfying the free Dirac equations. In the following, we write

$$\lim_{\alpha \rightarrow 0} g(p^2, a^2) = g(p^2), \dots$$

Using eq. (1), we obtain

$$Z_1^{-1} = d(-m^2), \quad (5)$$

$$Z_2^{-1} = g(-m^2) + 2m^2[f'(-m^2) - g'(-m^2)], \quad (6)$$

where

$$f'(-m^2) = [\partial f(p^2) / \partial p^2]_{p^2+m^2=0}, \dots$$

In the limit of  $\alpha=0$ , the propagation function (with self-mass term subtracted) and the vertex operator are given by

$$\begin{aligned} S'^{-1}(p) - \Sigma_0 = & Z_2^{-1} [i\gamma p G(p^2) + m(F(p^2) - F(-m^2) \\ & + G(-m^2))], \end{aligned} \quad (7)$$

$$\begin{aligned} \Gamma_{5k}(p, p) = & \lim_{\alpha \rightarrow 0} 1/i \cdot \partial / \partial a_k S'^{-1}(p, a) \\ = & Z_1^{-1} [\tau_k \gamma_5 D(p^2) + (i\gamma p + m) \tau_k \gamma_5 (i\gamma p + m) A(p^2) \\ & + \{i\gamma p + m, \tau_k \gamma_5\} H(p^2)], \end{aligned} \quad (8)$$

and further,

\* Numerical results in A, page 331, are erroneous.

$$\lim_{a \rightarrow 0} 1/i \cdot \partial/\partial a_i \Gamma_{ik}(p, p; a) = -2\delta_{ik} Z_2^{-1} [i\gamma^0 \dot{G}(p^2) + m \dot{F}(p^2)], \quad (9)$$

where  $G(p^2)$ ,  $F(p^2)$ ,  $D(p^2)$ ,  $A(p^2)$  and  $H(p^2)$  are finite functions defined by

$$\begin{aligned} G(p^2) &= Z_2 g(p^2), \quad F(p^2) = Z_2 f(p^2), \\ D(p^2) &= Z_1 d(p^2), \quad A(p^2) = Z_1 a(p^2), \\ H(p^2) &= Z_1 h(p^2), \end{aligned} \quad (10)$$

and

$$\dot{G}(p^2) = Z_2 \lim_{a \rightarrow 0} \partial_g(p^2, a^2)/\partial a^2 \dots$$

Applying the above formalism to the meson-nucleon scattering in the limit of vanishing mass and momentum of external mesons, we obtain the scattering amplitude:

$$\begin{aligned} T_{ji} &= g_1^2 1/m \cdot [(1 + 2mH(-m^2))^2/G(-m^2) + \\ &\quad 2m^2(\dot{F}(-m^2) - \dot{G}(-m^2))] \\ &\quad \times \phi_i^{(-)}(j) \phi_i^{(+)}(i) \bar{\psi}(p) \psi(p), \end{aligned} \quad (11)$$

where  $i$  and  $j$  denote the charge states of the incident and scattered mesons and  $g_1$  the renormalized coupling constant.

In eq. (11) we see that, contrary to the D.T.G.'s treatment, the damping factor  $G(-m^2)$  originated from the virtual nucleon-pair effect is not canceled by the vertex contribution  $1 + 2mH(-m^2)$ . Expanding the above quantities up to the second-order of the coupling constant, we obtain the expression

$$\begin{aligned} T_{ji} &= g^2 \cdot 1/m \cdot [(1 + 2g^2/(4\pi)^2)/\{1 + 3g^2/(4\pi)^2\} \\ &\quad - 6g^2/(4\pi)^2] \phi_i^{(-)}(j) \phi_i^{(+)}(i) \bar{\psi}(p) \psi(p) \end{aligned} \quad (12)$$

in exact agreement with the ordinary perturbation theory<sup>2)</sup>.

In the same way, we can treat the photo-production of mesons at threshold and the Compton scattering in the Thomson limit. We have obtained the result that the transition matrices for these processes are exactly the same as the second-order perturbation results, as shown previously by Kroll and Ruderman<sup>3)</sup>.

1) Deser, Thirring and Goldberger, Phys. Rev. **94** (1954), 711

- 2) Brueckner, Gell-Mann and Goldberger, Phys. Rev. **90** (1953), 476; Ashkin, Simon and Marshak, Prog. Theor. Phys. **5** (1950), 634.
- 3) Kroll and Ruderman, Phys. Rev. **93** (1954), 233

## Zero-energy Mesonic Processes and Renormalization in Schwinger's Formalism

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In the preceding note<sup>1)</sup> two of us (C.N.) have pointed out that the procedure of renormalization given by Deser, Thirring and Goldberger<sup>2)</sup> is unsatisfactory in view of the fundamental concept of charge renormalization established by Dyson, and that even in their formalism one can obtain the reasonable results by faithful applications of Dyson's prescription. Here we shall propose an alternative method directly based on the formalism of Schwinger<sup>3)</sup> rather than D.T.G. and on the Lagrangian<sup>4)</sup> written in terms of the formally renormalized Heisenberg operators and the renormalization constants. By the finite propagator obtained in this way, one may calculate the matrix elements for meson-nucleon scattering, threshold photomeson production and Compton scattering in the limit of vanishing mass and momentum of external mesons.

We shall start with the following equations of the renormalized Heisenberg operators  $\psi$  and  $\phi_i$  containing the observed mass  $m$  and charge  $g$  and the renormalization constants  $\delta m$ ,  $Z_1$  and  $Z_2$ :



$$\begin{aligned}
& (\gamma\partial + m - \delta m)\psi = i g^2(Z_1/Z_2)\gamma_5 \tau_i \phi_i \psi, \\
& \{\psi_\alpha(x, t), \psi_\beta(x', t)\} = 1/Z_2 \cdot \gamma_{4\alpha\beta} \delta(x - x'), \\
\end{aligned} \quad (1)$$

where the conventional notations are used. The finite propagators are defined by

$$\begin{aligned}
& S(x, x') = i \langle T(\psi(x) \bar{\psi}(x')) \rangle, \\
& A_{ij}(x, x') = i \langle T(\phi_i(x) \phi_j(x')) \rangle \\
& - \langle \phi_i(x) \rangle \langle \phi_j(x') \rangle.
\end{aligned} \quad (2)$$

By using Schwinger's technique of the functional differentiation, it is easy to obtain the inverse function of the nucleon propagator from (1) and (2):

$$\begin{aligned}
S^{-1} &= Z_2(i\gamma\partial + m) - Z_2\delta m + Z_1 i g^2 \gamma_5 \tau_i \langle \phi_i \rangle + Z_1 \Sigma, \\
\Sigma &= -i g^2 \gamma_5 \tau_i S A_{ij} \Gamma_{5j},
\end{aligned} \quad (3)$$

Here we have used the matrix notation. Let us confine ourselves to the case of  $\langle \phi_i \rangle = \text{constant}$ . (The removal of the restriction  $\langle \phi_i \rangle = \text{constant}$  introduces no essential difficulties but makes the expression corresponding to eq. (4) more complicated. For our present purposes of discussing the phenomena at low-energy limit and the unique determination of renormalization constants, it suffices to treat the case of constant  $\langle \phi_i \rangle$ .) If we develop  $\Sigma$  in power series of  $(i\gamma\partial + m)$ , one gets the following form from Lorentz and charge invariances;

$$\begin{aligned}
\Sigma &= Z_1^{-1} [A(\langle \phi \rangle^2) + i g^2 \gamma_5 \tau_i \langle \phi_i \rangle E(\langle \phi \rangle^2) \\
&+ 1/2 \cdot \{i\gamma\partial + m, B(\langle \phi \rangle^2) + i g^2 \gamma_5 \tau_i \langle \phi_i \rangle \\
&\times F(\langle \phi \rangle^2)\} + (i\gamma\partial + m) [C(p^2, \langle \phi \rangle^2) \\
&+ i\gamma\partial D(p^2, \langle \phi \rangle^2) + i g^2 \gamma_5 \tau_i \langle \phi_i \rangle G(p^2, \langle \phi \rangle^2)] \\
&\times (i\gamma\partial + m),
\end{aligned} \quad (4)$$

where we have used the momentum representation. The functions  $A$ ,  $E$  and  $B$  are infinite, while  $C$ ,  $D$ ,  $F$  and  $G$  are finite. Thus the requirement of renormalization is fulfilled with the conditions

$$\begin{aligned}
\delta m &= 1/Z_2 \cdot A(0), \\
Z_2 &= 1 - B(0), \\
Z_1 &= 1 - E(0).
\end{aligned} \quad (5)$$

For the case of  $\langle \phi_i \rangle = 0$ , we obtain the finite propagator and vertex operator from (3), (4) and (5), that is,

$$\begin{aligned}
S^{-1}(p) &= (i\gamma\partial + m) + (i\gamma\partial + m) [C(p^2, 0) \\
&+ i\gamma\partial D(p^2, 0)] (i\gamma\partial + m), \\
\Gamma_{5j}(p, p') &= \tau_j \gamma_5 + 1/2 \{i\gamma\partial + m, \tau_j \gamma_5\} F(0) \\
&+ (i\gamma\partial + m) \tau_j \gamma_5 G(p^2, 0) (i\gamma\partial + m).
\end{aligned} \quad (6)$$

One may infer from (6) that  $C(p^2, 0)$  and  $D(p^2, 0)$  represent the damping effect in propagator and  $F(0)$  and  $G(p^2, 0)$  mean the corrections of vertex operator.

We can now investigate the scattering of a meson with vanishing mass and momentum by a nucleon. After straightforward calculations we get

$$\begin{aligned}
T_{ji} &= \frac{N^2}{m} \phi_i^{(-)}(j) \phi_i^{(+)}(i) \bar{\psi}(p, 0) \psi(p, 0) \\
&\times \left[ \frac{[1 + mF(0)]^2}{1 + 2m[C(-m^2, 0) + mD(-m^2, 0)]} + \frac{2m\dot{A}(0)}{g^2} \right]
\end{aligned} \quad (7)$$

where  $\dot{A}(0) = [\partial A(\langle \phi \rangle^2) / \partial \langle \phi \rangle^2]_{\langle \phi \rangle = 0}$ . This has the same form as the preceding results<sup>1)</sup>. If we introduce the constant external electromagnetic potential  $a_\mu$  by D.T.G.'s gauge transformation, one can obtain the three-field propagator whose derivative with respect to  $ia_\mu$  gives us the (zero energy) photon vertex operator with all corrections of virtual mesons but without any correction of virtual photons. With these quantities one can show that, in the limit of vanishing mass and momentum of external meson, all corrections in the matrix element for threshold photomeson production are cancelled and the results are just in agreement with those given by Kroll and Ruderman<sup>5)</sup>. For the Compton scattering of a photon with zero energy by a nucleon, we can obtain the classical Thomson formula without all corrections. The detailed reports will be published in this journal in near future.

- 1) Chiba and Namiki, Prog. Theor. Phys. **12** (1954), 693.
- 2) Deser, Thirring and Goldberger, Phys. Rev. **94** (1954), 711.
- 3) Schwinger, Proc. Nat. Acad. **37** (1951), 452.
- 4) Takeda, Prog. Theor. Phys. **7** (1952), 359.
- 5) Kroll and Ruderman, Phys. Rev. **93** (1954), 233.



# Velocity-dependent Potential and Gauge Invariance of the First Kind

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October 11, 1954

It is known that in quantum mechanics the physical situation is invariant under gauge transformation of the first kind:  $\psi \rightarrow \psi e^{i\Lambda}$ . This property can be applied to transformation of a velocity-dependent potential energy into a velocity-independent one and vice versa. We shall restrict ourselves to the nonrelativistic Schrödinger equation.

Let us first examine the one-dimensional case in which the potential energy is assumed to be of the form  $V + Jp$  where  $V$  and  $J$  are functions of  $x$ , and  $p$  is the momentum. The Schrödinger equation is then

$$(\hat{p}^2/2m + V + Jp)\Psi = E\Psi.$$

Now, if we write  $\Psi = \psi e^{i\Lambda}$ , and choose  $\Lambda$  such that  $d\Lambda/dx = -mJ/\hbar$  then  $\psi$  satisfies

$$(\hat{p}^2/2m + V')\psi = E\psi$$

where  $V' = V - mJ^2/2 + (i\hbar/2)dJ/dx$

and the energy eigenvalue  $E$  remains the same. It is seen the expression of  $\Lambda$  is not needed if we are only interested in  $\psi$  or  $|\psi|^2 = |\Psi|^2$ .

However, in the three dimensional case

$$(\hat{p}^2/2m + V + \mathbf{J} \cdot \hat{\mathbf{p}})\Psi = E\Psi \quad (1)$$

the velocity-dependent potential energy can be transformed away only if  $\nabla \times \mathbf{J} = 0$ . In that case  $\Lambda$  satisfies the equation

$$\nabla^2 \Lambda = - (m/\hbar) \nabla \cdot \mathbf{J}$$

and  $\psi$  satisfies the equation

$$(\hat{p}^2/2m + V')\psi = E\psi,$$

where

$$V' = V - (m/2)\mathbf{J}^2 + (i\hbar/2)\nabla \cdot \mathbf{J}.$$

On the other hand, if there exists a  $\Lambda$  such that

$$\Delta^2 \Lambda + i(\nabla \Lambda)^2 + i(m/\hbar^2)(\hbar \mathbf{J} \cdot \Delta \Lambda + V) = 0$$

then the reduced problem has only a velocity-dependent potential.

An example is the Schrödinger equation for a non-relativistic electron in the electromagnetic field:

$$[(\hat{\mathbf{p}} - e\mathbf{A})^2/2m + e\phi]\Psi = E\Psi.$$

This can be reduced to the form (1) and then to the following form

$$(\hat{p}^2/2m + e\phi)\psi = E\psi \quad (2)$$

provided that  $\nabla \times \mathbf{A} = 0$ . We see that (2) describes an electron in a pure electrostatic field  $\phi$  without an external magnetic field which checks with the condition imposed on  $\mathbf{A}$ :  $\mathbf{H} = \nabla \times \mathbf{A} = 0$ .

A special case is that in which both  $V$  and  $\mathbf{J}$  are zero. Then our problem reduces to the one discussed by Morel-Viard<sup>1)</sup> in which  $\Psi$  and  $\psi$  are called  $\Lambda$ -correspondent.

1) Morel-Viard, Comptes Rendus **238** (1954), 992.

## ERRATA

## The Classical Equations of a Point Particle in a Symmetric Meson Field.

R. C. MAJUMDAR, *University of Delhi, Delhi,*S. GUPTA, *University of Aligarh, Aligarh,*S. K. TREHAN, *University of Delhi, Delhi.*

(Progress of Theoretical Physics, Vol. 12, No. 1, p. 31-48, July 1954)

- Page 31: The title should read "The Classical Equations of Motion of a Point Particle in a Charged Symmetric Meson Field."
- Page 34: In the equation (10) both  $\alpha-2/2$  and  $\alpha-4/2$  should be replaced by  $(\alpha-2)/2$  and  $(\alpha-4)/2$  respectively.
- Page 35: In the equation (15b) the subscript on  $U^{(\alpha)}(x)$  should be  $k$  instead of  $h$ .
- Page 37: The last term of the equation (30a) should read  $G_{kv}^{(\alpha=2)p} S_{\alpha p \mu}$ . In the equations (30b) and (31b) read  $\sum_{k \neq \alpha}$  instead of  $\sum$  for the summation.
- Page 38: In the first sentence of the fourth paragraph read ".....Lagrangian multipliers  $\xi^i$ s.....". On the left hand side of the equation (36a) read  $v_{\alpha\gamma} \dot{S}_{\alpha\mu p} v_{\alpha}^p$  instead of  $v_{\alpha\gamma} \dot{S}_{\alpha\mu\nu} v_{\alpha}^p$ .
- Page 39: In the equation (35b)  $e^{\mu\nu\rho\sigma}$  should be replaced by  $\epsilon_{\mu\nu\rho\sigma}$ . The left hand side of the equation (36b) should be read as  $I_{\alpha} \epsilon_{\mu\nu\rho\sigma} v_{\alpha}^p \dot{S}_{\alpha}^{\sigma}$ .
- Page 40: The third term of the third line of the equation (41a) should read  $2v_{\nu} \dot{T}_{\mu}^{\nu}$ . The second term in the first line and the second term in the fourth line of the equation (42a) should read  $\frac{2}{3} \dot{T}_{\mu\nu} v^2$  and  $\frac{1}{2} [v_{\mu} \dot{T}_{\nu\sigma} v^{\sigma}]$  respectively.
- Page 41: The third and the last terms in the fourth line of the equation (42b) should read  $2v_{\mu} \dot{Z}_{\sigma} v^{\sigma}$  and  $\frac{2}{3} v_{\mu} Z_{\sigma} v^{\sigma}$  respectively.  
In the last line of the same equation the factor outside the curly bracket should be  $\chi^2/2$  in place of  $\chi^2/3$ . In the first term of the third line of the equation (45a) read  $s^{\mu} v_{\mu}$  instead of  $s_{\mu} v_{\mu}$ .
- Page 43: In the last term of the first line of the equation (49a) read  $\chi^2$  in place of  $\chi_2$ .
- Page 44: In the equation (52) read  $\xi = (\omega_0^2 - \chi^2)^{3/2} / \omega_0$  instead of  $\xi = (\omega_0^2 - \chi^2) / \omega_0$ .
- Page 46: In the equations (59), (62) and (65),  $\sigma^2$  should be replaced by  $\sigma_2$ .  
In line 20,  $(1/2\chi^2)\epsilon^2 \omega_0(\omega_0^2 - \chi^2)^{3/2}$  should read  $(1/2\chi^2)\epsilon^2 g^2 \omega_0(\omega_0^2 - \chi^2)^{3/2}$ .  
In the denominator of the last factor of the equation (66) read  $\frac{1}{\lambda^2 \chi^4}$  instead of  $\frac{1}{\lambda^2 \chi^3}$ .
- Page 47: The equation (67) should read as follows:

$$\sigma_v^i = \frac{4\pi}{\chi^4} \frac{(\omega_0^2 - \chi^2)^2}{\omega_0^2} \cdot \frac{g^2 \left( g^2 + \frac{2}{3} f^2 \chi^2 \right)}{\lambda^2} - \frac{1}{1 + \frac{1}{\lambda^2 \chi^4} \left( g^2 + \frac{2}{3} f^2 \chi^2 \right)^2 \frac{(\omega_0^2 - \chi^2)^3}{\omega_0^2}}$$

In the first equation of (71),  $\sigma^2$  should be replaced by  $\sigma_2$ . The equation (72) should read as follows:

$$\sigma_s = \frac{4\pi}{\lambda^2 \omega_0^2} \frac{g'^4 + \frac{2}{3} g'^2 f'^2 (\omega_0^2 - \chi^2) + \frac{1}{9} f'^4 (\omega_0^2 - \chi^2)^2}{1 + \frac{1}{\lambda^2} \frac{(\omega_0^2 - \chi^2)}{\omega_0^2} \left\{ g'^4 - \frac{2}{3} g'^2 f'^2 (\omega_0^2 - \chi^2) + \frac{1}{9} f'^4 (\omega_0^2 - \chi^2)^2 \right\}}$$

- Page 48: In the numerator of the equation (72') read  $g'^4$  instead of  $g'^3$ .

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## The Cut Off Method in Meson Theory

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In the meson theory we adopt the cut-off method not only for convenience but also from the physical ground, considering the recent success of the application of the method to the analysis of the low energy meson processes. From this standpoint, the anomalous magnetic moments of the nucleons and the gamma decay of the neutral pions have been reexamined. The results have much improved those which have been obtained hitherto. The anomalous magnetic moment of the electron is also reexamined for the purpose of comparison of meson theory with quantum electrodynamics.

### § 1. Introduction\*

As is well known, one cannot make even the qualitative interpretation of various meson processes, i. e. nuclear forces, meson-nucleon scattering, magnetic moments, etc., if one uses the relativistic perturbation theory and adopts the renormalization technique. On the other hand, it has become clear that one can explain them rather well, if one uses the static approximation and apply appropriate approximation methods. However, one always encounters the divergence difficulties in the latter treatment and is usually obliged to adopt the so-called "cut-off" method. This fact seems to show that the higher frequency components of the field should be considerably cut down, while in the usual perturbation treatment those components which correspond to energies of the order of or higher than the nucleon mass are very appreciable even after the mass and charge renormalization has been performed. This difficulty is in fact overcome to some extent by taking into account higher order radiative interactions and making use of appropriate approximation methods (e. g. the Tamm-Dancoff method by Bethe and Dyson<sup>1)</sup>), but, as shown by Lehmann<sup>2)</sup>, the order of singularities of the propagation functions is never reduced in the exact solution. If this is the case, two difficulties may arise. First, the present meson theory is not closed by itself even in the low energy region, since the effects of all the heavy particles such as V-particles which interact rather strongly with the nucleon cannot be ignored. Secondly, any weak coupling approximations (including the Tamm-Dancoff method) may have no region of its applicability, because the interaction of the higher frequency components becomes stronger and stronger as one goes to higher order approximations. This situation is also justified by direct calculations. These difficulties seem to contradict the success of

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\* Considerations described in this section are due to Mr. N. Fukuda, who suggested, with Mr. S. Tani, the investigation of the anomalous magnetic moments of the nucleons. Detailed accounts will be published in this journal by Mr. N. Fukuda.



many recent theoretical analyses of the experiments on low energy meson processes. Thus one is forced to admit a new assumption that the "cut-off" method is necessary not only for convenience but also from physical ground.

Moreover, the interaction between the meson and nucleon fields can be treated as rather weak, so far as we adopt the cut-off method. This fact is illustrated by the investigation of nuclear forces in which the higher order contributions are effective only in the inner "core" and converge rapidly in the outer range<sup>3</sup>, and also seen in the calculations of the Tamm-Dancoff method by which many investigations have been performed restricting the virtual meson number up to two<sup>11</sup>. It has been proved by these investigations that one could get reasonable and consistent results by the weak coupling treatment; only the strong interaction in the S-state and the resonance state ( $I=3/2$ ,  $J=3/2$ ) should be treated carefully. Under these circumstances it seems interesting to reexamine the results which are obtained in the usual covariant meson theory in the light of the cut-off method. In this connection the anomalous magnetic moments of the nucleons and the gamma decay of the neutral mesons will be treated.

The former phenomenon, according to the calculation performed by Case<sup>5</sup>), cannot be explained by assuming the pseudoscalar interaction between the meson and nucleon fields on account of the large contribution of the nucleon current compared with the meson current. This circumstance is due to the concentration of the meson cloud within the range of only about the nucleon Compton wave length which is characteristic of the pseudoscalar interaction. Although this defect can be removed to some extent by including the higher order effects<sup>6</sup>), the convergence of the perturbation expansion is doubtful in this case, and from the above standpoint this problem should be reexamined by using the cut-off method of the high frequency parts. Thus the momentum dependence of the magnetic moments of the nucleons is analysed in § 2 and the effectiveness of the cut-off method is discussed.

The gamma decay of the neutral meson cannot be explained by the perturbation theory with the value of the coupling constant which is predicted from the analysis of the nuclear forces and the pion-nucleon scattering, etc<sup>7</sup>), and hence this process is also investigated by the cut-off method in § 3. The discrepancies between the theory and experiments are diminished for both phenomena by using the cut-off method, as expected.

In § 4, the anomalous magnetic moment of the electron is calculated in the same way. As is well-known, we can obtain good agreement with the experiment, if the cut-off energy is assumed to be about the electron mass for many phenomena in quantum electrodynamics<sup>8</sup>) and this shows that the high frequency parts of virtual photons do not contribute appreciably. The anomalous magnetic moment of the electron had been explained by Schwinger<sup>9</sup>), but the Welton's model<sup>10</sup>) could not succeed by the cut-off method, while Koba<sup>10</sup>) had pointed out that one can explain this phenomenon by improving the Welton's model. Thus the high frequency parts do not also contribute to this process, and so it is interesting to compare the features of meson theory and of quantum electrodynamics in this connection.

In § 5, the summary and discussion of the results obtained are described.



## § 2. The anomalous magnetic moments of the nucleons

### (i) Abstract of the calculation with the Feynman-Dyson method

The essential part of the contribution of nucleon current to the magnetic moment, which is the process illustrated in Fig. 1a, is represented by the integral

$$I_N = \int (dk) \frac{\gamma_5 [i(\gamma \cdot p_2 - k) - \kappa] (\gamma \cdot A) [i(p_1 - k \cdot \gamma) - \kappa] \gamma_5}{[(p_2 - k)^2 + \kappa^2] [(p_1 - k)^2 + \kappa^2] (k^2 + \mu^2)}. \quad (1)$$

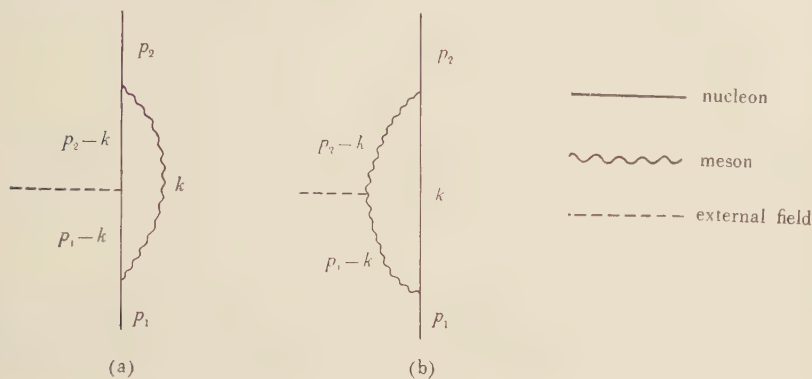


Fig. 1

If we perform the charge renormalization procedure and retain only the magnetic moments of the nucleons, this integral can be expressed, according to Feynman's method, as follows :

$$I_N \simeq -i\pi^2/\kappa \cdot (\sigma \cdot [A \times \Delta p]) \cdot J_N, \quad \Delta p = p_2 - p_1, \quad (2)$$

where

$$J_N = \int_0^1 dx \frac{\kappa^2 x^3}{\kappa^2 x^2 + \mu^2 (1-x)}. \quad (3)$$

In this calculation we have put  $A_4=0$  and used the symbol " $\simeq$ " in (2), which means that we have picked out only the magnetic moment, and this symbol will be frequently used henceforth.

Also, the contribution of meson current in the Fig. 1(b), is represented by the next integral :

$$I_M = \int (dk) \frac{\gamma_5 [i(\gamma \cdot k) - \kappa] \gamma_5 (p_1 - k \cdot A)}{(k^2 + \kappa^2) [(p_1 - k)^2 + \mu^2] [(p_2 - k)^2 + \mu^2]}, \quad (4)$$

which becomes

$$I_M \simeq \pi^2/2\kappa \cdot (\sigma \cdot [A \times \Delta p]) \cdot J_M, \quad (5)$$

where

$$J_M = \int_0^1 dx \frac{\kappa^2 x^2 (1-x)}{\kappa^2 x^2 + \mu^2 (1-x)}. \quad (6)$$

The anomalous magnetic moments of the nucleons can be expressed with these  $J_N$  and  $J_M$  in the unit of the Bohr magneton as follows:

$$J\mu_p = 1/\pi \cdot (f^2/4\pi) \left( -\frac{J_N}{2} + J_M \right) \quad \text{for proton,} \quad (7)$$

$$J\mu_n = 1/\pi \cdot (f^2/4\pi) (-J_N - J_M) \quad \text{for neutron,}$$

in which  $f$  is the pseudoscalar coupling constant. The integrals  $J_N$  and  $J_M$  can be easily obtained by elementary calculations and we have

$$J_N = 0.47, \quad J_M = 0.34. \quad (8)$$

If we assume  $f^2/4\pi = 15$ , then (7) becomes

$$J\mu_p = 0.52 \quad \text{or} \quad |J\mu_p/J\mu_n| = 0.13, \quad (9)$$

$$J\mu_n = -3.9$$

which contradict violently the experimental results, and this circumstance is due to the fact that the contribution of the nucleon current  $J_N$  is too large compared with that of the meson current  $J_M$ .

(ii) *The nucleon current contribution and its momentum dependence*

Now, we analyse the momentum dependence of the virtual meson. We first perform the integration with respect to  $k_0$  and, for this purpose, it is convenient to use the identity

$$i(\gamma \cdot k) - \kappa = [(E_k - k_0) A^-(\mathbf{k}) - (E_k + k_0) A^+(\mathbf{k})] \gamma_4, \quad (10)$$

where  $k$  is the internal momentum-energy of the nucleon,  $E_k = \sqrt{k^2 + \kappa^2}$  and

$$A^+(\mathbf{k}) = [E_k + (\boldsymbol{\alpha} \cdot \mathbf{k}) + \kappa\beta]/2E_k, \quad (11)$$

$$A^-(\mathbf{k}) = [E_k - (\boldsymbol{\alpha} \cdot \mathbf{k}) - \kappa\beta]/2E_k,$$

which are the projection operators to the positive and negative states respectively. Thus

$$I_N = \int (dk) \gamma_5 \left[ \frac{A^-(\mathbf{q})}{E_q + q_0} - \frac{A^+(\mathbf{q})}{E_q - q_0} \right] \gamma_4 (\gamma \cdot A) \left[ \frac{A^-(\mathbf{p})}{E_p + p_0} - \frac{A^+(\mathbf{p})}{E_p - p_0} \right] \gamma_4 \gamma_5$$

$$= I_N^{++} + I_N^{--} - I_N^{+-} - I_N^{-+}, \quad (12)$$

where

$$I_N^{++} = \int (dk) \frac{\gamma_5 A^+(\mathbf{q}) \gamma_4 (\gamma \cdot A)^+ A(\mathbf{p}) \gamma_4 \gamma_5}{(E_q - q_0)(E_p - p_0)(E_k^2 - k_0^2)} \quad \text{etc.,} \quad (13)$$

$$p = p_1 - k, \quad q = p_2 - k, \quad E_k = \sqrt{k^2 + \mu^2}.$$

When we perform the  $k_0$  integration, we have to remember that the small negative imaginary mass must be added to the mass in the denominator of the propagation function, and that the contour for  $I_N^{++}$  is such as to surround the lower half of the complex plane.

Thus, we obtain

$$I_N^{++} = i\pi \int d\mathbf{k} \frac{\gamma_5 A^+(\mathbf{q}) \gamma_4 (\gamma \cdot A) A^+(\mathbf{p}) \gamma_4 \gamma_5}{\varepsilon_k (E_q - \kappa + \varepsilon_k) (E_p - \kappa + \varepsilon_k)}. \quad (14)$$

In the same way

$$I_N^{--} = i\pi \int d\mathbf{k} \frac{\gamma_5 A^-(\mathbf{q}) \gamma_4 (\gamma \cdot A) A^-(\mathbf{p}) \gamma_4 \gamma_5}{\varepsilon_k (E_q + \kappa + \varepsilon_k) (E_p + \kappa + \varepsilon_k)}, \quad (15)$$

$$I_N^{+-} = i\pi \int d\mathbf{k} \frac{\gamma_5 A^+(\mathbf{q}) \gamma_4 (\gamma \cdot A) A^-(\mathbf{p}) \gamma_4 \gamma_5}{\varepsilon_k (E_p + E_q)} \left[ \frac{1}{E_q - \kappa + \varepsilon_k} + \frac{1}{E_p + \kappa + \varepsilon_k} \right] \quad (16)$$

$$\equiv I_N^{+-a} + I_N^{+-b}.$$

Each integral represents the contribution from the next time-ordered diagrams:

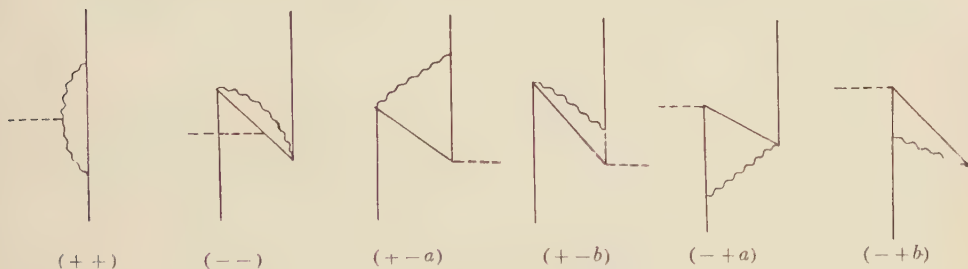


Fig. 2

The contributions from the processes corresponding to  $I_N^{+-}$  are equal to those from  $I_N^{+-}$ , because both diagrams can be obtained by the "time inversion" from each other. Hence we can write

$$I_N = I_N^{++} + I_N^{--} - 2(I_N^{+-a} + I_N^{+-b}). \quad (17)$$

We illustrate the calculation of  $I_N^{++}$ , as an example, in order to obtain the magnetic moment from it. The numerator of  $I_N^{++}$  (14) is, putting  $A_4 = 0$ ,

$$I^{++} \equiv i\gamma_5 A^+(\mathbf{q}) \gamma_4 (\gamma \cdot A) A^+(\mathbf{p}) \gamma_4 \gamma_5 \simeq -\tilde{A}^+(\mathbf{q}) (\mathbf{a} \cdot \mathbf{A}) \tilde{A}^+(\mathbf{p}) \gamma_4, \quad (18)$$

where

$$\tilde{A}^+(\mathbf{p}) = \gamma_5 A^+(\mathbf{p}) \gamma_5 = [E_p + (\mathbf{a} \cdot \mathbf{p}) - \kappa\beta]/2E_p,$$

$$\tilde{A}^-(\mathbf{p}) = \gamma_5 A^-(\mathbf{p}) \gamma_5 = [E_p - (\mathbf{a} \cdot \mathbf{p}) + \kappa\beta]/2E_p.$$

For the magnetic moment, only the linear terms with respect to  $\mathbf{p}_1$  and  $\mathbf{p}_2$  do contribute. Hence in this approximation we may put

$$(\mathbf{a} \cdot \mathbf{p}_1) \phi(\mathbf{p}_1) \simeq 0, \quad \bar{\psi}(\mathbf{p}_2) (\mathbf{a} \cdot \mathbf{p}_2) \simeq 0, \quad (19)$$

$$\gamma_4 \phi(\mathbf{p}_1) \simeq \phi(\mathbf{p}_1), \quad \bar{\psi}(\mathbf{p}_2) \gamma_4 \simeq \bar{\psi}(\mathbf{p}_2),$$

and thus

$$I^{++} \simeq -[E_q - (\mathbf{a} \cdot \mathbf{k}) - \kappa] (\mathbf{a} \cdot \mathbf{A}) [E_p - (\mathbf{a} \cdot \mathbf{k}) - \kappa]/4E_p E_q$$

$$\simeq [(E_q - \kappa) (\mathbf{a} \cdot \mathbf{A}) (\mathbf{a} \cdot \mathbf{k}) + (E_p - \kappa) (\mathbf{a} \cdot \mathbf{k}) (\mathbf{a} \cdot \mathbf{A})] / 4 E_p E_q.$$

The last modification is due to the facts that the terms proportional to  $(\mathbf{a} \cdot \mathbf{A})$  and  $(\mathbf{a} \cdot \mathbf{k}) (\mathbf{a} \cdot \mathbf{A}) (\mathbf{a} \cdot \mathbf{k})$  do not contribute to the magnetic moment, but they do to the charge renormalization and the higher moments.

Expanding into power series with respect to  $\mathbf{p}_1$  and  $\mathbf{p}_2$  and retaining only the zeroth power and first power, we obtain

$$\begin{aligned} I^{++} \simeq & [2(E_k - \kappa) (\mathbf{A} \cdot \mathbf{k}) + (E_k - 2\kappa) (\mathbf{p}_1 + \mathbf{p}_2 \cdot \mathbf{k}) (\mathbf{A} \cdot \mathbf{k}) / E_k^2] \\ & + i(\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{k}]) (\mathbf{p}_1 - \mathbf{p}_2 \cdot \mathbf{k}) / E_k \quad (20) \end{aligned}$$

in which we made use of the identity

$$(\mathbf{a} \cdot \mathbf{a}) (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) + i(\boldsymbol{\sigma} \cdot [\mathbf{a} \times \mathbf{b}]).$$

However, since we must finally perform the angular integrations of  $\mathbf{k}$ -vector in  $I^{++}$  and since the relation

$$\int d\Omega_k (\mathbf{k} \cdot \mathbf{c}) (\mathbf{k} \cdot \mathbf{d}) = (4\pi/3) \cdot k^2 (\mathbf{c} \cdot \mathbf{d})$$

does hold for arbitrary vectors  $\mathbf{c}$  and  $\mathbf{d}$ , we may write the above  $I^{++}$ , after carrying out the angular integrations, as follows:<sup>11)</sup>

$$\begin{aligned} I^{++} \simeq & [2(E_k - \kappa) (\mathbf{A} \cdot \mathbf{k}) + k^2 (E_k - 2\kappa) (\mathbf{A} \cdot \mathbf{p}_1 + \mathbf{p}_2) / 3 E_k^2] \\ & - i k^2 (\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{J}\mathbf{p}]) / 3 E_k \quad (4 E_k^2), \quad \mathbf{J}\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1. \end{aligned}$$

But

$$\begin{aligned} (\mathbf{A} \cdot \mathbf{p}_1 + \mathbf{p}_2) &= [(\mathbf{a} \cdot \mathbf{p}_1 + \mathbf{p}_2) (\mathbf{a} \cdot \mathbf{A}) + (\mathbf{a} \cdot \mathbf{A}) (\mathbf{a} \cdot \mathbf{p}_1 + \mathbf{p}_2)] / 2 \\ &\simeq [(\mathbf{a} \cdot \mathbf{p}_1 - \mathbf{p}_2) (\mathbf{a} \cdot \mathbf{A}) + (\mathbf{a} \cdot \mathbf{A}) (\mathbf{a} \cdot \mathbf{p}_1 - \mathbf{p}_2)] / 2 \\ &= i(\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{J}\mathbf{p}]) \end{aligned}$$

because of the Lorentz condition,  $(\mathbf{A} \cdot \mathbf{J}\mathbf{p}) = 0$ , and the relations (19). On account of these modifications we obtain finally

$$I^{++} \simeq [(E_k - \kappa) (\mathbf{A} \cdot \mathbf{k}) - i \kappa k^2 (\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{J}\mathbf{p}]) / 3 E_k^2] / 2 E_k^2. \quad (21)$$

Accordingly, (14) becomes

$$\begin{aligned} I_N^{++} &= \pi \int d\mathbf{k} \frac{I^{++}}{\varepsilon_k (E_q - \kappa + \varepsilon_k) (E_p - \kappa + \varepsilon_k)} \\ &\simeq \frac{\pi}{2} \int d\mathbf{k} \frac{[(E_k - \kappa) (\mathbf{A} \cdot \mathbf{k}) - i \kappa k^2 (\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{J}\mathbf{p}]) / 3 E_k^2]}{\varepsilon_k E_k^2 (E_q - \kappa + \varepsilon_k) (E_p - \kappa + \varepsilon_k)} \end{aligned}$$

Again, expanding the denominator into power series with respect to  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , retaining only the magnetic moment and performing the angular integrations, we obtain finally

$$I_N^{++} \simeq -i\pi^2 / \kappa \cdot (\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{J}\mathbf{p}]) \cdot \mathbf{J}^{++}, \quad (22)$$

where

$$J^{++} = \frac{2\kappa}{3} \int_0^\infty dk \frac{k^4}{\varepsilon_k E_k^3 L_1^3} \left[ -(E_k - \kappa) + \frac{\kappa L_1}{E_k} \right], \quad (23)$$

$$L_1 = E_k - \kappa + \varepsilon_k.$$

By similar calculations, we also obtain the following results:

$$I_N^{--} \simeq -i\pi^2/\kappa \cdot (\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{J}_1]) \cdot J^{--}$$

$$I_N^{+-} \simeq +i\pi^2/\kappa \cdot (\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{J}_1]) (J^{+-a} + J^{+-b}), \quad (24)$$

where

$$J^{--} = \frac{2\kappa}{3} \int_0^\infty dk \frac{k^4}{\varepsilon_k E_k^3 L_2^3} \left[ (E_k + \kappa) + \frac{\kappa L_2}{E_k} \right],$$

$$J^{+-a} = \frac{\kappa}{6} \int_0^\infty dk \frac{k^4}{\varepsilon_k E_k^4 L_1^2} \left[ -(E_k - \kappa) + \frac{3\kappa L_1}{E_k} \right], \quad (25)$$

$$J^{+-b} = \frac{\kappa}{6} \int_0^\infty dk \frac{k^4}{\varepsilon_k E_k^4 L_2^2} \left[ (E_k + \kappa) + \frac{3\kappa L_2}{E_k} \right],$$

$$L_2 = E_k + \kappa + \varepsilon_k.$$

Thus we obtain from (17), (22) and (24), the result

$$I_N \simeq -i\pi^2/\kappa \cdot (\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{J}_1]) [J^{++} + J^{--} + 2(J^{+-a} + J^{+-b})]. \quad (26)$$

If we compare it with (2), we obtain

$$J_N = J^{++} + J^{--} + 2(J^{+-a} + J^{+-b}). \quad (27)$$

In the integrand of the integrals  $J^{++}$  etc., the momentum distribution of the virtual mesons is plotted in Fig. 3a. As is expected, the high frequency parts of the virtual mesons contribute very much even after we have performed the charge renormalization. The results of numerical integrations of the integrals  $J^{++}$  etc. without and with the cut-off procedure at the nucleon mass, i. e.  $k_{\max} = \kappa$ , are shown in Table I. Thus nucleon pair formation processes are more diminished than the no-pair process with the cut-off method.

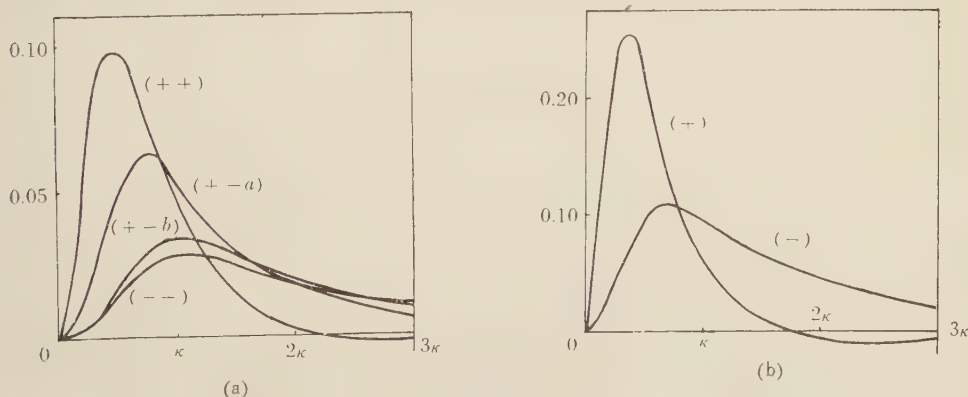


Fig. 3

Writing  $J = \int_0^\infty (dk/\kappa) F(k/\kappa)$ ,  $F(k/\kappa)$ 's are plotted as functions of  $k$ , for Figs. 3, 5 and 6, provided that  $\kappa$  is replaced by  $m$  for Fig. 6.



Table I

	$J_N$	$J^{++}$	$J^{--}$	$J^{+-a}$	$J^{+-b}$	$J_M$	$J_M^+$	$J_M^-$
without the cut-off	0.47	0.069	0.078	0.081	0.081	0.34	0.12	0.22
with the cut-off at $x$	0.19	0.066	0.014	0.038	0.016	0.22	0.15	0.074

(iii) *The meson current contribution and the magnetic moments of the nucleons*

By making use of the identity (10), the calculation of the integral  $I_M$  expressed in (4) is entirely analogous to that of the nucleon current in the preceding sub-section (ii). Then we can separate the Feynman diagram, Fig. 1b, into the processes of Fig. 4, i. e., the no-pair (+) and pair (-) processes.

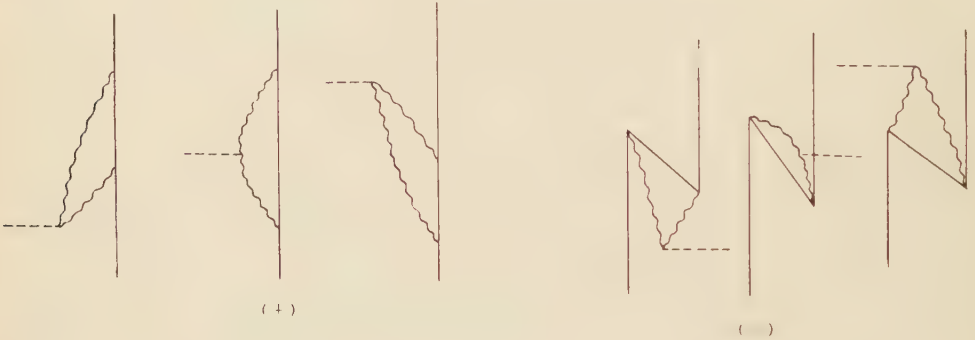


Fig. 4

Transforming the integration variable to  $k = p_1 - k'$  and rewriting  $k'$  as  $k$ , we obtain the following results :

$$\begin{aligned}
 I_M &= I_M^+ + I_M^-, \\
 I_M^+ &\simeq \pi^2/2\kappa \cdot (\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{A}p]) \cdot J_M^+, \\
 I_M^- &\simeq \pi^2/2\kappa \cdot (\boldsymbol{\sigma} \cdot [\mathbf{A} \times \mathbf{A}p]) \cdot J_M^-,
 \end{aligned}
 \tag{28}$$

in which

$$\begin{aligned}
 J_M^+ &= \frac{\kappa}{3} \int_0^\infty dk \frac{k^4}{E_k^2 \varepsilon_k^3} \left[ \frac{1}{E_k} - \frac{\varepsilon_k^2}{L_1^2} \left( \frac{1}{E_k} + \frac{2}{L_1} \right) \right], \\
 J_M^- &= \frac{\kappa}{3} \int_0^\infty dk \frac{k^4}{E_k^2 \varepsilon_k^3} \left[ \frac{1}{E_k} - \frac{\varepsilon_k^2}{L_2^2} \left( \frac{1}{E_k} + \frac{2}{L_2} \right) \right].
 \end{aligned}
 \tag{29}$$

Thus we can write, comparing with (5),

$$J_M = J_M^+ + J_M^-.
 \tag{30}$$

The momentum distributions of the virtual mesons are plotted in Fig. 3b, and we encounter the same circumstance as in the case of nucleon current. Namely, the contribution from the high frequency virtual mesons is again large : this may be also seen from the numerical values listed in Table I.

As shown in (8) and (9), the discrepancy between theory and experiment is very large when we use the exact values of  $f_N$  and  $f_Y$ . However, when we use the values with the cut-off at  $\kappa$  which are given in Table I, then the anomalous magnetic moments of the nucleons (7) become

$$\begin{aligned} \mu_p &= 0.61 \\ \text{or } |\mu_p / \mu_N| &= 0.32, \\ \mu_N &= -1.9 \end{aligned} \quad (31)$$

assuming  $f^2/4\pi = 15$ . Thus the result is much better than (9), although the proton moment is yet too small. Of course the excellent agreement for the neutron moment must not be taken seriously. In either case it is remarkable that the pair formation processes are very much damped when we adopt the cut-off method.

### § 3. The gamma decay of the neutral pions

We cannot obtain agreement of the theory with experiments for this process by assuming the value of the coupling constant as  $f^2/4\pi = 10 \sim 15$ , which is determined so as to explain the phenomena of nuclear forces and meson nucleon scattering. Thus we reexamine this process by the cut-off method.

As is well known, this process is represented by a closed loop of the nucleon line in the Feynman diagram so that it is necessary to produce a nucleon pair at any vertex point. Thus we do not separate the processes, a procedure which has been taken in the preceding calculations of the nucleon magnetic moments. At first sight, if we adopt the cut-off method for this process, one may say that it does not correspond to cut-off the high frequency parts of the virtual mesons. We consider, however, that it is adequate to suppress the pair formation of the nucleons and so, in this connection, the cut-off procedure makes sense.

Now, when we first perform the "spur" calculation, the diagonal sum, which comes from the closed loop of the nucleon line, then the essential integral for the internal lines becomes as follows:

$$I = \int \frac{(dk)}{[(p-k)^2 + \kappa^2][(q-k)^2 + \kappa^2][k^2 + \kappa^2]} \quad (32)$$

in which  $q$  is the momentum-energy of the neutral meson and  $p$  is that of the emitted photon. This integral can be easily performed by the Feynman method and it becomes

$$I = i\pi^2/2\kappa^2 \quad (33)$$

in the approximation in which the contributions of the order  $(\mu/\kappa)^2$  or higher are neglected. This corresponds to the life-time  $\tau$  of the neutral meson, with  $\alpha = c^2/4\pi$ ,

$$1/\tau = (\alpha^2/16\pi^2) (f^2/4\pi) (\mu/\kappa)^2 \cdot \mu \quad (34)$$

which is equal to  $\tau = 4.2 \times 10^{-17}$  sec. with  $f^2/4\pi = 15$ .

In order to see the momentum distribution of the virtual nucleon, we must first perform the  $k_0$  integration in (32). But in the approximation in which the contributions of the order  $(\mu/\kappa)^2$  or higher are neglected, we can put  $p=q=0$  in (32), i. e.

$$I \doteq \int \frac{dk}{(k^2 + \kappa^2)^3}.$$

Then we can easily perform the  $k_0$ -integration and angular integrations of  $\mathbf{k}$  resulting in

$$I = 3i\pi^2/2\kappa^2 \cdot J_0, \tag{35}$$

in which

$$J_0 = \kappa^2 \int_0^\infty dk \, k^2/E_k^5. \tag{36}$$

We can easily ascertain  $J_0=1/3$  by elementary integration so that (35) agrees with (33).

The integrand of  $J_0$  is plotted in Fig. 5 and the high frequency parts are effective in this process too. The value of  $J_0$  is shown in Table II with the cut-off procedure, in which the life-time is also tabulated for the corresponding values of  $J_0$ . Thus the discrepancy above mentioned is much improved also in this process.

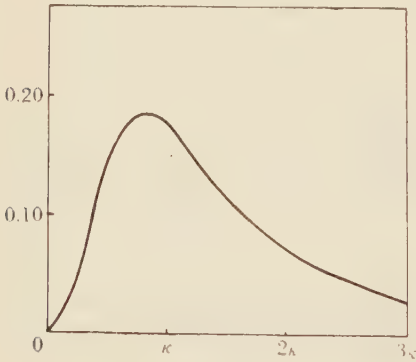


Fig. 5

Table II

$k_{\max}$	$\infty$	$\kappa$	$\kappa/2$
$J_0$	$1/3$	$1/6\sqrt{2}$	$1/15\sqrt{5}$
$\tau(\text{sec})$	$4.2 \times 10^{-17}$	$3.4 \times 10^{-16}$	$5.2 \times 10^{-15}$

#### § 4. The anomalous magnetic moment of the electron

As was described in introduction, this famous phenomenon had already been explained as well as other phenomena in the quantum electrodynamics. We adopted this process for the comparison of the meson theory with the quantum electrodynamics in order to see how the higher frequency parts of the virtual fields behave. Of course we know by many investigations that we can obtain good agreement by the cut-off method at about the electron mass  $m$  in the quantum electrodynamics.

We may think of many reasons why the different behavior of the high frequency part in the meson theory and the quantum electrodynamics exists: (a) the difference in the type of the interactions, i. e.  $\gamma_5$  and  $\gamma_\mu$ ; (b) the meson mass is finite contrary to

the zero mass of the photon; and (c) the nucleon structure is more complicate than the electron and so we may not be able to explain all the characters of the nucleon only by the meson theory, etc. Of course these would all be related with one another. But we do not consider (c) in this paper, since this is a very difficult problem and would be solved only by future investigations. Thus we attempt to examine the features of the higher frequency parts owing to the circumstances (a) and (b).

We want to make here a remark with respect to (a). The  $\gamma_5$  matrix is the "odd" interaction<sup>12)</sup> and so the nucleon pair formation is essential for this interaction. Thus we may expect that the high frequency parts would play an important role in this case and in fact we have shown in the preceding sections that this is true. This may, however, not be explained only by the reason of the "odd" interaction, since  $\gamma_\nu$  involves the odd matrix as its space component and this component should have the same character as the meson nucleon interaction. Thus it will be worth-while to reexamine the feature of the high frequency parts in quantum electrodynamics from this point too.

Since the calculation of this process is the same as for the nucleons except that the photon is a neutral particle, we describe here only the results. The essential part of this phenomenon is represented by the following integral:

$$I_e = \int (dk) \frac{\gamma_\nu [i(\gamma \cdot p_2 - k) - m] (\gamma \cdot A) [i(\gamma \cdot p_1 - k) - m] \gamma_\nu}{[(p_2 - k)^2 + m^2] [(p_1 - k)^2 + m^2] \cdot k^2}, \quad (37)$$

which, by the Feynman technique, equals

$$I_e \simeq i\pi^2/m \cdot (\sigma \cdot [A \times \Delta p]), \quad (38)$$

so that the anomalous magnetic moment of the electron, in the unit of the Bohr magneton, turns out to be  $\Delta\mu_e = \alpha/2\pi$ . If we put, in (38),

$$I_e \simeq i\pi^2/m \cdot (\sigma[A \times \Delta p]) \cdot J \quad (39)$$

then  $J$  must, of course, be equal to one and  $\Delta\mu_e = (\alpha/2\pi)J$ .

Now, for convenience we separate (37) into two parts, namely,

$$I_e = I_e^{(t)} + I_e^{(x)}, \quad (40)$$

corresponding to the time- and the space-components in the summation of  $\gamma_\nu$  in  $I_e$ . Then  $J$  in (39) becomes

$$J = J^{(t)} + J^{(x)} \quad (41)$$

and by the Feynman method we can easily ascertain that

$$J^{(t)} = -3/2, \quad J^{(x)} = 5/2.$$

In the same way as for the nucleons, we obtain the following results:

$$J^{(t)} = J_{++}^{(t)} + J_{--}^{(t)} + 2(J_{+-a}^{(t)} + J_{+-b}^{(t)}), \quad (42)$$

$$J_{++}^{(t)} = -\frac{2m}{3} \int_0^\infty dk \frac{k^3}{E^3 M_1^3} \left[ (E+m) + \frac{mM_1}{E} \right],$$





Table III

	$J^{(t)}$	$J_{++}^{(t)}$	$J_{--}^{(t)}$	$J_{+-a}^{(t)}$	$J_{+-b}^{(t)}$	$J^{(x)}$	$J_{++}^{(x)}$	$J_{--}^{(x)}$	$J_{+-a}^{(x)}$	$J_{+-b}^{(x)}$
without cut-off	-1.5	-0.94	-0.0078	-0.23	-0.041	2.5	-0.78	0.69	1.03	0.26
with cut-off at $m$	-0.95	-0.72	-0.0066	-0.099	-0.013	2.2	-0.40	0.34	0.95	0.17

Thus in the quantum electrodynamics, as expected, we see that the high frequency parts are damped in itself. This damping is, however, due to two reasons: First, the low frequency part is very large compared with the high frequency part and the latter does not play an essential role. Second, the high frequency parts cancel each other, especially in the space component. The first circumstance is due to the zero photon mass apart from the smallness of the high frequency part itself, because when we attribute the mass  $\lambda$  to the photon in the propagation function as  $(k^2 + \lambda^2)$  instead of  $k^2$ ,\* we can see easily that the momentum distributions of Fig. 6 do not start from a finite value at  $k=0$ , but from zero. In this sense we may conclude that the zero photon mass is essential in the quantum electrodynamics contrary to the meson theory. The second is due to characteristic behavior of the  $\gamma_\mu$  interaction in spite of the odd matrix involved. In this respect  $\gamma_5$  and  $\gamma_\mu$ -interactions are essentially different, while we cannot consider any plausible explanations for this.

It is interesting to speak here of the Welton's model. He had treated the electron as the classical particle although the electromagnetic field is fluctuating in quantum manner and so his investigation of the anomalous magnetic moment corresponds to inclusion of the no-pair processes only; he had only calculated  $J_{++}^{(t)}$  and  $J_{++}^{(x)}$  with the cut-off at  $m^\dagger$ . Then we obtain just  $\Delta\mu_e = -(\alpha/2\pi)$  as seen from Table III.

## § 5. Summary and discussion

We have started from the standpoint that the cut-off method has the physical meaning which is more than for convenience, as discussed in the introduction, and have calculated the anomalous magnetic moments of the nucleons and the gamma decay of the neutral meson. In order to compare the meson theory with the quantum electrodynamics, we have also calculated the anomalous magnetic moment of the electron. It has become clear in this way that the cut-off procedure has much improved the former theoretical results in the phenomena relating to the mesons. Of course, we cannot expect quantitative agreements because of the uncertainties in these calculations: the maximum momentum  $k_{\max}$ , higher order contributions and the effects of the other existing fields etc. We must rather put

\* Strictly speaking, we might assume the vector meson theory for the finite photon mass. However, it is the purpose of this investigation only to see the effect of the zero photon mass and so we adopt this procedure for convenience.

† We are indebted to Prof. Z. Koba for his helpful discussions in these points.

emphasis on qualitative aspects. If the existence of the cut-off momentum had to be expected in the theory itself, the value of the  $k_{\text{max}}$  would be much smaller in the meson theory than in the quantum electrodynamics, because the large radiative reactions are to be expected in the meson theory on account of the peculiar character of the coupling. From the considerations described in the introduction one might conclude that the nature of the cut-off should be found in a complete unified theory of elementary particles in future, rather than in the usual covariant renormalization theory.

In conclusion I wish to express my cordial thanks to Messrs. N. Fukuda, K. Sawada, S. Tani and D. Ito for their valuable discussions. Especially Mr. N. Fukuda's considerations as to the present status of meson theory has promoted me to examine the cut-off method. I should like also to express my hearty thanks to Messrs. K. Hasegawa and S. Matsuyama, who had also performed the same calculation of the magnetic moments of the nucleons, for their permission to publish this paper independently.

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## Note on the Angular Distribution of Photo-Reaction

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General formulae for the angular distribution of photo-reactions are obtained in a similar way to the case of particle reactions. Expressions thus obtained are more convenient in practical use than that obtained by Simon.

### § 1. Introduction

The general expression for the differential cross section in terms of the reaction matrix was obtained by Blatt and Biedenharn<sup>1)</sup> for the reaction in which photons are not involved as participants. In their method all summations over magnetic quantum numbers are performed and the final expression contains Racah coefficients or associated coefficients which are now tabulated<sup>2)</sup>. These formulae are, however, not applicable to the processes in which photons take part because the photon is different from other particles in two respects. Firstly, by virtue of the gauge condition, a photon has only two degrees of freedom, whereas its spin is unity, that is, the photon field should be transversal. Secondly, in particle reactions, the matrix element of a reaction matrix is usually specified by the channel spin which is the vector addition of the spins of target (residual) and incident (outgoing) particles and by the relative *orbital* angular momentum. On the other hand, in photo-reactions, the matrix element is specified by the *total* angular momentum and the parity of a photon, that is, we speak of electric or magnetic radiation with multipole order  $L$ .

Simon<sup>3)</sup> obtained the differential cross section and the polarization distribution for photo-reactions by performing a suitable transformation on the formulae for "particle" reactions. His results, however, contain some errors<sup>4)</sup> and further they are not always convenient for practical applications. We derived the differential cross section for photo-reactions directly by decomposing the incident plane wave of a photon into multipoles. The resulting formulae seem to be more convenient than Simon's which are shown to become equivalent to ours after suitable corrections.

### § 2. General expression for the photo-reaction

The reaction treated in this paper is of the type

$$a + X \rightarrow Y + b, \quad (1)$$

where  $a$  or  $b$  is either a photon or a particle other than photons, and  $X$  and  $Y$  are target and residual systems, respectively. Accordingly, to fix our problem, we call  $a$  and  $b$  reaction particles and  $X$  and  $Y$  nuclei, the first two being the particles on which observations are made. In this section, we derive the differential cross section for the particle  $b$  emerging at an angle  $\theta$  to the direction of the incident particle  $a$ . All quantities are measured in the center of mass system.

A system composed of a photon and a nucleus is described by the intrinsic spin  $I$  of the nucleus, the *total* angular momentum  $L$  and the parity  $\pi_\gamma$  of the photon, and channel index  $\alpha$  which defines the energy state of the nucleus. On the other hand, a system that contains a particle is usually described by the channel spin  $s$ , the *orbital* angular momentum  $l$ , and the channel index  $\alpha$ , which, in this case, defines the type of the reaction particle as well as the energy state of the nucleus. Such quantum numbers that refer to the final system will be primed.

We now consider the case where the incident particle is a photon and the outgoing one is a particle. The reverse process of this and the scattering of a photon can be treated in the same way. The wave function that describes a photon is taken to be the vector potential  $\mathbf{A}$ . This choice stems from the standpoint of quantized field theory and is found to be convenient for practical use. To write down the differential cross section in terms of reaction matrices whose matrix elements are specified by angular momenta and parities, we have first to decompose the incident wave into partial waves of suitable transformation characters, that is, into multipole radiations. The eigenfunction which corresponds to the orbital angular momentum  $l$ , and total angular momentum  $L$  of the photon has the asymptotic form

$$\phi_{LL}^{mL(\pm)} = \sum_{m_l} (l \ 1 \ m_l - m_l | l \ 1 \ L \ m_L) Y_{l \ m_l}^{mL}(\vartheta) \chi_1^{m_l} \exp [\pm i(kr - \pi l/2)], \quad (2)$$

where  $\chi_1^{m_l}$  is the spin wave function of the photon.  $Y_l^m(\vartheta)$  is the normalized spherical harmonics with the phase relation defined by Condon and Shortley<sup>5</sup>,  $(j_1 j_2 \ m_1 \ m_2 | j_1 j_2 \ j \ m)$  is the Clebsch-Gordan coefficient, and  $k$  is the wave number of the photon in the center of mass system. Since  $\phi_{LL}^{mL(\pm)}$  is transversal by itself but other two are not, we must construct a transversal wave by taking a suitable linear combination of the two nontransversal ones. This is carried out by Goertzel<sup>6</sup> and Blatt and Weisskopf<sup>7</sup>. Then, the multipole radiation can be expressed in the following compact form

$$\mathbf{A}_{Lp}^{mL(\pm)} = -\sqrt{2} \sum_l (L1-11 | L10) \delta(l, p) i^{l-L-p} \phi_{LL}^{mL(\pm)}, \quad (3)$$

where  $p$  is a quantum number that distinguishes electric and magnetic radiations. The electric  $2^L$ -pole radiation corresponding to  $\mathbf{A}$  has total angular momentum  $L$  and parity  $(-)^{1+L}$ , while the magnetic  $2^L$ -pole one has the same angular momentum but parity  $(-)^L$ . The parity change of the nucleus due to the emission or the absorption of a photon is opposite to the parity of the radiation concerned because the interaction Hamiltonian is either  $e\mathbf{p} \cdot \mathbf{A}$  or  $\mu \boldsymbol{\sigma} \cdot \text{curl} \mathbf{A}$ . We define  $p$  by  $\pi_\gamma = (-)^{p+L}$ , thus  $p=1$  and  $0$  represent the electric and magnetic radiations,



respectively. Accordingly,  $\delta(l, 1)$  and  $\delta(l, 0)$  specify  $l$  as  $l=L \pm 1$  and  $l=L$ , respectively. This definition of multipole radiations (3) is identical to that of Simon for  $p=0$  but differs by a phase factor  $i$  for  $p=1$ . With this choice of the phase, the reaction matrix element for the photo-reaction is proportional to the multipole moment  $Q$  or  $M$  derived by Blatt and Weisskopf<sup>7)</sup>, with a real proportional factor.

The total angular momentum  $J$  of the entire system is composed of the intrinsic spin  $I$  of the nucleus and the total angular momentum  $L$  of the photon. The eigenfunction that corresponds to  $J$  and its  $z$ -component  $M$  is written asymptotically as

$$\Psi'_{JLp}^{M(\pm)} = \sum_{m_I} (LIM - m_I \ m_I | L I J M) A_{Lp}^{M-m_I(\pm)} \chi_I^{m_I}. \quad (4)$$

Now we are prepared to decompose the incident plane wave of a photon into multipoles. The incident wave with left or right circular polarization on the target system with spin state  $\chi_I^{m_I}$  is given by  $\mp \exp(ikz) \chi_I^m \chi_I^{m_I}$ , ( $m = \pm 1$ ), which is expanded as follows:

$$\begin{aligned} & \mp \exp(ikz) \chi_I^m \chi_I^{m_I} \\ &= \mp \frac{i\pi^{1/2}}{kr} \sum_{l=0}^{\infty} \sum_{LJ} i^l (2l+1)^{1/2} (l \ 1 \ 0 \ m | l \ 1 \ L \ m) (LIm m_I | LIJM) [\Phi_{JLl}^{M(-)} - \Phi_{JLl}^{M(+)}], \\ &= \frac{i}{kr} (\pi/2)^{1/2} \sum_{LJ,p} i^{l+p} (2L+1)^{1/2} (m)^p (LIm m_I | LIJM) [\Phi_{JLp}^{M(-)} - \Phi_{JLp}^{M(+)}] \end{aligned} \quad (5)$$

in which  $\Phi_{JLl}^{M(\pm)}$  is obtained by replacing  $A_{Lp}^{m(\pm)}$  in (4) by  $\phi_{Ll}^{m(\pm)}$ . In the calculation of (5), use has been made of the orthonormality of the Clebsch-Gordan coefficients and the symmetry relations among them<sup>8)</sup>.

To describe the process which leads to a "particle" emission, it is necessary to define the eigenfunction in the channel spin formalism. The eigenfunction with total angular momentum  $J$ ,  $z$ -component thereof  $M$ , orbital angular momentum  $l$ , and channel spin  $s$  is defined by

$$\Psi_{Jls}^{M(\pm)} = \sum_{m_s} (lsM - m_s \ m_s | lsJM) Y_l^{M-m_s}(\varrho) \chi_s^{m_s} \exp[\pm i(kr - \pi l/2)], \quad (6)$$

where  $\chi_s^{m_s}$  is the channel spin wave function. Then, the most general wave function of the entire system with definite  $J$  and  $M$ , is represented asymptotically by a linear combination over  $L, p, l$  and  $s$  of the wave functions

$$\phi_{\alpha l p}(JM) = \frac{1}{r_{\alpha}(c)^{1/2}} f_{\alpha} [A_{\alpha l p}^{JM} \Phi_{JLp}^{M(-)} + B_{\alpha l p}^{JM} \Phi_{JLp}^{M(+)}], \quad (7)$$

$$\phi_{\alpha ls}(JM) = \frac{1}{r_{\alpha}(v_{\alpha})^{1/2}} g_{\alpha} [C_{\alpha ls}^{JM} \Psi_{Jls}^{M(-)} + D_{\alpha ls}^{JM} \Psi_{Jls}^{M(+)}], \quad (7')$$

in which  $f_{\alpha}$  represents the internal wave function of the nucleus,  $g_{\alpha}$  the product of those of the particle and the nucleus, and  $v_{\alpha}$  is the velocity of the reaction particle in channel  $\alpha$ . When the incident particle is a photon and the energy state of the target system is



specified by  $\alpha_0$ , the amplitudes of the incoming wave are given, comparing (5) and (7), by

$$A_{\alpha l p}^{JM} = i \hat{\sigma}_{\alpha \alpha_0} k_{\alpha}^{-1} (\pi c/2)^{1/2} i^{L+p} (2L+1)^{1/2} (m)^p (LIm m_l | LJM), \quad (8)$$

$$C_{\alpha l s}^{JM} = 0. \quad (8')$$

$B_{\alpha l p}^{JM}$  and  $D_{\alpha l s}^{JM}$  are determined from  $A_{\alpha l p}^{JM}$  with the reaction matrix :

$$B_{\alpha' l' p'}^{JM} = \sum_{\alpha l p} [R(\alpha' l' p', \alpha l p; J\pi) - \hat{\sigma}_{\alpha' \alpha} \hat{\sigma}_{l' l} \hat{\sigma}_{p' p}] A_{\alpha l p}^{JM}, \quad (9)$$

$$D_{\alpha' l' s'}^{JM} = \sum_{\alpha l p} R(\alpha' l' s', \alpha l p; J\pi) A_{\alpha l p}^{JM}, \quad (9')$$

where  $\pi$  is the parity of the entire system,  $\pi = \pi_{\gamma} \pi_X = (-)^{p+l+1} \pi_X = \pi_l \pi_Y (-)^{l'}$ .

The asymptotic wave function in the region of the configuration space corresponding to channel  $(\alpha', s')$  can be written analogous to BB<sup>1)</sup>,

$$\psi_{\text{reac}}(\alpha' s') = i k_{\alpha'}^{-1} \left( \frac{c}{v_{\alpha'}} \right)^{1/2} \frac{\exp(i k_{\alpha'} r_{\alpha'})}{r_{\alpha'}} \mathcal{G}_{\alpha'} q_{\alpha' s' m_{s'}; \alpha m_{l'm}}(\mathcal{Q}) \chi_{s'}^{m_{s'}} \quad (10)$$

where the "reaction amplitude"  $q_{\alpha' s' m_{s'}; \alpha m_{l'm}}$  is given by

$$q_{\alpha' s' m_{s'}; \alpha m_{l'm}}(\mathcal{Q}) = \sum_{Jl p l'} i^{L-l'+p} (\pi/2)^{1/2} (2L+1)^{1/2} (m)^p (LIm m_l | LJM) \quad (11)$$

$$\times (l' s' \mu' m_{s'} | l' s' J M) R(\alpha' l' s', \alpha l p; J\pi) Y_{l'}^{\mu'}(\mathcal{Q}).$$

The differential cross section for the process  $\alpha_0 \rightarrow \alpha'$  can be written as

$$d\sigma_{\alpha' \alpha_0} = \frac{1}{2(2L+1) k_{\alpha_0}^2} \sum_{s'} \sum_{m_{s'} m_{l'm}} |q_{\alpha' s' m_{s'}; \alpha m_{l'm}}(\mathcal{Q})|^2 d\mathcal{Q}. \quad (12)$$

The sum over all magnetic quantum numbers is geometrical in character since the reaction matrix, which alone bears the dynamical nature of the system, is independent of the magnetic quantum numbers. This summation will be done in the next section along the same line as BB.

### § 3. Reduction of the differential cross section

As  $q$  is linear in  $Y_{l'}^{\mu'}$ ,  $d\sigma$  is bilinear in it, but we can develop the product of two spherical harmonics by single  $Y$ 's. After this reduction is performed,  $d\sigma$  is reduced to

$$\begin{aligned} d\sigma_{\alpha' \alpha_0} = & \frac{1}{2(2L+1) k_{\alpha_0}^2} \sum_{\substack{l_1' l_2' l_1 l_2 m_1 m_2 \\ l_1' l_2' s' l_1 l_2}} i^{-l_1+l_2-l_1'+l_2'+p_2+l_1'-l_2'} \\ & \times [(2L_1+1)(2L_2+1)(2l_1'+1)(2l_2'+1)/(2L+1)]^{1/2} \\ & \times R^*(1) R(2) (l_1' l_2' 00 | l_1' l_2' L 0) \sum_{m_{s'} m_{l'm}} (-)^{\mu'} (m)^{p_1+p_2} (L_1 l m m_l | L_1 l J_1 M) \\ & \times (L_2 l m m_l | L_2 l J_2 M) (l_1' s' \mu' m_{s'} | l_1' s' J_1 M) (l_2' s' \mu' m_{s'} | l_2' s' J_2 M) \\ & \times (l_1' l_2' -\mu' \mu' | l_1' l_2' L 0) (\pi^{1/2}/4) Y_{L,0}(\mathcal{Q}). \end{aligned} \quad (13)$$

In the magnetic sum, the independent variables are  $m = \pm 1$ ,  $m_1$  and  $m_{s'}$ . The sum over  $m_1$  and  $m_{s'}$  can be performed with the same technique as BB and we get

$$\begin{aligned}
 d\sigma_{\alpha' \alpha_0} = & \frac{1}{2(2I+1)k_{\alpha_0}^2} \sum_{m=\pm 1} \frac{(-)^{s'-I-1}}{8} i^{-L_1+L_2-p_1+p_2+l_1'-l_2'} R^*(1) R(2) \\
 & \times (2J_1+1)(2J_2+1) [(2L_1+1)(2L_2+1)(2l_1'+1)(2l_2'+1)]^{1/2} \\
 & \times (l_1' l_2' 0 0 | l_1' l_2' L 0) W(l_1' J_1 l_2' J_2; s' L) W(L_1 J_1 L_2 J_2; I L) \\
 & \times \sum_{m=\pm 1} (-)^{L-L_1+L_2} (m)^{p_1+p_2} (L_1 L_2 - mm | L_1 L_2 L 0) P_L(\cos \theta) d\Omega.
 \end{aligned} \tag{14}$$

Since  $(L_1 L_2 m - m | L_1 L_2 L 0) = (-)^{L_1+L_2-L} (L_1 L_2 - mm | L_1 L_2 L 0)$ , the sum over  $m$  is trivial and the final result is

$$\begin{aligned}
 d\sigma_{\alpha' \alpha_0} = & \frac{1}{2(2I+1)k_{\alpha_0}^2} \sum_{m=\pm 1} \frac{(-)^{s'-I-1}}{4} R^*(1) R(2) \\
 & \times i^{L-L_1+L_2-p_1+p_2} [(2L_1+1)(2L_2+1)(2J_1+1)(2J_2+1)]^{1/2} \\
 & \times (L_1 L_2 1 - 1 | L_1 L_2 L 0) \\
 & \times W(L_1 J_1 L_2 J_2; I L) Z(l_1' J_1 l_2' J_2; s' L) P_L(\cos \theta) d\Omega,
 \end{aligned} \tag{15}$$

with the restriction

$$p_1 + p_2 + L_1 + L_2 - L = \text{even}. \tag{15'}$$

The sum is over  $J_1 J_2 L_1 L_2 p_1 p_2 l_1' l_2' s'$  and  $L$ . The restriction (15') is fulfilled automatically if we consider the parity selection rule imposed on  $R^*(1) R(2)$  and  $Z$ , since this rule implies  $p_1 + p_2 + L_1 + L_2 + l_1' + l_2' = \text{even}$  and  $l_1' + l_2' + L = \text{even}$ .

Simon's result for the corresponding process contains an extra factor  $f(q_1 q_2 q \kappa)$  by virtue of which the interference term between electric and magnetic radiations disappears. His argument for the necessity of this factor, however, is hard to understand (cf. the paragraphs preceding eq. (6.5) of his paper).<sup>\*</sup> In Appendix I, we will show that our result (15) is equal to the following expression

$$\begin{aligned}
 d\sigma_{\alpha' \alpha_0} = & \frac{1}{2(2I+1)k_{\alpha_0}^2} \sum_{m=\pm 1} \frac{(-)^{s'-I-1}}{2} R^*(1) R(2) [(2J_1+1)(2J_2+1)]^{1/2} (-)^{L-L_1+L_2} \\
 & \times \left[ \sum_{h l_2} (L_1 1 - 1 1 | L_1 1 l_1 0) (L_2 1 - 1 1 | L_2 1 l_2 0) \delta(l_1 p_1) \delta(l_2 p_2) i^{L_2-L_1-p_1+p_2+l_1'-l_2} \right. \\
 & \left. \times Z(l_1 L_1 l_2 L_2; 1 L) \right] W(L_1 J_1 L_2 J_2; I L) Z(l_1' J_1 l_2' J_2; s' L) P_L(\cos \theta) d\Omega,
 \end{aligned} \tag{16}$$

which is identical to Simon's result after the correction mentioned above, considering the

<sup>\*</sup>  $T_{\eta}^{\kappa}$  there is defined not in terms of the total angular momentum of the reaction particle but in terms of the spin angular momentum, whereas the multipole expansion of the plane wave means that it is expanded by eigenfunctions of the total angular momentum of the photon. The probability amplitude of the state with definite multipolarity and parity is not to be confused with that of the spin state.

difference of factor  $i^p$  in the definition of  $A(L, M, p)$ .

The differential cross section for the process in which the role of photon and the other particles is interchanged or both  $a$  and  $b$  are photons can be obtained in the same way. In this case,  $\Phi_{J, L, M}^{(1)}$  of the photon eigenfunction in the final state must be reduced to  $Y_{l_1'}^{m_1'}(\Omega) \chi_{l_2'}^{m_2'}$ , since the observed quantities in experiments are  $\theta$  and  $\varphi$  rather than  $J, L, M$ , and  $p$ . Then, the summation is over all magnetic quantum numbers and  $l_1'$  and  $l_2'$ . The latter can be performed as we did above, to show the equivalence of the revised Simon's result and ours (cf. Appendix I).

These cross sections are obtained for :

Photon emission caused by a "particle"

$$\begin{aligned} d\sigma_{\alpha\alpha'} = & \frac{1}{(2i+1)(2f+1)k_a^2} \sum \frac{(-)^{p+1-s}}{4} R^*(1)R(2) \\ & \times [(2L_1'+1)(2L_2'+1)(2J_1+1)(2J_2+1)]^{1/2} \\ & \times i^{L_1'-L_2'-L+p_1'-p_2'} (L_1' L_2' 1-1 | L_1' L_2' L 0) W(L_1' J_1 L_2' J_2; I' L) \\ & \times Z(L_1 J_1 L_2 J_2; sL) P_L(\cos \theta) d\Omega, \end{aligned} \quad (17)$$

where  $i$  is the spin of the incident particle.

Scattering of photon

$$\begin{aligned} d\sigma_{\alpha\alpha'} = & \frac{1}{2(2I+1)k_a^2} \sum \frac{(-)^{p'-I}}{4} R^*(1)R(2) \\ & \times [(2L_1+1)(2L_2+1)(2L_1'+1)(2L_2'+1)(2J_1+1)^2(2J_2+1)^2]^{1/2} \\ & \times i^{L_1'-L_2'-L_1+L_2+p_1'-p_2'-p_1+p_2} (L_1 L_2 1-1 | L_1 L_2 L 0) (L_1' L_2' 1-1 | L_1' L_2' L 0) \\ & \times W(L_1 J_1 L_2 J_2; I L) W(L_1' J_1 L_2' J_2; I' L) P_L(\cos \theta) d\Omega, \end{aligned} \quad (18)$$

with the restriction of the relation (15').

The expectation values of a general tensor moment can be obtained by the similar method to that used for differential cross sections. The resulting expressions can be shown to be identical to those obtained by Simon<sup>31</sup>, summing over possible orbital angular momenta of the  $\gamma$ -ray, provided that the correction discussed above is taken into account. A similar but more general procedure has been performed independently by Kennedy and Sharp<sup>11</sup>. As an example we show the result for the polarization of a particle produced by an unpolarized photon,

$$\begin{aligned} \langle T_{\kappa'}^1 \rangle = & \frac{1}{2(2I+1)k_a^2} \left[ \frac{2i'+1}{48} \right]^{1/2} \sum (-)^{i'-I'+1-I+p_3+s_2'} i^{L_2-L_1+p_2-p_1} \\ & \times R^*(1)R(2) [(2L_1+1)(2L_2+1)(2J_1+1)(2J_2+1)]^{1/2} (L_1 L_2 1-1 | L 0) \\ & \times W(L_1 J_1 J_2 L_2; I L) W(i' s_1' i' s_2'; I' 1) \\ & \times G_{\kappa'}(J_1 L_1 s_1'; L-1; J_2 L_2 s_2') \left[ \frac{4\pi}{2L+1} \right]^{1/2} Y_{L,\kappa'}(\theta 0) d\Omega, \end{aligned}$$

where  $G_{s'}(J_1 l_1' s_1'; L-1; J_2 l_2' s_2')$  is the function which is defined by Simon<sup>3)</sup>, and  $i'$  is the spin of the outgoing particle.

For the practical application of the formulae, it is desirable to write them in terms of such functions of angular momenta which are available in published tables. We have no tables of the Clebsch-Gordan coefficient  $(L_1 L_2 1-1 | L_1 L_2 L 0)$ , but this can be expressed by  $Z$ -coefficient as follows<sup>9)</sup>.

$$\sqrt{(2L_1+1)(2L_2+1)} i^{L-L_1+L_2-p_1+p_2} (L_1 L_2 1-1 | L_1 L_2 L 0) = Z(L_1 L_1 L_2 L_2; 1 L) \quad \text{for } L_1+L_2+L=\text{even}, \quad (19)$$

$$\begin{aligned} & \sqrt{(2L_1+1)(2L_2+1)} i^{L-L_1+L_2-p_1+p_2} (L_1 L_2 1-1 | L_1 L_2 L 0) \\ &= (-)^{n_1} \sqrt{\frac{2L_2+1}{L_2}} Z(L_1 L_1 L_2+1 L_2; 1 L) \quad \text{for } L_1+L_2+L=\text{odd}. \end{aligned} \quad (19')$$

#### § 4. Selection rules and symmetry properties

The selection rules pertinent to the photon is contained in the Clebsch-Gordan coefficient  $(L_1 L_2 1-1 | L_1 L_2 L 0)$  and the selection rule (15'). Thus, if there is the largest multipole order  $L_{max}$  in the process, the maximum angular complexity of the angular distribution is given by  $L \leq 2L_{max}$ . Further, the permissible values of  $L$  are restricted by  $L_1+L_2+L=\text{odd}$  for the interference terms between electric and magnetic radiations and  $L_1+L_2+L=\text{even}$  otherwise.

If the subscripts 1 and 2 are interchanged in eq. (15), (17), or (18), the resulting expressions are restored to their original forms by considering the symmetry properties of the Clebsch-Gordan,  $W$ , and  $Z$  coefficients.<sup>8)</sup> The element of the reaction matrix is always written in the form  $R^*(1)R(2)+R^*(2)R(1)$  and, as  $i^{L-L_1+L_2-p_1+p_2}$  is real, the cross section is manifestly real.

When the channel spin of the final system is  $1/2$ , there exists an interesting symmetry property which was first clarified by an explicit calculation of Hayakawa, Kawaguchi and Minami<sup>10)</sup>, that is, if the matrix elements for electric and magnetic radiations with the same  $J$ 's and  $L$ 's are interchanged for every  $J$  and  $L$ , the resulting expression does not change at all. To prove this theorem from our general formula (17), it is to be noted that  $l'$  takes only two values  $J \pm 1/2$  in this case and the interchange of  $p=1$  and  $p=0$  requires that  $l'=J+1/2$  should be replaced by  $l'=J-1/2$  and *vice versa* to fulfill the parity condition. Then, by virtue of the identities

$$Z(J-1/2 J J'-1/2 J'; 1/2 L) = Z(J+1/2 J J'+1/2 J'; 1/2 L), \quad (20)$$

$$Z(J+1/2 J J'-1/2 J'; 1/2 L) = -Z(J-1/2 J J'+1/2 J'; 1/2 L), \quad (20')$$

which are proved in Appendix II, the differential cross section is invariant under the transformation

$$R(\alpha' J \pm 1/2 \ 1/2, \alpha L 1; J\pi) \rightleftharpoons (-) R(\alpha' J \mp 1/2 \ 1/2, \alpha L 0; J\pi),$$

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### Appendix I. Comparison with Simon's formula

Comparing (16) and (15), the following identity is to be proved if they are equivalent to one another.

$$\sum_{l_1 l_2} (L_1 1 - 11 | L_1 1 \ l_1 0) (L_2 1 - 11 | L_2 1 \ l_2 0) \delta(l_1 p_1) \delta(l_2 p_2) [(2L_1 + 1)(2L_2 + 1)]^{1/2} \\ \times (l_1 l_2 00 | l_1 l_2 L_0) W(l_1 L_1 l_2 L_2; 1L) = (1/2) (-)^{L-L_1+L_2} (L_1 L_2 1 - 1 | L_1 L_2 L_0). \quad (\text{A1})$$

Here, the  $W$ -coefficient is rewritten with more familiar quantities according to its definition. First, we consider the case  $p_1 = p_2 = 0$ . Then, the left hand side of (A1) is reduced to, using the identity  $(L 1 - 11 | L 1 L 0) = -1/\sqrt{2}$ ,

$$(1/2) [(2L_1 + 1)(2L_2 + 1)]^{1/2} (L_1 L_2 00 | L_1 L_2 L_0) W(L_1 L_1 L_2 L_2; 1L). \quad (\text{A2})$$

This is expressible as a sum of three Clebsch-Gordan coefficients (BBR (19)).

$$(1/2) (-)^{1+L-L_1-L_2} \sum_m (L_1 1 0 m | L_1 1 L_1 m) (L_1 L_2 m - m | L_2 L_2 L_0) (1 L_2 m - m | 1 L_2 L_2 0) \quad (\text{A3})$$

By virtue of the symmetry properties of the Clebsch-Gordan coefficient, the sum over  $m$  results in twice the term with  $m=1$  and we have

$$(-)^{1+L-L_1-L_2} (L_1 1 0 1 | L_1 1 L_1 1) (L_1 L_2 1 - 1 | L_1 L_2 L_0) (1 L_2 1 - 1 | 1 L_2 L_2 0) \\ = (1/2) (-)^{L-L_1+L_2} (L_1 L_2 1 - 1 | L_1 L_2 L_0). \quad (\text{A4})$$

Secondly, we prove that the left hand side of (A1) for  $p_1 = p_2 = 1$  is equal to that for  $p_1 = p_2 = 0$ . An equivalent form of this equality is

$$\sum_{l_1 l_2} (L_1 1 - 11 | L_1 1 \ l_1 0) (L_2 1 - 11 | L_2 1 \ l_2 0) (l_1 l_2 00 | l_1 l_2 L_0) [(2L_1 + 1)(2L_2 + 1)]^{1/2} \\ \cdot W(l_1 L_1 l_2 L_2; 1L) \\ - \sum_{l_1} (L_1 1 - 11 | L_1 1 \ l_1 0) (L_2 1 - 11 | L_2 1 \ l_2 0) (l_1 L_2 00 | l_1 L_2 L_0) [(2L_1 + 1)(2L_2 + 1)]^{1/2} \\ \cdot W(l_1 L_1 L_2 L_2; 1L) \quad (\text{A5}) \\ - \sum_{l_2} (L_1 1 - 11 | L_1 1 \ l_1 0) (L_2 1 - 11 | L_2 1 \ l_2 0) (l_1 L_2 00 | l_1 L_2 L_0) [(2L_2 + 1)(2L_1 + 1)]^{1/2} \\ \cdot W(L_1 L_1 L_2 L_2; 1L) = 0.$$

For the sum over  $l_1$  or  $l_2$ , the eq. (18) of BBR should be applied, and then, the sum over remaining  $L$  is nothing but the orthonormality relation of the Clebsch-Gordan coefficient and (A5) is reduced to



$$\begin{aligned}
 & (-)^{l-l_1-l_2} (I_1 L_2 1 - 1 | L_1 L_2 L 0) \\
 & - (-)^{l-l_1-l_2} (I_2 1 - 1 | I_2 1 L_1 0) (I_2 1 0 1 | I_2 1 L_2 1) (L_1 L_2 1 - 1 | L_1 L_2 L 0) \\
 & - (-)^{l-l_1-l_2} (L_1 1 - 1 | L_1 1 L_1 0) (L_1 1 0 1 | L_1 1 L_1 1) (L_1 L_2 1 - 1 | L_1 L_2 L 0) = 0.
 \end{aligned} \tag{A6}$$

Remembering the identity given just before (A2), we have the desired equality.

Lastly, in the case of  $p_1=0$ ,  $p_2=1$  (or  $p_1=1$ ,  $p_2=0$ ) ( $L_1+L_2+L=\text{odd}$ ), the sum over  $l_2=L_2\pm 1$  can be extended to include  $l_2=L_2$ , because  $(L_1 L_2 0 0 | L_1 L_2 L 0)=0$  in this case. Thus, the calculation is carried out in the same way as that for reducing the left hand side of (A5).

$$\begin{aligned}
 & \sum_{l_2} (L_1 1 - 1 | L_1 1 L_1 0) (L_2 1 - 1 | L_2 1 L_2 0) (L_1 L_2 0 0 | L_1 L_2 L 0) [(2L_1+1)(2L_2+1)]^{1/2} \\
 & \quad \cdot W(L_1 L_1 L_2 L_2; 1 L) \\
 & = (-)^{l-l_1-l_2} (L_1 1 - 1 | L_1 1 L_1 0) (L_1 1 0 1 | L_1 1 L_1 1) (L_1 L_2 1 - 1 | L_1 L_2 L 0) \\
 & = (1/2) (-)^{l-l_1-l_2} (L_1 L_2 1 - 1 | L_1 L_2 L 0).
 \end{aligned} \tag{A7}$$

This completes the proof.

## Appendix II.

Expressing the  $Z$ -coefficient by the  $W$ -coefficients and using a symmetry property of  $W$ , eq. (20) becomes

$$\begin{aligned}
 & \sqrt{J J'} (J-1/2 J'-1/2 0 0 | J-1/2 J'-1/2 L 0) W(J-1/2 1/2 L J'; J J'-1/2) \\
 & = \sqrt{(J+1)(J'+1)} (J+1/2 J'+1/2 0 0 | J+1/2 J'+1/2 L 0) \\
 & \quad \cdot W(J+1/2 1/2 L J'; J J'+1/2).
 \end{aligned}$$

According to eq. (19) of BBR, this is written as

$$\begin{aligned}
 & \sqrt{J J'} \sum_{\beta=\pm 1/2} (J-1/2 1/2 0 \beta | J-1/2 1/2 J \beta) (J J' \beta - \beta | J J' L 0) \\
 & \quad \cdot (J'-1/2 1/2 0 \beta | J'-1/2 1/2 J' \beta) (-)^{1/2-\beta} \\
 & = - \sqrt{(J+1)(J'+1)} \sum_{\beta=\pm 1/2} (J+1/2 1/2 0 \beta | J+1/2 1/2 J \beta) (J J' \beta - \beta | J J' L 0) \\
 & \quad \cdot (J'+1/2 1/2 0 \beta | J'+1/2 1/2 J' \beta) (-)^{1/2-\beta}.
 \end{aligned}$$

Inserting the explicit form for the Clebsch-Gordan coefficients which contain  $1/2$ ,

$$\begin{aligned}
 (J-1/2 1/2 0 \beta | J-1/2 1/2 J \beta) & = \sqrt{(J+1/2)/2J}, \\
 (J+1/2 1/2 0 \beta | J+1/2 1/2 J \beta) & = (-)^{1/2+\beta} \sqrt{(J+1/2)/2(J+1)}.
 \end{aligned}$$

We observe that the identity holds.

The proof of eq. (20') can be done analogously.

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# On the Meaning of the Solution to the One-Body Dirac Equation

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In this paper we try to give another demonstration of a theorem concerning a fermion interacting with an external field; this theorem, which is first shown by Furry, states that in the treatment of the interaction with *external* fields it is sufficient for one to know the complete set of the solutions to the one-body Dirac equation, even when the fermion field is quantized according to the hole theory for negative energy states. Our demonstration rests on the examination of the effects which are generated by a special type of canonical transformation on the Hamiltonian of the system.

But the knowledge of the one-body solution is insufficient to predict the results of the hole theory, when the fermion is interacting with a quantized boson field. By the further examination of the transformation similar to the above, we can find to what extent the one-body solution is efficient in the whole scheme of quantized fields.

Using some techniques of the operator calculus, a way is shown by which we can compute exactly, at least formally, the correction terms coming from the effects of the negative energy sea of fermions. The effects of the hole-theoretical vacuum may be classified into two kinds; one is due to the genuine vacuum polarization which comes into play in our calculation in the form of spur expressions (and is represented by closed loops in Feynman diagrams), and the other is due to the exclusion principle which acts on the negative energy intermediate states (and plays its role in the process represented by connected one-line Feynman diagrams). Formal procedures to compute the latter type of corrections to the one-body solution are explained by the example of nucleon field interacting with neutral pseudoscalar meson field.

Four-dimensional aspect of the problem and the invariance to the charge conjugation are also considered.

## § 1. Introduction

It is well known that the expression of the second order self-energy for a fermion interacting with a quantized boson field is different from the hole-theoretical one when we calculate with the one body theory; the contribution of a negative energy state is of opposite sign in the one-body theory as compared with that in the hole theory. On the other hand, it is also well known that both theories give the same result in the calculation of the Compton scattering (up to the second order). In view of these situations, we may naturally bring forward the following questions; "How far can we proceed with the one-body theory?" and "Are there any simple relations between the results of both theories and can we expect some simple modifications of the one-body-theoretical results which

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would enable us to predict the hole-theoretical results? ". In this note we try to throw some light on these questions.

In this connection it is noticeable that Furry introduced the bound-state representation of spinor operators and established<sup>1)</sup> a theorem which states that in the treatment of fermions interacting with an *external* field it is sufficient for one to know the complete set of the solutions to the (unquantized) one-body Dirac equation in order that one may predict the results of the quantized theory. First we try to prove this theorem from a new point of view, i.e. that of canonical transformations performed on the Hamiltonian of the system. This way of approach enables us to make some discussions in more general cases than those of interaction with external fields.

## § 2. The proof of Furry's theorem

Our starting point is to examine the results of the canonical transformation performed on any field operator, when the transformation function is given by an operator of the form

$$U = \exp \left[ i \int \psi^*(x) A(x) \psi(x) dx \right]. \quad (1)$$

Here  $\psi$  and  $\psi^*$  is the spinor operator for the fermion and its Hermitian conjugate respectively, and their mutual commutation relation is

$$\{\psi_\alpha(x), \psi_\beta^*(x')\} = \delta_{\alpha\beta} \delta(x-x'), \quad (2)$$

where the curled bracket means to take the anticommutator\*. In eq. (1) the operator  $A(x)$  (which must be Hermitian) is given by a certain linear combination of Dirac matrices,  $\rho_i \sigma_j$  ( $i, j = 0, 1, 2, 3$ ;  $\rho_0 = \sigma_0 = 1$ ), where their respective coefficients are generally functions of the coordinate  $x$  and the linear momentum  $-i\nabla$  of the fermion. In the next section we shall consider the cases where these coefficients are functions of quantized operators for the boson. It is implied by the transformation of the type (1) that all fermions present in the system behave as if they were independently subjected to the change of state governed by the same law; this will be explained at the end of the next section.

To see the difference between the one-body-theory and the hole-theory, it is necessary to discriminate the spinor operators for positive energy states from those for negative energy states. In the one-body theory the quantization is made in such a way that, irrespective of the sign of energy,  $\psi$ -operator annihilates fermions, while  $\psi^*$ -operator creates them. On the other hand, in the hole theory the quantization is made according to

$$\psi_\alpha = \psi_{+\alpha} + \psi_{-\alpha}^\dagger, \quad \psi_\alpha^* = \psi_{+\alpha}^\dagger + \psi_{-\alpha}, \quad (3)$$

where the operators with dagger are creation operators and the sign  $\pm$  denotes that of energy. The commutation relations for component operators are given by

$$\begin{aligned} \{\psi_{+\alpha}(x), \psi_{+\beta}^\dagger(x')\} &= 1/2 \cdot (1 + \varepsilon)_{\alpha\beta} \delta(x-x'), \\ \{\psi_{-\alpha}^\dagger(x), \psi_{-\beta}(x')\} &= 1/2 \cdot (1 - \varepsilon)_{\alpha\beta} \delta(x-x'), \end{aligned} \quad (4)$$

\* All operators are those of the Schroedinger representation.

where  $\mathcal{E}$  is the operator of the sign of energy, i.e.,

$$\mathcal{E} = (M\rho_3 - i\rho_1(\boldsymbol{\sigma}\nabla)) / \sqrt{M^2 - \nabla^2}. \quad (5)$$

As can be shown easily, in both cases of quantization we have the same transformation formulae for the spinor operators by means of eq. (2) ;

$$\begin{aligned} \psi_\alpha(x) &\rightarrow U^{-1}\psi_\alpha(x)U = (e^{iA(x)})_{\alpha\beta}\psi_\beta(x), \\ \psi_\alpha^*(x) &\rightarrow U^{-1}\psi_\alpha^*(x)U = \psi_\beta^*(x) (e^{-iA(x)})_{\beta\alpha}. \end{aligned} \quad (6)$$

Now, in the case of fermions interacting with an external field, the Hamiltonian of the system is of the form

$$H = \int \psi^*(x) \{M\rho_3 - i\rho_1(\boldsymbol{\sigma}\nabla) + V(x)\} \psi(x) dx. \quad (7)$$

(We assume here that the external field is of the same type as the electrostatic potential, since other types can be treated similarly.) Then the transformation of the form given by eq. (1) changes the Hamiltonian of the system into the form,

$$U^{-1}HU = \int \psi^*(x) e^{-iA(x)} \{M\rho_3 - i\rho_1(\boldsymbol{\sigma}\nabla) + V(x)\} e^{iA(x)} \psi(x) dx. \quad (8)$$

This result is the consequence of the transformation properties of the spinor operators, eq. (6), as well as of the following properties of the operators which appear in the original Hamiltonian (7), e.g.  $\rho_3\sigma_j$ ,  $\boldsymbol{\sigma}\nabla$  and  $-i\nabla$ . These operators are such that every matrix element (matrix notations are here concerned with the four spinor indices) is commutable with the generating function of the transformation,

$$[B_{\alpha\beta}, \int \psi^*(x) A(x) \psi(x) dx] = 0 \quad (9)$$

( $B$ : any one of the above mentioned operators).

It is true that these are not commutable with the density operator  $\psi^*(x) A(x) \psi(x)$  itself, but on integration over the volume enclosing the system the result tends to be infinitesimally small as the volume becomes infinitely large; and we may put these commutators zero in the limit. According to the normalization of the system, it is only when the commutator with the density of the transformation generator contains a three dimensional delta function, i.e.

$$[O(x), \psi^*(x') A(x') \psi(x')] = \delta(x - x') \text{ (other factors)}, \quad (10)$$

that the operator  $O(x)$  gets changed after the transformation of the type (1). In the case now considered  $\psi$  and  $\psi^*$  only have this property; and the transformation formulae for them have been given by eq. (6).

Then we can make the total Hamiltonian of the system diagonal, if we can make the operator diagonal which operates on the spinors in eq. (8), namely,

$$e^{-iA(x)} \{M\rho_3 - i\rho_1(\boldsymbol{\sigma}\nabla) + V(x)\} e^{iA(x)} = \text{a diagonal operator}, \quad (11)$$

and if at the same time we can have the complete set of state vectors for this operator,



The diagonalization of the operator (11) is nothing but the solution of the unquantized one-body Dirac equation by means of a canonical transformation. This procedure may be regarded in principle as feasible, though the canonical transformation of scattering states is not yet fully worked out. So long as there is a one-to-one correspondence of states before and after the transformation, we can safely assume that the vacuum fermions in the presence of the external field are constituted by those which are in negative energy states when the system is free from the external field; and we can easily state the final results according to the hole theory. Since the transformed Hamiltonian is of the form

$$U^{-1}HU = \int \psi^*(x) \{ \text{a diagonal operator} \} \psi(x) dx \quad (12)$$

where  $\psi$  and  $\psi^*$  are operators given by eq. (3), the ordering of the Hamiltonian in a way to fit to the hole theory, i.e. the interchange of  $\psi_-$  and  $\psi_-^\dagger$ , is an easy task. This is our demonstration of Furry's theorem.

### § 3. The limitation of the one-body solution in the scheme of quantized fields

In this section we will investigate the case of interaction with a quantized boson field; in this case it is no longer sufficient for us only to know the solution to the one-body problem. In order to avoid confusions caused by using too much general terms, we specialize our considerations to the problem of nucleons interacting with neutral pseudoscalar mesons by pseudoscalar coupling. The Hamiltonian of the system is then given by

$$H = \int \psi^*(x) \{ M\rho_3 - i\rho_1(\boldsymbol{\sigma}\boldsymbol{\nabla}) + g\rho_2\phi(x) \} \psi(x) dx + H_M, \quad (13)$$

where  $H_M$  is the Hamiltonian for the free meson field,

$$H_M = 1/2 \cdot \int \{ \pi^2(x) + (\boldsymbol{\nabla}\phi(x))^2 + \mu^2\phi^2(x) \} dx,$$

$\phi(x)$  and  $\pi(x)$  are the meson field operator and its canonical conjugate respectively, and  $g$  is the coupling constant.

To find the limitation of the one-body theory, we continue to examine the effects which are generated by a canonical transformation with transformation function of the form (1). The statements at the beginning of the preceding section (which are accompanied by eqs. (1)–(6)) are valid also in this section, except that here  $A(x)$  is a certain linear combination of Dirac matrices,  $\rho_i\sigma_j$ , where their respective coefficients are generally functions of  $\phi(x)$  and  $\pi(x)$  as well as their derivatives.

Every operator gets changed by the canonical transformation (1), if its commutator with the density of generating function  $\psi^*(x)A(x)\psi(x)$  contains a three-dimensional delta function, as shown in eq. (10). In the case discussed in the preceding section the operators which have the mentioned property are only  $\psi$  and  $\psi^*$ ; and, as it was discussed there, this fact is the guarantee for the validity of the one-body solution even when the fermion field is quantized according to the hole-theory. However, in the treat-

ment of interactions with a quantized boson field it is not only  $\psi$  and  $\psi^*$  but in general also  $\phi$  and  $\pi$  that get changed by the transformation (1). The transformation formulae for  $\psi$  and  $\psi^*$  are the same as in eq. (6);  $\phi$  and  $\pi$  are transformed according to

$$U^{-1}B(x)U = \sum_{n=0}^{\infty} B_n(x)/n! \quad (14)$$

$$B_n(x) = \int \psi^*(x') [e^{-iA(x')} B_{n-1}(x) e^{iA(x')} - B_{n-1}(x)] \psi(x') dx'$$

for  $n \geq 1$ ,

where  $B(x)$  is either  $\phi(x)$  or  $\pi(x)$ , and  $B_0(x) = B(x)$ .

The proof for the result (14) is given in the same way as that given to prove eq. (18) below\*. It is possible that, when  $A(x)$  depends only on  $\phi(x)$ , then  $\phi$  remains unchanged and the transformation formula for  $\pi$  is very much simplified, but the restriction of  $A(x)$  to this type is not suitable for the general investigation of the dynamical problem given by the Hamiltonian (12). The Foldy transformation is one of these restricted types, and if we want to investigate the effective Hamiltonian for Ps-Ps theory in the nonrelativistic limit more closely than we have been able hitherto, we must treat less simple situations.

By means of the transformation formulae for basic operators, eqs. (6) and (14), we have the transformed Hamiltonian given by

$$U^{-1}HU = H_M$$

$$+ \int \psi^*(x) \{ e^{-iA(x)} [M\rho_3 - i\rho_1(\sigma \nabla) + g\rho_2\phi(x) + H_M] e^{iA(x)} - H_M \} \cdot \psi(x) dx$$

$$+ \iint \psi^*(x) \psi^*(x') \{ e^{-iA(x')} g\rho_2 (e^{-iA(x')} \phi(x) e^{iA(x')} - \phi(x)) e^{iA(x)} \quad (15)$$

$$+ e^{-iA(x)} (e^{-iA(x')} H_M e^{iA(x')} - H_M) e^{iA(x)} - e^{-iA(x')} H_M e^{iA(x')} + H_M \} \cdot \psi(x') \psi(x) dx' dx$$

$$+ \dots$$

If the negative energy states are not treated according to the hole-theory, and if we assume the case where only one nucleon is present, then those parts of the transformed Hamiltonian may be put out of our considerations which contain more than one pair of  $\psi$  and  $\psi^*$ ,

\* Since

$$U^{-1}B(x)U = (e^{-i \int \psi^* \vec{A} \psi dx} \cdot e^{+i \int \psi^* \overleftarrow{A} \psi dx}) \cdot B(x)$$

$$= e^{-i \int \psi^* (\vec{A} - \overleftarrow{A}) \psi dx} \cdot B(x),$$

we can proceed in the same way as in eq. (18). The exponential operator can be transcribed as

$$e^{-i \int \psi^* (\vec{A} - \overleftarrow{A}) \psi dx} = \exp \int [\psi^* ; (e^{-i (\vec{A} - \overleftarrow{A})} - 1) \psi] dx.$$

This equation can be proved by putting  $A = \vec{A} - \overleftarrow{A}$  in eq. (18). Consequently we have

$$U^{-1}B(x)U = \exp \int [\psi^* ; (e^{-i (\vec{A} - \overleftarrow{A})} - 1) \psi] dx \cdot B(x)$$

which is the transcription of eq. (14) by the notation of operator calculus used in this paper. As for the notation, cf. eqs. (18) and (29).

because they vanish when operated on the state vector for the one-body state. That is to say, in this case we have only to deal with the second term on the right-hand side of eq. (15). Since the operator  $\{\psi^*(x)H_M\psi(x)dx=H_M\{\psi^*(x)\psi(x)dx$  is already diagonal, we must choose such a suitable operator  $A(x)$  as to reduce the rest to a diagonal operator, namely

$$e^{-iA(x)}[M_{03}-i\rho_1(\sigma\mathcal{V})+g\rho_2\phi(x)+H_M]e^{iA(x)}=\text{a diagonal operator.} \tag{16}$$

The choice of  $A(x)$  to make the left-hand member of eq. (16) diagonal is nothing but the solution of the one-body problem by means of a suitable transformation. Consequently, if we know the complete set of solutions to the one-body problem, we are able to have the transformed Hamiltonian of the form

$$\begin{aligned} U^{-1}HU=H_M+\int \psi^*(x) \{ \text{a diagonal operator} \} \psi(x)dx \\ +\iint \psi^*(x)\psi^*(x') \{ \dots\dots\dots \} \psi(x')\psi(x)dx'dx \\ +\dots\dots\dots, \end{aligned} \tag{17}$$

which is useful when the negative energy states are not treated according to the hole-theory.

The transformation formula for the meson operators (14) or that for the Hamiltonian (15) is valid even when the negative energy states are treated according to the hole theory; these formulae are based on the commutator relation (2) for  $\psi$ - and  $\psi^*$ -operators as a whole and not on those for component operators (4). However, when the hole-theory is used, the ordering of fermion operators is incomplete with respect to the negative energy component, because the creation operator  $\psi^*_+$  stands on the right-hand side and the annihilation operator  $\psi_-$  stands on the left side. Thus it is necessary for us to perform an additional procedure of ordering negative energy components in order that we may proceed with the hole-theory. The actual calculation of the hole-theoretical formulae will be given in the next section.

In concluding this section, we shall give a remark which helps us to understand the meaning of the canonical transformation considered in this note. It is found that the transformation function (1) can be transcribed as follows;

$$\begin{aligned} e^{i\int \psi^* A \psi dx} &= e^{\int \psi^*(x) (e^{iA(x)}-1) \psi(x) dx} \\ &= 1 + \int \psi^*(x) (e^{iA(x)}-1) \psi(x) dx \\ &\quad + 1/2! \cdot \iint \psi^*(x') \psi^*(x) (e^{iA(x)}-1) \psi(x) (e^{iA(x')}-1) \psi(x') dx' dx + \dots^*, \end{aligned} \tag{18}$$

where  $e^{\lambda(P;\mathcal{Q})}$  is the notation originated by Schwinger<sup>2)</sup> and it means that

\* We must pay attention to the spinor suffixes. They must be contracted starting from the both ends and proceeding one by one inwards, as can be seen in the third term of the next expression;

$$e^{(\psi^* M \psi)} = e^{(\psi^* M \psi)} = 1 + \psi^* M \psi + (1/2!) \psi^*_{\alpha} \psi^*_{\tau} M_{\alpha\tau} M_{\tau\delta} \psi_{\delta} \psi_{\beta} + \dots$$

$$e^{\lambda(P; Q)} = \sum_{n=0}^{\infty} (\lambda^n/n!) P^n Q^n, \quad (19)$$

and this operator has the property that

$$de^{\lambda(P; Q)}/d\lambda = P e^{\lambda(P; Q)} Q, \quad (20)$$

whereas the usual exponential operator has the property that

$$de^{\lambda(PQ)}/d\lambda = PQ e^{\lambda(PQ)} = e^{\lambda(PQ)} PQ. \quad (21)$$

The proof of eq. (18) is given by the following procedures. First we introduce a parameter  $\lambda$  and put

$$U(\lambda) = e^{i\lambda \int \psi^* A \psi dx} = e^{\int \psi^* F(\lambda) \psi dx}. \quad (22)$$

By differentiation of the second member with respect to  $\lambda$ , we have

$$dU/d\lambda = i \int \psi^* A \psi dx \cdot U. \quad (23)$$

On the other hand, by differentiation of the third member of (22), we have

$$dU/d\lambda = \int \psi^*(x) \cdot U \cdot dF/d\lambda \cdot \psi(x) dx = \int \psi^* dF/d\lambda \cdot e^{-i\lambda A} \psi dx \cdot U. \quad (24)$$

By comparison of (23) with (24) we obtain the operator differential equation

$$dF/d\lambda = i A e^{i\lambda A}, \quad (25)$$

which must be integrated under the initial condition

$$F(0) = 0. \quad (26)$$

After all,  $F(\lambda=1)$  is actually  $(e^{iA} - 1)$ .

The formula (18) shows that every nucleon present in our system is subjected to the change\* caused by the transformation function in a manner as if all nucleons were independent of each other. When  $n$ -nucleons are present the terms  $1/n! \cdot (\psi_a^*; (e^{iA} - 1)_{\alpha\beta} \psi_{\beta})^m$  (with  $m \leq n$ ) play the role; and the factorial in the denominator takes into account the interchange of  $m$ -nucleons.

When we treat the interaction with external field this type of transformation function is sufficient for our purpose, because all nucleons present in our system are subjected to the effect of the external field independently of each other. On the other hand, when we treat the interaction with quantized meson field, the situation is less simple; because of the zero point fluctuation of meson field all nucleons cannot be independent of each other, or in other words, virtual mesons are exchanged among the nucleons and they interact with each other. Mathematically this fact is reflected in that the effect of the multiple product of the factor  $(e^{iA} - 1)$  is different from the product of the effects caused

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\* The subtraction of 1 can be regarded as the indication of the fact that we should speak of the net change of a state caused by the operation of transformation function (1).



by a single factor  $(e^{iA}-1)$ : thus the result of the transformation with most simple structure given by (1) is in general complicated as shown by eqs. (14) or (15). The above discussions are based on the one-body-theoretical ordering of spinor operators. When we resort to the hole-theory, further ordering of operators concerning negative energy states are necessary. The third term and so forth in (14) and (15) or in the expansion of right-hand side of (18) give contributions even when we are concerned with one-nucleon states.

In the next section we will perform the ordering of negative energy components and compute expressions which are useful when the hole-theory is applied.

#### § 4. Derivation of formulae in accordance with the hole-theory

First, we modify the formula (18) of the preceding section in order that it may be in accordance with the hole-theory. The desired modification, by which the transformation function itself is ordered in accordance with the hole-theory, can be achieved if we put the transformation function of the form

$$U = e^{i \int \psi^\dagger A \psi dx} = e^{i \int (\psi_+^\dagger A \psi_-^\dagger + \psi_+^\dagger A \psi_+ + \psi_- A \psi_-^\dagger + \psi_- A \psi_+) dx} \quad (27)$$

$$= e^{\int \psi_+^\dagger G_1 \psi_-^\dagger dx} \cdot \text{Exp} \int_0^1 d\lambda \left( \int \psi_+^\dagger F_1(\lambda) \psi_+ dx + \int \psi_-^\dagger F_2(\lambda) \psi_-^\dagger dx \right) \cdot e^{\int \psi_- G_2 \psi_+ dx}$$

and adopt Fujiwara's method<sup>3)</sup> to obtain the explicit expressions for  $F$ 's and  $G$ 's which are functions of  $A$  and  $\varepsilon$ . The result obtained is

$$U = e^{\int \psi_+^\dagger \varepsilon \psi_-^\dagger dx} \cdot e^{\int \psi_+^\dagger \varepsilon \psi_+ dx} \cdot e^{\int \psi_- \varepsilon \psi_-^\dagger dx} \cdot e^{\int \psi_- \varepsilon \psi_+ dx} \quad (28)$$

where

$$G = (e^{iA} - 1) / [1 + 1/2 \cdot (1 - \varepsilon) (e^{iA} - 1)] .$$

The proof can be given in the way quite similar to that used in proving the following eqs. (30) and so forth; by this reason, we do not give the proof of eq. (28) here. (Indeed we can obtain eq. (28) by putting  $\vec{A} = 0$  and  $\vec{A} = A$  in the following eq. (43).)

Our next task is to modify the transformation formula (14) for the meson operators. Again the technique of operator calculus is very useful<sup>1)</sup>. We introduce here an operator which multiplies  $\phi$  or  $\pi$  by  $A$  from the right- or left-hand side, (which is denoted by  $\vec{A}$  or  $\overleftarrow{A}$ ) and regard  $\phi$  or  $\pi$  (which is denoted by  $B$ ) as the operand for these new operators. Then we may consistently write

$$U^{-1} B U = e^{-i \int \psi^\dagger A \psi dx} e^{i \int \psi^\dagger \vec{A} \psi dx} B$$

$$= e^{i \int \psi^\dagger (A - \vec{A}) \psi dx} B, \quad (29)$$

since  $\vec{A}$  and  $\overleftarrow{A}$  may be regarded as commutable with each other in the course of calculation. As the operator  $\phi$  or  $\pi$  is free from spinors we have no need to make use of an artifice of this kind with regard to spinor operators. The ordering of spinor operators can be done in a similar way to that used in eq. (28); we have only to perform



ordering procedures in the exponential expressions. Again, assuming that the result is in the form

$$\begin{aligned} e^{i\int \psi^\dagger (\overleftarrow{A}-\overrightarrow{A}) \psi dx} = e^{i\int \psi_+^\dagger G_1 \psi_- dx} \cdot \text{Exp} \int_0^1 d\lambda \left( \int \psi_+^\dagger F_1(\lambda) \psi_+ dx + \int \psi_-^\dagger F_2(\lambda) \psi_- dx \right) \\ \cdot e^{i\int \psi_- G_2 \psi_+ dx}, \end{aligned} \quad (30)$$

we can determine the functions  $G$ 's and  $F$ 's by adopting Fujiwara's method. In more detail, first we introduce a parameter  $\lambda$  and put

$$\begin{aligned} I^\lambda(\lambda) = e^{i\lambda \psi^\dagger (\overleftarrow{A}-\overrightarrow{A}) \psi} \\ = e^{\psi_+^\dagger G_1(\lambda) \psi_-} \text{Exp} \int_0^\lambda d\mu (\psi_+^\dagger F_1(\mu) \psi_+ + \psi_-^\dagger F_2(\mu) \psi_-) \cdot e^{\psi_- G_2(\lambda) \psi_+}. \end{aligned} \quad (31)$$

Hereafter we shall omit integration over  $x$ ; the desired answer is given when we put  $\lambda=1$ . We use the abbreviations

$$i(\overleftarrow{A}-\overrightarrow{A}) = \mathfrak{A}, \quad (1 \pm \varepsilon)/2 = s_\pm. \quad (32)$$

By differentiating both sides of eq. (31) with respect to  $\lambda$ , we have

$$\begin{aligned} dV/d\lambda = [\psi^\dagger \mathfrak{A} \psi] V = [\psi_+^\dagger \mathfrak{A} \psi_- + \psi_-^\dagger \mathfrak{A} \psi_+ + \psi_-^\dagger \mathfrak{A} \psi_- + \psi_+^\dagger \mathfrak{A} \psi_+] V \\ = \left[ \psi_+^\dagger \frac{dG_1}{d\lambda} \psi_- + e^{\psi_+^\dagger G_1 \psi_-} (\psi_+^\dagger F_1 \psi_+ + \psi_-^\dagger F_2 \psi_-) e^{-\psi_+^\dagger G_1 \psi_-} \right. \\ \left. + e^{\psi_+^\dagger G_1 \psi_-} \text{Exp} \int_0^\lambda d\mu (\psi_+^\dagger F_1 \psi_+ + \psi_-^\dagger F_2 \psi_-) \psi_-^\dagger \frac{dG_2}{d\lambda} \psi_+ \text{Exp} \int_\lambda^0 d\mu ( \dots ) e^{-\psi_+^\dagger G_1 \psi_-} \right] V. \end{aligned} \quad (33)$$

Now the second and the third terms in the last member of (33) become

$$\psi_+^\dagger F_1 \psi_+ + \psi_-^\dagger F_2 \psi_- - \psi_+^\dagger F_1 s_+ G_1 \psi_- + \psi_+^\dagger G_1 s_- F_2 \psi_-,$$

and

$$\begin{aligned} e^{\psi_+^\dagger G_1 \psi_-} \psi_-^\dagger R \psi_+ e^{-\psi_+^\dagger G_1 \psi_-} \\ = \psi_-^\dagger R \psi_+ - \psi_-^\dagger R s_+ G_1 \psi_- + \psi_+^\dagger G_1 s_- R \psi_+ - \psi_+^\dagger G_1 s_- R s_+ G_1 \psi_-^2, \end{aligned}$$

respectively, where we have put

$$R = \text{Exp} \int_0^\lambda d\mu (F_2 s_-) dG_2/d\lambda \cdot \text{Exp} \int_\lambda^0 d\mu (s_+ F_1).$$

Inserting these results into eq. (33) and by comparing the operators of the same type\* with each other we have the (matrix)\*\* differential equations

$$\begin{aligned} R &= \mathfrak{A}, \\ F_1 + G_1 s_- R &= \mathfrak{A}, \quad F_2 - R s_+ G_1 = \mathfrak{A}, \\ dG_1/d\lambda - F_1 s_+ G_1 + G_1 s_- F_2 - G_1 s_- R s_+ G_1 &= \mathfrak{A}. \end{aligned} \quad (34)$$

\* We here proceed with the same argument and technique as we used in our previous work<sup>(5)</sup>.

\*\* So we must keep the order of terms in solving these differential equations.

We must solve these eqs. under the initial conditions,

$$G_1(0) = G_2(0) = F_1(0) = F_2(0) = 0.$$

These equations are reduced to a single equation for  $G_1$

$$dG_1/d\lambda = (1 - G_1 s_-) \mathfrak{A} (1 + s_+ G_1). \quad (35)$$

In the right-hand side member of this equation  $s_-$  appears on the left, while  $s_+$  appears on the right; this asymmetry causes trouble in the solution of the differential equation. To overcome this difficulty we change the variables in the following way; rewriting eq. (35) we have

$$-dG_1^{-1}/d\lambda = (G_1^{-1} - s_-) \mathfrak{A} (G_1^{-1} + s_+). \quad (36)$$

Next, if we put

$$G_1^{-1} + 1/2 \cdot (s_+ - s_-) = 1/2 \cdot (s_+ + s_-) \tilde{G}, \quad (37)$$

and

$$\mathfrak{A} \cdot 1/2 \cdot (s_+ + s_-) = \tilde{\mathfrak{A}}$$

we have

$$-d\tilde{G}/d\lambda = (\tilde{G} - 1) \tilde{\mathfrak{A}} (\tilde{G} + 1). \quad (38)$$

If we put

$$\tilde{G} = (1 - W) / (1 + W)$$

eq. (38) can be integrated, since we have

$$2 \frac{1}{1+W} \frac{dW}{d\lambda} \frac{1}{1+W} = -2 \frac{W}{1+W} \tilde{\mathfrak{A}} \frac{2}{1+W},$$

or

$$dW/d\lambda = -2W\tilde{\mathfrak{A}}.$$

Hence

$$W = -e^{-2\lambda\tilde{\mathfrak{A}}}, \quad (39)$$

where we have used the initial condition  $G_1(0) = 0$ , or  $\tilde{G}(0) = \infty$ . In this way we obtain the answer

$$G_1 = G_1(\lambda) = \left[ -1/2 \cdot (s_+ - s_-) + 1/2 \cdot (s_+ + s_-) \frac{1 + e^{-\lambda\mathfrak{A}(s_+ + s_-)}}{1 - e^{-\lambda\mathfrak{A}(s_+ + s_-)}} \right]^{-1}, \quad (40)$$

or

$$= (e^{\lambda\mathfrak{A}} - 1) / (1 + s_- (e^{\lambda\mathfrak{A}} - 1)) \quad (41)$$

from  $s_+ + s_- = 1$ .

And if we put\*

$$C = e^{\lambda \mathfrak{A}} - 1 = e^{-i\lambda \vec{A}} \cdot e^{i\lambda \vec{A}} - 1 = -C^T / (1 + C^T), \quad (42)$$

$C^T$  is found to be given by

$$C^T = e^{-i\lambda \vec{A}} \cdot e^{+i\lambda \vec{A}} - 1 = -C / (1 + C). \quad (42')$$

By comparing the second statements of eq. (42') with the third one of eq. (42), we find that  $C^T$  means to take the transpose of  $C$ . There exists a relation between  $C$  and  $C^T$  which is important in the consideration of the charge conjugation; this relation is in a sense reciprocal to each other. We can write

$$G_1 = C / (1 + s_- C) = -C^T / (1 + s_+ C^T). \quad (43)$$

Other terms can be computed by means of (43) and (34), and we find

$$\begin{aligned} F_1 &= (1 - G_1 s_-) \mathfrak{A} = (1 + s_- C)^{-1} \mathfrak{A} = -(1 + C^T s_+)^{-1} \frac{dC^T}{d\lambda} \\ F_2 &= \mathfrak{A} (1 + s_+ G_1) = \mathfrak{A} (1 + s_+ C^T)^{-1} = (1 + C s_-)^{-1} \frac{dC}{d\lambda} \end{aligned} \quad (44)$$

and

$$\begin{aligned} \text{Exp} \int_0^\lambda d\mu (F_2 s_-) &= (1 + C s_-)^{-1} = (1 - G_1 s_-)^{-1} \\ \text{Exp} \int_\lambda^0 d\mu (s_+ F_1) &= (1 + s_+ C^T)^{-1} = (1 + s_+ G_1)^{-1}. \end{aligned}$$

and finally

$$\begin{aligned} dG_2/d\lambda &= (1 - G_1 s_-) \mathfrak{A} (1 + s_+ G_1) = dG_1/d\lambda \\ \therefore G_2 &= G_1 \equiv G. \end{aligned} \quad (45)$$

Lastly if we put

$$\text{Exp} \int_0^\lambda d\mu (\psi_+^\dagger F_1 \psi_+) = \exp(\psi_+^\dagger; u_+(\lambda) \psi_+),$$

we can easily show (see the proof of eq. (18).)

$$\begin{aligned} u_+(\lambda) &= 1/s_+ \cdot (\text{Exp} \int_0^\lambda d\mu (s_+ F_1) - 1) \\ &= G, \end{aligned} \quad (46)$$

\* To this point we have proceeded formally and the result is expressed in a concise form by using the notation  $\exp[\lambda \mathfrak{A}]$ . So long as we are concerned with the operator  $\vec{A}$  or  $\vec{A}$  themselves, this procedure is permissible. But when we let them operate on the operand it is necessary for us to decompose  $\exp[\lambda \mathfrak{A}]$  into  $\exp[-i\lambda \vec{A}] \cdot \exp[i\lambda \vec{A}]$ , anticipating the operand put in the middle. As for further remarks on the notation, see the next section.

where we have used eq. (44). Similarly

$$\begin{aligned} & \text{Exp} \int_0^\lambda d\mu (\psi_- F_2 \psi_-^\dagger) \\ &= \text{Exp} \int_0^\lambda d\mu (\text{Sp} F_2 s_-) \cdot \text{Exp} \left( - \int_0^\lambda d\mu (\psi_-^\dagger F_2^T \psi_-) \right) \\ &= C_1 \cdot \exp(-\psi_-^\dagger; G^T \psi_-), \end{aligned}$$

where  $C_1$  is the vacuum factor coming from contributions of closed loops in the Feynman graphs, which is

$$C_1 = e^{\text{Sp} \log(1+s_-C)} = \det|1+s_-C|, \quad (48)$$

and

$$-G^T = C/(1+s_+C), \quad (49)$$

from (43).

The final result is written down as

$$U^{-1}BU = \{ e^{\psi_+^\dagger G \psi_-^\dagger} e^{(\psi_+^\dagger; G \psi_+)} e^{-(\psi_-^\dagger; G^T \psi_-)} e^{\psi_- G \psi_+} \cdot C_1 \} \cdot B \quad (50)$$

where  $G=G(1)$ ,  $C=C(1)$  and are defined by eqs. (42), (43) and (49). Formula (14) is a special case of eq. (50) and can be obtained by putting  $s_- = 0$  in the positive energy part  $e^{(\psi_+^\dagger; G \psi_+)}$  of eq. (50), as is required in the one-body theory where we should put  $\varepsilon = +1$ .

In concluding this section, we write the transformed Hamiltonian having disregarded the spur expressions. This expression can be obtained if we perform further ordering of negative energy spinors in the expression (15) of the preceding section. By means of the formula (50) we have

$$\begin{aligned} U^{-1}HU &= H_M + \int \bar{\psi}^* \left\{ e^{-iA} [M_{\rho_3} - i\rho_1(\sigma \nabla)] e^{iA} \right. \\ &\quad + \frac{C}{1+s_-C} \cdot H_M + i \frac{1}{1+C s_-} e^{-iA} \rho_2 \partial_\nu e^{iA} \frac{1}{1+s_-C} \left. \right\} \bar{\psi} dx \\ &\quad + \iint \bar{\psi}^* \bar{\psi}^{*'} \dots \bar{\psi}' \bar{\psi} dx' dx + \dots \end{aligned} \quad (51)$$

where the bar means to take the ordered pair of spinor operators, i.e.,

$$\bar{\psi}_\alpha^* M_{\alpha\beta} \bar{\psi}_\beta = \psi_\alpha^\dagger M_{\alpha\beta} \psi_{-\beta}^\dagger + \psi_\alpha^\dagger M_{\alpha\beta} \psi_{+\beta} - \psi_{-\beta}^\dagger M_{\alpha\beta} \psi_{-\alpha} + \psi_{-\alpha} M_{\alpha\beta} \psi_{+\beta}. \quad (52)$$

By comparing the above equation (51) with eq. (15) of the one-body-theoretical result, we see that the effect of the exclusion principle which acts on the intermediate states of negative energy is taken into account by the appearance of the factor  $(1+s_-C)^{-1}$  in the meson energy part and in the both sides of interaction energy. In the calculation of the interaction energy the boson operator contained in  $e^{-iA}$  should be placed on the left end and those contained in  $e^{iA}$  on the right end, while others contained in  $C s_-$  should be subjected to all permutations of their order in the manner which will be explained in

the next section; but the Dirac matrices should be arranged in the order as written down in this equation. In Appendix, we will explain this rule of calculation in more detail by an example.

In the forthcoming paper we will report the detailed calculation by taking  $A=1/2 \cdot \tan^{-1}(\lambda\rho_1\phi-\mu\rho_2\pi)$  which is a generalization of the Foldy transformation.

### § 5. Remark on the application of formulae and the four-dimensional aspect

a) The transformation of the fermion field discussed in this paper is analogous to the Bloch-Nordsieck transformation of the boson field. In the latter transformation the generating function is a linear function of the boson operator, and the bosons are subjected to the change of state independently of each other. Here, for the fermion, the generating function is given correspondingly by a bilinear form of the fermion operator.

The formulae of the preceding sections are applicable to other problems than the treatment of the vacuum effects. For example, if we make suitable modifications in them, they are useful in the formulation of the Hartree approximation for a many-fermion system. For these purposes we have only to redefine the projection operator  $s_{\pm}$  and let  $s_+$  or  $s_-$  denote the projection operator to states above or below the Fermi surface respectively. The effect of the exclusion principle is represented by the appearance of the factor  $(1+C s_-)^{-1}$  in some parts of the transformed Hamiltonian.

b) The formulae of Sec. 4, where the hole-theoretical ordering of operators is discussed, are exact so far as their formal derivation is concerned. But in the application of them some remark is necessary as regards the performance of the operation prescribed by these formulae. On the other hand, the formulae of Sec. 3 is unambiguous in their application. They are derived on the basis of the commutation relation between  $\psi$  and  $\psi^*$  as a whole, and only the operators  $\exp(-i\vec{A}) \cdot \exp(i\vec{A})$  or  $C=\exp(-i\vec{A}) \cdot \exp(i\vec{A})-1$  are contained in them; since  $\vec{A}$  and  $\vec{A}$  are commutable with themselves and each other, we are sure not to encounter ambiguities.

However, in Sec. 4 not only  $C=\exp(-i\vec{A}) \cdot \exp(i\vec{A})-1$  but also  $s_-$  are contained in the formulae.

It should be noticed that in operating the multiple product of operator  $s_-C$ , we must pay attention to the order of operation of each single operator  $s_-C$ . For example, in the triple operation of  $s_-C$ , the formal expression is written as

$$\begin{aligned} (s_-C)^3 \cdot B &= [s_- (e^{-i\vec{A}}, e^{i\vec{A}} - 1)]^3 \cdot B \\ &= [s_- (e^{-i\vec{A}}, e^{i\vec{A}} - 1)]^2 \cdot (s_- e^{-i\vec{A}} B e^{i\vec{A}} - s_- B) \text{ etc.} \end{aligned} \tag{53}$$

The Dirac matrices should be arranged in the order written above. However, with respect to the boson operators contained in  $s_-C$ , we must take all permutations of the order of their operation. The order of operation will be shown by figures attached to  $s_-C$ . Here one must understand that the expression (53) is used by us in the meaning of the following expression



$$1/3! \cdot \{ (s_- \overset{1}{C}) [ (s_- \overset{2}{C}) (s_- \overset{3}{C}) + (s_- \overset{2}{C}) (s_- \overset{2}{C}) ] + [ (s_- \overset{2}{C}) (s_- \overset{3}{C}) + (s_- \overset{3}{C}) (s_- \overset{2}{C}) ] (s_- \overset{1}{C}) \\ + (s_- \overset{2}{C}) (s_- \overset{1}{C}) (s_- \overset{3}{C}) + (s_- \overset{3}{C}) (s_- \overset{1}{C}) (s_- \overset{2}{C}) \} \cdot B. \quad (54)$$

The correctness of this statement can be checked, if we remind ourselves that the result of the Sec. 3 must be modified so far as the ordering of negative energy spinor is incomplete. Here we search for a closed expression in which the ordering of negative energy components is complete.

In the example mentioned above (eqs. (53) and (54)), we start from the fourth term of the equation (14) of the preceding section,

$$1/3! \cdot (\psi^* (e^{-i\vec{A} \cdot \vec{\sigma}} - 1) \psi)^3 \cdot B \\ = 1/3! (\psi^* C \psi^* C \psi \psi^* C \psi) \cdot B, \quad (55)$$

where

$$\psi = \psi_+ + \psi_-^\dagger, \quad \psi^* = \psi_+^\dagger + \psi_-.$$

In this expression the ordering of the positive energy components is finished, but the negative energy components are not yet ordered. In the course of ordering negative energy components, a pair of negative energy spinor operator is replaced by the matrix element of the matrix  $s_- = (1 - \varepsilon)/2$ , and we obtain expressions with fewer spinor operators. That part which retains a single pair of spinors and does not contain any spur expression turns out to be given by

$$1/3! \cdot \bar{\psi}^* \{ C [ (s_- \overset{2}{C}) (s_- \overset{3}{C}) + (s_- \overset{3}{C}) (s_- \overset{2}{C}) ] + [ (s_- \overset{2}{C}) (s_- \overset{3}{C}) + (s_- \overset{3}{C}) (s_- \overset{2}{C}) ] C \\ + C s_- C s_- C + C s_- C s_- C \} \bar{\psi} \cdot B. \quad (54')$$

On the other hand, the corresponding expression obtainable by the formula (43) in Sec. 4 is

$$\bar{\psi}^* C (s_- C)^2 \bar{\psi} \cdot B \quad (53')$$

where the bars mean to take the ordered pair of spinor operators as was shown by eq. (52). In this way we have checked our statement.

c) In the formulae of Sec. 4 the fermion and the anti-fermion are treated in a symmetrical way, as they should be. In this connection, it may be interesting to notice the following relations between  $C$  and  $C^T$ . From the property of Schwinger exponential (19)

$$e^{(\psi^*; X \psi)} = e^{\psi^* \log(1 + X) \psi} \quad (X; \text{arbitrary}), \quad (56)$$

we first notice the fact

$$[e^{(\psi^*; C \psi)}]^{-1} = e^{-\psi^* \log(1 + C) \psi} = e^{(\psi^*; \frac{-C}{1+C} \psi)} \\ = e^{(\psi^*; C^T \psi)} = e^{-\text{Sp} \log(1 + C)} e^{(\psi; C^T \psi^*)},$$

where we have used the relation (43) in Sec. 4.

Next we find in a similar manner

$$e^{(\psi_+^\dagger; G\psi_+)} = e^{(\psi_+^\dagger; \frac{C}{1+s_-C}\psi_+)} = e^{\text{Sp}(s_+ \log(1+G))} \cdot e^{(\psi_+; \frac{C}{1-s_-C}\psi_+^\dagger)}$$

and we see that when we interchange the creation and annihilation operator while we change  $s_+ \rightarrow -s_+$ ,  $s_- \rightarrow -s_-$ , our formula is invariant except for the closed loop factor. Moreover our formula is invariant to the change of sign of energy:  $\psi_\pm^\dagger \rightarrow \psi_\mp^\dagger$ ,  $\psi_\pm \rightarrow \psi_\mp$ ,  $s_\pm \rightarrow s_\mp$ , since

$$\psi_+^\dagger G \psi_-^\dagger = \psi_+^\dagger \frac{C}{1+s_-C} \psi_-^\dagger = \psi_-^\dagger (-G^T) \psi_+^\dagger = \psi_-^\dagger \frac{C}{1+s_+C} \psi_+^\dagger,$$

$$e^{(\psi_+^\dagger; G\psi_+)} e^{-(\psi_-^\dagger; G^T\psi_-)} = e^{(\psi_+^\dagger; \frac{C}{1+s_-C}\psi_+)} e^{(\psi_-^\dagger; \frac{C}{1+s_+C}\psi_-)}.$$

Here again eq. (43) has been used.

d) We want to add some comments about the ordering of spinor operators in the four-dimensional case, which was treated in our previous work<sup>5)</sup>.

The well known transformation function is in the form

$$U(t_1, t_2) = \text{Exp} \left( - \int_{t_2}^{t_1} dx \psi^*(x) A(x) \psi(x) \right),$$

where  $x$  stands for the four dimensional coordinate and all the operators are those of interaction representation and satisfy the following commutation relations

$$\{\psi_+(x), \psi_+^\dagger(x')\} = -iS_+(x-x') = K_+(x-x'), \quad \text{if } (x-x')_0 > 0^*,$$

$$\{\psi_-(x), \psi_-^\dagger(x')\} = -iS_-(x-x') = -K_+(x-x'), \quad (x-x')_0 < 0.$$

The ordering of  $U(t_1, t_2)$  proceeds quite in the same way as in Sec. 4 and we get eq. (40), where we replace  $\mathfrak{A}$  by  $-A$ . In place of  $s_+ + s_- = 1$  in Sec. 4, we must now put  $s_+ = -s_- = K_+$  in eq. (40), where the necessary time relations come automatically from the time ordering of  $U(t_1, t_2)$ . And putting

$$(s_+ - s_-)/2 \rightarrow K_+, \quad (s_+ + s_-)/2 \rightarrow 0,$$

in eq. (40), we obtain

$$G = - \int dt \frac{A}{1 + \int dt K_+ A}$$

which is just eq. (A.3) of reference 5. And thus our formula (40) can be used both in the three-dimensional case, where  $s_+ + s_- = 1$ , and in the four-dimensional case, where  $s_+ + s_- = 0$ .

If we use the same argument which leads to eq. (46), the positive energy propagation part can be rewritten as follows,

\* Our functions differ from the usual notations by a factor  $\gamma_4$ .

$$\text{Exp} \left\{ - \int_{t_2}^{t_1} dt \left( \phi_+^\dagger \frac{A}{1 + \int_{t_2}^t dt' AK_+} \phi_+ \right) \right\} = e^{\langle \phi_+^\dagger; G\phi_+ \rangle}. \quad (57)$$

In the one-body-theoretical treatment, where  $\{\phi(x), \phi^*(x')\} = -iS(x-x')$ , the transformation function is in the form

$$U^0(t_1, t_2) = \text{Exp} \left( - \int_{t_2}^{t_1} dt \phi^* A \phi \right). \quad (58)$$

We now want to show the fact that in spite of the apparent difference between (57) and (58) where  $A$  appears in place of  $\frac{A}{1 + \int_{t_2}^t AK_+}$  in (57), the result of ordering of (58) is obtained by putting  $K$  in place of  $K_+$  in (57).  $K$  is defined by the following equation,

$$\left. \begin{aligned} K(x) &= -iS(x) & \text{if } x_0 > 0 \\ &= 0 & x_0 < 0 \end{aligned} \right\}. \quad (59)$$

By a now familiar computation we obtain

$$U^0(t_1, t_2) = e^{\langle \phi^*; \frac{i}{N} [\text{Exp}(\int_{t_2}^{t_1} dt SA) - 1] \phi \rangle}.$$

But from the definition (59) of  $K(x)$  we get

$$\begin{aligned} \text{Exp} \left( +i \int_{t_2}^{t_1} dt SA \right) &= 1 + i \int_{t_2}^{t_1} dt SA - \int_{t_2}^{t_1} dt \int_{t_2}^t dt' SASA' - \dots \\ &= 1 - \int_{t_2}^{t_1} dt KA + \int_{t_2}^{t_1} dt \int_{t_2}^t dt' KAKA' - \dots \\ &= \frac{1}{1 + \int_{t_2}^{t_1} dt KA}. \end{aligned}$$

And finally we get

$$U^0(t_1, t_2) = e^{\langle \phi^*; G^0 \phi \rangle},$$

where

$$G^0 = - \int dt \frac{A}{1 + \int dt KA}.$$

Thus in the four-dimensional case, too, we have established the relation between the one-body-theory and the hole-theory.

## § 6. Conclusions

In this paper we have examined the results to be obtained by the canonical transformation, which is generated by the exponential function of a bilinear form of  $\phi^*$  and  $\phi$ . It is shown that this type of transformation yields the change of fermion's states in a manner as if all fermions were independent of each other.

By means of this type of transformation, another proof has been given to the theorem,

which is concerned with fermions interacting with an external field and has been first established by Furry. (Sec. 2.).

The transformation formulae have been derived which are useful for one to treat the interaction of fermions with the quantized boson field. By the observation of the transformed Hamiltonian, it is possible to find the efficiency of the one-body solution in the whole scheme of quantized fields. (Sec. 3.).

These formulae are further developed so that they are applicable when the hole-theory of the Fermi gas is applied with regard to the states below the Fermi surface. It is possible to see how large is the effects of the exclusion principle which is valid for the intermediate states below the Fermi surface. (Sec. 4.).

The formulae in accordance with the hole theory have been derived by means of some techniques of operator calculus; and the meaning of the formal prescriptions in these formulae has been explained. The application to the nucleon interacting with pseudoscalar meson field by pseudoscalar coupling will be made in the forthcoming paper. (cf. Ref. 6).

The symmetry between the fermion and the anti-fermion has been examined. It has been shown that the results obtained in this paper can be utilized in the four-dimensional consideration of fermion's propagation in space-time. (Sec. 5.).

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## Appendix

In this Appendix we will explain how to make use of the formulae given in Sec. 4. As an example, we will continue to consider the nucleon field interacting with Ps-meson field by Ps-coupling. Akiba and Sawada<sup>7)</sup> considered the effective Hamiltonian for this system which is obtained after the Foldy transformation and the ordering of creation-annihilation operators. They have shown that the Dyson transformation used by Drell and Henley<sup>8)</sup> is valid only when the coupling constant is small. As briefly reported recently in this journal<sup>6)</sup>, the analysis of the one-body problem shows us that the transformation function

$$U = \exp \left[ \frac{i}{2} \int \psi^*(x) \tan^{-1}(\lambda \rho_1 \phi(x) - \mu \rho_2 \pi(x)) \psi(x) dx \right] \quad (\text{A} \cdot 1)$$

is suitable for our purpose. In the derivation of the Foldy transformation the meson field is treated classically, but in our new type of transformation the quantum-mechanical nature of the meson field is taken into account to a certain extent.

Now, by this example we will explain how to deal with the factors  $[1 + C S_-]^{-1}$  which are due to the exclusion principle in negative energy states.

a) For convenience' sake we introduce here the following notations;

$$\begin{aligned} S(x) &\equiv \exp \left[ \frac{i}{2} \tan^{-1}(\lambda \rho_1 \phi(x) - \mu \rho_2 \pi(x)) \right], \\ S^{-1}(x) &\equiv \exp \left[ -\frac{i}{2} \tan^{-1}(\lambda \rho_1 \phi(x) - \mu \rho_2 \pi(x)) \right], \end{aligned} \quad (\text{A} \cdot 2)$$

$$G_+ \equiv \lambda\phi - i\mu\pi, \quad G_- \equiv \lambda\phi + i\mu\pi. \quad (\text{A}\cdot 3)$$

The Dirac matrices are represented by the following projection operators,

$$\begin{aligned} {}_+\rho_- &= (\rho_1 + i\rho_2)/2, & -\rho_+ &= (\rho_1 - i\rho_2)/2, \\ {}_+\rho_+ &= (1 + \rho_3)/2, & -\rho_- &= (1 - \rho_3)/2, \end{aligned} \quad (\text{A}\cdot 4)$$

where the multiplication law for these operators is given by

$$\alpha\rho_\beta \gamma\rho_\delta = \delta_{\beta\gamma} \alpha\rho_\delta \quad (\alpha, \beta, \gamma, \delta = \pm). \quad (\text{A}\cdot 5)$$

The operator  $S(x)$  is expressed in terms of the following abbreviations;

$$S(x) \equiv {}_+\rho_+ P_1 + {}_-\rho_- P_0 + i{}_+\rho_- (Q_1 G_-) + i{}_-\rho_+ (G_+ Q_1), \quad (\text{A}\cdot 6)$$

where we have put

$$P_n \equiv \sqrt{(1 + M_n)/2M_n}, \quad Q_n \equiv \sqrt{1/2M_n(1 + M_n)}, \quad (\text{A}\cdot 7)$$

with

$$M_n \equiv \sqrt{1 + G_+ G_-} + 2n\lambda\mu\Delta. \quad (\text{A}\cdot 8)$$

The quantity  $\Delta$  appears when we take the commutator of  $G_-$  with  $G_+$

$$[G_-, G_+] = 2\lambda\mu\Delta. \quad (\text{A}\cdot 9)$$

Then the operator  $C$  is expressed in terms of the above quantities as

$$\begin{aligned} C &= \vec{S}^{-1} \overleftarrow{S} - 1 = [{}_+\rho_+ \vec{P}_1 + {}_-\rho_- \vec{P}_0 - i{}_+\rho_- (\overrightarrow{Q_1 G_-}) - i{}_-\rho_+ (\overrightarrow{G_+ Q_1})] \\ &\quad \times [{}_+\rho_+ \overleftarrow{P}_1 + {}_-\rho_- \overleftarrow{P}_0 + i{}_+\rho_- (\overleftarrow{Q_1 G_-}) + i{}_-\rho_+ (\overleftarrow{G_+ Q_1})] - 1 \\ &\equiv {}_+\rho_+ \tilde{X} + {}_-\rho_- X + i{}_+\rho_- \tilde{Z} - i{}_-\rho_+ Z, \end{aligned} \quad (\text{A}\cdot 10)$$

where we have put

$$\begin{aligned} \tilde{X} &\equiv \vec{P}_1 \overleftarrow{P}_1 + (\overrightarrow{Q_1 G_-}) \cdot (\overleftarrow{G_+ Q_1}) - 1, \quad X \equiv \vec{P}_0 \overleftarrow{P}_0 + (\overrightarrow{Q_0 G_+}) \cdot (\overleftarrow{G_- Q_0}) - 1, \\ \tilde{Z} &\equiv \vec{P}_1 \cdot (\overrightarrow{Q_1 G_-}) - (\overrightarrow{Q_1 G_-}) \cdot \overleftarrow{P}_0, \quad Z \equiv (\overrightarrow{G_+ Q_1}) \cdot \overleftarrow{P}_1 - \vec{P}_0 \cdot (\overleftarrow{G_+ Q_1}). \end{aligned} \quad (\text{A}\cdot 11)$$

In the following computations the recoil effects are neglected, and we assume that

$$s_- = -\rho_-. \quad (\text{A}\cdot 12)$$

b) First we consider that part of the transformed Hamiltonian which has its origin in the energy of the free mesons. If we disregard the effects coming from closed loops of the Feynman diagrams, the desired expression is written down formally by eq. (50) of Sec. 4;



$$\begin{aligned}
& U^{-1} H_M U \\
&= \left\{ \exp \left[ \int \psi^\dagger_+ G \psi^\dagger_- \right] \exp \left[ \int \psi^\dagger_+ ; G \psi_+ \right] \exp \left[ - \int \psi^\dagger_- ; G^T \psi_- \right] \exp \left[ \int \psi_- G \psi_+ \right] \right\} \cdot H_M \\
&= H_M + \int \bar{\psi}^* G \bar{\psi} \cdot H_M + \dots
\end{aligned} \tag{A.13}$$

In the consideration of one-nucleon state, we must deal with the operator

$$G \cdot H_M = \frac{1}{1 + C_{S-}} C \cdot H_M = C \cdot H_M - (C_{S-} C) \cdot H_M + (C_{S-} C_{S-} C) \cdot H_M - \dots \tag{A.14}$$

As it has been stated in the text, we must not calculate the second term in the way shown by

$$\overset{2}{C}_{S-} \overset{1}{C} \cdot H_M$$

where the superscript indicates the order of operation of boson operators  $\tilde{X}$ ,  $X$ ,  $\tilde{Z}$ , and  $Z$ . Actually if we calculate in this way, namely if we calculate

$$\begin{aligned}
\overset{2}{C}_{S-} \overset{1}{C} \cdot H_M &= [-\rho_- \overset{2}{X} + i_+ \rho_- \overset{2}{\tilde{Z}}] [-\rho_- \overset{1}{X} - i_- \rho_+ \overset{1}{Z}] \cdot H_M \\
&= {}_+ \rho_+ \tilde{Z} \cdot (Z \cdot H_M) + {}_- \rho_- X \cdot (X \cdot H_M) \\
&\quad + i_+ \rho_+ \tilde{Z} \cdot (X \cdot H_M) - i_- \rho_+ X \cdot (Z \cdot H_M),
\end{aligned} \tag{A.15}$$

the result is not Hermitian and unsymmetrical with respect to the sign of energy (i.e. with respect to the interchange of  ${}_+ \rho_- \longleftrightarrow {}_- \rho_+$ ). In order to find the way to get rid of this defect, we go back to the formulae given in Sec. 3. There the corresponding part of the transformed Hamiltonian is given by

$$\begin{aligned}
U^{-1} H_M U &= \exp \left[ \int \psi^* ; (\vec{S}^{-1} \overleftarrow{S} - 1) \psi \right] \cdot H_M \\
&= H_M + \int \psi^* (S^{-1} H_M S - H_M) \psi \\
&\quad + \frac{1}{2!} \int \psi^* [S^{-1} \int \psi'^* (S'^{-1} H_M S' - H_M) \psi' S \\
&\quad - \int \psi'^* (S'^{-1} H_M S' - H_M) \psi'] \psi \\
&\quad + \dots
\end{aligned} \tag{A.16}$$

After the ordering of the negative energy components appearing in the third term we have

$$\frac{1}{2!} \psi^*_\alpha \psi^*_\beta \{ (S^{-1})_{\alpha\mu} (S^{-1} H_M S - H_M)_{\beta\delta} (S)_{\mu\bar{\gamma}} - (S^{-1} H_M S - H_M)_{\beta\delta} \partial_{\alpha\bar{\gamma}} \} \psi_\delta \psi_{\bar{\gamma}}$$



$$\begin{aligned} \frac{\tilde{Z}}{1+X} Z &= \sum_{n=0}^{\infty} \tilde{Z} (-X)^n Z \rightarrow \frac{1}{2} \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^{n-i} \frac{1}{n+1} \frac{1}{i+j+1} \\ &\times \{ (-X)^i \tilde{Z} (-X)^j Z (-X)^{n-i-j} + (-X)^i Z (-X)^j \tilde{Z} (-X)^{n-i-j} \} \\ &= \frac{1}{2} \sum_{i,j,k=0}^{\infty} \frac{1}{(i+j+k+1)(i+j+1)} (-X)^i \{ \tilde{Z} (-X)^j Z + Z (-X)^j \tilde{Z} \} (-X)^k. \end{aligned}$$

This replacement can be achieved by means of the following artifice: in place of (A.19) we calculate

$$\begin{aligned} &\left[ \rho_+ \left( \tilde{X} - \frac{1}{2} \int_0^1 dx \int_0^1 dy \frac{1}{1+xyX} \left[ \tilde{Z} \frac{1}{1+xyX} Z + Z \frac{1}{1+xyX} \tilde{Z} \right] \frac{1}{1+X} \right) \right. \\ &\left. + \rho_- \left( -\frac{X}{1+X} \right) + i \rho_- \left( \int_0^1 dx \frac{1}{1+X} \tilde{Z} \frac{1}{1+X} \right) - i \rho_+ \left( \int_0^1 dx \frac{1}{1+X} Z \frac{1}{1+X} \right) \right] \cdot H_M. \end{aligned} \quad (\text{A.21})$$

In (A.21) the boson operators operate on  $H_M$  in the order from right to left. Actually the result of the  $n$ -time operation of  $X$  on  $H_M$ , i.e. the expression formally represented by  $X^n \cdot H_M$  can be computed systematically as functions of  $n$ , and we can get the answer in a closed form.

c) Next we will consider that part of the transformed Hamiltonian which has its origin in the meson-nucleon interaction energy.

Here we must deal with

$$U^{-1} \phi^*_{\rho_2} \phi U = (\phi^* S^{-1})_{\alpha} (\rho_2)_{\alpha\beta} (U^{-1} \phi U) (S \phi)_{\beta}. \quad (\text{A.22})$$

Our first task is to obtain the expression in which formally the exclusion principle in the negative energy intermediate states are taken into account. We will restrict our considerations to the term which is responsible for the transition from the vacuum state into the state where only one nucleon pair is present. Other terms can be dealt with similarly. Now the desired term should be derived from the expression

$$\begin{aligned} &{}_1 \langle (\phi^* S^{-1} \rho_2)_{\alpha} U^{-1} \phi U (S \phi)_{\alpha} \rangle_0 \\ &= {}_1 \langle \{ (\phi^{\dagger}_+ + \phi_-) S^{-1} \rho_2 \}_{\alpha} \{ (1 + \phi^{\dagger}_+ G \phi^{\dagger}_-) (1 - \phi^{\dagger}_- G^T \phi_-) \cdot \phi \} \{ S \phi^{\dagger}_- \}_{\alpha} \rangle_0, \end{aligned} \quad (\text{A.23})$$

in which  $U^{-1} \phi U$  is computed by the formula (50) of Sec. 4 up to the second order. No other terms give contributions, since they contribute only to processes in which more fermions play their role. Here  $\rangle_0$  denotes the vacuum state and  $\rangle_1$  the one-pair state.

The term (A.23) can be transformed as follows,

$$\begin{aligned} &{}_1 \langle \{ (\phi^{\dagger}_+ S^{-1} \rho_2)_{\alpha} \{ (1 - \phi^{\dagger}_- G^T \phi_-) \cdot \phi \} \{ S \phi^{\dagger}_- \}_{\alpha} \} \rangle_0 \\ &+ {}_1 \langle \{ (\phi_- S^{-1} \rho_2)_{\alpha} \{ (\phi^{\dagger}_+ G \phi^{\dagger}_-) (1 - \phi^{\dagger}_- G^T \phi_-) \cdot \phi \} \{ S \phi^{\dagger}_- \}_{\alpha} \} \rangle_0 \\ &= {}_1 \langle (\phi^{\dagger}_+ S^{-1} \rho_2)_{\alpha} \left( \frac{1}{1 + S_- C} \cdot \phi \right)_{\mu\nu} (S)_{\alpha\mu} \phi^{\dagger}_{-\nu} \rangle_0 \\ &- {}_1 \langle (S^{-1} \rho_2)_{\mu\alpha} (\phi^{\dagger}_+ G S_-)_{\mu} \left( \frac{1}{1 + S_- C} \cdot \phi \right)_{\nu\beta} (S)_{\alpha\nu} \phi^{\dagger}_{-\beta} \rangle_0 \end{aligned} \quad (\text{A.24})$$

$$= \psi_{-}^{\dagger} S^{-1} \rho_{2}^{\leftarrow} \left( \frac{1}{1+s_{-}C} \cdot \phi \right) \psi_{-}^{\dagger} - \psi_{-}^{\dagger} G s_{-} \vec{S}^{-1} \rho_{2}^{\leftarrow} \left( \frac{1}{1+s_{-}C} \cdot \phi \right) \psi_{-}^{\dagger},$$

where use is made of the following relation,  $\rangle_{00}$  representing the vacuum state and  $\rangle_{01}$  the state where only one anti-nucleon is present,

$$\begin{aligned} {}_{01}\langle U^{-1} \phi U (S\psi)_{\alpha} \rangle_{00} &= {}_{01}\langle \{ (1 - \psi_{-}^{\dagger} G^T \psi_{-}) \cdot \phi \} \{ S\psi_{-}^{\dagger} \}_{\alpha} \rangle_{00} \\ &= {}_{01}\langle (\phi_{-} - \psi_{-}^{\dagger} (G_{\nu\mu} \cdot \phi) \psi_{-}) (S\psi_{-}^{\dagger})_{\alpha} \rangle_{00} = {}_{01}\langle \phi (S\psi_{-}^{\dagger})_{\alpha} - \psi_{-}^{\dagger} G_{\nu\mu} \phi S_{\alpha\beta} (s_{-})_{\beta\nu} \rangle_{00} \quad (\text{A} \cdot 25) \\ &= (\vec{S}(1 - s_{-}G) \cdot \phi \psi_{-}^{\dagger})_{\alpha} = \left( \vec{S} \cdot \frac{1}{1+s_{-}C} \cdot \phi \psi_{-}^{\dagger} \right)_{\alpha}. \end{aligned}$$

Thus we have

$$\begin{aligned} {}_1\langle (\psi^* S^{-1} \rho_2)_{\alpha} U^{-1} \phi U (S\psi)_{\alpha} \rangle_0 &= \psi_{+}^{\dagger} \left( \frac{1}{1+C s_{-}} \vec{S}^{-1} \rho_2^{\leftarrow} \frac{1}{1+s_{-}C} \right) \cdot \phi \psi_{-}^{\dagger} \\ &= \psi_{+}^{\dagger} (S^{-1})_{\mu\lambda} \left( \frac{1}{1+C s_{-}} \right)_{\alpha\mu} (\rho_2)_{\lambda\nu} \phi \left( \frac{1}{1+s_{-}C} \right)_{\nu\beta} (S)_{\nu\mu} \psi_{-}^{\dagger}, \end{aligned} \quad (\text{A} \cdot 26)$$

from which the result given in the text eq. (51) is obtained.

Then we go back to the formula of Sec. 3 to find the way of dealing with the multiple products of  $s_{-}C$ . By the formula (14) of Sec. 3 we have to calculate

$$\begin{aligned} {}_1\langle (\psi^* S^{-1} \rho_2)_{\alpha} (\exp [\psi^* ; C\psi] \cdot \phi) (S\psi)_{\alpha} \rangle_0 \\ = {}_1\langle (\psi^* S^{-1} \rho_2)_{\alpha} \{ \phi + \psi^* (S^{-1} \phi S - \phi) \psi \\ + \frac{1}{2!} \psi^* [S^{-1} \psi'^* (S'^{-1} \phi S' - \phi) \psi' S - \psi'^* (S'^{-1} \phi S' - \phi) \psi'] \psi \\ + \cdots \} (S\psi)_{\alpha} \rangle_0. \end{aligned} \quad (\text{A} \cdot 27)$$

We will examine the third term of the expansion of (A·27). It can be transformed as

$$\begin{aligned} & \frac{1}{2} \langle (\psi^* S^{-1} \rho_2)_{\lambda} \psi_{\alpha}^* \psi_{\beta}^* \{ S_{\alpha\mu}^{-1} (C \cdot \phi)_{\beta\tau} S_{\mu\delta} - (C \cdot \phi)_{\beta\tau} \partial_{\alpha\delta} \} \psi_{-\tau}^{\dagger} \psi_{-\delta}^{\dagger} (S\psi_{-}^{\dagger})_{\lambda} \rangle_0 \\ &= \frac{1}{2} \langle \{ \psi_{-} (S^{-1} \rho_2)_{\nu\lambda} (\psi_{-\alpha} \psi_{+\beta}^{\dagger} + \psi_{+\alpha}^{\dagger} \psi_{-\beta}) + (\psi_{+}^{\dagger} S^{-1} \rho_2)_{\lambda} \psi_{-\alpha} \psi_{-\beta} \} \\ & \quad \times \{ S_{\alpha\mu}^{-1} (C \cdot \phi)_{\beta\tau} S_{\mu\delta} - (C \cdot \phi)_{\beta\tau} \partial_{\alpha\delta} \} \psi_{-\tau}^{\dagger} \psi_{-\delta}^{\dagger} (S)_{\lambda\rho} \psi_{-\rho}^{\dagger} \rangle_0 \\ &= \frac{1}{2} \{ \psi_{+\beta}^{\dagger} (S^{-1} \rho_2)_{\nu\lambda} [S_{\alpha\mu}^{-1} (C \cdot \phi)_{\beta\tau} S_{\mu\delta} - (C \cdot \phi)_{\beta\tau} \partial_{\alpha\delta}] (S)_{\lambda\rho} \\ & \quad \times [(s_{-})_{\rho\alpha} (s_{-})_{\tau\nu} \psi_{-\delta}^{\dagger} + (s_{-})_{\tau\alpha} (s_{-})_{\delta\nu} \psi_{-\rho}^{\dagger}] \\ & \quad + \psi_{+\alpha}^{\dagger} (S^{-1} \rho_2)_{\nu\lambda} [ \quad \quad \quad ] (S)_{\lambda\rho} \\ & \quad \times [(s_{-})_{\rho\beta} (s_{-})_{\delta\nu} \psi_{-\tau}^{\dagger} + (s_{-})_{\delta\beta} (s_{-})_{\tau\nu} \psi_{-\rho}^{\dagger}] \\ & \quad + (\psi_{+\alpha}^{\dagger} S^{-1} \rho_2)_{\lambda} [ \quad \quad \quad ] (S)_{\lambda\rho} \\ & \quad \times [(s_{-})_{\rho\alpha} (s_{-})_{\delta\beta} \psi_{-\tau}^{\dagger} + (s_{-})_{\rho\beta} (s_{-})_{\tau\alpha} \psi_{-\delta}^{\dagger}] \} \end{aligned}$$

$$\begin{aligned}
&= 1/2 \cdot \{ \phi_+^\dagger (C \cdot \phi) (s_- \overrightarrow{S}^{-1} \rho_2 \overleftarrow{S} s_-) C \phi_-^\dagger + \phi_+^\dagger (C \cdot \phi) (s_- C s_- \overrightarrow{S}^{-1} \rho_2 \overleftarrow{S}) \phi_-^\dagger \\
&\quad + \phi_+^\dagger C (s_- \overrightarrow{S}^{-1} \rho_2 \overleftarrow{S} s_-) (C \cdot \phi) \phi_-^\dagger + \phi_+^\dagger C s_- (C \cdot \phi) s_- \overrightarrow{S}^{-1} \rho_2 \overleftarrow{S} \phi_-^\dagger \\
&\quad + \phi_+^\dagger (\overrightarrow{S}^{-1} \rho_2 \overleftarrow{S} s_- C s_-) (C \cdot \phi) \phi_-^\dagger + \phi_+^\dagger (\overrightarrow{S}^{-1} \rho_2 \overleftarrow{S} s_-) (C \cdot \phi) s_- C \phi_-^\dagger \}.
\end{aligned} \quad (\text{A} \cdot 28)$$

Other terms contain a spur expression and are discarded. This result shows us that we should calculate according to the following rule

$$\begin{aligned}
1/2 \cdot \phi_+^\dagger [ & (\overline{C\phi}) s_- \overrightarrow{\rho_2} s_- \overleftarrow{C} + \overline{C} s_- \overrightarrow{\rho_2} s_- (\overline{C\phi}) + (\overline{C\phi}) s_- \overrightarrow{C} s_- \overleftarrow{\rho_2} \\
& + \overline{C} s_- (\overline{C\phi}) s_- \overrightarrow{\rho_2} + \overrightarrow{\rho_2} s_- \overleftarrow{C} s_- (\overline{C\phi}) + \overrightarrow{\rho_2} s_- (\overline{C\phi}) s_- \overleftarrow{C} ] \phi_-^\dagger,
\end{aligned} \quad (\text{A} \cdot 29)$$

where we have put

$$\overrightarrow{\rho_2} = \overrightarrow{S}^{-1} \rho_2 \overleftarrow{S}, \quad (\text{A} \cdot 30)$$

and boson operators contained in it should be put on the both ends. The Dirac matrices should be arranged in the order written formally, while as regards the boson operators contained in  $\overleftarrow{C}$ , the permutation of their order should be made. This result is equivalent to that obtained from the third term of (A·26) if we obey the rule stated earlier. Thus we have checked our statements made in the text, "The boson operator contained in  $\overrightarrow{S}^{-1}$  or  $\overleftarrow{S}$  should stand on the left- or right-hand ends respectively."

Consequently we should calculate

$$\begin{aligned}
& {}_+\rho_+ \left[ ((\overrightarrow{Q_1 G_-}) \overleftarrow{P_1} + \overrightarrow{P_1} (\overleftarrow{Q_1 G_-})) - ((\overrightarrow{G_+ Q_1}) \cdot \overleftarrow{P_0} + \overrightarrow{P_0} \cdot (\overleftarrow{Q_1 G_-})) \right. \\
& \quad \times \frac{1}{2} \int_0^1 dx \frac{1}{1+xX} \left\{ Z \frac{1}{1+xX} \tilde{Z} + \tilde{Z} \frac{1}{1+xX} Z \right\} \frac{1}{1+X} \\
& \quad + (\overrightarrow{P_1} \overleftarrow{P_0} - (\overrightarrow{Q_1 G_-}) (\overleftarrow{Q_1 G_-})) \int_0^1 dx \frac{1}{1+xX} Z \frac{1}{1+xX} \\
& \quad \left. + (\overrightarrow{P_0} \overleftarrow{P_1} - (\overrightarrow{G_+ Q_1}) (\overleftarrow{G_+ Q_1})) \int_0^1 dx \frac{1}{1+xX} \tilde{Z} \frac{1}{1+xX} \right] \cdot \phi \\
& - {}_-\rho_- \left[ ((\overrightarrow{G_+ Q_1}) \overleftarrow{P_0} + \overrightarrow{P_0} (\overleftarrow{Q_1 G_-})) \left( \frac{1}{1+X} \right)^2 \right] \cdot \phi \\
& - i {}_+\rho_- \left[ - ((\overrightarrow{G_+ Q_1}) \overleftarrow{P_0} + \overrightarrow{P_0} (\overleftarrow{Q_1 G_-})) \frac{1}{1+X} \tilde{Z} \frac{1}{1+X} \right. \\
& \quad \left. + (\overrightarrow{P_1} \overleftarrow{P_0} - (\overrightarrow{Q_1 G_-}) (\overleftarrow{Q_1 G_-})) \frac{1}{1+X} \right] \cdot \phi \\
& + i {}_-\rho_+ \left[ - ((\overrightarrow{G_+ Q_1}) \overleftarrow{P_0} + \overrightarrow{P_0} (\overleftarrow{Q_1 G_-})) \frac{1}{1+X} Z \frac{1}{1+X} \right. \\
& \quad \left. + (\overrightarrow{P_0} \overleftarrow{P_1} - (\overrightarrow{G_+ Q_1}) (\overleftarrow{G_+ Q_1})) \frac{1}{1+X} \right] \cdot \phi,
\end{aligned} \quad (\text{A} \cdot 31)$$



in the place of the formal expression

$$\begin{aligned}
 & \frac{1}{1+C_{s-}} \vec{S}^{-1} \rho_2 \phi \overleftarrow{S} \frac{1}{1+s_- C} \\
 &= \left\{ \left[ 1 - \rho_- \frac{X}{1+X} - i_+ \rho_- \tilde{Z} \frac{1}{1+X} \right] [ {}_+\rho_+ \vec{P}_1 + {}_-\rho_- \vec{P}_0 - i_+ \rho_- (\vec{Q}_1 \vec{G}_-) - i_- \rho_+ (\vec{G}_+ \vec{Q}_1) ] \right. \\
 & \quad \times \left( \frac{{}_+\rho_- - {}_-\rho_+}{i} \right) [ {}_+\rho_+ \overleftarrow{P}_1 + {}_-\rho_- \overleftarrow{P}_0 + i_+ \rho_- (\overleftarrow{Q}_1 \overleftarrow{G}_-) + i_- \rho_+ (\overleftarrow{G}_+ \overleftarrow{Q}_1) ] \\
 & \quad \times \left[ 1 - \rho_- \frac{X}{1+X} + i_- \rho_+ \frac{1}{1+X} Z \right] \Big\} \cdot \phi \\
 &= \left\{ ((\vec{Q}_1 \vec{G}_-) \overleftarrow{P}_1 + \overleftarrow{P}_1 (\overleftarrow{G}_+ \vec{Q}_1)) {}_+\rho_+ \right. \\
 & \quad - ((\vec{G}_+ \vec{Q}_1) \overleftarrow{P}_0 + \overleftarrow{P}_0 (\vec{Q}_1 \vec{G}_-)) \left( {}_+\rho_+ \tilde{Z} \left( \frac{1}{1+X} \right)^2 Z + {}_-\rho_- \left( \frac{1}{1+X} \right)^2 \right. \\
 & \quad \left. \left. - i_+ \rho_- \tilde{Z} \left( \frac{1}{1+X} \right)^2 + i_- \rho_+ \left( \frac{1}{1+X} \right)^2 Z \right) \right. \\
 & \quad + (\overleftarrow{P}_1 \overleftarrow{P}_0 - (\vec{Q}_1 \vec{G}_-) (\overleftarrow{Q}_1 \vec{G}_-)) \left( {}_+\rho_+ \frac{1}{1+X} Z - i_+ \rho_- \frac{1}{1+X} \right) \\
 & \quad \left. + (\overleftarrow{P}_0 \overleftarrow{P}_1 - (\vec{G}_+ \vec{Q}_1) (\overleftarrow{G}_+ \vec{Q}_1)) \left( {}_+\rho_+ \tilde{Z} \frac{1}{1+X} + i_- \rho_+ \frac{1}{1+X} \right) \right\} \cdot \phi.
 \end{aligned} \tag{A.32}$$

In eq. (A.31) the boson operators operate in the order from the right to the left-hand side.

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## Beta-ray Spectra, III

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In the general expression of the correction factor, there appear the integrals which contain the product of the nucleon wave functions. In this paper these integrals are expressed in terms of the radial integrals, by making use of the  $jj$  coupling model, and their non-relativistic approximation is discussed.

## § 1. Introduction

In the first paper<sup>1)\*</sup> of this series, a general expression of the correction factor is obtained. This correction factor contains the integrals  $\mathfrak{N}_{su}^{a,b}(J, J')$ , which correspond to the nuclear matrix elements in the usual formalism but differ from these in having the part of the lepton part in them. In the second paper<sup>2,\*\*)†</sup>, the integral  $\mathfrak{N}_{su}^{a,b}(J, J')$  for the many nucleon configuration is reduced to that of the single nucleon transition  $\mathfrak{N}_{su}^{a,b}(K, K')$ . In the present paper we have calculated the angular part of  $\mathfrak{N}_{su}^{a,b}(K, K')$ . As the result of this calculation, there remains nothing unknown in the correction factor, except the products of the coupling constants and the radial integrals in  $\mathfrak{N}_{su}^{a,b}(K, K')$  and the mixing ratio of the configuration as is mentioned in II. Unfortunately, at present, we have no sufficient information on the nuclear structure, so that the radial integrals can not be calculated.

In the third section the non-relativistic approximation is discussed.

§ 2. Relativistic treatment of  $\mathfrak{N}_{su}^{a,b}(K, K')$ 

The general expression of the correction factor is given by (I. 24). This expression contains the unknown parameters  $\mathfrak{N}_{su}^{a,b}(J, J')$ . As is mentioned in I, these parameters reduce to the nuclear matrix elements, when the lepton part is evaluated at the nuclear radius. If we can calculate these parameters completely, and are able to determine the coupling constants, there remains no ambiguity any more.

The coupling constants should be determined experimentally. On the other hand, the parameters  $\mathfrak{N}_{su}^{a,b}(J, J')$  are to be calculated theoretically when we know the nuclear structure completely. To our regret, however, at present the sufficient information on the nuclear structure has been obtained neither theoretically nor experimentally, and we have to make

\* This will be referred to as I.

\*\*) This will be referred to as II.

some assumptions on nuclear model to calculate these parameters. In the following, we shall assume the  $j\bar{j}$  coupling model.

As is shown in II, the parameter  $\bar{\mathfrak{Y}}_{suw}^{a,b}(J, J')$  for the many nucleon configuration can be reduced to that of the single nucleon transition,  $\bar{\mathfrak{Y}}_{suw}^{a,b}(K, K')^*$ , where  $K$  or  $K' = \pm(j_P \text{ or } N + 1/2)$  for  $j_P \text{ or } N = l_F \text{ or } N \mp 1/2$ . Therefore, in the most part of this paper, we shall confine ourselves to the single nucleon transition.

$\bar{\mathfrak{Y}}_{suw}^{a,b}(K, K')$  is defined as follows\*\*:

$$\bar{\mathfrak{Y}}_{suw}^{a,b}(K, K') \equiv \int (r-R)^{a_r b} \mathfrak{T}_{suw}(K, K') dr, \quad (1)$$

with

$$\mathfrak{T}_{suw}(K, K') = (2k+1)^{-1/2} (1/2 \| \mathcal{Y}_k(\sigma) \| 1/2) (K \| Q\omega_s \times \mathcal{Y}_{u(kr)}(\sigma, \nu) \| K') \quad (2a)$$

for odd  $\omega$ , and

$$\mathfrak{T}_{suw}(K, K') = -i(2k+1)^{-1/2} (1/2 \| \mathcal{Y}_k(\sigma) \| 1/2) (K \| Q\omega_s \times \mathcal{Y}_{u(kr)}(\sigma, \nu) \| K') \quad (2b)$$

for even  $\omega$ . The reduced matrix element is given by

$$\langle \Psi^* Q\omega \times \mathcal{Y}_{u(kr)}^{\nu-\mu, -\mu}(\sigma, \nu) \phi d\Omega = (j_N u j_P | m' m - m' m) (K \| Q\omega_s \times \mathcal{Y}_{u(kr)}(\sigma, \nu) \| K'), \quad (3)$$

where  $\phi$  and  $\Psi$  are the initial neutron and the final proton wave function, respectively,\*\*

In an arbitrary central field the proton wave function is

$$\Psi = \begin{pmatrix} -i F_K^P \chi_{-K}^m \\ G_K^P \chi_K^m \end{pmatrix}. \quad (4)$$

The neutron wave function has the same form with  $F_{K'}^N (G_{K'}^N)$  in place of  $F_K^P (G_K^P)$ .  $F_K^P$  and  $G_K^P$  are the radial wave functions, and the angular parts are

$$\chi_K^m = \sum_T (l_K 1/2 j_P | m - T T m) Y_{l_K}^{m-T}(\nu) \chi_{1/2}^T. \quad (5)$$

Inserting (4) in to (3), this can be written as follows:

$$\begin{cases} \langle \Psi^* Q\omega \times \mathcal{Y}_{u(kr)}^{\nu-\mu, -\mu}(\sigma, \nu) \phi d\Omega \\ = i [ F_K^P (G_{K'}^N (\chi_{-K}^m \mathcal{Y}_{u(kr)}^{\mu-\nu, -\mu}(\sigma, \nu) \chi_{K'}^{m'}) - \epsilon_s (F_K^P (G_{K'}^N (\chi_K^m \mathcal{Y}_{u(kr)}^{-\mu-\nu, -\mu}(\sigma, \nu) \chi_{-K'}^{m'}) ] \quad (6a) \\ \text{for odd } \omega, \end{cases}$$

\* Cf. (II. 12). In the following, for the sake of convenience, to distinguish proton and neutron we shall use prime for neutron and no prime for proton. The exceptions are  $j_l$ ,  $j_N$ ,  $l_P$  and  $l_N$ . As  $j$  or  $l$  is used in I to represent the total and the orbital angular momentum of electron, we use  $j_P$  etc. instead of  $j$  etc..

\*\* Cf. (I. 25). We use the same notations as in I and II.

\*\*\* Note that the integration is only over the direction, and not extended over the whole volume.

$$\left\{ \begin{aligned} &= F_K^P F_{K'}^N (\chi_{-K}^m, \mathcal{J}_{u(k\nu)}^{j-\mu_\nu-\mu}(\sigma, \nu) \chi_{-K'}^{m'}) + \varepsilon_s G_K^P G_{K'}^N (\chi_K^m, \mathcal{J}_{u(k\nu)}^{j-\mu_\nu-\mu}(\sigma, \nu) \chi_{K'}^{m'}) \\ &\quad \text{for even } \omega. \end{aligned} \right. \quad (6b)$$

By making use of Racah's method\*, the matrix element of  $\mathcal{J}_{u(k\nu)}^{j-\mu_\nu-\mu}(\sigma, \nu)$  can be written as

$$\begin{aligned} &(\chi_K^m, \mathcal{J}_{u(k\nu)}^{j-\mu_\nu-\mu}(\sigma, \nu) \chi_{K'}^{m'}) \\ &= (-1)^{j_P+j_N+l_P+v+1} (2j_P+1)^{-1/2} \left( \frac{1}{2} \|\mathcal{J}_k\| \frac{1}{2} \right) (j_N u j_P | m' m - m' m) B_{uvk}(KK'), \end{aligned} \quad (7)$$

where

$$\begin{aligned} B_{uvk}(KK') &= [(2l_P+1)(2l_N+1)(2j_P+1)(2j_N+1)(2u+1)(2v+1)/2\pi]^{1/2} \\ &\quad \times \begin{pmatrix} l_P & l_N & v \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_P & l_N & v \\ j_P & j_N & u \\ 1/2 & 1/2 & k \end{pmatrix}. \end{aligned} \quad (8)$$

From (3), (6) and (7), the reduced matrix elements are:

$$\begin{aligned} &(K \| Q\omega_s \times \mathcal{J}_{u(k\nu)}^{j-\mu_\nu-\mu}(\sigma, \nu) \| K') \\ &= \begin{cases} (-1)^{j_P+j_N+l_P+v+1/2} (2j_P+1)^{-1/2} \left( \frac{1}{2} \|\mathcal{J}_k\| \frac{1}{2} \right) \\ \quad \times [F_K^P G_{K'}^N B_{uvk}(-K, K') + \varepsilon_s G_K^P F_{K'}^N B_{uvk}(K, -K')] & \text{for odd } \omega, \quad (8a) \\ (-1)^{j_P+j_N+l_P+v} (2j_P+1)^{-1/2} \left( \frac{1}{2} \|\mathcal{J}_k\| \frac{1}{2} \right) \\ \quad \times [F_K^P F_{K'}^N B_{uvk}(-K, -K') - \varepsilon_s G_K^P G_{K'}^N B_{uvk}(K, K')] & \text{for even } \omega. \quad (8b) \end{cases} \end{aligned}$$

where  $l_P \equiv l_P(K)$  and not  $l_P(-K)$ .

From (2), (3) and (8), we have for odd  $\omega$

$$\begin{aligned} \mathfrak{T}_{su\nu}(KK') &= (-1)^{j_P+j_N+l_P+v+1/2} [(2k+1)(2j_P+1)]^{-1/2} (1/2 \|\mathcal{J}_k(\sigma)\| 1/2)^2 \\ &\quad \times [F_K^P G_{K'}^N B_{uvk}(-K, K') + \varepsilon_s G_K^P F_{K'}^N B_{uvk}(K, -K')] \end{aligned} \quad (9a)$$

and for even  $\omega$

$$\begin{aligned} \mathfrak{T}_{su\nu}(KK') &= (-1)^{j_P+j_N+l_P+v-1/2} [(2k+1)(2j_P+1)]^{-1/2} (1/2 \|\mathcal{J}_k(\sigma)\| 1/2)^2 \\ &\quad \times [F_K^P F_{K'}^N B_{uvk}(-K, -K') - \varepsilon_s G_K^P G_{K'}^N B_{uvk}(K, K')]. \end{aligned} \quad (9b)$$

The transition probability amplitude  $\int H d\tau$  for single nucleon transition can be written from (I, 12, 13) and (8) as follows:

$$\begin{aligned} &\int H d\tau = (-1)^{i+j_\nu+l+l(\kappa_\nu)+j_N+l_P+v+1/2} \\ &\quad \times \left[ \sum_{su\nu} \begin{pmatrix} j & j_\nu & u \\ -\mu & -\mu_\nu & \mu+\mu_\nu \end{pmatrix} \begin{pmatrix} j_P & j_N & u \\ m & -m' & m'-m \end{pmatrix} (2k+1)^{-1/2} (1/2 \|\mathcal{J}_k\| 1/2)^2 \right. \\ &\quad \left. \times \{ f_\kappa f_{\kappa_\nu}^\nu B_{uvk}(-\kappa, -\kappa_\nu) + \varepsilon_s g_\kappa g_{\kappa_\nu}^\nu B_{uvk}(\kappa, \kappa_\nu) \} \right] \end{aligned}$$

\* As to the references on this method, see I.

$$\begin{aligned}
& \times \{F_K^P G_{K'}^N B_{uvk}(-K, K') + \varepsilon_s G_K^P F_{K'}^N B_{uvk}(K, -K')\} r^2 dr \\
& - \sum_{s, s'} \left( \begin{matrix} j & j_v & u \\ -\mu & -\mu_v & \mu + \mu_v \end{matrix} \right) \left( \begin{matrix} j_P & j_{K'} & u \\ m & m' & m' - m \end{matrix} \right) (2\bar{k} + 1)^{-1/2} (1/2 \|\mathcal{J}_k\| 1/2)^2 \\
& \times \{f_\kappa g_{\kappa_v}^v B_{uvk}(-\kappa, \kappa_v) - \varepsilon_s g_\kappa f_{\kappa_v}^v B_{uvk}(\kappa, -\kappa_v) \\
& \times \{F_K^P F_{K'}^N B_{uvk}(-K, -K') - \varepsilon_s G_K^P G_{K'}^N B_{uvk}(K, K')\} r^2 dr \}. \quad (10)
\end{aligned}$$

Here  $s$  and  $\bar{s}$  mean odd and even operator, respectively.

Inserting (9a, b) into (1), we get

i) for odd  $\omega$

$$\begin{aligned}
\bar{\mathfrak{S}}_{suv}^{a,b}(K, K') &= (-1)^{j_P + j_{K'} + l_P + v + 1/2} [(2\bar{k} + 1)(2j_P + 1)]^{-1/2} (1/2 \|\mathcal{J}_k\| 1/2)^2 \\
&\times [\mathcal{B}_{uvk}(-K, K') \mathfrak{A}^{a,b}(K, K') + \varepsilon_s B_{uvk}(K, -K') \mathfrak{B}^{a,b}(K, K')], \quad (11a)
\end{aligned}$$

and ii) for even  $\omega$

$$\begin{aligned}
\bar{\mathfrak{S}}_{suv}^{a,b}(K, K') &= (-1)^{j_P + j_{K'} + l_P + v - 1/2} [(2\bar{k} + 1)(2j_P + 1)]^{-1/2} (1/2 \|\mathcal{J}_k\| 1/2)^2 \\
&\times [B_{uvk}(-K, -K') \mathfrak{C}^{a,b}(K, K') - \varepsilon_s B_{uvk}(K, K') \mathfrak{D}^{a,b}(K, K')]. \quad (11b)
\end{aligned}$$

$\mathfrak{A}^{a,b}(K, K')$ ,  $\mathfrak{B}^{a,b}(K, K')$ ,  $\mathfrak{C}^{a,b}(K, K')$  and  $\mathfrak{D}^{a,b}(K, K')$  are defined by

$$\mathfrak{A}^{a,b}(K, K') = \int (r - R)^a r^b F_K^P G_{K'}^N dr, \quad (12a)$$

$$\mathfrak{B}^{a,b}(K, K') = \int (r - R)^a r^b G_K^P F_{K'}^N dr, \quad (12b)$$

$$\mathfrak{C}^{a,b}(K, K') = \int (r - R)^a r^b F_K^P F_{K'}^N dr, \quad (12c)$$

$$\mathfrak{D}^{a,b}(K, K') = \int (r - R)^a r^b G_K^P G_{K'}^N dr. \quad (12d)$$

When we substitute (11a, b) for  $\bar{\mathfrak{S}}_{suv}^{a,b}(K, K')$  in the correction factor (I. 24), the phase factor

$$(-1)^{j_P + j_{K'} + l_P} \quad (13)$$

drops out. Therefore,  $\bar{\mathfrak{S}}_{suv}^{a,b}(J, J')$  in (I. 24) is to be replaced by

$$(-1)^{v + 1/2} B(u) [(2\bar{k} + 1)(2j_P + 1)]^{-1/2} (1/2 \|\mathcal{J}_k\| 1/2)^2 \mathfrak{R}_{suv}^{a,b}(K, K'), \quad (14)$$

where  $B(u)$  is the ratio of  $\bar{\mathfrak{S}}_{suv}^{a,b}(J, J')$  to  $\bar{\mathfrak{S}}_{suv}^{a,b}(K, K')$  given by (II. 11).  $\mathfrak{R}_{suv}^{a,b}(K, K')$  is given by

$$\mathfrak{R}_{suv}^{a,b}(K, K') = B_{suv}(-K, K') \mathfrak{A}^{a,b}(K, K') + \varepsilon_s B_{suv}(K, -K') \mathfrak{B}^{a,b}(K, K') \quad (15a)$$

for odd  $\omega$ ,

and for even  $\omega$

$$\mathfrak{R}_{suv}^{a,b}(K, K') = -B_{suv}(-K, -K') \mathfrak{C}^{a,b}(K, K') + \varepsilon_s B_{suv}(K, K') \mathfrak{D}^{a,b}(K, K'). \quad (15b)$$



By performing this replacement, the correction factor (I. 24) becomes as follows :

$$\begin{aligned}
 C = & \frac{4\pi^2}{p^2 q^2 F_1(R)} \frac{2J+1}{(2J'+1)(2j_P+1)} \sum'_{\substack{ss'uu'v' \\ \kappa\kappa_v i j m n}} \lambda_s \lambda_{s'}^* \frac{(-1)^{v+v'}}{(2u+1)^2 (2k+1)^{1/2} (2k'+1)^{1/2}} \\
 & \times B^2(u) (1/2 \|\mathcal{O}_k^j\| 1/2)^2 (1/2 \|\mathcal{O}_k^j\| 1/2)^{*2} \\
 & \times [\alpha_{ss'uu'v'}^{(I)}(\kappa, \kappa_v) \varphi_{I,ss'\kappa_v} \mathfrak{L}_{\kappa nm} \delta_{ij}^{(I,ss'\kappa_v)} \\
 & \quad \times \mathfrak{R}_{su\nu}^{(I,ss'\kappa_v)+i+1}(K, K') \mathfrak{R}_{s'u'\nu'}^{(I,ss'\kappa_v)+j+1}(K, K')^* \\
 & + \alpha_{ss'uu'v'}^{(M)}(\kappa, \kappa_v) \varphi_{M,ss'\kappa_v} \mathfrak{M}_{\kappa nm} \delta_{ij}^{(M,ss'\kappa_v)} \\
 & \quad \times \mathfrak{R}_{su\nu}^{(M,ss'\kappa_v)+i+1}(K, K') \mathfrak{R}_{s'u'\nu'}^{(M,ss'\kappa_v)+j+1}(K, K')^* \\
 & + \alpha_{ss'uu'v'}^{(N_1)}(\kappa, \kappa_v) \varphi_{N_1,ss'\kappa_v} \mathfrak{N}_{\kappa nm} \delta_{ij}^{(N_1,ss'\kappa_v)} \\
 & \quad \times \mathfrak{R}_{su\nu}^{(N_1,ss'\kappa_v)+i+1}(K, K') \mathfrak{R}_{s'u'\nu'}^{(N_1,ss'\kappa_v)+j+1}(K, K')^* \\
 & + \alpha_{ss'uu'v'}^{(N_2)}(\kappa, \kappa_v) \varphi_{N_2,ss'\kappa_v} \mathfrak{N}_{\kappa nm} \delta_{ij}^{(N_2,ss'\kappa_v)} \\
 & \quad \times \mathfrak{R}_{su\nu}^{(N_2,ss'\kappa_v)+i+1}(K, K') \mathfrak{R}_{s'u'\nu'}^{(N_2,ss'\kappa_v)+j+1}(K, K')^* ] \quad (16)
 \end{aligned}$$

$(-1)^{v+v'}$  is 1 (or  $-1$ ) when the even-odd characters of  $\omega$  and  $\omega'$  are the same (or different).<sup>1)</sup> In this formula,  $B(u)$  connects the matrix element for the many-nucleon configuration to that of the single-nucleon.  $\alpha_{ss'uu'v'}^{(I)}$  etc. are the geometrical factors for the leptons.  $\mathfrak{L}_{\kappa nm}$ ,  $\mathfrak{M}_{\kappa nm}$  and  $\mathfrak{N}_{\kappa nm}$  are the electron radial parts, and the factors  $\varphi_{N,ss'\kappa_v} \delta_{ij}^{(N,ss'\kappa_v)}$  are the neutrino radial parts. The factors  $\mathfrak{R}_{su\nu}^{a,b}(K, K')$  are the nucleon parameters.

In I we have taken the factors  $\mathfrak{Y}_{su\nu}^{\alpha,b}(J, J')$  as unknown parameters. In (16) we consider the integrals  $\mathfrak{R}_{su\nu}^{a,b}(K, K')$ , or  $\mathfrak{V}^{a,b}(K, K')$ ,  $\mathfrak{V}^{a,b}(K, K')$ ,  $\mathfrak{U}^{a,b}(K, K')$  and  $\mathfrak{D}^{a,b}(K, K')$ , as the parameters. As is seen from (12), to evaluate these parameters we must know the nuclear radial wave functions. But these wave functions are not known as yet, and we have left these integrals as unknown.

The integrals  $\mathfrak{R}_{su\nu}^{a,b}(K, K')$  depend on  $u$  and  $v$ . But the integrals  $\mathfrak{V}^{a,b}(K, K')$  etc. are independent of  $u$  and  $v$ . Accordingly, by the use of  $\mathfrak{V}^{a,b}(K, K')$  etc. in place of  $\mathfrak{R}_{su\nu}^{a,b}(K, K')$  the total number of the parameters are reduced. This reduction facilitates the comparison of the theoretical curve with the experimentally obtained spectrum.

As the angular parts are already calculated in this paper, we shall be able to determine the coupling constants when the radial wave functions are known. Or inversely, when we know the coupling constants, nuclear configuration and the radial wave functions of the nucleon, the half-life and the spectrum shape can be predicted from (16).

By taking the non-relativistic approximation Rose and Osborn<sup>3)</sup> have obtained the relation between the various radial parts of the nuclear matrix elements. The analogous procedure may be applicable to our calculation. But as will be seen in the next section, if we want to discuss in more detail, such a procedure leads to complexity. Therefore, we prefer to treat the nucleon relativistically, though the reduction of the number of the parameters is not sufficient. When there are many parameters to be determined in the correction

factor, the determination may be done by the method of least squares<sup>4)</sup>.

Although  $\mathfrak{M}^{a,b}(K, K')$  and  $\mathfrak{B}^{a,b}(K, K')$  are of the same order,  $\mathfrak{S}^{a,b}(K, K')$  is usually of the order of  $1/M^2$  and can be neglected as compared with  $\mathfrak{D}^{a,b}(K, K')$ .

### § 3. Non-relativistic approximation

The Hamiltonian of the nucleon field coupled to the lepton field is

$$H = H_N + H_\beta + H_l, \quad (17)$$

where

$$H_N = -\beta M + \mathfrak{E}_N + \mathcal{O}_N \quad (18a)$$

is the nucleon Hamiltonian, and

$$H_\beta = \mathfrak{E}_\beta + \mathcal{O}_\beta \quad (18b)$$

is the interaction Hamiltonian while  $H_l$  is the lepton Hamiltonian.  $\mathfrak{E}$  and  $\mathcal{O}$  are the even and the odd operators:

$$\mathfrak{E}_N = V(e) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}, \quad (19a)$$

$$\mathcal{O}_N = -\mathbf{aP} + V(o) = -\mathbf{aP} + \begin{pmatrix} 0 & U_1 \\ U_2 & 0 \end{pmatrix}, \quad (19b)$$

$$\begin{aligned} \mathfrak{E}_\beta &= \sum_s \lambda_s \omega_s \times \mathcal{J}_{u(k\nu)}^{-\mu_\nu - \mu}(\boldsymbol{\sigma}, \mathbf{r}) L_s(r) \\ &= \sum_s \lambda_s \begin{pmatrix} L_s(r) \mathcal{J}_{u(k\nu)}^{-\mu_\nu - \mu}(\boldsymbol{\sigma}, \mathbf{r}) & 0 \\ 0 & \varepsilon_s L_s(r) \mathcal{J}_{u(k\nu)}^{-\mu_\nu - \mu}(\boldsymbol{\sigma}, \mathbf{r}) \end{pmatrix}, \end{aligned} \quad (19c)$$

$$\begin{aligned} \mathcal{O}_\beta &= \sum_{\bar{s}} \lambda_{\bar{s}} \omega_{\bar{s}} \times \mathcal{J}_{u(k\nu)}^{-\mu_\nu - \mu}(\boldsymbol{\sigma}, \mathbf{r}) L_{\bar{s}}(r) \\ &= \sum_{\bar{s}} \lambda_{\bar{s}} \begin{pmatrix} 0 & L_{\bar{s}}(r) \mathcal{J}_{u(k\nu)}^{-\mu_\nu - \mu}(\boldsymbol{\sigma}, \mathbf{r}) \\ \varepsilon_{\bar{s}} L_{\bar{s}}(r) \mathcal{J}_{u(k\nu)}^{-\mu_\nu - \mu}(\boldsymbol{\sigma}, \mathbf{r}) & 0 \end{pmatrix}. \end{aligned} \quad (19d)$$

$V(r)$  and  $V(o)$  are the even and the odd potentials.  $L_e(r)$  and  $L_s(r)$  are the lepton radial parts multiplied by certain factors\*:

$$\begin{aligned} L_s(r) &= (-1)^{j+j_\nu+l+l(\kappa_\nu)+3/2} (2k+1)^{-1/2} (1/2 \|\mathcal{J}_k\| 1/2) \begin{pmatrix} j & j_\nu & n \\ -\mu & -\mu_\nu & \mu + \mu_\nu \end{pmatrix} \\ &\quad \times [f_\kappa g_{\kappa_\nu}^\nu B_{u\kappa}(-\kappa, \kappa_\nu) - \varepsilon_s g_\kappa f_{\kappa_\nu}^\nu B_{u\kappa}(\kappa, -\kappa_\nu)] Q, \end{aligned} \quad (20a)$$

$$\begin{aligned} L_{\bar{s}}(r) &= (-1)^{j+j_\nu+l+l(\kappa_\nu)} (2k+1)^{-1/2} (1/2 \|\mathcal{J}_k\| 1/2) \begin{pmatrix} j & j_\nu & n \\ -\mu & -\mu_\nu & \mu + \mu_\nu \end{pmatrix} \\ &\quad \times [f_\kappa f_{\kappa_\nu}^\nu B_{u\bar{\kappa}}(-\kappa, -\kappa_\nu) + \varepsilon_{\bar{s}} g_\kappa g_{\kappa_\nu}^\nu B_{u\bar{\kappa}}(\kappa, \kappa_\nu)] Q. \end{aligned} \quad (20b)$$

\* Cf. (I. 7a, b).  $s(\bar{s})$  is used to specify the even (odd) character of  $\omega_s$  ( $\omega_{\bar{s}}$ ).

Rose and Osborn<sup>3</sup> have considered the many-nucleon Hamiltonian, and taken as the potential the sum of the two-nucleon potentials. In this paper, however, for the sake of simplicity we assume the single particle model. The Hamiltonian (18a) is the one nucleon Hamiltonian, and the potentials  $V(r)$  and  $V(0)$  are the nuclear potential for one nucleon. The mutual interaction between two nucleons may be treated as the perturbation, and the state with the mutual interaction can be constructed by the superposition of the states determined by the Hamiltonian (18a).

When we eliminate the odd operators of the order lower than  $1/M^2$  by repeated application of a canonical transformation<sup>5(a)</sup>, the following expression for the transformed Hamiltonian, approximate to the order  $1/M^2$ , is obtained:

$$H'_N = -\beta M + \mathcal{E}_N - (\beta/2M) \mathcal{O}_N^2 + (1/4M^2) \mathcal{O}_N \mathcal{E}_N \mathcal{O}_N - (1/8M^2) [\mathcal{E}_N, \mathcal{O}_N^2]_+, \quad (21a)$$

and

$$H'_\beta = \mathcal{E}_\beta - (\beta/2M) [\mathcal{O}_N, \mathcal{O}_\beta]_+ + (1/4M^2) (\mathcal{O}_\beta \mathcal{E}_N \mathcal{O}_N + \mathcal{O}_N \mathcal{E}_\beta \mathcal{O}_N + \mathcal{O}_N \mathcal{E}_N \mathcal{O}_\beta) - (1/8M^2) ([\mathcal{E}_N, [\mathcal{O}_\beta, \mathcal{O}_N]_+]_+ + [\mathcal{E}_\beta, \mathcal{O}_N^2]_+) \quad (21b)$$

where  $[A, B]_\pm = AB \pm BA$ . The terms quadratic in the nucleon-lepton coupling constants are neglected.

By setting  $\beta = -1$  in (21), we obtain the non-relativistic Hamiltonian. From (19) and (21), the explicit forms are:

$$H'_N = M + V_2 \quad (22a)$$

$$+ (1/2M) \{ P^2 - (U_1 + U_2) \boldsymbol{\sigma} \mathbf{P} + i U_1' \boldsymbol{\sigma} \mathbf{r} / r + U_1 U_2 \} \quad (22b)$$

$$+ (1/4M^2) [(I_1' - I_2') P^2 - i (I_1' - I_2') \mathbf{r} \mathbf{P} / r + (I_1' - I_2') \boldsymbol{\sigma} [\mathbf{r} \times \mathbf{P}] / r + V_2' / (2r) + V_2'' / 2] \quad (22c)$$

$$- (V_1 - V_2) (U_1 + U_2) \boldsymbol{\sigma} \mathbf{P} + i \{ (U_1 V_1)' - (V_2 U_1' + (U_1 V_2)' + U_2 V_2') / 2 \} \boldsymbol{\sigma} \mathbf{r} / r, \quad (22d)$$

and

$$H'_\beta = \sum_{suvt} \lambda_s [\mathcal{E}_s L_s \mathcal{Y}_{suvt} \quad (23a)$$

$$+ (1/4M^2) \{ (1 - \mathcal{E}_s) U_1 U_2 L_s \mathcal{Y}_{suvt} - (1 - \mathcal{E}_s / 2) \boldsymbol{\sigma} \mathbf{P} U_1 L_s \mathcal{Y}_{suvt} + \mathcal{E}_s L_s \mathcal{Y}_{suvt} \boldsymbol{\sigma} \mathbf{P} U_1 / 2 - (1 - \mathcal{E}_s / 2) U_2 L_s \mathcal{Y}_{suvt} \boldsymbol{\sigma} \mathbf{P} + \mathcal{E}_s U_2 \boldsymbol{\sigma} \mathbf{P} L_s \mathcal{Y}_{suvt} / 2 \} \quad (23b)$$

$$+ \boldsymbol{\sigma} \mathbf{P} L_s \mathcal{Y}_{suvt} \boldsymbol{\sigma} \mathbf{P} - \mathcal{E}_s L_s \mathcal{Y}_{suvt} P^2 / 2 - \mathcal{E}_s P^2 L_s \mathcal{Y}_{suvt} / 2 \} \quad (23c)$$

$$+ \sum_{suvt} (\lambda_s / 2M) [(U_2 + \mathcal{E}_s U_1) L_s^- \mathcal{Y}_{suvt}^- \quad (23d)$$

$$- \boldsymbol{\sigma} \mathbf{P} L_s^- \mathcal{Y}_{suvt}^- - \mathcal{E}_s^- L_s^- \mathcal{Y}_{suvt}^- \boldsymbol{\sigma} \mathbf{P} \quad (23e)$$

$$+ (1/2M) \{ (\mathcal{E}_s^- V_1 U_1 + (1 - \mathcal{E}_s^- / 2) U_2 V_1 - \mathcal{E}_s^- V_2 U_1 / 2 - U_2 V_2) L_s^- \mathcal{Y}_{suvt}^- \quad (23f)$$

$$- \mathcal{E}_s^- (V_1 - V_2 / 2) L_s^- \mathcal{Y}_{suvt}^- \boldsymbol{\sigma} \mathbf{P} + \mathcal{E}_s^- L_s^- \mathcal{Y}_{suvt}^- \boldsymbol{\sigma} \mathbf{P} V_1 - \boldsymbol{\sigma} \mathbf{P} (V_1 - V_2 / 2) L_s^- \mathcal{Y}_{suvt}^- + V_2 \boldsymbol{\sigma} \mathbf{P} L_s^- \mathcal{Y}_{suvt}^- / 2 \} \quad (23g)$$

with the assumptions that the  $l_i^*$  and  $U_i$  ( $i=1, 2$ ) are the functions of  $r$  only and

$$[\sigma, V_i]=0, \quad [\sigma, U_i]=0. \quad (24)$$

In the above expression the following abbreviations are used :

$$\mathcal{J}_{su\bar{v}} = \mathcal{J}_{n(k^{(v)})}^{-\frac{1}{2}, -\frac{1}{2}}(\sigma, \mathbf{r}),$$

$$V_i' = dV_i/dr, \quad U_i' = dU_i/dr.$$

If we assume the  $\beta$ -decay of RaE as  $0 \rightarrow 0$  (yes) transition, and want to explain its spectrum by the linear combination  $(T, P)^T$ , the potential  $V(0)$  is important<sup>(10)</sup>. But recently the spin of RaE was determined by Smith<sup>(11)</sup> as 1, and we have no immediate need of this potential. There is, however, no evidence that this potential does not exist, and so we have retained them. Konopinski's spin-orbit term<sup>(11)</sup> corresponds to (22e) and (22g). To explain both the spin-orbit energy and the RaE  $\beta$ -spectrum Konopinski has assumed that these terms are large. His argument is based on the assumption of the spin 0 for RaE, and as the result of Smith's experiment the support from  $\beta$ -spectrum has vanished. But from the consideration on the spin-orbit energy, these terms might be large.

The matrix element of  $H'_3$  is the sum of the integrals of the following four types :

$$\int (\mathcal{R}_K \chi_K^M)^* u(r) \mathcal{J}_{su\bar{v}}(r) \mathcal{R}'_{K'} \chi_{K'}^{M'} d\tau, \quad (24a)$$

$$\int (\mathcal{R}_K \chi_K^M)^* u(r) \sigma \mathbf{P} \bar{v}(r) \mathcal{J}_{su\bar{v}}(r) \mathcal{R}'_{K'} \chi_{K'}^{M'} d\tau, \quad (24b)$$

$$\int (\mathcal{R}_K \chi_K^M)^* u(r) \mathcal{J}_{su\bar{v}} \sigma \mathbf{P} \bar{v}(r) \mathcal{R}'_{K'} \chi_{K'}^{M'} d\tau, \quad (24c)$$

$$\int (\mathcal{R}_K \chi_K^M)^* u(r) \sigma \mathbf{P} \bar{v}(r) \mathcal{J}_{su\bar{v}} \sigma \mathbf{P} \bar{v}(r) \mathcal{R}'_{K'} \chi_{K'}^{M'} d\tau, \quad (24d)$$

where  $\mathcal{R}_K \chi_K^M$  and  $\mathcal{R}'_{K'} \chi_{K'}^{M'}$  are the eigenfunctions of  $H'_{N'}$  for the final and the initial states.

By the same method with § 2, these integrals are expressed as :

$$\begin{aligned} & \int (\mathcal{R}_K \chi_K^M)^* u(r) \mathcal{J}_{su\bar{v}}(r) \mathcal{R}'_{K'} \chi_{K'}^{M'} d\tau \\ &= (-1)^{j_N + l_P + m_P + v + 1} \begin{pmatrix} j_P & j_N & u \\ m_P & -m_N & m_N - m_P \end{pmatrix} (1/2 \|\mathcal{J}_k\| 1/2) B_{uvk}(KK') \\ & \quad \times \int \mathcal{R}_K^* u(r) \bar{v}(r) \mathcal{R}'_{K'} r^2 dr, \end{aligned} \quad (25a)$$

$$\begin{aligned} & \int (\mathcal{R}_K \chi_K^M)^* u(r) \sigma \mathbf{P} \bar{v}(r) \mathcal{J}_{su\bar{v}}(r) \mathcal{R}'_{K'} \chi_{K'}^{M'} d\tau \\ &= (-1)^{j_N + l_P + l_N + m_P + 1} \begin{pmatrix} j_P & j_N & u \\ m & -m_N & m_N - m_P \end{pmatrix} (1/2 \|\mathcal{J}_k\| 1/2) (1/2 \|\mathcal{J}_k\| 1/2) \\ & \quad \times [C_{uvk}(KK' l_P - 1) \{\mathcal{R}_K^* u(r)\} (\partial/\partial r - (l_P - 1)/r) \bar{v}(r) \mathcal{R}'_{K'} \{r^2 dr \\ & \quad + C_{wvk}(KK' l_P + 1) \{\mathcal{R}_K^* u(r)\} (\partial/\partial r + (l_P + 2)/r) \bar{v}(r) \mathcal{R}'_{K'} \{r^2 dr\}], \end{aligned} \quad (25b)$$

$$\int (\mathcal{R}_K \chi_K^M)^* u(r) \mathcal{J}_{su\bar{v}} \sigma \mathbf{P} \bar{v}(r) \mathcal{R}'_{K'} \chi_{K'}^{M'} d\tau$$

$$\begin{aligned}
&= (-1)^{l_N + l_P + m_P} \begin{pmatrix} j_P & j_N & n \\ m_P & -m_N & m_P \end{pmatrix} (1/2 \| \mathcal{Y}_1 \| 1/2) (1/2 \| \mathcal{Y}_k \| 1/2) \\
&\quad \times [C_{wk}(KK'l_N - 1) \int \mathcal{R}_{k'l}^*(r) \{ (\partial/\partial r + (l_N + 1)/r) v(r) \mathcal{R}'_{k'l}(r^2) dr \\
&\quad + C_{wk}(KK'l_N + 1) \int \mathcal{R}_{k'l}^*(r) \{ (\partial/\partial r - l_N/r) v(r) \mathcal{R}'_{k'l}(r^2) dr], \quad (25c) \\
&\int (\mathcal{R}_K \chi_K^u)^* u(r) \sigma P v \mathcal{Y}_{\varepsilon w} \sigma P v(r) \mathcal{R}'_{K'l}(r^2) \chi_{K'l}^{u'} d\tau
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{j_N + l_P - m_P} \begin{pmatrix} j_P & j_N & n \\ m_P & -m_N & m_P \end{pmatrix} (1/2 \| \mathcal{Y}_1 \| 1/2) (1/2 \| \mathcal{Y}_k \| 1/2) \\
&\quad \times [D_{wk}(KK'l_P - 1, l_N - 1) \int \mathcal{R}_{k'l}^*(r) \{ (\partial/\partial r - (l_P - 1)/r) v(r) (\partial/\partial r + (l_N + 1)/r) \\
&\quad \quad \quad \times v(r) \mathcal{R}'_{k'l}(r^2) dr \\
&\quad + D_{wk}(KK'l_P + 1, l_N - 1) \int \mathcal{R}_{k'l}^*(r) \{ (\partial/\partial r + (l_P + 2)/r) v(r) (\partial/\partial r + (l_N + 1)/r) \\
&\quad \quad \quad \times v(r) \mathcal{R}'_{k'l}(r^2) dr \\
&\quad + D_{wk}(KK'l_P - 1, l_N + 1) \int \mathcal{R}_{k'l}^*(r) \{ (\partial/\partial r - (l_P - 1)/r) v(r) (\partial/\partial r - l_N/r) \\
&\quad \quad \quad \times v(r) \mathcal{R}'_{k'l}(r^2) dr \\
&\quad + D_{wk}(KK'l_P + 1, l_N + 1) \int \mathcal{R}_{k'l}^*(r) \{ (\partial/\partial r + (l_P + 2)/r) v(r) (\partial/\partial r - l_N/r) \\
&\quad \quad \quad \times v(r) \mathcal{R}'_{k'l}(r^2) dr]. \quad (25d)
\end{aligned}$$

$C_{wk}(KK'P)$  and  $D_{wk}(KK'PR)$  are defined as follows :

$$\begin{aligned}
C_{wk}(KK'P) &= 2 \left[ \frac{(2j_P + 1)(2j_N + 1)(2l_P + 1)(2l_N + 1)(2u + 1)(2v + 1)}{3} \right]^{1/2} \\
&\quad \times (2P + 1) \begin{pmatrix} l_N & 1 & P \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v & P & l_P \\ 0 & 0 & 0 \end{pmatrix} \left\{ \begin{matrix} l_P & P & v \\ j_P & j_N & u \\ 1/2 & 1/2 & k \end{matrix} \right\} W(l_N - 1/2, 1/2; 1j_N), \\
D_{wk}(KK'PR) &= 2 \left[ \frac{2(2j_P + 1)(2j_N + 1)(2l_P + 1)(2l_N + 1)(2u + 1)(2v + 1)}{3} \right]^{1/2} \\
&\quad \times (2P + 1) \begin{pmatrix} P & 1 & l_P \\ 0 & 0 & 0 \end{pmatrix} W(1/2, 1j_P, l_P; 1/2, P)(2R + 1) \begin{pmatrix} l_N & 1 & R \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad \times W(1/2, 1j_N, l_N; 1/2, R) \begin{pmatrix} v & R & P \\ 0 & 0 & 0 \end{pmatrix} \left\{ \begin{matrix} P & R & v \\ j_P & j_N & u \\ 1/2 & 1/2 & k \end{matrix} \right\},
\end{aligned}$$

$H'_\beta$  contains too many terms, and moreover it is very difficult to obtain the relations between the radial integrals of (25)\* from  $H'_N$ . But when there is no odd potential and the terms of the order higher than  $(1/M)^2$  are discarded, the same treatment with Rose and Osborn<sup>(3)</sup> can be performed. In this case (22) and (23) become as follows :

\* Cf. eg. (29) of the reference 3).



$$H'_N = M + V_u + P^2/2M, \quad (26)$$

$$H''_{\beta} = \sum_{suv} \lambda_s \varepsilon_s L_s \mathcal{J}_{suv} \quad (27a)$$

$$+ \sum_{suv} \bar{\lambda}_s / 2M (-\sigma \mathbf{P} L_s \mathcal{J}_{suv} - \varepsilon_s L_s \mathcal{J}_{suv} \sigma \mathbf{P}). \quad (27b)$$

The matrix element of  $H''_{\beta}$  for single nucleon transition is given by virtue of (25a, b, c) and (I, 7a, b).

$$\begin{aligned} & \int \Psi^* H''_{\beta} \Phi d\tau \\ &= \sum_{suv} \lambda_s (-1)^{j_N + l_P + m_P + j + j_v + l + l_v + v} i \varepsilon_s \begin{pmatrix} j & j_v & u \\ -\mu & -\mu_v & \mu + \mu_v \end{pmatrix} \begin{pmatrix} j_P & j_N & u \\ m_P & -m_N & m_N - m_P \end{pmatrix} \\ & \times (2k+1)^{-1/2} (1/2 \|\mathcal{J}_k\| 1/2)^2 B_{uvk}(KK') \\ & \times [B_{uvk}(-\kappa, \kappa_v) \int r^2 dr \mathcal{R}_K^* f_{\kappa} g_{\kappa_v}^v \mathcal{R}'_{K'} \\ & - \varepsilon_s B_{uvk}(\kappa, -\kappa_v) \int r^2 dr \mathcal{R}_K^* g_{\kappa} f_{\kappa_v}^v \mathcal{R}'_{K'}] \\ & + \sum_{suv} (\bar{\lambda}_s / 2M) \begin{pmatrix} j_P & j_N & u \\ m_P & -m_N & m_N - m_P \end{pmatrix} \begin{pmatrix} j & j_v & u \\ -\mu & -\mu_v & \mu + \mu_v \end{pmatrix} \\ & \times (2\bar{k}+1)^{-1/2} (1/2 \|\mathcal{J}_k\| 1/2) (1/2 \|\mathcal{J}_{\bar{k}}\| 1/2)^2 \\ & \times [(-1)^{j_N + j_P - l_N - m_P + j + j_v + l + l_v} \\ & \times \{ C_{uv\bar{k}}(KK'l_P - 1) B_{uv\bar{k}}(-\kappa, -\kappa_v) \int r^2 dr \mathcal{R}_K^* (\partial/\partial r - (l_P - 1)/r) f_{\kappa} f_{\kappa_v}^v \mathcal{R}'_{K'} \\ & + \varepsilon_s C_{uv\bar{k}}(KK'l_P - 1) B_{uv\bar{k}}(\kappa, \kappa_v) \int r^2 dr \mathcal{R}_K^* (\partial/\partial r - (l_P - 1)/r) g_{\kappa} g_{\kappa_v}^v \mathcal{R}'_{K'} \\ & + C_{uv\bar{k}}(KK'l_P + 1) B_{uv\bar{k}}(-\kappa, -\kappa_v) \int r^2 dr \mathcal{R}_K^* (\partial/\partial r + (l_P + 2)/r) f_{\kappa} f_{\kappa_v}^v \mathcal{R}'_{K'} \\ & + \varepsilon_s C_{uv\bar{k}}(KK'l_P + 1) B_{uv\bar{k}}(\kappa, \kappa_v) \int r^2 dr \mathcal{R}_K^* (\partial/\partial r + (l_P + 2)/r) g_{\kappa} g_{\kappa_v}^v \mathcal{R}'_{K'} \} \\ & + (-1)^{l_N + m_P + j + j_v + l + l_v + 1} \varepsilon_s \\ & \times \{ C_{uv\bar{k}}(KK'l_N - 1) B_{uv\bar{k}}(-\kappa, -\kappa_v) \int r^2 dr \mathcal{R}_K^* f_{\kappa} f_{\kappa_v} (\partial/\partial r + (l_N + 1)/r) \mathcal{R}'_{K'} \\ & + \varepsilon_s C_{uv\bar{k}}(KK'l_N - 1) B_{uv\bar{k}}(\kappa, \kappa_v) \int r^2 dr \mathcal{R}_K^* g_{\kappa} g_{\kappa_v}^v (\partial/\partial r + (l_N + 1)/r) \mathcal{R}'_{K'} \\ & + C_{uv\bar{k}}(KK'l_N + 1) B_{uv\bar{k}}(-\kappa, -\kappa_v) \int r^2 dr \mathcal{R}_K^* f_{\kappa} f_{\kappa_v} (\partial/\partial r - l_N/r) \mathcal{R}'_{K'} \\ & + \varepsilon_s C_{uv\bar{k}}(KK'l_N + 1) B_{uv\bar{k}}(\kappa, \kappa_v) \int r^2 dr \mathcal{R}_K^* g_{\kappa} g_{\kappa_v}^v (\partial/\partial r - l_N/r) \mathcal{R}'_{K'} \} ]. \quad (28) \end{aligned}$$

The correction factor can be obtained by making use of the method of I and the differential equations for the lepton radial wave functions. But as may be expected from (28), this treatment is very complicated, and at present we shall no more meddle with this approximation,

#### § 4. Concluding remarks

The correction factor (I. 24) serves to examine whether Fermi theory can explain the experimental results or not, or to see what kind of the linear combination of the interactions is profitable. This correction factor does not depend on any particular nuclear model.

If the  $\bar{j}j$  coupling model is a good approximation, the ambiguity of the parameters of the correction factor is much reduced by the results of II and III. Or inversely, if the experimental results can be explained with the correction factor (I. 24) by adjusting the parameters suitably, and on the other hand when these parameters are reduced to the integrals  $\mathfrak{M}^{a,b}(\Lambda^-, \Lambda')$  etc. by the method of II and III, the values of these integrals are unreasonable, this unreasonableness can be a criticism of the  $\bar{j}j$  coupling model.

The application of our argument to the experimental results is now in progress.

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## Kinematical Investigation of the S-matrix

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A general method is proposed to investigate the kinematical properties of the S-matrix in a covariant manner, independent of the method of approximation and the type of interaction. The S-matrix is separated into a definite number of independent *bases*, i. e., the kinematical parts and their dynamical coefficients. The number and types of such kinematical bases are obtained for one fermion problems and boson decay processes. As an application we can derive some absolute selection rules for three body decays. For example, a spin 0 particle cannot decay into three photons, any two of which making an angle of  $120^\circ$  with each other. The same result holds for the decay of a spin 1 particle into three identical spin 0 particles. Some of our results are useful for analysing the decay of  $\tau$ -meson.

### § 1. Introduction

The kinematical investigation of the S-matrix based on the mere assumptions of i) Lorentz invariance, ii) conservation laws, iii) current field equations and supplementary conditions for the external lines, has given important results to meson theory. We pay our attention firstly to the absolute selection rules discussed by Yang<sup>1</sup> and Peaslee<sup>2</sup> which have served in determining the type of pion, and secondly to the work of Wolfenstein and Ashkin<sup>3,4</sup> to find out the most general form of the transition matrix elements of nucleon-nucleon scattering in a nonrelativistic way. The arguments on these problems are purely kinematical in a three dimensional version, but are not systematic.

It is the aim of this paper to observe the kinematical properties of the S-matrix in a covariant manner from a general point of view. Our method is an extension of our previous work<sup>1)\*</sup> which has been used for the radiative processes. That is, a transition matrix element is expanded into a definite number of independent bases constructed by the four-momenta of participating particles and the  $\gamma$ -matrices. The combination of these bases are irrelevant to the detailed type of interaction and the method of approximation. We explain how to get the combination and the number of the bases by taking the examples of one fermion problem. The argument in § 2 is a generalization of the interesting observation in the fourth order calculation of the photomeson production by Koba et al.<sup>5)</sup> (§ 2)

The result of § 2 is applied to boson decay processes. In our previous paper<sup>4)</sup> we have reformulated the absolute selection rules for radiative processes in a covariant manner.

\* M. Kawaguchi and N. Mugibayashi., Prog. Theor. Phys. 8 (1952), 212, referred to as I.

We have given, for instance, the four dimensional version of Yang's<sup>1)</sup> proof that a meson of spin 1 cannot decay into two photons. The characteristic of our method consists essentially in inquiring whether or not we can construct the transition matrix element satisfying all the invariance properties and the conditions imposed on. Thus, if the number of independent bases of a process is zero this process is *absolutely forbidden*, and otherwise we must say this process is *not absolutely forbidden*. Although our method does not give any new result, it seems worth while to ascertain that some decay modes are not absolutely forbidden and our description is preferable for this purpose. (§3).

Finally the absolute selection rules for three body decays are discussed in more detail. It is found that a scalar or pseudoscalar particle cannot decay into three photons, any two of which making an angle of  $120^\circ$  with each other in the centre of mass system. The same selection rule holds for the decay of a spin 1 particle into three identical spin 0 particles. Further if a vector (pseudovector) particle decays into two identical and one unlike pseudoscalar (scalar) particles the like particles cannot have momenta of the same magnitude. This is found to be useful for specifying the type of the  $\tau$ -meson<sup>6,7)</sup> and the recent experimental data show that the  $\tau$ -meson would be unlikely to be a vector particle. It is also noted that there is a definite relation between the direction of the polarization of the parent particle and the decay plane. (§4)

## § 2. Base expansion of the S-matrix

We use throughout this paper the momentum representation for practical purpose. A transition matrix element  $M$  is specified by an operator part  $\chi$  and a c-number coefficient  $R$ ,

$$M = R\chi. \quad (2.1)$$

Here  $\chi$  is a product of free field operators or unquantized external fields. The c-number part  $R$  is expanded in terms of kinematical bases  $(A_l, T_m)$  and their dynamical coefficients  $c_{lm}$ :

$$R = \sum_{l,m} c_{lm} (A_l, T_m). \quad (2.2)$$

Here  $T_m$ 's have the same transformation property as  $\chi$ . They are built of available four-momenta and  $\gamma$ -matrices and are independent of one another.  $A_l$ 's are, on the other hand, four dimensional scalar quantities containing  $\gamma$ -matrices.  $A_l = 1$  always, and if there is no external fermion line  $A_l = 0$  for  $l \neq 1$ , while if there are fermion lines  $A_l$  is a product of any number of factors  $(\gamma \mathbf{k}_1), (\gamma \mathbf{k}_2), \dots$  where  $\mathbf{k}_1, \mathbf{k}_2, \dots$  are external four-momenta. Therefore  $A_l$  plays no role in a boson-boson transition. Each  $A_l$  does not contain two or more identical factors and further the order of different factors in individual  $A_l$  is unimportant. And moreover, the order of  $A_l$  and  $T_m$  is inessential for our consideration even if they are non-commutative. The dynamical coefficients  $c_{lm}$ 's are scalar quantities not containing  $\gamma$ -

matrix. The summation is to be taken over all independent bases\*.

We apply this expansion theorem to the following two cases: there are besides two external fermion lines\*\* satisfying the Dirac equation either i)  $n$  spin 0 particles, or ii)  $p$  photons. Denoting the number of independent  $T_m$ 's and  $A_l$ 's by  $N_T$  and  $N_A$  respectively, the total number of independent bases is  $N = N_T \times N_A$ . Both examples i) and ii) are typical in getting the number and the form of the bases. In fact the former example leads us to the way of finding  $N_A$  and the latter, on the other hand, is the best example of getting  $N_T$ . These examples suggest the method to obtain the bases in more general cases.

The knowledge on the bases certainly simplifies practical calculations. For example, if we want to obtain the differential cross section from a matrix element, one has only to calculate at most  $1/2 N(N+1)$  traces to take summation over all possible spin states. Furthermore, each base has a definite physical meaning, so that the interpretation of it will

\* We shall illustrate this cumbersome circumstance by an example of the Compton scattering in its lowest order of perturbation, where current notations are used. Then the matrix element is, apart from uninteresting factors, given by

$$M = e^2 \bar{\psi}(\mathbf{p}') A_\nu(\mathbf{k}') R_{\mu\nu}(\mathbf{k}, \mathbf{k}', \mathbf{p}, \mathbf{p}', \gamma) A_\mu(\mathbf{k}) \psi(\mathbf{p}),$$

where

$$\begin{aligned} R_{\mu\nu} &= \gamma_\nu S_F(\mathbf{p} + \mathbf{k}) \gamma_\mu + \gamma_\mu S_F(\mathbf{p} - \mathbf{k}') \gamma_\nu \\ &= i(\mathbf{p}\mathbf{k} \cdot (\mathbf{p}\mathbf{k}')^2)^{-1} \cdot (k_\nu \mathbf{p}\mathbf{k}' - \delta_{\nu\gamma} \mathbf{k}\mathbf{k}') (\gamma_\mu \mathbf{p}\mathbf{k} - \delta_{\mu\gamma} \gamma\mathbf{k}) \\ &\quad + i(\mathbf{p}\mathbf{k} \cdot (\mathbf{p}\mathbf{k}')^2)^{-1} \cdot (k_\mu \mathbf{p}\mathbf{k} - \delta_{\mu\gamma} \mathbf{k}\mathbf{k}') (\gamma_\nu \mathbf{p}\mathbf{k}' - \delta_{\nu\gamma} \gamma\mathbf{k}') \\ &\quad - 1/2 \cdot ((\mathbf{p}\mathbf{k})^2 \cdot \mathbf{p}\mathbf{k}')^{-1} \cdot (\gamma_\nu \mathbf{p}\mathbf{k}' - \delta_{\nu\gamma} \gamma\mathbf{k}') (\gamma_\mu \mathbf{p}\mathbf{k} - \delta_{\mu\gamma} \gamma\mathbf{k}) i\gamma\mathbf{k} \\ &\quad - 1/2 \cdot (\mathbf{p}\mathbf{k} \cdot (\mathbf{p}\mathbf{k}')^2)^{-1} \cdot (\gamma_\mu \mathbf{p}\mathbf{k} - \delta_{\mu\gamma} \gamma\mathbf{k}) (\gamma_\nu \mathbf{p}\mathbf{k}' - \delta_{\nu\gamma} \gamma\mathbf{k}') i\gamma\mathbf{k}. \end{aligned}$$

If we put

$$\begin{aligned} A_1 &= i, \quad A_2 = i\gamma\mathbf{k}; \\ T_1 &= (k_\nu \mathbf{p}\mathbf{k}' - \delta_{\nu\gamma} \mathbf{k}\mathbf{k}') (\gamma_\mu \mathbf{p}\mathbf{k} - \delta_{\mu\gamma} \gamma\mathbf{k}), \\ T_2 &= (k_\mu \mathbf{p}\mathbf{k} - \delta_{\mu\gamma} \mathbf{k}\mathbf{k}') (\gamma_\nu \mathbf{p}\mathbf{k}' - \delta_{\nu\gamma} \gamma\mathbf{k}'), \\ T_3 &= (\gamma_\nu \mathbf{p}\mathbf{k}' - \delta_{\nu\gamma} \gamma\mathbf{k}') (\gamma_\mu \mathbf{p}\mathbf{k} - \delta_{\mu\gamma} \gamma\mathbf{k}), \\ T_4 &= (\gamma_\mu \mathbf{p}\mathbf{k} - \delta_{\mu\gamma} \gamma\mathbf{k}) (\gamma_\nu \mathbf{p}\mathbf{k}' - \delta_{\nu\gamma} \gamma\mathbf{k}'), \end{aligned}$$

then

$$R_{\mu\nu} = c_{11}(T_1 A_1) + c_{12}(T_2 A_1) + c_{23}(T_3 A_2) + c_{24}(T_4 A_2),$$

where

$$c_{11} = c_{12} = i(\mathbf{p}\mathbf{k} \cdot (\mathbf{p}\mathbf{k}')^2)^{-1}, \quad c_{23} = -1/2 \cdot ((\mathbf{p}\mathbf{k})^2 \cdot \mathbf{p}\mathbf{k}')^{-1}, \quad c_{24} = -1/2 \cdot (\mathbf{p}\mathbf{k} \cdot (\mathbf{p}\mathbf{k}')^2)^{-1}.$$

$(T_1 A_1)$ ,  $(T_2 A_1)$ ,  $(T_3 A_2)$  and  $(T_4 A_2)$  are independent bases which satisfy gauge invariance. See also § 2 and § 3 of I.

\*\* Two external fermion lines do not mean two body problem but one body problem.



make facile the analysis of calculations.

It is to be noted that the number of the bases has nothing to do with the nature of external lines such as the mass, the charge and the arrangement, that is, which line is incoming or outgoing. If there are identical bosons in a state, the symmetry property due to the Bose statistics sometimes diminishes the number of the bases. Consideration of the parity may also reduce the number. These circumstances are shown in the examples of boson decays. It is almost unnecessary to say that if we specify the type of interaction and the method of approximation, some of the bases do not appear.

The present method is certainly convenient to investigate the absolute selection rules but is not always the best one. In fact, the kinematical study of the photomeson production by Brueckner and Watson<sup>8)</sup>, which is intimately related to the general method in the angular momentum representation<sup>9)</sup>, is more familiar. The relation between these two methods will be discussed elsewhere.

### 1. Two fermion lines and *n* spin 0 particles

We have only one  $T_m$  which is either 1 or  $\gamma_5$  according as even or odd number of pseudoscalar particles participate in the process\*. Next we classify  $A_i$ 's according to the number of  $\gamma$ -matrices contained. The number of the available four-momenta, which combine with  $\gamma$ -matrices to build up  $A_i$ 's, is  $(n-1)$ . We have  $(n+2)$  momenta corresponding to each external lines and the energy-momentum conservation law reduces them by one, and further two momenta  $p_1$  and  $p_2$  of the fermion lines must be excluded due to the Dirac equation. Indeed  $i\gamma p_1$  and  $i\gamma p_2$  are expressed by universal constants not containing  $\gamma$ -matrix. We notice that each  $A_i$  cannot have two or more identical factors such as  $(\not{k}) \cdot (\not{k})$ , which is immediately changed into a form not containing  $\gamma$ 's. The possible form and the number of combinations are shown in Table 1.

Table 1.

Number of $\gamma$ -matrices	0	1	2	3	—	$n-1$	$n$
Number of $A_i$	1	$n-1C_1$	$n-1C_2$	$n-1C_3$	—	$n-1C_{n-1}$	0
Typical form	1	$\not{k}_i$	$\not{k}_i \not{k}_j$	$\not{k}_i \not{k}_j \not{k}_l$	—	$\not{k}_1 \not{k}_2 \dots \not{k}_{n-1}$	0

As is well-known, we have 16 independent combinations of  $\gamma$ -matrices including unity, so that the independent  $A_i$ 's are only those which contain  $\gamma$ 's four times or less. Thus we get the result\*\*

$$N_A = 1 + n-1C_1 + n-1C_2 + n-1C_3 + n-1C_4 \equiv f(n). \quad (2.3)$$

\* If one of the fermion lines is pseudospinor, this form of  $T_m$  must be reversed.

\*\* The terms having five or more  $\gamma$ 's can actually be reduced to another form in principle, though it is rather complicated. Even if one does not try this reduction  $N_A$  does not exceed

$$1 + n-1C_1 + n-1C_2 + \dots + n-1C_{n-1} = 2^{n-1}.$$

2. Two fermion lines and  $p$  photon lines

In this case  $N_\Lambda(p)$  is just the same as in the preceding example. Next,  $T_m$ 's, whose transformation property is a tensor of rank  $p$ , consist of  $\gamma$ -matrices,  $(p+2)$  momenta and  $\hat{\partial}_{\mu\nu}$ . Of  $(p+2)$  momenta one is eliminated by the conservation law. The Lorentz condition rules out  $k_\mu$  when it is combined with  $A_\mu(k)$  and the gauge invariance banishes one arbitrary momentum as shown in § 2 of I. Then  $p$  vectors,  $\gamma_\mu$  and  $(p-1)$  momenta, are available to combine with each suffix of the tensor of rank  $p$ . Instead of taking into account the order of  $\gamma$ 's in individual  $T_m$ , it is convenient to take  $\hat{\partial}_{\mu\nu}$  as specifying quantities. In Table 2  $T_m$ 's are classified according to the number of  $\hat{\partial}_{\mu\nu}$  contained.

Table 2.

Number of $\hat{\partial}_{\mu\nu}$ contained	0	1	2	—	$p/2$ for even $p$	$(p-1)/2$ for odd $p$
Number of independent $T_m$	$p^n$	$p C_2 p^{p-2}$	$\frac{1}{2}! p C_4 C_2 p^{p-4}$ $\times p^{p-4}$	—	$\frac{1}{(p/2)!} p C_2 p^{p-2} \dots$ $\dots C_2$	$\frac{1}{((p-1)/2)! p C_2 p^{p-1} C_2 p^{p-3}}$ $\dots C_2$

Thus we get the final result :

$$\begin{aligned}
 N_T = & p^n + p C_2 p^{p-2} + 1/2! p C_4 C_2 p^{p-4} + \dots \\
 & + \left\{ \begin{array}{ll} \frac{1}{(p/2)!} p C_2 p^{p-2} \dots C_2 & \text{for even } p, \\ \frac{1}{((p-1)/2)!} p C_{p-1} C_{p-3} \dots C_2 p & \text{for odd } p. \end{array} \right. \quad (2.4)
 \end{aligned}$$

For the sake of convenience let us define the following function,

$$\begin{aligned}
 F(x, n) \equiv & x^n + n C_2 x^{n-2} + 1/2! n C_4 C_2 x^{n-4} + \dots \\
 & + \left\{ \begin{array}{ll} \frac{1}{(n/2)!} n C_n C_{n-2} \dots C_2 & \text{for even } n, \\ \frac{1}{((n-1)/2)!} n C_{n-1} C_{n-3} \dots C_2 x & \text{for odd } n \end{array} \right. \quad (2.5)
 \end{aligned}$$

or the compact form\*

$$F(x, n) = \frac{1}{2^{n/2} i^n} \cdot H_n(ix/\sqrt{2}), \quad (2.6)$$

where  $H_n$  means the Hermitian polynomial of the  $n$ -th order,

$$H_n(x) = (-1)^n e^{x^2} d^n e^{-x^2} / dx^n.$$

For some simple cases we arrange the number of bases in Table 3 with the aid of (2.6).

\* The author expresses his thanks to Mr. Y. Kawasaki who kindly informed him of this expression.

Table 3.

External lines	2 fermions $n$ spin 0 particles	2 fermions $p$ photons	2 fermions $v$ spin 1 particles	2 fermions $p$ photons $n$ spin 0 particles	$p$ photons $n$ spin 0 particles
$N$	$f(n)$	$f(p)F(p, p)$	$f(v)F(v+1, v)$	$f(p+n) \times F(p+n, p)$	$F(p+n-3, p)$

The practical applications to the photomeson production and the Compton scattering have been done originally by Koba et al.<sup>(1)</sup> and also in I. In the present paper we shall emphasize the absolute selection rules for boson decays, particularly for three body decays.

### § 3. Absoluteness of selection rules

From the general point of view stated in § 2 the meaning of absolute selection rules should be understood by our language.

#### *Absolute selection rules*

If we have no such base that satisfies all conditions imposed on the external lines, the process is called absolutely forbidden. By the conditions we mean well established ones that are based on the invariances under Lorentz transformations and the gauge transformation and also on statistics.

#### *Relative selection rules*

This is the case that the dynamical coefficients vanish, even if there remain a number of bases. For example, the selection rules due to the conservation of the isotopic spin and the invariance under the charge conjugation are relative ones.

The four dimensional derivation of absolute selection rules by our theorem is different in appearance from the usual ones<sup>(1,2)</sup> but essentially equivalent to them. Therefore nothing new can be expected. Since our method is a covariant reformulation, we have only to mention a typical example of forbiddenness and some processes which are not absolutely forbidden.

#### 1. Selection rules when $n$ scalar particles and $p$ photons participate

One can immediately see from Table 3 the number of bases is

$$N = F(p+n-3, p). \quad (3.1)$$

From (2.5) we get

$$N \leq 0 \text{ if } p=\text{odd and } p+n-3 \leq 0. \quad (3.2)$$

Hence this case is absolutely forbidden. Only one which is not meaningless\* is that a

\*  $n=0$  and  $p=3$  is also forbidden. It must be noted that our argument is quite different in its nature from Furry's theorem that owes to the charge conjugation.

meson of spin 0 cannot decay into a photon and a spin 0 meson.

## 2. Processes not absolutely forbidden

Though, following our method, we can enumerate all possible bases, it is enough to show at least one matrix element for the present purpose.  $F_{\mu\nu}$  and  $\varphi$  denote the electromagnetic field and the meson field operators respectively, and  $k, l, m$  and  $n$  are the four-momenta of decay products.

a) The decay of a scalar meson into three photons<sup>10)</sup>

$$M = F_{\mu\nu}(k) F_{\nu\rho}(l) F_{\rho\mu}(m) (kl - lm) (mk - kl) (lm - mk) \\ \times \varphi(k + l + m). \quad (3.3)$$

b) The decay of a pseudovector meson into three photons

$$M = F_{\mu\nu}(k) F_{\nu\rho}(l) F_{\rho\mu}(m) \varepsilon(\lambda\omega\sigma\tau) k_\omega l_\sigma m_\tau \varphi_\lambda(k + l + m). \quad (3.4)$$

These two matrix elements are evidently gauge invariant and symmetrical with respect to three photons.

c) The decay of a pseudoscalar meson into four identical spin 0 mesons

$$M = \varphi(k) \varphi(l) \varphi(m) \varphi(n) (k - l) \cdot (m - n) (m - k) \cdot (l - n) \\ \times (n - k) \cdot (l - m) \varepsilon(\mu\nu\rho\sigma) k_\mu l_\nu m_\rho n_\sigma \varphi(k + l + m + n). \quad (3.5)$$

Thus this process is not absolutely forbidden, though the pseudoscalar meson is not allowed to decay into three scalar mesons.<sup>11)</sup>

d) The decay of a vector meson into three identical scalar mesons

$$M = \varphi(k) \varphi(l) \varphi(m) (lm k_\mu + mk l_\mu + kl m_\mu) \varphi_\mu(k + l + m). \quad (3.6)$$

As stated above, other principles may rule out *relatively* some of these processes. Indeed, in the process b) the lowest order calculation gives vanishing matrix element for both pseudovector and tensor couplings in virtue of the charge conjugation and the divergence theorem.<sup>12)13)</sup>

## § 4. Absolute selection rules on the angular correlation for three body decays

The kinematical restrictions on three body decays are obtained in this section by studying the common features of possible bases when at least two of the decay products are identical. Some of the results, which have never been pointed out, are based on the symmetry property for identical particles. All discussions are given in the centre of mass system.

1. A spin 0 particle cannot decay into three photons at an angle of  $120^\circ$  with each other

\* We shall omit the dynamical coefficients  $c_{lm}$  hereafter, which are symmetric with respect to the identical particles. (This footnote refers to\* in Line 2, p. 765.)

According to (3.1) the number of bases for this process is four, one of which is of such a type as in (3.3) and the remaining three are\*

$$M = F_{\mu\nu}(\mathbf{k}) F_{\mu\nu}(\mathbf{l}) F_{\rho\sigma}(\mathbf{m}) k_\rho l_\sigma (\mathbf{k}\mathbf{m} - \mathbf{m}\mathbf{l}) \varphi(\mathbf{k} + \mathbf{l} + \mathbf{m}) \quad (4.1)$$

and cyclic,

if the parent particle is scalar. Here  $(\mathbf{k}\mathbf{l} - \mathbf{l}\mathbf{m})$   $(\mathbf{m}\mathbf{k} - \mathbf{k}\mathbf{l})$   $(\mathbf{l}\mathbf{m} - \mathbf{m}\mathbf{k})$  in (3.3) and  $(\mathbf{k}\mathbf{m} - \mathbf{m}\mathbf{l})$  etc. in (4.1) are simple and typical scalar quantities which make matrix elements symmetric with respect to all photons. However, we cannot construct such scalar quantities\*\* when three photons are produced at an angle of  $120^\circ$  with each other. Thus this must be ruled out. For a pseudoscalar case the argument is quite similar.

2. *A spin 0 particle cannot decay into a photon and two identical spin 0 particles with momenta of equal magnitude*

From (3.1) we get only one base, that is,

$$M = \varphi(\mathbf{k}) \varphi(\mathbf{l}) F_{\mu\nu}(\mathbf{m}) k_\mu l_\nu (\mathbf{k}\mathbf{m} - \mathbf{m}\mathbf{l}) \varphi(\mathbf{k} + \mathbf{l} + \mathbf{m}), \quad (4.2)$$

if the initial particle is scalar, and

$$M = \varphi(\mathbf{k}) \varphi(\mathbf{l}) A_\mu(\mathbf{m}) (\mathbf{k}\mathbf{m} - \mathbf{m}\mathbf{l}) \varepsilon(\mu\rho\sigma) m_\nu k_\rho l_\sigma \varphi(\mathbf{k} + \mathbf{l} + \mathbf{m}), \quad (4.3)$$

if the initial one is pseudoscalar. In either example there must be a scalar factor such as  $(\mathbf{k}\mathbf{m} - \mathbf{m}\mathbf{l})$  antisymmetric with respect to  $\mathbf{k}$  and  $\mathbf{l}$ , in order to make the matrix element symmetric. This factor makes the matrix element zero if two identical spin 0 particles are emitted symmetrically.

Next, we consider more general case where two spin 0 particles in the final state are not identical. Now the form of matrix element is the same as in (4.2) or (4.3) but a factor  $(\mathbf{k}\mathbf{m} - \mathbf{m}\mathbf{l})$ . When the photon is emitted in the direction of  $z$  axis,  $A_\mu(\mathbf{m})$  has two independent components along  $x$  and  $y$  axes. If the final state is collinear,  $\mathbf{k}$  and  $\mathbf{l}$  as well as  $\mathbf{m}$  have only two components along  $z$  and  $t$  directions. Since no terms of  $F_{\mu\nu}$   $k_\mu$   $l_\nu$  remain under such a condition, the matrix element has to vanish. Further, in the case of a pseudoscalar initial particle the collinear final state is obviously ruled out in virtue of  $\varepsilon(\mu\rho\sigma)$   $m_\nu k_\rho l_\sigma$ . Thus we get the result: *If a spin 0 particle decays into a photon and two arbitrary spin 0 particles the final state is not allowed to be collinear.* This may be presumed by the previously mentioned rule that a spin 0 particle can not decay into a photon and a spin 0 particle.

3. *A spin 1 particle cannot decay into three identical spin 0 particles at an angle of  $120^\circ$  with each other*

When a vector (pseudovector) particle decays into three identical scalar (pseudoscalar) particles the possible matrix element is only of the form like (3.6), where the scalar factor  $\mathbf{l}\mathbf{m}$ ,  $\mathbf{m}\mathbf{k}$  and  $\mathbf{k}\mathbf{l}$  are generalized into any symmetrical combination. These factors forbid the case where the three scalars are emitted quite symmetrically due to the subsidiary condition on the vector field  $(\mathbf{k} + \mathbf{l} + \mathbf{m})_\mu \varphi_\mu(\mathbf{k} + \mathbf{l} + \mathbf{m}) = 0$ . It is also noted that the decay

\*\* These factors do not appear in a denominator, because the matrix element must be finite in all cases.



plane cannot be perpendicular to the direction of the polarization of the vector (pseudovector) particle.

Next if a vector (pseudovector) particle decays into three identical pseudoscalar (scalar) particles the matrix element is expressed by

$$M = \varphi(\mathbf{k}) \varphi(\mathbf{l}) \varphi(\mathbf{m}) \varepsilon(\rho\nu\rho\sigma) k_\nu l_\nu m_\sigma \\ \times (kl - lm) (mk - kl) (lm - mk) \varphi_\mu(\mathbf{k} + \mathbf{l} + \mathbf{m}). \quad (4.4)$$

In this case the antisymmetrical scalar factor such as  $(kl - lm) (mk - kl) (lm - mk)$  excludes the symmetrical case, as in the example 1. Similar to the above case, the decay does not occur in such a plane that contains the direction of the polarization of the parent particle.

Further, contrary to the former case, if two of the pseudoscalar (scalar) particles are identical and one is different, the like particles cannot have the momenta of the same magnitude. If  $\varphi(\mathbf{k})$  and  $\varphi(\mathbf{l})$  are identical, in fact, the matrix element is the same as (4.4) but the scalar factor  $(kl - lm) (mk - kl)$ . The remaining factor  $(lm - mk)$  gives this result, which has been pointed out by Dalitz<sup>(6)</sup> and used to analyse  $\tau$ -meson decay. The recent experiments seem to indicate that two like pions are of nearly equal momenta in considerable cases. This looks unfavorable to the vector theory of  $\tau$ -meson though not conclusive yet. It is almost needless to say that even if there are arbitrary three pseudoscalar (scalar) particles in the final state they cannot be collinear.

#### 4. Further remarks

One may try to discuss the three photon decay of  $^3S$  state of a positronium or a vector particle, but our method gives no definite result because of too many bases of various characters. Indeed, we have 14 bases symmetric and antisymmetric with respect to three photons.

We have no insight as to whether our method is useful or not for explaining the behaviour of  $A$ -particle.

In conclusion the author would like to express his cordial thanks to Profs. H. Yukawa and S. Hayakawa and Mr. N. Mugibayashi for their encouragement and valuable discussions.

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## Pion-Nucleon Scattering in the Tamm-Dancoff Approximation

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The pion-nucleon scattering is treated in the Tamm-Dancoff approximation using the symmetrical ps-ps theory, with emphasis particularly on the renormalization procedure. The number of mesons and nucleon-pairs is restricted up to two and one, respectively, in every intermediate state. In order to perform the renormalization consistently, it is required to adopt the five-box approximation of Bethe and Dyson. But even in this case, the meson self-energy term is not treated consistently and should be discarded. A method is proposed to remove the overlapping divergences and it is shown that the Salam's procedure can be carried out in a closed form. The scattering phase shifts are evaluated and compared with experiments.

### § 1. Introduction

The experimental and theoretical analysis of the pion-nucleon ( $\pi$ -N) scattering has attracted our attention for the past few years, in expectation that it would give an essential knowledge about the elementary interaction between the pion and the nucleon. Recent results of the experiments up to 210 Mev by Fermi and co-workers<sup>1)</sup> together with those of the low energy scattering at Rochester and Columbia<sup>2)</sup> have come to a fairly reliable stage. Furthermore, the experimental data for the high energy  $\pi$ -N scattering are going to be arranged at Brookhaven,<sup>3)</sup> and we shall gain much more knowledge about the  $\pi$ -N interaction in the near future.

At present, however, it should be noted that the measurements of the angular distribution, in particular those of forward and backward scattering are somewhat inaccurate and consequently the phase shift analysis has many ambiguities. It is not always certain that we may believe the strong dependence of the  $S$ -wave phase shifts on energy and isotopic spin. Further, the  $D$ -wave contribution may not be neglected even in the low energy region. But it would be certain that the  $(3/2, 3/2)$  state (the first number  $3/2$  represents the isotopic spin state of the  $\pi$ -N system and the second the spacial spin state) has a resonance-like phenomenon and that the  $S$ -wave contribution is considerably large.\*

Now, we would like to give an outline of the theoretical analysis of the  $\pi$ -N interaction performed so far. First, while the type of the pion is definitely pseudoscalar, the type of the coupling between pion and nucleon has two possibilities; pseudoscalar (p.s.) and pseudovector (p.v.). It is possible that both types of coupling mix together, but it is convenient, for the sake of simplicity, to treat either of the two. (An assumption that a greater part

\* Recent analysis by Glicksman seems to be more favorable from the theoretical point of view, (private communication)

of the coupling is p.v. and the remaining is p.s. seems to be attractive.) The p.s.-coupling theory is renormalizable in principle, while the p.v.-coupling theory is favorable in that the static approximation can be applied. But even in the former theory this approximation is not impossible (Tani-Foldy transformation,<sup>1</sup> strong coupling theory by Wentzel,<sup>5</sup> etc.), and so this point is not so essential for the theory. In order to make a definite conclusion about the type of the  $\pi$ -N interaction we have to investigate all mesonic processes systematically. Using the non-relativistic p.v.-coupling theory based on the cut-off hypothesis, we could interpret, at least qualitatively, various phenomena such as magnetic moment,  $\gamma$ - $\pi$  reaction, nuclear forces, etc. and also  $P$ -wave  $\pi$ -N scattering.<sup>6</sup> On the other hand, the p.s.-coupling, as is clear from the Tani-Foldy transformation, is composed of a strong  $S$ -wave pair interaction and a weak p.v.-coupling. Therefore, we might not rely too much upon this theory, so far as we adopt the conventional perturbation theory. For example, the  $S$ -wave contribution is predominant in the  $\pi$ -N scattering, and the anomalous magnetic moment of the nucleon is not in agreement with the experiment because of the fact that the recoil current of the nucleon is so large.<sup>7)\*</sup> But, afterwards, it was found that the  $S$ -wave contribution is damped strongly on account of the field reaction,<sup>8</sup> and at present it is expected that the p.s.-theory could also explain experimental data including the  $S$ -wave interaction, at least qualitatively, in addition to its renormalizability. (Recent conclusions of Kroll-Ruderman<sup>9</sup>) and Deser-Thirring-Goldberger<sup>10</sup>) do not seem to be significant, since it is clear from the beginning that, at the threshold energy, essential higher order effects would disappear, because the wave length of the incident meson is very large compared with the nuclear dimension.) Since the Tamm-Dancoff (T.D.) method<sup>11</sup>) was applied to nuclear forces<sup>12</sup>) and to the  $\pi$ -N scattering,<sup>13</sup>) the meson theory has come to a new stage. The intermediate coupling theory developed by Tomonaga et al.<sup>14</sup>) has been considered so far to be appropriate and powerful in treating meson processes, but in this theory, it is difficult to take into account the nucleon recoil and to perform the mass and charge renormalization. Accordingly, the T.D. approximation method in which the number of mesons and nucleon-pairs is limited in every intermediate state, seems to correspond to the actual situation of the p.s. meson theory. Sawada-approximation using a canonical transformation<sup>15</sup>) is also worth mentioning by the same reason.

There are two different standpoints for the application of the T.D. method to the  $\pi$ -N scattering. One is a standpoint in which the T.D. method is applied directly without performing any transformation, as that of Bethe and Dyson.<sup>16</sup>) (This method was independently proposed by N. Fukuda at summer school, in Kyoto, 1952.) From this point of view, Lévy and Marshak<sup>17</sup>) have recently made the analysis of the  $S$ -wave scattering and obtained reasonable results. The other is a standpoint in which the T.D. method is applied after the elimination of odd terms by the Tani-Foldy transformation, as was performed by Fukuda, Sawada et al.<sup>18)\*\*</sup> The latter one seems to be appropriate for a qualitative discussion,

\* The strong coupling theory cannot give the anomalous magnetic moment correctly as well, though it has succeeded in explaining the resonance phenomena in the  $\pi$ -N scattering.

\*\* The Dyson transformation used by Drell and Hestley (Phys. Rev. **88** (1952), 1053.) is not sufficient to eliminate the odd interaction, because the ratio of the effective coupling to the mass is almost the same as the original one.



because the odd interaction is very strong in the  $\gamma_5$  interaction. On the other hand, the first standpoint is consistent in that the characteristic features of the primary  $\gamma_5$  interaction are always retained, particularly if we try to carry out the renormalization program. (The Bethe-Salpeter method is close to this one.)

In this paper, we keep the first standpoint and treat the  $\pi$ -N scattering with the renormalization program. We shall adopt the five-box approximation of Bethe and Dyson and make use of the cut-off hypothesis for the higher momentum in order to avoid divergences caused by the non-relativistic approximation. This procedure will not change the qualitative nature of our results, because we might expect that the higher momentum components would be cut down considerably by recoil effects and the mass and charge renormalization. The renormalization has been treated so far only for the  $P$ -wave,<sup>19)</sup> but here this problem will be considered including the  $S$ -wave as well. We shall show that the renormalization program can be carried out consistently except for the meson self-energy part and that the overlapping divergences can also be subtracted in a closed form, according to the prescriptions by Salam.<sup>20)</sup>

In conclusion, our results are in agreement with those of Fukuda, Sawada et al. for the  $P$ -wave scattering, but the sufficient separation between the two isotopic spin states in the  $S$ -wave scattering was not obtained, probably because the effect of the  $\tau \cdot \phi \times \pi$  term is too small to give rise to this separation.

In Sec. 2 we derive the scattering equations in five-box approximation taking into account of counter terms. In Sec. 3, using the approximate Fredholm solution of the integral equation, we derive the formulas of phase shifts for scattering waves and discuss the method of the renormalization for the overlapping divergences. In Sec. 4 the results of calculations are compared with the experimental data.

## § 2. Scattering equations in five-box approximation

We adopt the symmetrical pseudoscalar meson theory with the pseudoscalar coupling between the pion and the nucleon. We shall restrict ourselves to the so-called five-box approximation in which at most two mesons and one nucleon-pair are simultaneously present in an intermediate state.<sup>16)</sup> We shall not consider any contribution from the state in which only one meson and one nucleon-pair exist simultaneously, because this state is not connected directly with the one meson and one nucleon state, and, in addition, the inclusion of this state destroys the consistent mass and charge renormalization, as we shall see later. The transition scheme is shown in Fig. 1, where the symbol  $(m, n)$  represents a state in the Fock space in which  $m$  mesons,  $n$  nucleon-pairs and one nucleon are present.

We start from the Schrödinger equation

$$i\partial\Psi(t)/\partial t = (H_0 + H')\Psi(t), \quad (1)$$

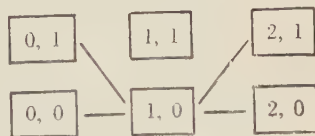


Fig. 1

where  $H_0$  is the Hamiltonian of the uncoupled system and  $H'$  is the interaction Hamiltonian with counter-terms which are added for the consistent

mass renormalization.  $H_0$  and  $H'$  are given as follows :

$$H_0 = \int \bar{\psi} (\gamma \cdot \nabla + m) \psi d\mathbf{r} + \frac{1}{2} \int [\pi_a^2 + \varphi_a (\mu^2 - \Delta) \varphi_a] d\mathbf{r}, \quad (2)$$

$$H' = H_1 - H_2 - H_3 - H_4, \quad (3)$$

where

$$\begin{aligned} H_1 &= if \int \bar{\psi} \gamma_4 \tau_a \varphi_a \psi d\mathbf{r}, & H_2 &= \delta m \int \bar{\psi} \psi d\mathbf{r}, \\ H_3 &= \delta \mu^2 \int \varphi_a^2 d\mathbf{r}, & H_4 &= \delta \lambda \int \varphi_a^2 \varphi_a^2 d\mathbf{r}. \end{aligned} \quad (4)$$

$H_2$ ,  $H_3$  and  $H_4$  are the counter-terms corresponding to the nucleon self-energy, meson self-energy and meson-meson scattering, respectively, but the last one may be discarded in our approximation.

Now we expand  $\Psi(t)$  in terms of the eigenfunctions of the uncoupled system,  $\Psi_\lambda^{(m,n)}$ ;

$$\Psi(t) = \sum_{\lambda, m, n} \alpha_\lambda^{(m,n)} \Psi_\lambda^{(m,n)} e^{-iW_0 t}, \quad (5)$$

where

$$H_0 \Psi_\lambda^{(m,n)} = E_\lambda^{(m,n)} \Psi_\lambda^{(m,n)}. \quad (6)$$

$m$  and  $n$  are the number of mesons and nucleon-pairs, and  $\lambda$  stands for the momentum, charge, spin etc. of the particles in an uncoupled state.  $H_0$  is the total energy of the system. Substituting eq. (5) into eq. (1), we obtain a set of equations to be satisfied by the amplitudes  $\alpha_\lambda^{(m,n)}$ 's :

$$\begin{aligned} (W_0 - E_\lambda^{(m,n)}) \alpha_\lambda^{(m,n)} &= \sum_{\mu} \sum_{\substack{p=m \pm 1 \\ q=n, n \pm 1}} (\lambda, m, n | H_1 | \mu, p, q) \alpha_\mu^{(p,q)} \\ &- \sum_{\mu} \sum_{q=n, n \pm 1} (\lambda, m, n | H_2 | \mu, m, q) \alpha_\mu^{(m,q)} - \sum_{\mu} \sum_{p=m, m \pm 1} (\lambda, m, n | H_3 | \mu, p, n) \alpha_\mu^{(p,n)}. \end{aligned} \quad (7)$$

Now it is easily seen that the necessary counter-terms in our case are only  $(\lambda, 1, 0 | H_2 | \lambda, 1, 0)$ ,  $(\lambda, 1, 0 | H_3 | \lambda, 1, 0)$  and  $(\mu, 0, 0 | H_2 | \mu, 0, 0)$  as a result of our restrictions. For example, the counter-term  $(\mu, 0, 0 | H_2 | \nu, 0, 1)$  is represented in Fig. 2 (a), and corresponds to the process shown in Fig. 2 (a') (and to the processes obtained by inserting various insertions in (a')), which includes the (1,1) state to be excluded in our case. Similarly,  $(\mu, 0, 0 | H_3 | \nu, 2, 0)$ , represented in Fig. 2(b), corresponds to the process (b') (and to those obtained by inserting various insertions in (b')), which also includes the (1,1) state to be excluded. Others are also excluded by the similar reasons. Thus only the above three counter terms remain and we need not any other counter-term in our five-box approximation (except for the case of the meson self-energy; see below). When

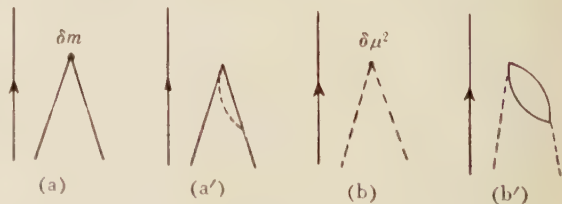


Fig. 2



we include the (1,1) state, the renormalization is more complicated, and moreover can not be performed consistently in our present approximation. If we want to include this state we should take into account higher configurations.

After dropping unnecessary counter-terms, we obtain the equation for the amplitude  $\alpha^{(1,0)}$  by eliminating  $\alpha^{(0,0)}$ ,  $\alpha^{(2,0)}$ ,  $\alpha^{(0,1)}$  and  $\alpha^{(2,1)}$ , namely

$$\begin{aligned}
 & (W_0 - E_p - \omega_k + \mathcal{D}_p^{(1,0)} + \tilde{\mathcal{D}}_k^{(1,0)}) \alpha(p, s; k) \\
 &= \sum_{k', s'} \frac{(p, s; k | H_1 | p - k', s'; k, k') (p - k', s'; k, k' | H_1 | p, s; k)}{W_0 - E_{p-k'} - \omega_k - \omega_{k'}} \alpha(p, s; k) \\
 &- \sum_{k', s', s''} \frac{(p, s; k | H_1 | p, p - \overline{p - k'}, s, s', s''; k, k')}{W_0 - 2E_p - E_{p-k'} - \omega_k - \omega_{k'}} \\
 &\quad \frac{(p, p, -\overline{p - k'}, s, s', s''; k, k' | H_1 | p, s'; k)}{ } \alpha(p, s'; k) \\
 &+ \sum_{p', s', s''} \frac{(p, s; k | H_1 | p, p', -\overline{p' + k}, s, s', s'')}{W_0 - E_p - E_{p'} - E_{p'+k}} \\
 &\quad \frac{(p, p', -\overline{p' + k}, s, s', s'' | H_1 | p, s; k)}{ } \alpha(p, s; k) \\
 &+ \sum_{p', s', s''} \frac{(p, s; k | H_1 | p, p', -\overline{p' - k}, s, s', s''; k, k)}{W_0 - E_p - E_{p'} - E_{p'+k} - 2\omega_k} \\
 &\quad \frac{(p, p', -\overline{p' - k}, s, s', s''; k, k | H_1 | p, s; k)}{ } \alpha(p, s; k) \\
 &+ \sum_{k', s', s''} \frac{(p, s; k | H_1 | p + k, s') (p + k, s' | H_1 | p + k - k', s''; k')}{W_0 - E_{p+k} + \mathcal{D}_{p+k}^{(0,0)}} \cdot \alpha(p + k - k', s''; k') \\
 &- \sum_{k', s', s''} \frac{(p, s; k | H_1 | p, p + k - k', -\overline{p - k}, s, s', s''; k, k')}{W_0 - E_p - E_{p+k-k'} - E_{p+k} - \omega_k - \omega_{k'}} \\
 &\quad \frac{(p, p + k - k', -\overline{p - k}, s, s', s''; k, k' | H_1 | p + k - k', s'; k')}{ } \cdot \alpha(p + k - k', s'; k') \\
 &+ \sum_{k', s', s''} \frac{(p, s; k | H_1 | p - k', s'; k, k') (p - k', s'; k, k' | H_1 | p + k - k', s''; k')}{W_0 - E_{p-k'} - \omega_k - \omega_{k'}} \cdot \alpha(p + k - k', s''; k') \\
 &- \sum_{k', s', s''} \frac{(p, s; k | H_1 | p, p + k - k', -\overline{p + k'}, s, s', s'')}{W_0 - E_p - E_{p+k-k'} - E_{p-k'}} \\
 &\quad \frac{(p, p + k - k', -\overline{p + k'}, s, s', s'' | H_1 | p + k - k', s'; k')}{ } \cdot \alpha(p + k - k', s'; k'), \quad (8)
 \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_p^{(1,0)} &= (\mathbf{p}, s; \mathbf{k} | H_2 | \mathbf{p}, s; \mathbf{k}), \\ \tilde{\mathcal{A}}_p^{(1,0)} &= (\mathbf{p}, s; \mathbf{k} | H_3 | \mathbf{p}, s; \mathbf{k}), \\ \mathcal{A}_{p+k}^{(0,0)} &= (\mathbf{p} + \mathbf{k}, s' | H_2 | \mathbf{p} + \mathbf{k}, s'). \end{aligned} \quad (9)$$

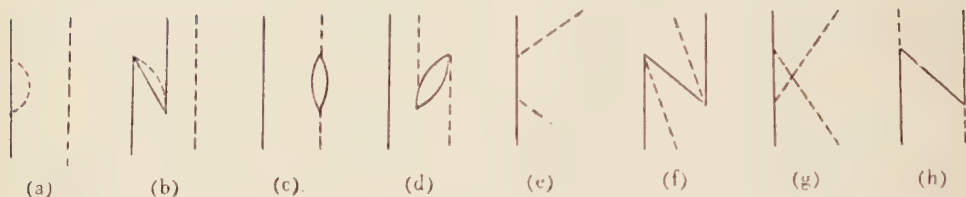


Fig. 3

Here we express the momentum, energy and spin of a nucleon by  $\mathbf{p}$  ( $\mathbf{p}$  for an anti-nucleon),  $E_p (= (\mathbf{p}^2 + m^2)^{1/2})$  and  $s$ , respectively, and the momentum and energy of a meson by  $\mathbf{k}$  and  $\omega_k (= (\mathbf{k}^2 + \mu^2)^{1/2})$ . We do not write down the charge variables explicitly in the equations for the sake of brevity, but these are always implied. Each term on the right hand side of eq. (8) corresponds, one by one, to the process shown in Fig. 3. In eq. (8) we need not consider the Pauli exclusion principle when the summations are carried out, since all the terms to be excluded by this principle are cancelled out.

In the center-of-mass system we have for eq. (8)

$$\begin{aligned} & (W_0 - E_k - \omega_k + \mathcal{A}_k^{(1,0)} + \tilde{\mathcal{A}}_k^{(1,0)}) \alpha(-\mathbf{k}, s; \mathbf{k}; T) \\ &= [\mathcal{A}^{(1,0)}(\mathbf{k}, W_0) + \tilde{\mathcal{A}}^{(1,0)}(\mathbf{k}, W_0)] \alpha(-\mathbf{k}, s; \mathbf{k}; T) \\ &+ \sum_{T', \mathbf{k}', s'} (T | \boldsymbol{\tau} \cdot \boldsymbol{\phi}^* \boldsymbol{\tau} \cdot \boldsymbol{\phi}' | T') [I_1(\mathbf{k}, s; \mathbf{k}', s') + I_2(\mathbf{k}, s; \mathbf{k}', s')] \alpha(-\mathbf{k}', s'; \mathbf{k}'; T') \\ &+ \sum_{T', \mathbf{k}', s'} (T | \boldsymbol{\tau} \cdot \boldsymbol{\phi}' \boldsymbol{\tau} \cdot \boldsymbol{\phi}^* | T') [I_3(\mathbf{k}, s; \mathbf{k}', s') + I_4(\mathbf{k}, s; \mathbf{k}', s')] \alpha(-\mathbf{k}', s'; \mathbf{k}'; T'), \end{aligned} \quad (10)$$

where  $\boldsymbol{\phi}^*$  and  $\boldsymbol{\phi}'$  are unit vectors in the meson charge-space and correspond to mesons with momenta  $\mathbf{k}$  and  $\mathbf{k}'$ , respectively, and  $T$  and  $T'$  represent the charge states of the meson-nucleon system. And

$$\begin{aligned} \mathcal{A}^{(1,0)}(\mathbf{k}, W_0) &= \frac{3}{2} f^2 \sum_{\mathbf{k}'} \frac{1}{\omega_{k'}} \left[ \frac{1}{W_0 - E_{k+k'} - \omega_k - \omega_{k'}} \left( \frac{\mathbf{k}}{E_k + m} - \frac{\mathbf{k} + \mathbf{k}'}{E_{k+k'} + m} \right)^2 \right. \\ &- \frac{1}{W_0 - 2E_k - E_{k+k'} - \omega_k - \omega_{k'}} \left( 1 + \frac{2\mathbf{k} \cdot (\mathbf{k} + \mathbf{k}')}{(E_k + m)(E_{k+k'} + m)} \right. \\ &\left. \left. + \frac{\mathbf{k}^2 (\mathbf{k} + \mathbf{k}')^2}{(E_k + m)^2 (E_{k+k'} + m)^2} \right) \right] \frac{1}{D_k^2 D_{k+k'}^2}, \end{aligned} \quad (11a)$$

$$\begin{aligned} \tilde{J}^{(1,0)}(\mathbf{k}, W_0) = & 2f^2 \sum_{k'} \frac{1}{\omega_k} \left( 1 + \frac{2\mathbf{k}' \cdot (\mathbf{k} + \mathbf{k}')}{(E_{k'} + m)(E_{k+k'} + m)} + \frac{\mathbf{k}'^2 (\mathbf{k} + \mathbf{k}')^2}{(E_{k'} + m)^2 (E_{k+k'} + m)^2} \right) \\ & \times \left( \frac{1}{W_0 - E_k - E_{k'} - E_{k+k'}} + \frac{1}{W_0 - E_k - E_{k'} - E_{k+k'} - 2\omega_k} \right) \frac{1}{D_{k'}^2 D_{k+k'}^2}, \end{aligned} \quad (11b)$$

$$I_1(\mathbf{k}, s; \mathbf{k}', s') = f^2 \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{1}{W_0 - m + \mathcal{A}^{(0,0)}} \frac{(\boldsymbol{\sigma} \cdot \mathbf{k} \boldsymbol{\sigma} \cdot \mathbf{k}')_{ss'}}{(E_k + m)(E_{k'} + m)} \frac{1}{D_k D_{k'}}, \quad (12a)$$

$$I_2(\mathbf{k}, s; \mathbf{k}', s') = -f^2 \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{1}{W_0 - E_k - E_{k'} - m - \omega_k - \omega_{k'}} \frac{1}{D_k D_{k'}} \delta_{ss'}, \quad (12b)$$

$$\begin{aligned} I_3(\mathbf{k}, s; \mathbf{k}', s') = & f^2 \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{1}{2E_{k+k'}(W_0 - E_{k+k'} - \omega_k - \omega_{k'})} \left[ (E_{k+k'} - m) \right. \\ & \left. - (E_k + E_{k'} - 2m) + \frac{E_{k+k'} - E_k - E_{k'} - m}{(E_k + m)(E_{k'} + m)} \boldsymbol{\sigma} \cdot \mathbf{k} \boldsymbol{\sigma} \cdot \mathbf{k}' \right] \frac{1}{\delta_{ss'} D_k D_{k'}}, \end{aligned} \quad (12c)$$

$$\begin{aligned} I_4(\mathbf{k}, s; \mathbf{k}', s') = & -f^2 \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{1}{2E_{k+k'}(W_0 - E_k - E_{k'} - E_{k+k'})} \left[ (E_{k+k'} + m) \right. \\ & \left. + (E_k + E_{k'} - 2m) + \frac{E_{k+k'} + E_k + E_{k'} + m}{(E_k + m)(E_{k'} + m)} \boldsymbol{\sigma} \cdot \mathbf{k} \boldsymbol{\sigma} \cdot \mathbf{k}' \right] \frac{1}{\delta_{ss'} D_k D_{k'}}, \end{aligned} \quad (12d)$$

where

$$D_k = [1 + k^2 / (E_k + m)^2]^{1/2}. \quad (13)$$

$\mathcal{J}^{(1,0)}(\mathbf{k}, W_0)$  and  $\tilde{\mathcal{J}}^{(1,0)}(\mathbf{k}, W_0)$  are the nucleon and meson self-energy terms, and correspond to the processes shown in Figs. 3 (a), (b) and (c), (d);  $I_1$ ,  $I_2$ ,  $I_3$  and  $I_4$  correspond to the processes (e), (f), (g) and (h), respectively. Here, we can determine the counter-terms,  $\mathcal{A}_k^{(1,0)}$  and  $\tilde{\mathcal{A}}_k^{(1,0)}$ , as follows:

$$\begin{aligned} \mathcal{A}_k^{(1,0)} &= \mathcal{A}^{(1,0)}(\mathbf{k}, E_k + \omega_k), \\ \tilde{\mathcal{A}}_k^{(1,0)} &= \tilde{\mathcal{A}}^{(1,0)}(\mathbf{k}, E_k + \omega_k). \end{aligned} \quad (14)$$

Now we define the amplitude  $\langle \mathbf{k} | R | \mathbf{k}_0 \rangle$  by the equation

$$i(-\mathbf{k}, s; \mathbf{k}; T) = i^*[-\mathbf{k}] T^*[\mathbf{k}] u^{(s)*}(-\mathbf{k}) \langle \mathbf{k} | R | \mathbf{k}_0 \rangle u^{(s_0)}(-\mathbf{k}_0) T[\mathbf{k}_0] i[-\mathbf{k}_0], \quad (15)$$

where  $i[-\mathbf{k}]$  and  $T[\mathbf{k}]$  represent the charge state functions of the nucleon and the meson with momenta  $-\mathbf{k}$  and  $\mathbf{k}$ , and  $u^{(s)}(-\mathbf{k})$  the spin function of the nucleon. (The suffix 0 refers to the initial state.) The amplitude  $\langle \mathbf{k} | R | \mathbf{k}_0 \rangle$  is evidently independent of the spin and isotopic spin states.

Since we treat the low energy scattering, we take into account only  $S$ - and  $P$ -waves and expand eq. (10) in powers of  $v/c$  ( $v$  is the nucleon velocity) neglecting terms

higher than  $(v'/v)^2$  for each wave. (As a result of this approximation, we must cut off the upper limit of every integration.) Then we obtain

$$\begin{aligned} & (W_0 - m - \omega_k + \mathcal{A}_k^{(1,0)} + \tilde{\mathcal{A}}_k^{(1,0)} - \mathcal{A}^{(1,0)}(\mathbf{k}, W_0) - \tilde{\mathcal{A}}^{(1,0)}(\mathbf{k}, W_0)) (\mathbf{k}|R|\mathbf{k}_0) \\ &= (\boldsymbol{\tau} \cdot \boldsymbol{\phi}^* \boldsymbol{\tau} \cdot \boldsymbol{\phi}') \sum_{k'} [I_1(\mathbf{k}, \mathbf{k}') + I_2(\mathbf{k}, \mathbf{k}')] (\mathbf{k}'|R|\mathbf{k}_0) \\ &+ (\boldsymbol{\tau} \cdot \boldsymbol{\phi}' \boldsymbol{\tau} \cdot \boldsymbol{\phi}^*) \sum_{k'} [I_3(\mathbf{k}, \mathbf{k}') + I_4(\mathbf{k}, \mathbf{k}')] (\mathbf{k}'|R|\mathbf{k}_0), \end{aligned} \quad (16)$$

where

$$I_1(\mathbf{k}, \mathbf{k}') = f^2 \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{1}{W_0 - m + \mathcal{A}^{(0,0)} 4m^2} (\mathbf{k} \cdot \mathbf{k}' + i\boldsymbol{\sigma} \cdot \mathbf{k} \times \mathbf{k}'), \quad (17a)$$

$$I_2(\mathbf{k}, \mathbf{k}') = -f^2 \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{1}{W_0 - 3m - \omega_k - \omega_{k'}}, \quad (17b)$$

$$I_3(\mathbf{k}, \mathbf{k}') = f^2 \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{1}{W_0 - m - \omega_k - \omega_{k'}} \frac{1}{4m^2} (\mathbf{k} \cdot \mathbf{k}' - i\boldsymbol{\sigma} \cdot \mathbf{k} \times \mathbf{k}'), \quad (17c)$$

$$I_4(\mathbf{k}, \mathbf{k}') = -f^2 \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{1}{W_0 - 3m} \left( 1 + \frac{1}{2m^2} i\boldsymbol{\sigma} \cdot \mathbf{k} \times \mathbf{k}' \right). \quad (17d)$$

The nucleon self-energy term is rewritten as

$$\mathcal{A}_k^{(1,0)} - \mathcal{A}^{(1,0)}(\mathbf{k}, W_0) = (W_0 - m - \omega_k) P - (W_0 - m - \omega_k)^2 Q(\mathbf{k}, W_0), \quad (18)$$

$$P = \frac{3}{2} f^2 \sum_{k'} \frac{1}{\omega_{k'}} \left[ \frac{1}{\omega_{k'}^2} \frac{k'^2}{4m^2} - \frac{1}{(2m + \omega_{k'})^2} \right], \quad (19)$$

$$\begin{aligned} Q(\mathbf{k}, W_0) = \frac{3}{2} f^2 \sum_{k'} \frac{1}{\omega_{k'}} & \left[ \frac{k'^2/4m^2}{\omega_{k'}^2 (W_0 - m - \omega_k - \omega_{k'})} \right. \\ & \left. - \frac{1}{(2m + \omega_{k'})^2 (W_0 - 3m - \omega_k - \omega_{k'})} \right], \end{aligned} \quad (20)$$

and the meson self-energy term as

$$\tilde{\mathcal{A}}_k^{(1,0)} - \tilde{\mathcal{A}}^{(1,0)}(\mathbf{k}, W_0) = (W_0 - m - \omega_k) \tilde{P}(k) - (W_0 - m - \omega_k)^2 \tilde{Q}(k, W_0), \quad (21)$$

$$\tilde{P}(k) = 2f^2 \frac{1}{\omega_k} \left[ \frac{1}{(2m - \omega_k)^2} + \frac{1}{(2m + \omega_k)^2} \right] \sum_{k'}, \quad (22)$$

$$\begin{aligned} \tilde{Q}(k, W_0) = 2f^2 \frac{1}{\omega_k} & \left[ \frac{1}{(2m - \omega_k)^2 (W_0 - 3m)} \right. \\ & \left. + \frac{1}{(2m + \omega_k)^2 (W_0 - 3m - 2\omega_k)} \right] \sum_{k'}. \end{aligned} \quad (23)$$

Thus the energy factor on the left hand side of eq. (16) becomes

$$\begin{aligned} & W_0 - m - \omega_k + \mathcal{A}_k^{(1,0)} + \tilde{\mathcal{A}}_k^{(1,0)} - \mathcal{A}^{(1,0)}(\mathbf{k}, W_0) - \tilde{\mathcal{A}}^{(1,0)}(\mathbf{k}, W_0) \\ &= (W_0 - m - \omega_k) [(1 + P + \tilde{P}(k)) - (W_0 - m - \omega_k) (Q(k, W_0) + \tilde{Q}(k, W_0))]. \end{aligned} \quad (24)$$

Now, we can perform the charge renormalization by putting

$$1 + P + \tilde{P}(k) \rightarrow 1.$$

It should be noticed, however, that  $\tilde{P}(k)$  depends on  $k$  and, in addition,  $\tilde{P}(k)$  and  $\tilde{Q}(k, W_0)$  are linear- and logarithmic-divergent, respectively, even in the rigorous expressions. These facts show that our approximation method in which the number of particles is limited is not consistent as regards the meson self-energy term. To avoid this difficulty, we must take into account the additional processes besides the above ones.

After the charge renormalization, the right hand side of eq. (24) becomes

$$(W_0 - m - \omega_k)[1 - (W_0 - m - \omega_k)(Q(k, W_0) + \tilde{Q}(k, W_0))].$$

As a result of our renormalization procedure which is not consistent for the meson self-energy term, the result of calculation depends seriously on the magnitude of the cut-off momentum. Thus we omit the term  $\tilde{Q}(k, W_0)$ . Furthermore, for the sake of simplicity, we shall not consider the term  $Q(k, W_0)$ , too.

Next we shall decompose  $(\mathbf{k}|R|\mathbf{k}_0)$  into eigenstates with respect to the total charge and the total angular momentum. We define  $T_{1/2}$  and  $T_{3/2}$  as the projection operators for the charge states with isotopic spin 1/2 and 3/2, respectively, and  $S$ ,  $P_{1/2}$  and  $P_{3/2}$  as the projection operators for the states with total angular momentum 1/2 ( $l=0$ ), 1/2 ( $l=1$ ) and 3/2 ( $l=1$ ), namely

$$\begin{aligned} T_{1/2} &= 1/3 \cdot (\phi^* \cdot \phi' + i\tau \cdot \phi^* \times \phi'), \\ T_{3/2} &= 1/3 \cdot (2\phi^* \cdot \phi' - i\tau \cdot \phi^* \times \phi'), \end{aligned} \quad (25)$$

and

$$\begin{aligned} S &= 1/4\pi, \\ P_{1/2} &= (1/4\pi) (\mathbf{k} \cdot \mathbf{k}_0 + i\boldsymbol{\sigma} \cdot \mathbf{k} \times \mathbf{k}_0) / k k_0, \\ P_{3/2} &= (1/4\pi) (2\mathbf{k} \cdot \mathbf{k}_0 - i\boldsymbol{\sigma} \cdot \mathbf{k} \times \mathbf{k}_0) / k k_0. \end{aligned} \quad (26)$$

Then we may put

$$\begin{aligned} k_0^2 (\mathbf{k}|R|\mathbf{k}_0) &= T_{1/2} S (k|W_1|k_0) + T_{1/2} P_{1/2} (k|W_{11}|k_0) + T_{1/2} P_{3/2} (k|W_{13}|k_0) \\ &\quad + T_{3/2} S (k|W_3|k_0) + T_{3/2} P_{1/2} (k|W_{31}|k_0) + T_{3/2} P_{3/2} (k|W_{33}|k_0). \end{aligned} \quad (27)$$

From eqs. (16), (17) and (27) we obtain the following integral equations for scattering amplitudes :

$$(k|W_1|k_0) = \delta(k - k_0) + \int dk' [K_2(k, k') + K_4(k, k')] (k'|W_1|k_0), \quad (28)$$

$$(k|W_{11}|k_0) = \delta(k - k_0) + \int dk' [3K_1(k, k') - K_3(k, k') - 2K_5(k, k')] (k'|W_{11}|k_0), \quad (29)$$



$$(k|W_{13}|k_0) = \delta(k-k_0) + \int dk' [2K_3(k, k') + K_5(k, k')] (k'|W_{13}|k_0), \quad (30)$$

$$(k|W_3|k_0) = \delta(k-k_0) - 2 \int dk' K_4(k, k') (k'|W_3|k_0), \quad (31)$$

$$(k|W_{31}|k_0) = \delta(k-k_0) + 2 \int dk' [K_3(k, k') + 2K_5(k, k')] (k'|W_{31}|k_0), \quad (32)$$

$$(k|W_{33}|k_0) = \delta(k-k_0) - 2 \int dk' [2K_3(k, k') + K_5(k, k')] (k'|W_{33}|k_0), \quad (33)$$

where

$$K_1(k, k') = \frac{f^2}{2\pi^2} \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{k k'^3}{4m^2 (W_0 - m - \omega_k) (W_0 - m + J)}, \quad (34a)$$

$$K_2(k, k') = -\frac{f^2}{2\pi^2} \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{3k'^2}{(W_0 - m - \omega_k) (W_0 - 3m - \omega_k - \omega_{k'})}, \quad (34b)$$

$$K_3(k, k') = -\frac{f^2}{2\pi^2} \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{k k'^3}{12m^2 (W_0 - m - \omega_k) (W_0 - m - \omega_k - \omega_{k'})}, \quad (34c)$$

$$K_4(k, k') = \frac{f^2}{2\pi^2} \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{k'^2}{(W_0 - m - \omega_k) (W_0 - 3m)}, \quad (34d)$$

$$K_5(k, k') = -\frac{f^2}{2\pi^2} \frac{1}{(2\omega_k)^{1/2} (2\omega_{k'})^{1/2}} \frac{k k'^3}{6m^2 (W_0 - m - \omega_k) (W_0 - 3m)}. \quad (34e)$$

In these equations the renormalization is required only for the amplitude  $(k|W_{11}|k_0)$ , whose equation has the kernel  $K_1$  including the counter-term. As to the amplitude  $(k|W_1|k_0)$ , we need not consider the renormalization at all.

### § 3. Phase shifts for scattering waves

Now we proceed to solve our integral equations. We cannot, however, solve these integral equations exactly, because the kernels are not generally degenerate, and so we shall make use of the approximate method of Fredholm to the integral equation.

According to Fredholm, the solution of the integral equation

$$(k|W|k_0) = \delta(k-k_0) + \int dk' K(k, k'; \omega_0) (k'|W|k_0) \quad (35)$$

is approximated by

$$(k|W|k_0) = \delta(k-k_0) + \frac{1}{D(\omega_0)} K(k, k_0; \omega_0), \quad (36)$$

$$D(\omega_0) = 1 - \int K(k, k; \omega_0) dk. \quad (37)$$

It should be noticed here that the integration on the right hand side of eq. (37), when applied to our scattering problem, is divergent even if the nucleon recoil is taken into

account, and so the approximate solution given by eq. (36) would be meaningless. But, if the cut-off procedure is used, this solution gives a rather good approximation.

In order to obtain the standing-wave solution of eq. (35), we take the principal-value for the integral in eq. (37), and if there is no need for renormalization, the phase shift is given by the formula

$$\tan \alpha = \pi \lim_{k \rightarrow k_0} (k - k_0) (1/D(\omega_0)) K(k, k_0; \omega_0). \quad (38)$$

In the following, we put

$$W_0 = m + \omega_0.$$

Before we calculate the phase shifts for scattering waves, we write the integral formulas, in which the principal-values are always taken and the interval of the integration extends from  $\mu$  to  $k_{\max}$ ;

$$F_0(\omega_0) = \int dk \frac{k^2}{\omega_k(\omega_k - \omega_0)} = k_{\max} - k_0 \log \frac{\omega_0 \omega_{\max} + k_0 k_{\max} - \mu^2}{\mu(\omega_{\max} - \omega_0)} + \omega_0 \log \frac{\omega_{\max} + k_{\max}}{\mu}, \quad (39a)$$

$$F_1(\omega_0) = \int dk \frac{k^4}{\omega_k - \omega_0} = G_4 + \omega_0 F_2(\omega_0), \quad (39b)$$

$$F_2(\omega_0) = \int dk \frac{k^4}{\omega_k(\omega_k - \omega_0)} = \frac{1}{3} k_{\max}^3 + \omega_0 G_0 + k_0^2 F_0(\omega_0), \quad (39c)$$

$$F_3(\omega_0) = \int dk \frac{k^4}{\omega_k^2(\omega_k - \omega_0)} = \frac{1}{\omega_0} (F_2(\omega_0) - G_5), \quad (39d)$$

where  $G_n$ 's are given by the formulas

$$G_0 = \int dk k^2 / \omega_k = \frac{1}{2} \omega_{\max} k_{\max} - \frac{\mu^2}{2} \log \frac{\omega_{\max} + k_{\max}}{\mu}, \quad (40a)$$

$$G_1 = \int dk k^2 / \omega_k^2 = k_{\max} - \mu \cos^{-1} \mu / \omega_{\max}, \quad (40b)$$

$$G_2 = \int dk k^2 / \omega_k^3 = -\frac{k_{\max}}{\omega_{\max}} + \log \frac{\omega_{\max} + k_{\max}}{\mu}, \quad (40c)$$

$$G_3 = \int dk k^4 = k_{\max}^5 / 5, \quad (40d)$$

$$G_4 = \int dk k^4 / \omega_k = \frac{1}{4} (\omega_{\max} k_{\max}^3 - 3\mu^2 G_0), \quad (40e)$$

$$G_5 = \int dk k^4 / \omega_k^2 = k_{\max}^3 / 3 - \mu^2 G_1, \quad (40f)$$

$$G_6 = \int dk k^4 / \omega_k^3 = G_0 - \mu^2 G_2, \quad (40g)$$

$$G_7 = \int dk \, k^4 / \omega_k^4 = G_1 + \frac{\mu^2}{2} \left( \frac{k_{\max}^2}{\omega_{\max}^2} - \frac{1}{\mu} \cos^{-1} \frac{\mu}{\omega_{\max}} \right), \quad (40h)$$

$$G_8 = \int dk \, k^5 / \omega_k^5 = \frac{1}{5} (\omega_{\max}^5 - \mu^5) - \mu^2 (\omega_{\max}^3 - \mu^3) + 3\mu^4 (\omega_{\max} - \mu) + \mu^6 \left( \frac{1}{\omega_{\max}} - \frac{1}{\mu} \right), \quad (40i)$$

$$G_9 = \int dk \, k^7 / \omega_k^4 = \frac{1}{4} (\omega_{\max}^4 - \mu^4) - \frac{3}{2} \mu^2 k_{\max}^2 + 3\mu^4 \log \frac{\omega_{\max}}{\mu} + \frac{1}{2} \mu^6 \left( \frac{1}{\omega_{\max}^2} - \frac{1}{\mu^2} \right). \quad (40j)$$

(1).  $T=1/2$ ,  $S$ -wave

From eq. (28) we obtain

$$\langle k | W_1 | k_0 \rangle = \delta(k - k_0) + \frac{f^2}{2\pi^2} \frac{1}{D_1(\omega_0)} \frac{1}{(2\omega_k)^{1/2} (2\omega_0)^{1/2}} \frac{k_0^2}{\omega_0 - \omega_k} \left( \frac{3}{2m + \omega_k} - \frac{1}{2m - \omega_0} \right). \quad (41)$$

Then we have

$$\tan \alpha_1 = - \frac{f^2}{4\pi} \frac{k_0}{D_1(\omega_0)} \left( \frac{3}{2m + \omega_0} - \frac{1}{2m - \omega_0} \right), \quad (42)$$

where

$$D_1(\omega_0) = 1 + \frac{f^2}{4\pi^2} \left[ \frac{3}{2m + \omega_0} \left\{ F_0(\omega_0) - F_0\left(\frac{\omega_0}{2} - m\right) \right\} - \frac{1}{2m - \omega_0} F_0(\omega_0) \right]. \quad (43)$$

(2).  $T=1/2$ ,  $P_{1/2}$ -wave

In this case, it is necessary to perform the renormalization for the solution of the integral equation. We define  $\langle k | K_{11} | k' \rangle$  as the solution of the following integral equation:

$$\langle k | K_{11} | k' \rangle = \delta(k - k') - \int dk'' [K_3(k, k'') + 2K_5(k, k'')] \langle k'' | K_{11} | k' \rangle. \quad (44)$$



Fig. 4

Since the kernel of this equation is not of "renormalization type", we can solve this without any consideration for renormalization. This solution corresponds to the scattering process such as

shown in Fig. 4. The approximate Fredholm solution of eq. (44) is

$$\langle k | K_{11} | k' \rangle = \delta(k - k') - (1/D_{11}(\omega_0)) [K_3(k, k') + 2K_5(k, k')], \quad (45)$$

where

$$D_{11}(\omega_0) = 1 - \frac{f^2}{4\pi^2} \frac{1}{12m^2} \left[ \frac{1}{\omega_0} \left\{ F_2(\omega_0) - F_2\left(\frac{\omega_0}{2}\right) \right\} + \frac{4}{2m - \omega_0} F_2(\omega_0) \right]. \quad (46)$$

Using this  $\langle k | K_{11} | k' \rangle$ , we can write the integral equation for  $\langle k | W_{11} | k_0 \rangle$  (eq. (29)) as

follows\* .

$$(k|W_{11}|k_0) = \int dk' (k|K_{11}|k') [\delta(k' - k_0) + 3 \int dk'' K_1(k', k'') (k''|W_{11}|k_0)]. \quad (47)$$

Because of eq. (34a), eq. (47) becomes

$$(k|W_{11}|k_0) = (k|K_{11}|k_0) + \frac{f^2}{2\pi^2} \frac{1}{W_0 - m + A} - \frac{3}{4m^2} \int dk' (k|K_{11}|k') \\ \times \frac{k'}{W_0 - m - \omega_{k'}} \frac{1}{(2\omega_{k'})^{1/2}} \int dk'' \frac{k''^3}{(2\omega_{k'})^{1/2}} (k''|W_{11}|k_0). \quad (48)$$

The kernel is degenerate in this integral equation, so we can easily solve this equation and obtain :

$$(k|W_{11}|k_0) = (k|K_{11}|k_0) + \frac{f^2}{2\pi^2} \frac{3}{4m^2} \frac{v(k; \omega_0) \bar{v}(k_0)}{W_0 - m + A - A(W_0)}, \quad (49)$$

where

$$v(k; \omega_0) = \int dk' (k|K_{11}|k') \frac{k'}{\omega_0 - \omega_{k'}} \frac{1}{(2\omega_{k'})^{1/2}}, \quad (50)$$

$$\bar{v}(k_0) = \lim_{k \rightarrow k_0} \bar{v}(k; \omega_0) = \lim_{k \rightarrow k_0} \int dk' \frac{k'^3}{(2\omega_{k'})^{1/2}} (k'|K_{11}|k), \quad (51)$$

$$A(W_0) = \frac{f^2}{2\pi^2} \frac{3}{4m^2} \int dk \frac{k^3}{(2\omega_k)^{1/2}} \int dk' (k|K_{11}|k') \frac{k'}{W_0 - m - \omega_{k'}} \frac{1}{(2\omega_{k'})^{1/2}}. \quad (52)$$

$v(k; \omega_0)$  and  $\bar{v}(k_0)$  correspond to such graphs as obtained by closing either of the two meson lines in Fig. 4, i.e., to the graphs shown in Figs. 5(a) and (b). They give the

\* The integral equation

$$\varphi(k) = f(k) + \int dk' A(k, k') \varphi(k') \quad (A1)$$

is equivalent to the following set of equations :

$$(k|K|k') = \delta(k - k') + \int dk'' A(k, k'') (k''|K|k'), \quad (A2)$$

$$\varphi(k) = \int dk' (k|K|k') f(k'), \quad (A3)$$

where  $f(k)$  may be some functional of  $\varphi(k)$ .  $(k|K|k')$  is the so-called "resolvent" of the kernel  $A(k, k')$ . The proof is as follows :

From eqs. (A2) and (A3) we have

$$\varphi(k) = f(k) + \int dk'' A(k, k'') \int dk' (k''|K|k') f(k').$$

Using eq. (A3) again, we obtain

$$\varphi(k) = f(k) + \int dk'' A(k, k'') \varphi(k').$$

contributions from the vertex parts with respect to the scattered and incident mesons.  $\Delta(W_0)$  corresponds to a graph obtained by closing both meson lines in Fig. 4, as shown in Fig. 5 (c). It gives the self-energy contribution for the nucleon. However,  $v(k; \omega_0)$ ,  $\tau_v(k_0)$  and  $\Delta(W_0)$  contain the so-called "overlapping divergences", and we must remove these divergences by some appropriate prescriptions. We shall follow the prescriptions given by Salam.<sup>(20)(21)</sup> From eqs. (44) and (50) it is found that  $v(k; \omega_0)$  satisfies the integral equation

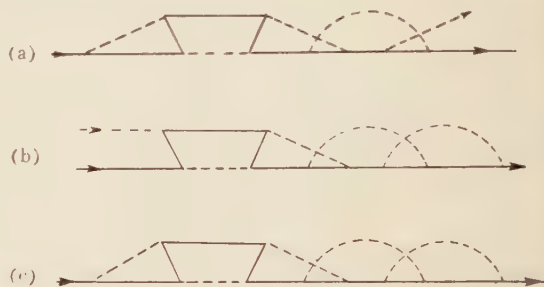


Fig. 5

$$v(k; \omega_0) = \frac{k}{\omega_0 - \omega_k} \frac{1}{(2\omega_k)^{1/2}} - \int dk' [K_3(k, k'; \omega_0) + 2K_5(k, k'; \omega_0)] \tau_v(k'; \omega_0), \quad (53)$$

where we write  $\omega_0$  explicitly as the argument of  $K_3$  and  $K_5$ , in order to show the  $\omega_0$ -dependence distinctly. By eqs. (34c) and (34e) we can rewrite eq. (53) as

$$v(k; \omega_0) = v_0(k; \omega_0) \left[ 1 + \int dk' V(k', \omega_k; \omega_0) \tau_v(k'; \omega_0) \right], \quad (54)$$

where

$$v_0(k; \omega_0) = \frac{k}{\omega_0 - \omega_k} \frac{1}{(2\omega_k)^{1/2}}, \quad (55)$$

$$V(k', \omega_k; \omega_0) = \frac{f^2}{2\pi^2 12m^2} \frac{k'^3}{(2\omega_{k'})^{1/2}} \left( \frac{1}{\omega_0 - \omega_k - \omega_{k'}} + \frac{4}{\omega_0 - 2m} \right). \quad (56)$$

The overlapping divergences evidently appear in solving this equation. In order to subtract the diverging parts for every subintegration, we replace eq. (54) by the following set of equations:

$$\tau_v(k; \omega_0) = v_0(k; \omega_0) \left[ 1 + \int dk' V(k', \omega_k; \omega_0) \tau_v(k'; \omega_0) - \int dk' V(k', 0; 0) \tau_v(k'; 0) \right], \quad (57)$$

$$v_c(k; 0) = v_0(k; 0) \left[ 1 + \int dk' \{ V(k', \omega_k; 0) - V(k', 0; 0) \} \tau_v(k'; 0) \right]. \quad (58)$$

$\tau_v(k; \omega_0)$  is just what is given as a result of subtractions of all the overlapping divergences from  $\tau_v(k; \omega_0)$ .<sup>\*</sup> This can be seen easily, if we solve eq. (57) in the iterated form. The true divergences are completely subtracted for every subintegration. Using the approximate solution of Fredholm, we obtain the solution of eq. (58) as

$$\tau_v(k; 0) = v_0(k; 0) \left[ 1 + \frac{1}{D} \int dk' \{ V(k', \omega_k; 0) - V(k', 0; 0) \} \tau_v(k'; 0) \right]$$

\* The same renormalization of the overlapping divergences was independently proposed by M.M. Lévy. (private communication)



$$=v_0(k;0)\left[1-\frac{1}{D}\frac{f^2}{4\pi^2}\frac{1}{12m^2}\left\{\frac{k^2}{\omega_k}F_0(-\omega_k)-\mu^2G_2+\frac{\mu^2}{\omega_k}G_1\right\}\right], \quad (59)$$

where

$$\begin{aligned} D &= 1 - \int dk v_0(k;0) [V(k, \omega_k;0) - V(k, 0;0)] \\ &= 1 + (f^2/4\pi^2) (1/24m^2) \cdot G_0. \end{aligned} \quad (60)$$

Next, we notice the following equation which comes from eqs. (44), (50) and (56) :

$$(k|K_{11}|k') = \delta(k-k') + v_0(k;\omega_0) \int dk'' V(k'', \omega_k; \omega_0) (k''|K_{11}|k'), \quad (61)$$

then the solution of eq. (57) is given as follows :

$$v_c(k;\omega_0) = \int dk' (k|K_{11}|k') v_0(k'; \omega_0) \left[1 - \int dk'' V(k'', 0;0) v_c(k'';0)\right].$$

Using eq. (50) we obtain

$$v_c(k;\omega_0) = (1-L)v(k;\omega_0), \quad (62)$$

where

$$L = \int dk V(k, 0;0) v_c(k;0). \quad (63)$$

$L$  and  $v(k;\omega_0)$  are both diverging quantities, but here they give finite contributions on account of the cut-off procedure.  $L$  is approximately given by

$$\begin{aligned} L &\simeq \frac{f^2}{4\pi^2} \frac{1}{12m^2} \left[ \left( G_6 + \frac{2}{m} G_5 \right) - \frac{1}{D} \frac{f^2}{4\pi^2} \frac{1}{12m^2} \right. \\ &\quad \times \left\{ \left( G_5 k_{\max} - G_4 \log \frac{\omega_{\max} + k_{\max}}{\mu} + G_9 \log \frac{\omega_{\max}}{\mu} + \mu^2 \frac{k_{\max} G_4}{\omega_{\max}} - \mu^3 G_7 \cos^{-1} \frac{\mu}{\omega_{\max}} \right) \right. \\ &\quad \left. \left. + \frac{2}{m} \left( G_4 k_{\max} - G_3 \log \frac{\omega_{\max} + k_{\max}}{\mu} + G_8 \log \frac{\omega_{\max}}{\mu} + \mu^2 \frac{k_{\max} G_5}{\omega_{\max}} - \mu^3 G_6 \cos^{-1} \frac{\mu}{\omega_{\max}} \right) \right\} \right], \end{aligned} \quad (64)$$

and the explicit expression for  $v(k;\omega_0)$  by

$$\begin{aligned} v(k;\omega_0) &= \int dk' (k|K_{11}|k') v_0(k'; \omega_0) \\ &= v_0(k;\omega_0) \left[ 1 + \frac{f^2}{4\pi^2} \frac{1}{D_{11}(\omega_0)} \frac{1}{6m^2} \left\{ \frac{1}{2\omega_k} (F_2(\omega_0) - F_2(\omega_0 - \omega_k)) \right. \right. \\ &\quad \left. \left. + \frac{2}{2m - \omega_0} F_2(\omega_0) \right\} \right]. \end{aligned} \quad (65)$$

Next, from eqs. (51) and (44) and using the reciprocal relation

$$\int dk'' K(k', k'') (k'' | K_{11} | k) = \int dk'' (k' | K_{11} | k'') K(k'', k) \quad (66)$$

where

$$K(k, k') = -[K_3(k, k') + 2K_5(k, k')],$$

we obtain the integral equation for  $\tau v(k; \omega_0)$ ,

$$\tau v(k; \omega_0) = k^3 / (2\omega_k)^{1/2} + \int dk' \tau v(k'; \omega_0) K(k', k). \quad (67)$$

From eqs. (34c), (34e) and (53) the following relation holds:

$$\tau v(k; \omega_0) = (\omega_0 - \omega_k) k^2 v(k; \omega_0). \quad (68)$$

We define  $\tau v_e(k; \omega_0)$  by

$$\begin{aligned} \tau v_e(k; \omega_0) &= (\omega_0 - \omega_k) k^2 \tau v(k; \omega_0) \\ &= (1 - L) \tau v(k; \omega_0), \end{aligned} \quad (69)$$

which expresses the finite vertex part obtained by subtracting all the overlapping divergences from  $\tau v(k; \omega_0)$ . Further, in order to remove the overlapping divergences from  $\Delta(W_0)$ , we write  $\Delta(W_0)$  as follows:

$$\Delta(W_0) = \frac{f^2}{2\pi^2} \frac{3}{4m^2} \int dk dk' dk'' \frac{k^3}{(2\omega_k)^{1/2}} \delta(k - k') (k' | K_{11} | k'') \frac{k''}{W_0 - m - \omega_{k''}} \frac{1}{(2\omega_{k''})^{1/2}}. \quad (70)$$

From eqs. (44) and (66), we see that the following relation holds:

$$\delta(k - k') = (k | K_{11} | k') - \int dl (k | K_{11} | l) K(l, k').$$

Substituting this equation into eq. (70) we obtain for  $\Delta(W_0)$

$$\Delta(W_0) = (f^2/2\pi^2) (3/4m^2) \int dk dk' \tau v(k; \omega_0) [\delta(k - k') - K(k, k')] \tau v(k'; \omega_0). \quad (71)$$

If we define  $\bar{\Delta}(W_0)$  as the self-energy part in which the overlapping divergences are removed out, then it is given by

$$\bar{\Delta}(W_0) = (f^2/2\pi^2) (3/4m^2) \int dk dk' \tau v_e(k; \omega_0) [\delta(k - k') - K(k, k')] \tau v_e(k'; \omega_0), \quad (72)$$

where  $\tau v_e(k; \omega_0)$  and  $\tau v(k; \omega_0)$  are given by eqs. (69) and (62). By expanding the right hand side of eq. (72) in powers of the coupling constant, we can easily justify that  $\bar{\Delta}(W_0)$  is just what is given as a result of subtracting all the overlapping divergences from  $\Delta(W_0)$ ; the true divergences are completely subtracted for every subintegration. From eqs. (72), (71), (69) and (62), we obtain

$$\bar{\Delta}(W_0) = (1 - L)^2 \Delta(W_0). \quad (73)$$

If we perform the substitution

$$\mathcal{A}(W_0) \rightarrow (1-L)^2 \mathcal{A}(W_0) \quad (74)$$

in the energy denominator of eq. (49), there remains only the final true divergence to be removed. We expand this equation in terms of  $(W_0 - m)$ , namely

$$(1-L)^2 \mathcal{A}(W_0) = (1-L)^2 \left[ \mathcal{A}(m) + (W_0 - m) \lim_{W_0 \rightarrow m} \frac{\partial}{\partial W_0} \mathcal{A}(W_0) + \mathcal{A}'(W_0) \right]. \quad (75)$$

The first and second terms on the right hand side of eq. (75) are the quantities to be renormalized into mass and charge and the remaining term is the effective part. Then, we can determine the counter-term  $\mathcal{A}$  as

$$\mathcal{A} = (1-L)^2 \mathcal{A}(m). \quad (76)$$

The energy denominator of eq. (49) becomes

$$(W_0 - m) \left[ 1 - (1-L)^2 \lim_{W_0 \rightarrow m} \partial \mathcal{A}(W_0) / \partial W_0 \right] - \mathcal{A}_c(W_0), \quad (77)$$

where

$$\mathcal{A}_c(W_0) = (1-L)^2 \mathcal{A}'(W_0).$$

In order to perform the charge renormalization, we make the substitution

$$1 - (1-L)^2 \lim_{W_0 \rightarrow m} \partial \mathcal{A}(W_0) / \partial W_0 \rightarrow 1. \quad (78)$$

Then (77) becomes

$$W_0 - m - \mathcal{A}_c(W_0). \quad (79)$$

By the approximate integration of (52)  $\mathcal{A}(W_0)$  is given as follows:

$$\begin{aligned} \mathcal{A}(W_0) \simeq & - (f^2/4\pi^2) (3/4m^2) F_2(\omega_0) - (f^2/4\pi^2)^2 (1/4m^2)^2 (1/D_{11}(\omega_0)) \\ & \times \left[ G_0 F_2(\omega_0) + k_0^2 F_0(\omega_0) F_3(\omega_0) - k_{\max} \{ F_1(\omega_0) - 2\omega_0 F_2(\omega_0) + k_0^2 F_3(\omega_0) \} \right. \\ & \left. + (G_3 - 2\omega_0 G_4 + k_0^2 G_5) \log \frac{2\omega_{\max} - \omega_0}{\omega_{\max} - \omega_0} + \frac{4}{2m - \omega_0} \{ F_2(\omega_0) \}^2 \right]. \end{aligned} \quad (80)$$

Using this result we can easily evaluate  $\mathcal{A}(m)$ ,  $\lim_{W_0 \rightarrow m} \frac{\partial}{\partial W_0} \mathcal{A}(W_0)$  and  $\mathcal{A}'(W_0)$ . Thus we obtain the final expression for  $\langle k | W_{11} | k_0 \rangle$ :

$$\langle k | W_{11} | k_0 \rangle = \langle k | K_{11} | k_0 \rangle + \frac{f^2}{2\pi^2} \frac{3}{4m^2} \frac{v_c(k; \omega_0) v_c(k_0)}{W_0 - m - \mathcal{A}_c(W_0)}. \quad (81)$$

The phase shift is given by

$$\begin{aligned} \tan \alpha_{11} = & \frac{f^2}{4\pi} \frac{1}{12m^2} \frac{1}{D_{11}(\omega_0)} k_0^3 \left( \frac{1}{\omega_0} + \frac{4}{2m - \omega_0} \right) - \frac{f^2}{4\pi} \frac{3}{4m^2} (1-L)^2 k_0^3 \\ & \times \frac{1}{\omega_0 - (1-L)^2 \mathcal{A}'(W_0)} \left[ 1 + \frac{f^2}{4\pi^2} \frac{1}{6m^2} \frac{1}{D_{11}(\omega_0)} \left\{ \frac{1}{2\omega_0} (F_2(\omega_0) - F_2(0)) + \frac{2}{2m - \omega_0} F_2(\omega_0) \right\}^2 \right]. \end{aligned} \quad (82)$$

(3),  $T=1/2$ ,  $P_{3/2}$ -wave

From eq. (30) we obtain

$$\langle k|W_{13}|k_0\rangle = \delta(k-k_0) + \frac{f^2}{2\pi^2} \frac{1}{6m^2 D_{13}(\omega_0)} \frac{1}{(2\omega_k)^{1/2} (2\omega_0)^{1/2}} \frac{k k_0^3}{\omega_0 - \omega_k} \left( \frac{1}{\omega_k} + \frac{1}{2m - \omega_0} \right). \quad (83)$$

The phase shift is given by

$$\tan \alpha_{13} = - \frac{f^2}{4\pi} \frac{1}{6m^2 D_{13}(\omega_0)} k_0^3 \left( \frac{1}{\omega_0} + \frac{1}{2m - \omega_0} \right), \quad (84)$$

where

$$D_{13}(\omega_0) = 1 + \frac{f^2}{4\pi^2} \frac{1}{6m^2} \left[ \frac{1}{\omega_0} \left\{ F_2(\omega_0) - F_2\left(\frac{\omega_0}{2}\right) \right\} + \frac{1}{2m - \omega_0} F_2(\omega_0) \right]. \quad (85)$$

(4),  $T=3/2$ ,  $S$ -wave

From eq. (31) we obtain

$$\langle k|W_{31}|k_0\rangle = \delta(k-k_0) + \frac{f^2}{2\pi^2} \frac{1}{D_3(\omega_0)} \frac{1}{(2\omega_k)^{1/2} (2\omega_0)^{1/2}} \frac{1}{\omega_0 - \omega_k} \frac{2k_0^2}{2m - \omega_0}. \quad (86)$$

The phase shift is given by

$$\tan \alpha_3 = - (f^2/4\pi) (1/D_3(\omega_0)) (2k_0/(2m - \omega_0)), \quad (87)$$

where

$$D_3(\omega_0) = 1 + (f^2/4\pi^2) (2/(2m - \omega_0)) F_0(\omega_0). \quad (88)$$

(5),  $T=3/2$ ,  $P_{1/2}$ -wave

From eq. (32) we obtain

$$\langle k|W_{31}|k_0\rangle = \delta(k-k_0) + \frac{f^2}{2\pi^2} \frac{1}{6m^2 D_{31}(\omega_0)} \frac{1}{(2\omega_k)^{1/2} (2\omega_0)^{1/2}} \frac{k k_0^3}{\omega_0 - \omega_k} \left( \frac{1}{\omega_k} + \frac{4}{2m - \omega_0} \right). \quad (89)$$

The phase shift is given by

$$\tan \alpha_{31} = - \frac{f^2}{4\pi} \frac{1}{6m^2 D_{31}(\omega_0)} k_0^3 \left( \frac{1}{\omega_0} + \frac{4}{2m - \omega_0} \right), \quad (90)$$

where

$$D_{31}(\omega_0) = 1 + \frac{f^2}{4\pi^2} \frac{1}{6m^2} \left[ \frac{1}{\omega_0} \left\{ F_2(\omega_0) - F_2\left(\frac{\omega_0}{2}\right) \right\} + \frac{4}{2m - \omega_0} F_2(\omega_0) \right]. \quad (91)$$

(6),  $T=3/2$ ,  $P_{3/2}$ -wave

From eq. (33) we obtain

$$\langle k|W_{33}|k_0\rangle = \delta(k-k_0) + \frac{f^2}{2\pi^2} \frac{1}{3m^2 D_{33}(\omega_0)} \frac{1}{(2\omega_k)^{1/2} (2\omega_0)^{1/2}} \frac{k k_0^3}{\omega_0 - \omega_k} \left( \frac{1}{\omega_k} + \frac{1}{2m - \omega_0} \right). \quad (92)$$

The phase shift is given by

$$\tan \alpha_{33} = \frac{f^2}{4\pi} \frac{1}{3m^2} \frac{1}{D_{33}(\omega_0)} k_0^3 \left( \frac{1}{\omega_0} + \frac{1}{2m - \omega_0} \right), \quad (93)$$

where

$$D_{33}(\omega_0) = 1 - \frac{f^2}{4\pi} \frac{1}{3m^2} \left[ \frac{1}{\omega_0} \left\{ F_2(\omega_0) - F_2\left(\frac{\omega_0}{2}\right) \right\} + \frac{1}{2m - \omega_0} F_2(\omega_0) \right]. \quad (94)$$

#### § 4. Results of calculation and discussion

From the formulas (42), (82), (84), (87), (90) and (93) we can evaluate the phase shifts for scattering waves. The calculations were carried out for nine cases from the combinations of  $f^2/4\pi = 10, 15, 20$  and  $k_{\max} = 4\mu, 5\mu, 6\mu$ . These results are shown in Figs. 6, 7 and 8. In Fig. 9 the experimental analysis by Fermi and others<sup>1</sup> is reproduced. (These are all referred to the center-of-mass system.)

As regards  $\alpha_{33}$ ,  $\alpha_{11}$  and  $\alpha_{31}$ , our results have the same signs and reasonable magnitudes compared with the experiments, but, as regards  $\alpha_1$ ,  $\alpha_{13}$  and  $\alpha_{31}$ , our results have opposite signs.

As to  $\alpha_{33}$ , however, the energy dependence is not good, i.e., the very steep energy dependence as shown in the experiments is not given. As to  $\alpha_{33}$ , the tendency is generally good, but the detailed points such as the position, magnitude and breadth of the resonance are not interpreted. As to  $\alpha_{11}^*$  in the case of  $f^2/4\pi = 20$  and  $k_{\max} = 6\mu$ , the attractive

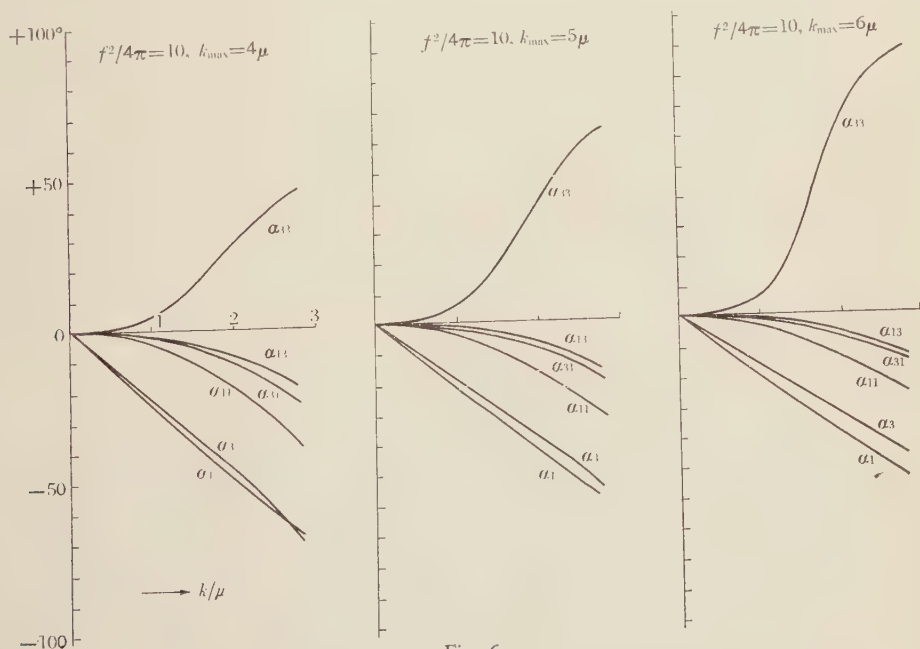


Fig. 6

\* Experimental values of  $\alpha_{11}$ ,  $\alpha_{13}$  and  $\alpha_{31}$  are not relied upon so much because of the ambiguities of the phase shift analysis.



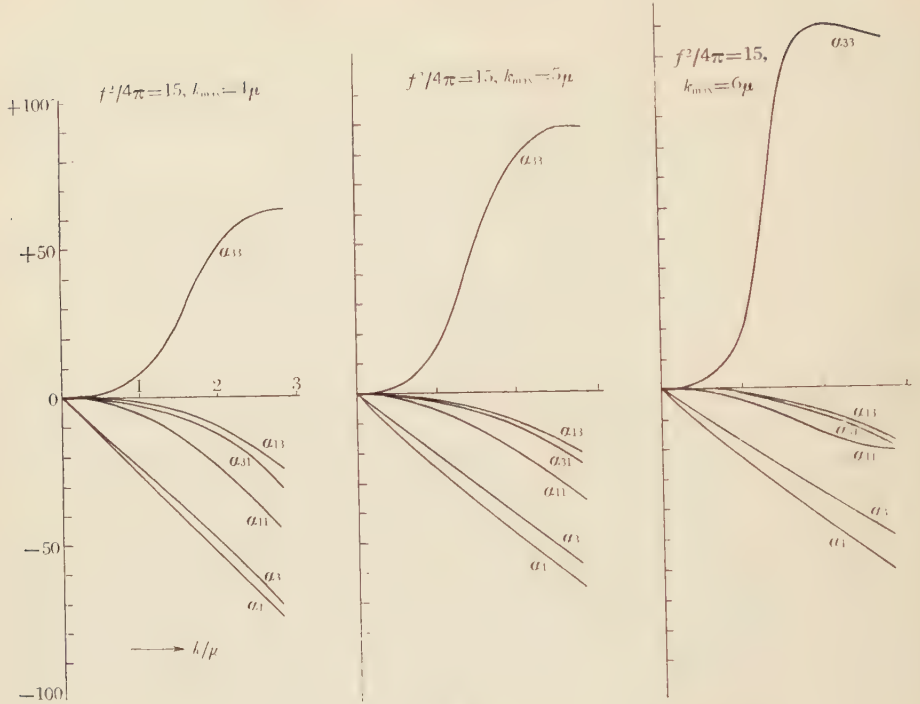


Fig. 7

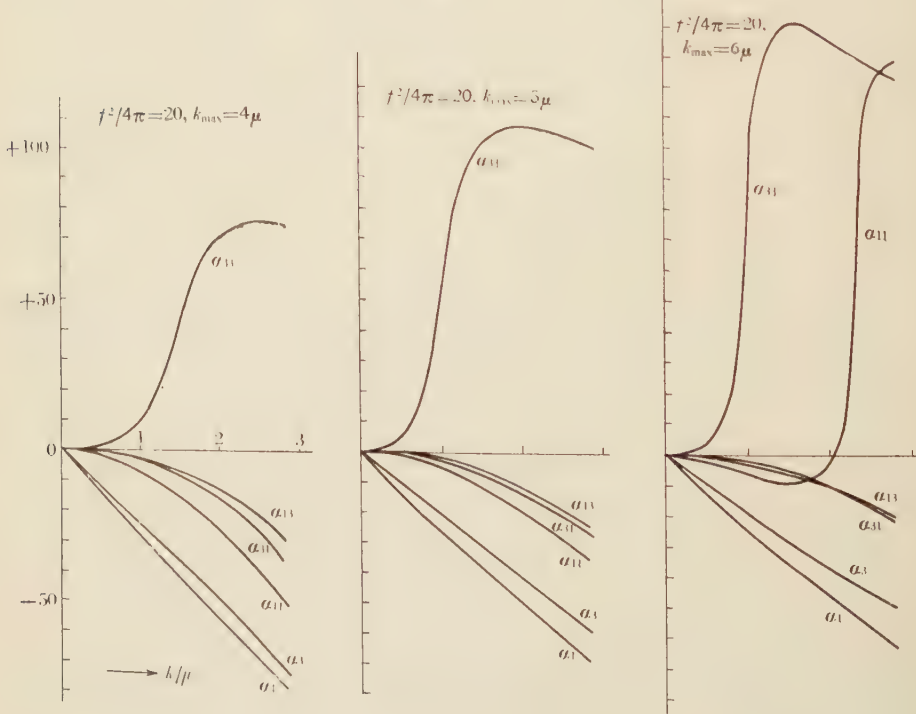


Fig. 8

part overcomes the repulsive one in the neighbourhood of  $k=2\mu$ , and the sign is reversed. Further, in our approximation, the  $S$ -wave scattering potential is nearly isotopic spin independent and represented by a short range repulsive core. On the other hand, the results by Fermi and others seem to require the existence of a strong attractive potential for the  $T=1/2$ ,  $S$ -state. In order to get over this discrepancy, the  $\tau \cdot \phi \times \pi$  term, which was first noticed by Wentzel and Drell-Henley, will be effective as shown by Fukuda, Sawada, et al., if the higher order effects of this type of interaction are included. Furthermore, as for  $\alpha_{13}$  and  $\alpha_{31}$ , our approximation does not give any appreciable difference between them, contrary to the experiments. At present, however, the experiment is not so accurate, especially as to the angular distribution, and moreover, the  $D$ -wave contribution is entirely neglected in the phase shift analysis. Thus it seems to be dangerous to try to compare the theory with experiments about such detailed points.

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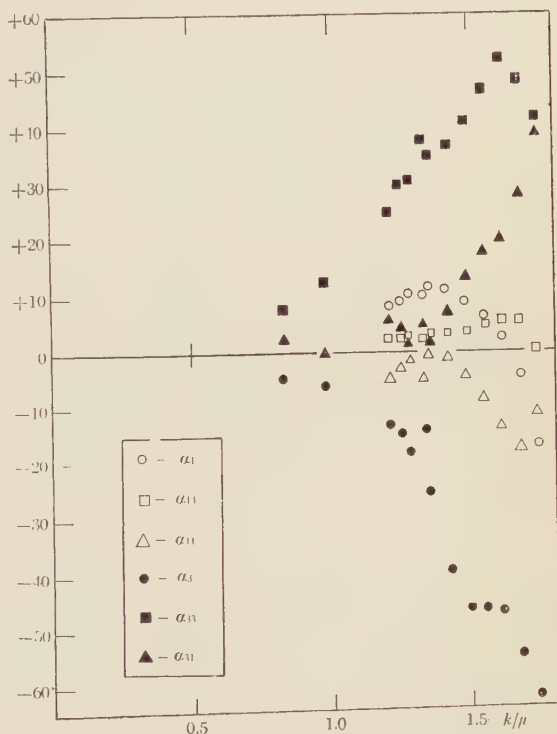


Fig. 9

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## On the Relation between Meson-Nucleon Scattering and Photomeson Production

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Using the unitarity of the S-matrix, Aizu and Watson have shown some relations between the photomeson production and the meson-nucleon scattering. Their theory is reformulated and generalized here in a summarized form, in order to make facile the analysis of experimental results. Experiments on the photomeson production near threshold energy lead us to suggest some limits on the magnitude of  $s$ -phase shifts of the meson-nucleon scattering and on the contribution of the nucleon current.

### § 1. Introduction

This paper aims to investigate the relation between the photomeson production and the meson-nucleon scattering from the kinematical point of view. This purpose has been achieved using the unitarity of the S-matrix first by Nakano and Nishijima<sup>1)</sup> in the momentum space and by Aizu<sup>2)</sup>, Ross<sup>3)</sup> and Watson<sup>4)</sup> with the partial wave analysis, some of which are related to specific models. Since these works are restricted to lower angular momentum states, it seems worth while to give a general formalism with wider range of applications. Our formalism is to present an explicit connection between the amplitudes of the photomeson production and the phase shifts of the meson-nucleon scattering in the angular momentum representation. The unitarity condition is combined with the detailed reversibility and further with the hypothesis of charge independence. Thus the S-matrix is separated not only into the angular momentum but also into the total isotopic spin, in order to make facile the analysis of experimental data. It is needless to say that our results agree with Aizu's and Watson's ones and include them.

In our previous paper<sup>5)</sup>\* we have emphasized that there exists a remarkable symmetry property in the angular distribution of the meson-nucleon scattering as well as of the photomeson production. Indeed, the angular distribution of the meson-nucleon scattering is invariant under the simultaneous interchange of every pair of the phase shifts belonging to the same total angular momentum  $J^{(6)}$  while that of the photomeson production is invariant under the interchange of the reaction amplitudes of the electric and magnetic radiations with the

\* S. Hayakawa, M. Kawaguchi and S. Minami, Prog. Theor. Phys. 12 (1954), 355. This paper will be cited as I.

same multipole order both belonging to the same  $J$ . Utilization of the unitarity of the  $S$ -matrix, which is yet only a kinematical theory, does not remove the degeneracy stated above, but it can only give the correspondence between a phase shift of the meson-nucleon scattering and an amplitude of the photomeson production. Another important result is that the phases of two amplitudes, which correspond to two different kinds of multipole radiations with the same total angular momentum  $J$  as well as the same parity  $II$ , are always equal apart from the inevitable ambiguity.

The differential cross section of the photomeson production is expressed in terms of the phase shifts of the meson-nucleon scattering and the absolute magnitudes of the amplitudes which are undetermined. From experiments on the photomeson production near threshold, where  $s$ -wave meson may be produced predominantly, we find limits on the  $s$ -phase shifts of the meson-nucleon scattering and on the magnitude of the nucleon current without any detailed assumption. In the present paper, contrary to Watson<sup>4)</sup> who has assumed various models, we start from only experimental data of considerable reliability and the established facts such as charge independence as a basic assumption. We get, for example, an upper limit on the difference of  $s$ -phase shifts  $|\delta_0 - \delta_1|$  from the ratio of the cross sections  $\sigma(\gamma + p \rightarrow p + \pi^0) / \sigma(\gamma + p \rightarrow n + \pi^+)$ . Further from  $\sigma(\gamma + n \rightarrow p + \pi^-) / \sigma(\gamma + p \rightarrow n + \pi^+)$  we can find out the magnitude of the nucleon current to be confined within narrow limits near  $\mu/m$  times of that of the meson current, where  $\mu$  and  $m$  are the mass of meson and nucleon respectively. These results were expected from an unsophisticated classical consideration.<sup>7), 8)</sup>

The fundamental formulation of the unitarity of the  $S$ -matrix in the angular momentum representation is developed in § 2, in order to combine the photomeson production with the meson-nucleon scattering. In § 3 the basic equation is separated into the eigenstates in the isotopic spin space for the purpose of practical applications. In § 4 we apply this theory to the photomeson production near threshold energy and get nontrivial restrictions on the phase shifts from experimental results. Next, the angular distribution of the photomeson production is tabulated in terms of the phase shifts of the meson-nucleon scattering up to  $p$ -wave of the dipole radiation. Detailed calculation of the time reversal used in § 2 is exhibited in Appendix.

## § 2. General formalism

Our fundamental equations are based on the unitarity of the  $S$ -matrix<sup>1)</sup> that is to mean the conservation of probability. It is convenient to express the unitarity condition in the following form:

$$SR^+ = -R, \text{ where } R = S - 1. \quad (2 \cdot 1)$$

Taking the matrix element for  $\gamma + N \rightarrow N + \pi$ , this yields,

$$(\pi|S|\pi)(\pi|R^+|\gamma) = -(\pi|R|\gamma). \quad (2 \cdot 2a)$$

We consider only the case where the energy of the system is too low to produce two or more mesons, that is, we neglect the matrix elements such as  $(\pi\pi|R|\gamma)$ . Moreover



the higher order terms in the fine structure constant are not taken into consideration. Below we use the following notations:  $(JIIb|S|JIIa)$  represents an eigenvalue of the  $S$ -matrix in the representation in which the total angular momentum  $J$  and parity  $II$  are diagonal. Here  $b$  and  $a$  are used in order to specify the states in addition to  $J$  and  $II$ . When only a state of given  $J$ ,  $II$  is concerned, this eigenvalue is simply written as  $S_{ba}$ . Then (2.2a) may be expressed as

$$S_{\pi\pi}R_{\pi\gamma}^{\dagger} = -R_{\pi\gamma}. \quad (2.2b)$$

Using the above notations, we are able to expand the  $S$ -matrix as follows:

$$(f|S|i) = \sum_{JII M} (f|JII M b) (JII b|S|JII a) (JII M a|i), \quad (2.3)$$

where  $M$  is the  $z$  component of  $\mathbf{J}$ . Here both the scattering amplitude for the meson-nucleon scattering and the reaction amplitude for the photomeson production are given in I. For the meson-nucleon scattering, the orbital angular momentum of a meson belonging to a definite  $J$  state is uniquely determined in virtue of the parity conservation. Then we are allowed to write down the eigenvalue of the  $S$ -matrix in the following form,

$$(JL|S|JL) = e^{2i\delta^+_{LJ}} \quad (L=J-1/2), \quad (2.4a)$$

$$(JL+1|S|JL+1) = e^{2i\delta^-_{L+1J}} \quad (L+1=J+1/2). \quad (2.4b)$$

Although the  $R$ -matrix for the photomeson production may be brought in the formula by decomposing the radiation field in terms of orbital angular momenta of photon, as has been carried out by Aizu<sup>2)</sup>, it is more natural and practical to refer to  $a_J(m)$  and  $b_J(m)$ , that have been introduced in I, with regard to the amplitude for multipole radiations. These quantities are dealt with on the same footing as the  $R$ -matrix. The relation between the multipole radiation and our notations is shown in table 1.

Table 1

Radiation	parity	Orbital angular mom. of pion.	$R$ -matrix
$EL$ $ML+1$	$(-1)^L$	$L+1$	$a_J(e)$ $b_J(m)$
$EL+1$ $ML$	$(-1)^{L+1}$	$L$	$b_J(e)$ $a_J(m)$

From this table, we see that there are two states of  $EL$  and  $ML+1$  for  $\gamma-\pi$  processes even if  $J=L+1/2$  and  $II=(-1)^L$  are specified, that is,  $(JII\pi|R|JII\gamma)$  has to be a matrix form and (2.2b) holds for each of its elements.

Let us decompose the  $R$ -matrix into a real amplitude and a phase, namely

$$a_J(m) = \alpha_J(m) e^{2i\eta_{J(m)}}, \quad (2.5a)$$

$$b_J(m) = \beta_J(m) e^{2i\eta_{J(m)}}. \quad (2.5b)$$

Thus, the problem to be solved is how to get  $(\pi|R^+|\gamma)$  from  $(\pi|R|\gamma)$ . By the

definition  $(f|R^+|i)$ ,

$$(f|R^+|i) = (i|R|f)^*, \quad (2.6)$$

where the notation  $*$  means the complex conjugate. Therefore we have only to know the expression of  $(i|R|f)$ . The time reversibility<sup>1)</sup> of the physical process under consideration yields

$$(f|R^+|i) = (f^*|R|i^*)^*, \quad (2.8)$$

where the notation  $i^*$  denotes the state which is produced from state  $i$  through the time reversal. Detailed calculation for this procedure will be explained in Appendix. Applying (A. 16) and (A. 17) to (2.2), we obtain the following relation.

$$S_{\pi\pi} R_{\pi\pi}^* = R_{\pi\pi}^{* \#}. \quad (2.9)$$

By the expression of (2.4) for the meson-nucleon scattering and (2.5) for the photomeson production, equation (2.9) can be written as follows:

$$e^{2i\delta_{L+1}^-} = e^{4i\epsilon_{J^+}} f_{0J^+} EL, \quad (2.10a)$$

$$e^{2i\delta_{L+1}^-} = e^{4i\epsilon_{J^+}^{(m)}} f_{0J^+} ML + 1, \quad (2.10b)$$

$$e^{2i\delta_L^+} = e^{4i\epsilon_{J^+}^{(e)}} f_{0J^+} EL + 1, \quad (2.10c)$$

$$e^{2i\delta_L^+} = e^{4i\epsilon_{J^+}^{(m)}} f_{0J^+} ML. \quad (2.10d)$$

These relations serve to know one of phases for the meson-nucleon scattering and the photomeson production from another. However, nothing can be said about  $\alpha$  and  $\beta$ . Moreover, the necessary quantities  $e^{2i\epsilon}$ , etc. are subject to the ambiguity in phase by  $\pi$ , as deduced from (2.10):

$$e^{2i\epsilon_{J^+}^{(e)}} = \pm e^{2i\delta_{L+1}^-} \text{ etc.} \quad (2.11)$$

There is no way to decide this sign in our framework. In Watson's paper<sup>1)</sup> this ambiguity seems to be overlooked.

Apart from this ambiguity, the phases for two different multipole radiations belonging to the same  $J$  and  $H$  are discriminated only by a constant phase, namely,

$$\bar{\epsilon}_J(e) = \epsilon_{J^+}(m) + (n/2)\pi, \quad (2.12a)$$

$$\bar{\epsilon}_J(m) = \epsilon_{J^+}(e) + (n/2)\pi, \quad (2.12b)$$

where  $n$  is any integer. These results agree with those of Aizu<sup>2)</sup> and Watson.<sup>4)</sup>

Finally, we discuss the symmetry property with respect to the phase shifts for the meson-nucleon scattering, which has been found by one of us (S.M.)<sup>6)</sup>. That is, the differential cross section for the meson-nucleon scattering is invariant under the simultaneous exchange of  $\delta_L^+$  and  $\delta_{L+1}^-$  belonging to the same  $J$  but different parities. Also we have shown in I that the differential cross section for the photomeson production is insensitive to the

#) The minus sign here given by Aizu<sup>2)</sup> does not arise. However this sign is irrelevant to the final result.

substitution  $a_J(m) \rightleftharpoons a_J(e)$  and  $b_J(m) \rightleftharpoons b_J(e)$ . If the substitution  $\delta_L^+ \rightleftharpoons \delta_{L+1}^-$  is carried out, it induces the substitution  $\xi_J(m) \rightleftharpoons \xi_J(e)$  and  $\eta_J(m) \rightleftharpoons \eta_J(e)$  and consequently  $a_J(m) \rightleftharpoons a_J(e)$  and  $b_J(m) \rightleftharpoons b_J(e)$ . With this substitution the angular distribution for the photomeson production is left unaltered. Thus, the degeneracy which we have emphasized can not be removed either in the application of the unitary property of the  $S$ -matrix. We get only the correspondence between the phase shifts of the meson-nucleon scattering and the amplitudes of the photomeson production.

### § 3. Separation of the $R$ -matrix in the isotopic spin space

For the purpose of practical applications the fundamental equations (2.10) are separated into the eigenstates in the isotopic spin space. By means of the assumption of charge independence the phase shifts for the meson-nucleon scattering are usually decomposed into the eigenstates not only of angular momenta but also of isotopic spins. The final state for the photomeson production must also be managed in the same manner.

The  $R$ -matrices for the meson-nucleon scattering are expressed in the current manner as in table 2.

Table 2

Process		R-matrix
$\pi^+ + p \rightarrow p + \pi^+$	$\pi^- + n \rightarrow n + \pi^-$	$R^{\pm}_{L,3}$
$\pi^- + p \rightarrow p + \pi^-$	$\pi^+ + n \rightarrow n + \pi^+$	$(1/3) (R^{\pm}_{L,3} + 2R^{\pm}_{L,1})$
$\pi^- + p \rightarrow n + \pi^0$	$\pi^+ + n \rightarrow p + \pi^0$	$(\sqrt{2}/3) (R^{\pm}_{L,3} - R^{\pm}_{L,1})$
$\pi^0 + n \rightarrow n + \pi^0$	$\pi^0 + p \rightarrow p + \pi^0$	$(1/3) (2R^{\pm}_{L,3} + R^{\pm}_{L,1})$

$R^{\pm}_{L,3}$  and  $R^{\pm}_{L,1}$  are the  $R$ -matrices for the states of  $I=3/2$  and  $I=1/2$  respectively. They are expressed in terms of phase shifts as follows:

$$\begin{aligned} R^{\pm}_{L,3} &= e^{2i\delta^{\pm}_{L,3}} - 1, \\ R^{\pm}_{L,1} &= e^{2i\delta^{\pm}_{L,1}} - 1. \end{aligned} \quad (3.1)$$

For the photomeson production there is a way as shown by Watson<sup>(10)</sup> to describe the matrix element with the third component of vector and scalar parts in the isotopic spin space. This treatment is very convenient to examine the contributions from the meson current and the recoil of a nucleon separately. However, it is subject to the defect that the quantities corresponding to the state of  $I=1/2$  in the final state are contained both in the third component of vector and in scalar parts. Another method, by which the matrix element is separated only into the isotopic spin of the final state, is sometimes convenient. In this circumstances we express in table 3 the  $R$ -matrix in reference to the both stand-points, Watson's and another one. But it must be noted that our notations are different from those of Watson.<sup>4),10)</sup>

Table 3

Process	$R$ -matrix (Watson's notion)	$R$ -matrix (another notion)
$\gamma + n \rightarrow n + \pi^0$	$\sqrt{\frac{2}{3}} a_{J,3} + \sqrt{\frac{1}{3}} \bar{a}_{J,1} + \sqrt{\frac{1}{3}} a_{J,8}$	$\sqrt{\frac{2}{3}} a_{J,3} + \sqrt{\frac{1}{3}} \bar{a}_{J,1}$
$\gamma + p \rightarrow p + \pi^0$	$\sqrt{\frac{2}{3}} a_{J,3} - \sqrt{\frac{1}{3}} \bar{a}_{J,1} + \sqrt{\frac{1}{3}} a_{J,8}$	$\sqrt{\frac{2}{3}} a_{J,3} - \sqrt{\frac{1}{3}} a'_{J,1}$
$\gamma + n \rightarrow p + \pi^-$	$\sqrt{\frac{1}{3}} a_{J,3} - \sqrt{\frac{2}{3}} \bar{a}_{J,1} - \sqrt{\frac{2}{3}} a_{J,8}$	$\sqrt{\frac{1}{3}} a_{J,3} - \sqrt{\frac{2}{3}} a_{J,1}$
$\gamma + p \rightarrow n + \pi^+$	$\sqrt{\frac{1}{3}} a_{J,3} + \sqrt{\frac{2}{3}} \bar{a}_{J,1} - \sqrt{\frac{2}{3}} a_{J,8}$	$\sqrt{\frac{1}{3}} a_{J,3} + \sqrt{\frac{2}{3}} a'_{J,1}$

$a_{J,3}$  is the  $R$ -matrix for the state of  $I=3/2$  and is defined in the same manner for  $a_J(m)$ ,  $a_J(e)$ ,  $b_J(m)$  and  $b_J(e)$  in table 1 respectively.  $\bar{a}_{J,1}$  is one for the third component of a vector belonging to  $I=1/2$ , while  $a_{J,8}$  is the one for the scalar part.  $a_{J,1}$  and  $a'_{J,1}$  are connected with  $\bar{a}_{J,1}$  and  $a_{J,8}$  by the following relations :

$$\begin{aligned} a_{J,1} &= \bar{a}_{J,1} + a_{J,8}, \\ a'_{J,1} &= \bar{a}_{J,1} - a_{J,8}. \end{aligned} \tag{3.2}$$

With this relationship we easily find that (2.10) is valid for respective isotopic spin state, for example

$$e^{2i\delta^-_{L+1,3}} = e^{4\frac{1}{2}\xi_{J,3}(e)} \tag{3.3a}$$

and

$$e^{2i\delta^-_{L+1,1}} = e^{4\frac{1}{2}\xi_{J,1}(e)} \quad \text{or} \quad e^{4\frac{1}{2}\xi'_{J,1}(e)} \tag{3.3b}$$

for  $EL, J=L+1/2$ .

where  $\xi_{J,3}(e)$  and  $\xi_{J,1}(e)$  or  $\xi'_{J,1}(e)$  denote the phase of  $a_{J,3}(e)$  and  $a_{J,1}(e)$  or  $a'_{J,1}(e)$  respectively. Therefore the phases of  $a_{J,1}$  and  $a'_{J,1}$  are equal apart from the inevitable ambiguities.

#### § 4. Photomeson production near threshold

As discussed in the above two sections, the cross section for the photomeson production is expressed in terms of the real amplitudes and of the phase shifts for the meson-nucleon scattering apart from the ambiguity in sign. In this section, we treat only such a low energy event ( $\sim 180$  Mev.) that mesons are produced in  $s$ -wave by absorbing incident photons through  $E1, J=1/2$ . In this case we may assume the following facts as firmly established from existing experiments.

- i) First, the charge independence hypothesis should hold apart from the small mass difference between  $\pi^\pm$  and  $\pi^0$ .
- ii) As for the  $s$ -phase shifts for the meson-nucleon scattering, let us assume  $|\partial_3 - \partial_1|$  to be less than  $\pi/2$ .
- iii)  $\sigma(0)/\sigma(+)$  is not larger than  $1/10$ ,<sup>11)</sup>

where  $\sigma(0)$  and  $\sigma(+)$  are the total cross sections for the processes  $\gamma + p \rightarrow p + \pi^0$  and  $\gamma + p \rightarrow n + \pi^+$  respectively, while the theoretical value of this ratio is estimated as the order of

$(\mu/M)^2$ .

iv)  $\sigma(-)/\sigma(+)$  is nearly equal to  $4/3^{12)}$ .

Here  $\sigma(-)$  is the total cross section for the process of  $\gamma+n \rightarrow p+\pi^-$ . On the other hand, the theoretical value<sup>7),8)</sup> of this ratio is expected to be about  $(1+\mu/M)^2$ .

v)  $\sigma(n+\pi^0)/\sigma(-) < \sigma(0)/\sigma(+)$ .

As  $\sigma(n+\pi^0)$ , which is the total cross section for the process  $\gamma+n \rightarrow n+\pi^0$ , may be obtained experimentally through  $\gamma+d \rightarrow d+\pi^0$ , we cannot have any accurate knowledge but qualitative one. This inequality is, however, considered to be correct.

Using the table 4 of I, the table 3 of this paper and the relation (3.3), the total cross sections for the photomeson production in the  $s$ -state are expressed as follows:

$$\sigma(0) = \frac{1}{3} \pi \lambda^2 \left[ \frac{2}{3} b_{1/2,3}^2(\epsilon) + \frac{1}{3} b_{1/2,1}^2(\epsilon) - \frac{2\sqrt{2}}{3} b_{1/2,3}(\epsilon) b'_{1/2,1}(\epsilon) \cos(\delta_3 - \delta_1 + n\pi) \right], \quad (4.1a)$$

$$\sigma(+)=\frac{1}{3} \pi \lambda^2 \left[ \frac{1}{3} b_{1/2,3}^2(\epsilon) + \frac{2}{3} b_{1/2,1}^2(\epsilon) + \frac{2\sqrt{2}}{3} b_{1/2,3}(\epsilon) b'_{1/2,1}(\epsilon) \cos(\delta_3 - \delta_1 + n\pi) \right], \quad (4.1b)$$

$$\sigma(-)=\frac{1}{3} \pi \lambda^2 \left[ \frac{1}{3} b_{1/2,3}^2(\epsilon) + \frac{2}{3} b_{1/2,1}^2(\epsilon) - \frac{2\sqrt{2}}{3} b_{1/2,3}(\epsilon) b'_{1/2,1}(\epsilon) \cos(\delta_3 - \delta_1 + n'\pi) \right], \quad (4.1c)$$

$$\sigma(n+\pi^0)=\frac{1}{3} \pi \lambda^2 \left[ \frac{2}{3} b_{1/2,3}^2(\epsilon) + \frac{1}{3} b_{1/2,1}^2(\epsilon) + \frac{2\sqrt{2}}{3} b_{1/2,3}(\epsilon) b'_{1/2,1}(\epsilon) \cos(\delta_3 - \delta_1 + n'\pi) \right], \quad (4.1d)$$

where  $b$ 's are the real amplitudes of (2.5) defined for respective isotopic spin states, the  $s$ -phase shifts  $\delta_3$  and  $\delta_1$  are  $\delta_{0,3}^+$  and  $\delta_{0,1}^+$  of (3.1) respectively and  $n$  and  $n'$  are arbitrary integers.

First of all, we can fix the ambiguity in sign, namely, the evenness or oddness of  $n$ , in reference to the evidences ii) and iii). If  $n$  were an odd number, the signs of the interference terms in (4.1a) and (4.1b) would result in  $\sigma(0) \geq (1/2) \sigma(+)$ , because  $\cos(\delta_3 - \delta_1 + n\pi)$  is negative. As  $n$  is thus fixed to be even, we can express the ratio of the cross sections as

$$\frac{\sigma(0)}{\sigma(+)} = \frac{2+x^2-2\sqrt{2}\rho x}{1+2x^2+2\sqrt{2}\rho x} \quad (4.2)$$

by introducing parameters

$$x = b'_{1/2,1}(\epsilon)/b_{1/2,3}(\epsilon) \quad \text{and} \quad \rho = \cos(\delta_3 - \delta_1).$$

Similarly  $n'$  is to be odd from the evidence v), then

$$\frac{\sigma(n+\pi^0)}{\sigma(-)} = \frac{2+y^2-2\sqrt{2}\rho y}{1+2y^2+2\sqrt{2}\rho y}, \quad (4.3)$$

where  $y = b_{1/2,1}(\epsilon)/b_{1/2,3}(\epsilon)$ .



In taking into account the evidence iii) we plot  $\sigma(0)/\sigma(+)$  in Fig. 1. This has a minimum at  $x = (1 + \sqrt{1 + 8\rho^2})/2\sqrt{2\rho^4}$ , where

$$\frac{\sigma(0)}{\sigma(+)} \Big|_{\min} = \frac{1 - 4\rho^2 + \sqrt{1 + 8\rho^2}}{2(1 + 2\rho^2 + \sqrt{1 + 8\rho^2})}. \quad (4.4)$$

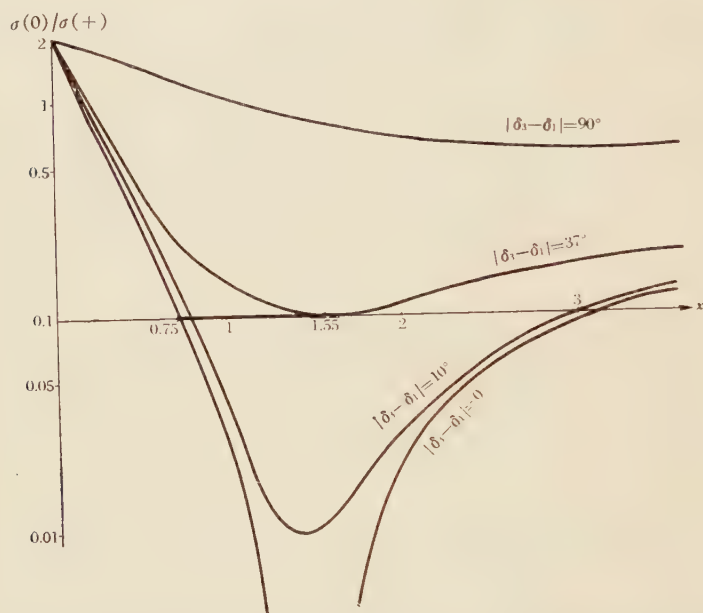


Fig. 1. The relation of  $x$  and  $\sigma(0)/\sigma(+)$  for various values of the parameter  $|\delta_3 - \delta_1|$ . All the curves lie between those of  $|\delta_3 - \delta_1| = 90^\circ$  and of  $|\delta_3 - \delta_1| = 0^\circ$ . The relation of  $y$  and  $\sigma(n + \pi^\circ)/\sigma(-)$  is also shown by this graph.

As  $\sigma(0)/\sigma(+)$  must be smaller than  $1/10$ , we find

$$|\delta_3 - \delta_1| \leq 37^\circ. \quad (4.5)$$

If the ratio is smaller than, say,  $1/50$  or  $1/100$ ,  $|\delta_3 - \delta_1|$  should be smaller than  $18^\circ$  or  $10^\circ$ . As a similar argument can be applied for  $\sigma(n + \pi^\circ)/\sigma(-)$ , the relation between  $y$  and  $\sigma(n + \pi^\circ)/\sigma(-)$  is expressed by the same curve of Fig. 1. If one could get this ratio experimentally,  $|\delta_3 - \delta_1|$  would have a severer upper limit from the evidence v).

The ratio  $\sigma(-)/\sigma(+)$  is expressed by parameters  $x$ ,  $y$  and  $\rho$  as

$$\frac{\sigma(-)}{\sigma(+)} = \frac{1 + 2y^2 + 2\sqrt{2}\rho y}{1 + 2x^2 + 2\sqrt{2}\rho x}. \quad (4.6)$$

The evidence iv) together with (4.6) yields

$$\frac{y}{x} = \frac{b_{1/2\ 1}(e) + b_{1/2\ 2}(e)}{b_{1/2\ 1}(e) - b_{1/2\ 2}(e)} = \kappa > 1, \quad (4.7)$$

where  $\bar{b}_{1/2\ 1}(e)$  and  $b_{1/2\ 8}(e)$  are the absolute magnitudes of  $\bar{b}_{1/2\ 1}(e)$  and  $b_{1/2\ 8}(e)$  given in table 3. Assuming  $\sigma(-)/\sigma(+)=4/3$  and  $\sigma(0)/\sigma(+)=1/10$ , (4.2) and (4.6) are combined to the following form,

$$8x^2 - 22\sqrt{2}\ \rho x + 19 = 0 \quad (4.2')$$

and

$$(8 - 6\kappa^2)x^2 + (8 - 6\kappa)\sqrt{2}\ \rho x + 1 = 0. \quad (4.6')$$

By eliminating the parameter  $\rho$  from these two equations we get

$$x = \left( \frac{29 - 19\kappa}{22\kappa^2 + 8\kappa - 40} \right)^{1/2} \quad (4.8)$$

As  $x$  must have its minimum at the smaller one of two intersecting points of the curve of  $|\partial_3 - \partial_1| = 0$  and  $\sigma(0)/\sigma(+)=1/10$  in Fig. 1,

$$x > 0.75. \quad (4.9)$$

When Fig. 1 is considered as the graph of  $\sigma(n+\pi^0)/\sigma(-)$ , which is assumed to be smaller than  $1/10$  from the evidence v), two solutions of  $y$  for a fixed value of  $\rho$  lie always between those of  $x$ . As  $y$  must, however, be larger than  $x$ , only the smaller one of the two solutions  $x$  fits our requirement. As a result, the maximum value of  $x$  is 1.55 where two solutions of  $x$  coincide.

$$x < 1.55. \quad (4.10)$$

From (4.8), (4.9) and (4.10) we get immediately

$$1.2 < \kappa < 1.3 \text{ or } 0.10 < b_{1/2\ 8}(e)/\bar{b}_{1/2\ 1}(e) < 0.13. \quad (4.11)$$

As  $b_{1/2\ 8}(e)$  represents the contribution from a nucleon recoil, (4.11) shows that the recoil effect is about 10% (the order of  $\mu/m$ ). This result is quite reasonable, when only the electric dipole radiation contributes to the photomeson production. In the table 4a, the values of  $\kappa$ ,  $\rho$ , etc. are given for  $\sigma(0)/\sigma(+)=1/10$  and  $\sigma(-)/\sigma(+)=4/3$ , and in the table 4b for  $\sigma(0)/\sigma(+)=1/50$  and  $\sigma(-)/\sigma(+)=4/3$ .

Table 4a

$b_{1/2\ 8}(e)/\bar{b}_{1/2\ 1}(e)$	$\kappa$	$\rho$	$ \partial_3 - \partial_1 $	v	y	$\sigma(n+\pi^0)/\sigma(-)$
0.11	1.25	0.838	33°	1.10	1.38	0.077
0.13	1.30	1	0°	0.75	0.975	0.035

Table 4b

$b_{1/2\ 8}(e)/\bar{b}_{1/2\ 1}(e)$	$\kappa$	$\rho$	$ \partial_3 - \partial_1 $	x	y	$\sigma(n+\pi^0)/\sigma(-)$
0.11	1.25	0.962	16°	1.14	1.42	0.018
0.12	1.27	0.997	4.5°	1.00	1.27	0.005

Thus we can find out the magnitude of the nucleon current to be confined within narrow limits near  $\mu/m$  times of that of the meson current. These results are consistent with the existing data.

Finally the differential cross sections for the photomeson production are given in terms of the phase shifts of the meson-nucleon scattering up to  $p$ -wave of the dipole radiation.

Table 5a

$36$ $\frac{d\sigma}{d\omega}$	$d\sigma(0)$ $d\omega$	$E$		$M1$		$M1$	
		$I=1/2$ $b'_{1/2\ 1}(e)$	$f=1/2$ $I=3/2$ $b_{1/2\ 3}(e)$	$I=1/2$ $b'_{1/2\ 1}(m)$	$f=1/2$ $I=3/2$ $b_{1/2\ 3}(m)$	$I=1/2$ $a'_{3/2\ 1}(m)$	$f=3/2$ $I=3/2$ $a_{3/2\ 3}(m)$
$E1$	$I=1/2$ $b'_{1/2\ 1}(e)$	1	$-\sqrt{2}(-)^{n_3-n_1}$ $\times \cos(\delta_3-\delta_1)$	$-\cos\theta(-)^{n_{11}-n_1}$ $\times \cos(\delta_{11}-\delta_1)$	$\sqrt{2}\cos\theta(-)^{n_{31}-n_1}$ $\times \cos(\delta_{31}-\delta_1)$	$\cos\theta$ $(-)^{n_{13}-n_1}$ $\times \cos(\delta_{13}-\delta_1)$	$-\sqrt{2}\cos\theta$ $(-)^{n_{33}-n_1}$ $\times \cos(\delta_{33}-\delta_1)$
$J=1/2$	$I=3/2$ $b_{1/2\ 3}(e)$	2		$\sqrt{2}\cos\theta(-)^{n_{11}-n_3}$ $\times \cos(\delta_{11}-\delta_3)$	$-2\cos\theta(-)^{n_{31}-n_3}$ $\times \cos(\delta_{31}-\delta_3)$	$-\sqrt{2}\cos\theta$ $(-)^{n_{13}-n_3}$ $\times \cos(\delta_{13}-\delta_3)$	$2\cos\theta(-)^{n_{33}-n_3}$ $\times \cos(\delta_{33}-\delta_3)$
$M1$	$I=1/2$ $b'_{1/2\ 1}(m)$	1		$-\sqrt{2}(-)^{n_{31}-n_{11}}$ $\times \cos(\delta_{31}-\delta_{11})$	$(1/2)(1-3\cos^2\theta)$ $\times (-)^{n_{13}-n_{11}}$ $\cos(\delta_{13}-\delta_{11})$	$-(\sqrt{2}/2)$ $(1-3\cos^2\theta)$ $\times (-)^{n_{33}-n_{11}}$ $\times \cos(\delta_{33}-\delta_{11})$	
$J=1/2$	$I=3/2$ $b_{1/2\ 3}(m)$	2			$-\sqrt{2}/2$ $(1-3\cos^2\theta)$ $\times (-)^{n_{13}-n_{31}}$ $\cos(\delta_{13}-\delta_{31})$	$(1-3\cos^2\theta)$ $\times (-)^{n_{33}-n_{31}}$ $\times \cos(\delta_{33}-\delta_{31})$	
$M1$	$I=1/2$ $a'_{3/2\ 1}(m)$				$(1/2)(5-3\cos^2\theta)$	$-(\sqrt{2}/2)$ $(5-3\cos^2\theta)$ $\times (-)^{n_{33}-n_{13}}$ $\times \cos(\delta_{33}-\delta_{13})$	
$J=3/2$	$I=3/2$ $a_{3/2\ 3}(m)$					$5-3\cos^2\theta$	

Each blank in the table is to be filled by the same one symmetrical to the diagonal figures.

Here  $n$ 's are arbitrary integers except the relation  $n_2-n_1=even$  as found in this section.  $\delta_1, \delta_3$ , etc. are the phase shifts for the meson-nucleon scattering. Similarly we can write down the expressions for  $d\sigma(-)/d\omega$  and  $d\sigma(n+\pi)/d\omega$  with the substitution  $b' \rightarrow -b$ ,  $a' \rightarrow -a$  and  $n_3-n_1=odd$ . Owing to the lack of experimental data for  $\gamma+n$  and ambiguity of  $n^*$ , it is impossible to give any remarkable result at present.

When they are fixed the analysis of experiments will be easy. Then, it may be expected to decide the superiority or inferiority of the number of phase shifts reported so far, for instance, Fermi's, Yang's and Glicksman's solutions.

Table 5b

$36 \frac{d\sigma(+)}{\hat{\kappa}^2 d\omega}$	$E \ 1$		$M \ 1$		$M \ 1$	
	$I=1/2$ $b'_{1/2 \ 1}(e)$	$I=3/2$ $b_{1/2 \ 3}(e)$	$I=1/2$ $b'_{1/2 \ 1}(m)$	$I=3/2$ $b_{1/2 \ 3}(m)$	$I=1/2$ $a'_{3/2 \ 1}(m)$	$I=3/2$ $a_{3/2 \ 3}(m)$
$E1$	$I=1/2$ $b'_{1/2 \ 1}(e)$	2	$\sqrt{2}(-)^{n_{13}-n_{11}}$ $\times \cos(\delta_{13}-\delta_{11})$	$-2\cos\theta(-)^{n_{11}-n_{13}}$ $\times \cos(\delta_{11}-\delta_{13})$	$-\sqrt{2}\cos\theta$ $(-)^{n_{13}-n_{11}}$ $\times \cos(\delta_{13}-\delta_{11})$	$\sqrt{2}\cos\theta$ $(-)^{n_{13}-n_{11}}$ $\times \cos(\delta_{13}-\delta_{11})$
$I=1/2$ $b_{1/2 \ 3}(e)$			$-\sqrt{2}\cos\theta$ $(-)^{n_{11}-n_{13}}$ $\times \cos(\delta_{11}-\delta_{13})$	$-\cos\theta(-)^{n_{13}-n_{11}}$ $\times \cos(\delta_{13}-\delta_{11})$	$\sqrt{2}\cos\theta(-)^{n_{13}-n_{11}}$ $\times \cos(\delta_{13}-\delta_{11})$	$\cos\theta(-)^{n_{13}-n_{11}}$ $\times \cos(\delta_{13}-\delta_{11})$
$M1$	$I=1/2$ $b'_{1/2 \ 1}(m)$		2	$\sqrt{2}(-)^{n_{13}-n_{11}}$ $\times \cos(\delta_{13}-\delta_{11})$	$(1-3\cos^2\theta)$ $\times (-)^{n_{13}-n_{11}}$ $\cos(\delta_{13}-\delta_{11})$	$(\sqrt{2}/2)$ $(1-3\cos^2\theta)$ $\times (-)^{n_{13}-n_{11}}$ $\cos(\delta_{13}-\delta_{11})$
$I=3/2$ $b_{1/2 \ 3}(m)$				1	$(\sqrt{2}/2)(1-3\cos^2\theta)$ $\times (-)^{n_{13}-n_{11}}$ $\cos(\delta_{13}-\delta_{11})$	$(1/2)(1-3\cos^2\theta)$ $\times (-)^{n_{13}-n_{11}}$ $\cos(\delta_{13}-\delta_{11})$
$M1$	$I=1/2$ $a'_{3/2 \ 1}(m)$				$(5-3\cos^2\theta)$	$(\sqrt{2}/2)$ $(5-3\cos^2\theta)$ $\times (-)^{n_{13}-n_{11}}$ $\cos(\delta_{13}-\delta_{11})$
$I=3/2$ $a_{3/2 \ 3}(m)$						$(1/2)(5-3\cos^2\theta)$

Each blank in the table is to be filled by the same one symmetrical to the diagonal figures.

In conclusion we would like to express our thanks to Prof. S. Hayakawa, Prof. K. Aizu and Dr. K. Nishijima for their suggestions on this problem and valuable discussions. One of us (S.M.) is indebted to the Yukawa Fellowship of Osaka University for financial aid.

### Appendix

Detailed calculation with regard to the time reversal is carried out in this appendix. The state which is induced by the time reversal from  $i$  or  $f$ , is reduced in the following way,

$$(f^\star | JHM) = (\Psi_f^\star, \Psi_{JHM}) = (U_f \Psi_f^*, \Psi_{JHM}). \quad (A.1)$$

Here we introduce a unitary matrix  $U$ , according to the definition given by Wigner,<sup>9)</sup>

$$\Psi_f^\star = U_f \Psi_f^*. \quad (A.2)$$

We bring (A.1) in

$$(f^\star | JHM) = (\Psi_f^\star, U_f^{-1} \Psi_{JHM}) = (\Psi_f^\star, U_f^{\star-1} U_f^{-1} \Psi_{JHM}^\star)^\star. \quad (\text{A} \cdot 3)$$

There appears an operator  $UU^\star$  that means twice of the time reversal,

$$\Psi^{\star\star} = UU^\star \Psi. \quad (\text{A} \cdot 4)$$

According as the state obeys Bose or Fermi statistics, there results  $\Psi^{\star\star} = \pm \Psi$ , so that

$$UU^\star = \pm 1 \equiv \varepsilon. \quad (\text{A} \cdot 5)$$

From (A.3) and (A.5), therefore, we have

$$(f^\star | JHM) = \varepsilon_f (JHM^\star | f). \quad (\text{A} \cdot 6)$$

Similarly

$$(JHM | i^\star) = \varepsilon_i (i | JHM^\star). \quad (\text{A} \cdot 7)$$

Thus

$$(f^\star | R | i^\star) = \sum_{JHM} \varepsilon_i \varepsilon_f (i | JHM^\star a) (JHb | R | JHa) (JHM^\star b | f). \quad (\text{A} \cdot 8)$$

The quantity for which we ask is obtained by taking complex conjugate of (A.8).

$$(f | R^\dagger | i) = \sum_{JHM} \varepsilon_i \varepsilon_f (f | JHM^\star b) (JHb | R | JHa)^\star (JHM^\star a | i). \quad (\text{A} \cdot 9)$$

Let us apply this general formula to the  $\gamma$ - $\pi$  process we are interested in.  $\varepsilon_i$  and  $\varepsilon_f$  are equal to  $-1$  respectively, because there is one nucleon both in the initial and the final states. If we consider, for example, the reaction amplitude for magnetic  $L$  pole radiation  $A_J^{\pi+}(m)$  given in § 3 of I,  $(f | R | i)$  is given by  $A_J^{\pi+}(m) \times (\alpha \mathcal{Y}_L^+)^*$ , etc.

$$\begin{aligned} & (\pi | J=L+1/2, L, M=3/2) (JH | R | JH) (JH M, M L | \gamma) \\ &= \sqrt{\pi} \left[ \alpha \sqrt{\frac{L+2}{2L+1}} Y_{L,1} + \beta \sqrt{\frac{L-1}{2L+1}} Y_{L,2} \right] A_J(m) \left( \sqrt{\frac{L+2}{2L+1}} \alpha \mathcal{Y}_L^+ \right)^*, \end{aligned} \quad (\text{A} \cdot 10)$$

where  $\mathcal{Y}_L^+$  shows the angular part of wave function which is specified by the  $ML$  pole and magnetic quantum number 1.

In order to obtain  $(\pi | R^\dagger | \gamma)$  following (A.9),

$$\begin{aligned} & (\pi | J=L+1/2, L, M=3/2^\star) (JH | R | JH)^\star (JH M M L^\star | \gamma) \\ &= -\sqrt{\pi} \sigma_y \left[ \alpha \sqrt{\frac{L+2}{2L+1}} Y_{L,1} + \beta \sqrt{\frac{L-1}{2L+1}} Y_{L,2} \right] A_J^\star(m) \left\{ \sigma_y \left( \sqrt{\frac{L+2}{2L+1}} \alpha \mathcal{Y}_L^+ \right)^* \right\}^\star, \end{aligned} \quad (\text{A} \cdot 11)$$

where we use  $U = \sigma_y$  in the current representation  $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ .

The minus sign in the right hand side is due to the property that the intrinsic parity of a meson against the time reversal is odd. By means of Wigner's theory<sup>9)</sup>

$$\alpha^\star = \sigma_y \alpha^\star = i\beta, \quad \beta^\star = \sigma_y \beta^\star = -i\alpha. \quad (\text{A} \cdot 12)$$

Taking advantage of the property of spherical harmonics  $Y_{l,m}$  and of polarization of photon



against the time reversal, that is

$$Y_{l,m}^{\star} = Y_{l,m}^* = (-)^m Y_{l,-m}, \quad (\text{A} \cdot 13)$$

and

$$\chi_{\pm}^{\star} = \chi_{\pm}^* = -\chi_{\mp}. \quad (\text{A} \cdot 14)$$

Our desired result can be obtained from (A·11),

$$\begin{aligned} & (\pi | J = L + 1/2, L, M = 3/2 | \gamma) (JH | R | JH)^* (JHMM | \gamma) \\ &= -\sqrt{\pi} \left[ \beta \sqrt{\frac{L+2}{2L+1}} Y_{L,-1} + \alpha \sqrt{\frac{L-1}{2L+1}} Y_{L,-2} \right] a_J^*(m) \left( \sqrt{\frac{L+2}{2L+1}} \beta \mathcal{J}_L^- \right)^*. \end{aligned} \quad (\text{A} \cdot 15)$$

For  $A_J(e)$ ,  $B_J(e)$  and  $B_J(m)$  we are able to perform our calculation in the same manner as this. As for the photomeson production (A·9) results in

$$(\pi | R^+ | \gamma) = -\sum_{J \Pi M} (\pi | JH - M) (JH | R | JH)^* (JH - M | \gamma).$$

Putting  $M = -M$ ,

$$(\pi | R^+ | \gamma) = -\sum_{J \Pi M} (\pi | JH M) (JH | R | JH)^* (JH M | \gamma). \quad (\text{A} \cdot 16)$$

Therefore

$$R_{\pi\gamma}^+ = -R_{\pi\gamma}^*. \quad (\text{A} \cdot 17)$$

It is remarkable that the relation between  $(\pi | R | \gamma)$  and  $(\pi | R^+ | \gamma)$  with respect to sign depends neither upon the order of multipole for photon nor upon the orbital angular momentum of meson as shown by Aizu.<sup>2)</sup>

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## Letters to the Editor

### The Relaxation Process in Ferromagnetic Resonance Absorption

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Concerning the problems of nuclear and paramagnetic resonance absorption, many researches have been done since Bloch<sup>1)</sup> and agreement between theory and experiment is fairly good. This agreement is owing in part to the simplicity of the mechanisms by which the line width, relaxation process, etc. occur. Namely, we can treat these problems on the following assumptions: (i) The energy spectra of these spin systems are mainly determined by the Zeeman energy, and the dipole-dipole interaction and the exchange interaction are so small that they can be treated as perturbations. That is to say, we can proceed on the picture of one body approximation. (ii) As the Zeeman energy is determined by the component of the magnetization in the direction of the external magnetic field,  $M_z$  is changed not by the spin-spin interaction but only by the spin-lattice interaction. By these reasons we can introduce the Bloch type damping forces  $(-M_x/T_2, -M_y/T_2, -(M_0 - M_z)/T_1)$ , where  $T_1$  and  $T_2$  are called respectively spin-lattice and spin-spin relaxation time. Physical meanings of these relaxation times are very clear by the above considerations.

In the case of ferromagnetic resonance absorption, however, the situation is completely different. As already pointed out by Anderson,<sup>2)</sup> the characteristic properties of ferromagnetic resonance are: (i) The main energy of the system is the exchange energy. Namely, the phenomenon of ferromagnetism is one of the most typical problems of the cooperative phenomena, which can not be treated on the basis of one body approximation.  $M_z$  is no longer a constant of motion in the spin-spin interaction, because the thermal equilibrium and the energy spectrum of the spin system are determined mainly by the exchange energy, and the Zeeman energy can change its value in the spin-spin interaction by cancelling the exchange energy. (ii) As the heat capacity of the

spin system is very large, the process of heat transfer from the spin system to the phonon system is not so important. As is well known, the exchange energy commutes with the perturbation energy by microwave field, and so we can consider roughly that the Zeeman energy is excited by microwave field and the exchange energy system behaves as if it be an external system.

At first we treat the system with the spin wave approximation. This model is applicable only to non-metallic ferromagnetism such as ferrimagnetism. In the spin wave approximation, the spin wave excited by microwave field is only that having zero wave vector, or in other words, that having the Zeeman energy in the first approximation. We use as the Hamiltonian of the spin system that of Holstein-Primakoff<sup>3)</sup> with a term of pseudo-dipolar interaction added to it. If we apply strictly the selection rule that only the spin waves having zero wave vector are excited, the effective processes for the relaxation are collisions of more than four spin waves. The relaxation time by these processes is calculated as follows:

(I) Collision of four spin waves

(4c) exchange term

$$\frac{1}{\tau} = \frac{10^6}{2 \times (2s)^2} \left( \frac{\chi T}{(\eta/3)sJ} \right)^5 \frac{T}{100} \text{ sec}^{-1} \quad (1)$$

where,  $\tau$  is the relaxation time,  $s$  the magnitude of the spin quantum number,  $\eta$  the number of the nearest neighbours,  $\chi$  the Boltzmann constant,  $T$  the absolute temperature, and  $J$  the exchange energy between nearest neighbours.

(4d) dipole term

$$1/\tau = 1.4 \times 10^{11} (10^{-8}/a)^6 100/T \cdot (\chi T/(\eta/3)sJ)^3 \text{ sec}^{-1} \quad (2)$$

where  $a$  is the distance between the nearest neighbours.

(4f) pseudo-dipole term

$$1/\tau = 10^{11}/2 \times (2s)^2 \cdot (D/J)^2 (\chi T/(\eta/3)sJ)^3 \text{ sec}^{-1} (3)$$

where  $D$  is the constant of pseudo dipole coupling, and by van Vleck<sup>4)</sup> the order of magnitude of the ratio  $(D/J)$  is about  $10^{-1} \sim 10^{-2}$  in room temperatures. The inverse relaxation times by these three terms are all less than  $10^6 \text{ sec}^{-1}$  and their temperature dependence is more than square of the absolute temperature. On the other hand, according to the experimental results, the magnitude of  $1/\tau$  is about

of the order of  $10^8 \sim 10^9 \text{ sec}^{-1}$  in room temperatures and proportional to the absolute temperature.

When the deviation of the magnetization from the equilibrium value becomes large, collisions of the five spin waves such as  $a_0 a_0 a_0 a_k^* a_{-k}^*$  become important. The results are as follows:

(II) Five spin wave collisions

(5d) dipole term

$$1/\tau = 2.5 \times 10^{10} (2s)^3 (10^{-8} a)^6 (100/f')^{1/2} \times (\chi T / (\eta/3) s f)^{3/2} (M_z/M)^2 \quad (4)$$

where  $M_z$  is the  $Z$  component of the magnetization  $M$ . Values of the other terms are fairly smaller than that of the dipole term. The magnitude of this term is also too small and  $M_z$  dependency of the relaxation time has not been found experimentally so far.

(III) Three spin wave collision

So far, we applied the selection rule strictly, but if we now consider that the spin waves having small but non-zero wave vector are also excited by microwave field, the three spin wave collision becomes important. The basis of the above assumption is as follows: The half widths of the resonance absorption in ferrites are almost independent of temperature, and greater than inverse relaxation times of  $M_z$ . These facts suggest that the mechanism causing the half width is different from that of the relaxation process. According to the facts that the line width is dependent on the sample and the process of cooling, it seems likely to me that the line width in ferrite is determined mainly by the lattice imperfection, by which the mixing between the spin waves having non-zero wave vectors and zero wave vector is caused. The largest wave vector to be mixed may have nearly the same kinetic energy as that of the line half width. According to Damon<sup>5)</sup>, we obtain  $k_{\max}$  as follows:

$$(\eta/3) s f a^2 k_{\max} = 2\beta \Delta H \sim 1.3 \times 10^{-8} \text{ erg.} \quad (\Delta H = 75 \text{ oe}) \quad (5)$$

$$a k_{\max} \sim 3 \times 10^{-3}. \quad (6)$$

On the other hand, for the smallest wave vector  $k_{\min}$  in the three spin wave collision, we have

$$(\eta/3) s f a^2 k_{\min}^2 = 1/4 \cdot (2\beta H)^2 / \chi T \sim 4.5 \times 10^{-20} 100/T \cdot \text{erg.} \quad (7)$$

$$a k_{\min} \sim 4 \times 10^{-4} \text{ in room temperatures.} \quad (8)$$

Therefore this process may take place even in sufficiently low temperatures. The results are:

(3d) dipole term

$$1/\tau = 0.9 \times 10^{10} (2s)^3 (10^{-8} a)^6 (\chi T / (\eta/3) s f) \times (10^3 a k_{\max}) \text{ sec}^{-1} \quad (9)$$

(3f) pseudo-dipole term

$$1/\tau = 4 \times 10^{11} 1/(2s) (D/f)^2 (\chi T / (\eta/3) s f)^3 \times (10^3 a k_{\max})^{-1} \text{ sec}^{-1}. \quad (10)$$

The pseudo-dipole term is smaller in ordinary case than the dipole term because of the factor  $(D/f)^2$ . The dipole term gives in room temperature the order of magnitude  $10^8 \text{ sec}^{-1}$ , which value agrees fairly well with the experimental value, and the temperature dependence is also satisfactory.

Our model of spin system is fairly simple and does not correspond strictly to that of ferrites; even in this case, however, essential change may not occur.

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## The Relaxation Process in Ferromagnetic Resonance Absorption

— The Effect of the Conduction Electron —

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According to the renewed estimation of Bloembergen-Wang,<sup>1)</sup> the relaxation time of non-metallic ferromagnetics such as ferrites agrees fairly well with our earlier results both in magnitude and temperature dependence. The relaxation time of metallic ferromagnetics such as Ni, however, is larger than that of ferrite and seems to be almost independent of temperature. This fact seems to imply that in metallic ferromagnetism, other relaxation processes may exist which are much stronger than that of collision between spin waves. As suggested by Bloembergen<sup>2)</sup> the interaction between  $d$ -electron and conduction electron would be the most important of such processes.

There are three types of such a  $c$ - $d$  interaction.

(1) The exchange interaction between  $c$ -electron and  $d$ -electron

This interaction can not be written in a simple form such as Dirac's vector model. But it is easily seen that the Hamiltonian of this interaction commutes with the magnetization  $\mathbf{M}$ , and consequently only exchanges the direction of spins of  $d$ -electron and  $c$ -electron. That is to say, it is the largest interaction, but has no direct relation to such relaxation processes that change the magnitude of  $M_z$ . As we shall see later, however, this indirect process also seems to be important to the relaxation process.

(2) The spin-dipole interaction between  $d$ - and  $c$ -electrons

This interaction is formed of two terms. The one part commutes with  $\mathbf{M}$  and may have nearly the same form as the exchange interaction, but is much smaller in magnitude. Therefore this term is not important. The other part does not commute with  $\mathbf{M}$  and has nearly the same form as the spin-orbit interaction which will be discussed immediately. But, if we treat  $d$ -electrons with spin wave approximation, the spin waves excited by microwave field have very small wave vector  $K$ . Therefore this interaction does not become important compared with the spin-orbit interaction since, as we can see easily, they are in the ratio  $K/k$ , where  $k$  is the wave vector of  $c$ -electron at the Fermi level.

(3) The interaction between the spin magnetic moment of  $d$ -electron and the orbital magnetic moment of  $c$ -electron.

The hamiltonian of this interaction can be written as follows:

$$H_i = \sum_i \sum_l 4\mu^2 \frac{m}{m^*} \frac{(\mathbf{S}_i \cdot (\mathbf{r}_i - \mathbf{R}_l)) \cdot \mathbf{P}_{l,i}}{|\mathbf{r}_i - \mathbf{R}_l|^3}, \quad (1)$$

where, suffix  $i$  refers to the  $c$ -electron,  $\mathbf{R}_l$  is the position of  $d$ -electron,  $\mathbf{P}_i$  the momentum of  $i$ -th electron, and  $\beta$  the Bohr magneton.

For the further treatment, we assume that  $d$ -electron is distributed equally to each lattice position in a ratio  $n_d/N$ , where  $n_d$  is the number of  $d$ -electrons and  $N$  is the number of lattice points per unit volume. We think that this model is not so wrong, and we treat this system with spin wave approximation, considering further  $c$ -electron as a free electron having a certain effective mass.

By this assumption, the eq. (1) becomes as follows:

$$\mathcal{H}_i = \sum_K \sum_K [A_K K^+ b_{K+K} + K^+ b_{K+K} a_{K'}^* a_{K+K'}]$$

$$+ A_K K^+ b_{K+K} K^+ b_{K+K} a_{K'}^* a_{K+K'}] + \sum_K \sum_K \sum_K$$

$$B_K K b_{K+K} + K^+ b_{K+K} a_{K'}^* a_{K+K'}, \quad (2)$$

where

$$A_K K^{\pm} = i8\pi (2s)^{1/2} N^{1/2} (\beta^2/V) \cdot (m/m^*) \cdot [K \times K]^{\pm} / K^2, \quad (3)$$

$$B_K K = i16\pi (\beta^2/V) \cdot (m/m^*) \cdot [K \times K] z / K^2. \quad (4)$$

The first term in the eq. (2) is the most important, but if we apply strictly the selection rule that only spin waves having zero wave vector are excited by microwave field, this process does not concern the relaxation process. As mentioned in the case of non metallic substances, however, this restriction seems to be too strict. The minimum wave vector concerning this process is of the order of  $10^3 \text{ cm}^{-1}$ . On the other hand, if we take the breadth of the region of mixing of spin waves having non-zero wave vectors with that of zero wave vector as of the order of magnitude of the absorption line width, the maximum wave vector of mixing  $K_{\text{max}}$  is about  $10^5 \text{ cm}^{-1}$ . In this respect, we think that this process seems to concern the relaxation process.

The result of calculation is as follows;

$$\frac{1}{\tau_K} = 1.3 \times 10^{10} (2s) \left( \frac{N}{V} 10^{-23} \right) (H_0 10^{-3})^2 \frac{(k 10^{-8})^2}{(K 10^{-5})^3} \times \left[ 1 - \left( \frac{K_{\text{min}}}{K} \right)^2 \right], \quad (5)$$

where

$$K_{\text{min}} = m^* g \beta H_0 / \hbar^2 k \sim 10^3 m^* / m \text{ cm}^{-1}. \quad (6)$$

If we take the value of  $K$  as of the order of  $10^5 \text{ cm}^{-1}$ , the eq. (5) gives the order of  $10^{10} \text{ sec}^{-1}$ . This value is larger than the value of our earlier paper by one or two in the order of magnitude, and is independent of temperature. These results agree fairly well with the experimental results both in the order of magnitude and the temperature dependence.

As to the causes of mixing of spin waves having non-zero wave vector, the simple explanation, which seems to be applicable to the cases of ferrites, does not seem to be applicable, because, although experimental data are not sufficient, the mechanism of the line width seems to be the same as that of the relaxation process of  $M_z$ . It seems likely that the second order perturbation by the exchange interaction may be the cause of this mixing. The study of this process is now in progress.



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## The Method of Generating Function Applied to Radiative and Non- Radiative Transitions of a Trapped Electron in a Crystal

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The radiative and non-radiative transitions of a trapped electron in a crystal are discussed with the aid of the generating function method.<sup>1)2)</sup> Assuming a linear form for the electron-lattice interaction Hamiltonian, one can calculate the generating functions of these two processes for a general electron-lattice system, that is: not only the equilibrium lattice positions but also the vibrational frequencies and the axes of the normal modes are different for the two electronic states between which the transition takes place.

The generating function for radiative transition, from which one can calculate the shape of the absorption band by the inverse formula for Laplace transformation, factorizes into three functions, the first of which represents the effect of the difference in equilibrium positions, the second due to the difference in frequency tensors ( $N$ -dimensional, in general, where  $N$  is the number of modes) and the third due to the variation of the transition dipole moment with lattice configuration. All these effects contribute additively to the moments of the absorption curve such as peak shift, broadening and asymmetry, and one can estimate the order of magnitude of each contribution for actual examples. The third factor is important when one discusses a transition which is forbidden but for the lattice vibration.

The generating function for non-radiative transition also turns out to be a product of three parts, the two of which are identical with the first two factors

for the radiative transition stated above. We can derive, in a general form, the low and high temperature features<sup>1)</sup> of the transition probability, that is: the temperature dependences of the probability are given by  $\exp(-\varepsilon_0/kT)$  and  $\exp(-\varepsilon^*/kT)$  for the two limiting cases, respectively, where  $\varepsilon_0$  is the energy difference of the minimum points of the two adiabatic potentials, while  $\varepsilon^*$  is the minimum point of the intersection of the two adiabatic potentials. This means that at low temperatures the transition occurs primarily as a tunneling effect while at high temperatures the dominant process is the jumps over the activated states. The result of Huang and Rhys<sup>3)</sup> is obtained as a special case. The method used here, however, permits one to calculate the thermal ionization probability of a trapped electron or hole in non-polar crystals such as silicon and germanium. The result at high temperatures is written in the form

$$W_i = 2V \sqrt{\frac{m^* c^2}{\hbar}} \gamma^2 \left( \frac{kT}{\varepsilon_0} \right)^2 \exp(-\varepsilon^*/kT),$$

where  $\varepsilon_0$  is the depth of trapping,  $m^*$  is the effective mass of the electron or the hole,  $c$  is the longitudinal sound velocity in the crystal and  $\gamma$  is a dimensionless constant which contains the well-known electron lattice interaction constant  $C$ . The activation energy  $\varepsilon^*$  is given by

$$\varepsilon^* = (1 + \gamma/2)^2 / 2\gamma \cdot \varepsilon_0 \geq \varepsilon_0.$$

For the crystals cited above  $\gamma$  ranges from 0.1 to 0.4. The cross-section  $\sigma_i$  for retrapping process is also calculable as we can relate it with  $W_i$  on the basis of detailed balance theorem. The numerical values of the ionization rate and the cross-section are very sensitive to  $\gamma$  which depends on the effective mass of the electrons or holes and on the interaction constant  $C$ . Thus it is impossible at present to give reliable values for the actual crystals, but examples of  $\sigma_i$  at 300°K in germanium are:  $\sigma_i$  (electron) =  $2 \times 10^{-16} \text{cm}^2$ ,  $\sigma_i$  (hole) =  $1 \times 10^{-15} \text{cm}^2$ .

If we confine ourselves *ab initio* to high temperature region, it is possible to discuss radiative and non-radiative transitions for a most general form of adiabatic potentials which are not necessarily harmonic. Thus we can derive, for instance, the well-known formula for the rate of transition

$$W = \kappa \frac{kT}{\hbar} \exp(-F^*/kT)$$

with appropriate definitions of  $F^*$  (the free energy of activation) and  $\kappa$  (transmission coefficient). It is a simple matter to derive the Franck-Condon



principle for radiative transition.

We have also discussed the broadening of the absorption band due to the degeneracy of the excited states, and the effect of resonance repulsion of the two adiabatic potentials near the activated states on the nonradiative transition. In the latter problem the transmission coefficient turns out to be strongly temperature-dependent. Detailed accounts will be given in a later issue of this journal.

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## Meson Theoretical Potentials in Triplet Odd State

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Nuclear forces derived from the symmetrical pseudoscalar meson theory have succeeded in explaining many experimental data of two nucleon system<sup>1)</sup>. However an important problem is left untouched concerning the triplet averaged *P*-wave phase shift at low energies. It is possible in this problem to compare theoretical results with experimental data quantitatively without uncertainty which the meson theoretical nuclear force problems often go with. The nuclear potentials derived meson-theoretically become unreliable when two nucleons come nearer than about 0.6 times the meson Compton wave length ( $1.40 \times 10^{-13}$  cm). However, effects due to this unreliable part of the potentials are surely much reduced as the energy is very low and the centrifugal force exists in this problem.

Presence of non-central forces separates the triplet *P*-wave phase shift into three,  $\delta_{J^L}$ ,  $J=0, 1, 2$  being the total angular momentum. However, at low energies *p-p* scattering experiments tell one only the averaged *P*-wave phase shift  $\Delta \equiv (1/9)(\delta_0^1 + 3\delta_1^1 + 5\delta_2^1)$ . Most of the experiments up to 5 Mev show either negative  $\Delta$  or too larger experimental errors to say something definite. The latest and the most precise one is the Wisconsin data<sup>2)</sup>, of which the only point at 4.203 Mev is not quite

consistent with the other points due to experimental difficulties, as was stated by the authors. The variation of  $\Delta$  with energy at the other 6 points from 1.855 to 3.899 Mev is consistent, so it would be sufficient to compare theoretical results with  $\Delta_{\text{exp}} = -0.109 \pm 0.020^\circ$  at 3.899 Mev.

It is expected that the representative phenomenological *p-p* potentials proposed so far can never reproduce negative  $\Delta$ . At low energies it is estimated that  $\delta_{J^L}(\text{exact}) > \delta_{J^L}(\text{Born})$  mathematically for monotonous potentials. As far as we confine ourselves to interactions involving no higher powers of the relative velocity than the first<sup>3)</sup>, the only allowed noncentral interactions for *p-p* scattering are tensor type or **L**·**S** type. For both types the effects of the non-central interactions to  $\Delta(\text{Born})$  cancel out as a whole, so that

$$\Delta(\text{exact}) > \Delta(\text{Born}) \propto V(\text{central potential}).$$

Thus the averaged effect to  $\Delta$  due to the non-central forces is usually attractive. As the triplet central potentials of Christian-Noyes<sup>4)</sup> and Jastrow<sup>5)</sup> are vanishing,  $\Delta(\text{Born})=0$ , so that  $\Delta > 0$ . This was actually calculated for the latter case by S. O-numa<sup>6)</sup>. The Case-Pais potentials<sup>7)</sup> are probably unfavourable as its triplet central force (nearly the same with the Christian-Hart potentials<sup>8)</sup>) is too weak, while  $\Delta_{\text{exp}}$  requires for the central Yukawa well the exchange character of  $(0.38 + 0.62 P_M)^{9)}$ . Presence of hard core is of no use. A hard core of the radius 0.4 times the meson Compton wave length contributes to  $\Delta$  only of the order  $-0.033^\circ$ .

It is to be noted that the triplet odd central force derived from the symmetrical pseudoscalar meson theory by both TMO<sup>10)</sup> and BW<sup>11)</sup> treatments is repulsive in the region  $x > 1.5$ ,  $x$  being the internucleon distance in the unit of the meson Compton wave length, as can be seen from Fig. 1. Emphasis must be laid on the fact that this repulsive force is due only to the 2nd order contribution and is not affected by the 4th order terms. Moreover, this central force is not so strong as those of the other states owing to the dependence of the 2nd order central force on  $(\mathbf{r}_1 \cdot \mathbf{r}_2)(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)$ . This condition has been shown to be very favourable to explain the intermediate *p-p* and *n-p* scattering.<sup>12)</sup>

Our purpose is then to see whether this 2nd order slight repulsive force in the triplet odd state can reproduce negative  $\Delta$ , overwhelming the contributions from the 4th order terms and non-central terms which are effective to make  $\Delta$  larger mathematically.

As the first step, we take into account only the 2nd order central and tensor forces, cutting the potentials all to be zero in the region  $x \leq 1$ . The purpose of this prescription is twofold. One is to study the effect of the 2nd order force alone which is expected to be responsible for negative  $\Delta$ . The other is that, unfortunately, the 4th order potentials of TMO and BW treatments are different from each other in the region  $x \leq 1$  in the triplet odd state as can be seen from Fig. 1, though this difference is not so large as in the triplet even state.  $\delta_{r^1}$ 's are calculated numerically in this case as the Born approximation is not suitable for such delicate problems. For the pseudo-vector coupling constant  $g^2/4\pi=0.10$ , the results are:  $\delta_0^1=1.502^\circ$ ,  $\delta_1^1=-0.694^\circ$ ,  $\delta_2^1=-0.280^\circ$  and the resultant  $\Delta=-0.207^\circ$ . Actually, negative  $\Delta$  is obtained.

This result is corrected for various cases, i.e., for the variation of  $g^2/4\pi$  or by the contribution from the 4th order potentials in the region  $x > 1$  and that from the 2nd plus 4th order potentials in the region  $x \leq 1$ . The resultant  $\Delta$ 's are tabulated in the table. The Born approximation is used to calculate these corrections. They are small compared with the 2nd order contribution. Results for FST potentials<sup>13)</sup> are expected to lie between TMO and BW results. The assumed presence of a hard core with radius 0.3 would contribute  $-0.015^\circ$  to  $\Delta$ . We see from this table that the symmetrical pseudoscalar meson-theoretical potentials can actually reproduce  $\Delta_{\text{exp}}$  for  $g^2/4\pi=0.08\sim 0.06$  and that this is mainly due to the 2nd order central potential that is repulsive far from the origin. This  $g^2/4\pi$  is comparable with that predicted from the singlet even state data, i.e.  $0.08\sim 0.10$ . We feel all these facts to be an experimental support of the meson theory of nuclear forces. Note that, though meson-theoretical potentials can reproduce negative  $\Delta$  at low energies, they will give

positive  $\Delta$  at intermediate energies owing to the 4th order central terms that are large and attractive in the region  $x < 1$  and to the tensor force effect.

We wish to express our thanks to Prof. S. Takagi for his encouragement.

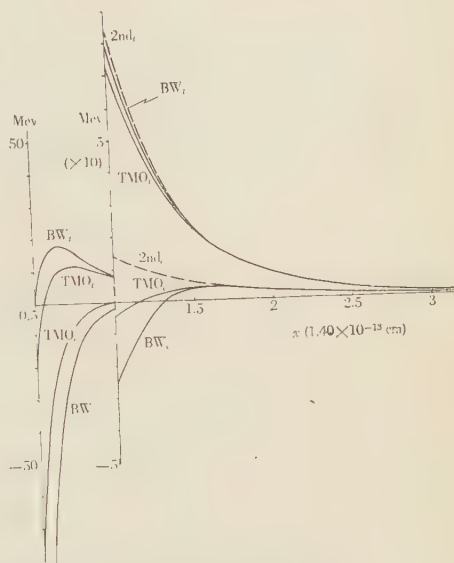


Fig. 1

Symmetrical pseudoscalar meson theoretical potentials of the triplet odd state of

TMO with  $g^2/4\pi=0.08$ ,

BW with equivalent  $G^2/4\pi(\mu/2M)^2=0.08$ ,  
the 2nd order term alone with  $g^2/4\pi=0.08$ .

The suffixes *c* and *t* mean central and tensor part respectively.

$g^2/4\pi$	0.10		0.08		0.06	
	TMO	BW	TMO	BW	TMO	BW
Case A	-0.19	-0.17	-0.16	-0.14	-0.12	-0.11
Case B	-0.18	-0.15	-0.15	-0.13	-0.12	-0.10
Case C	-0.16	-0.11	-0.14	-0.10	-0.11	-0.09

Table.  $\Delta$  in degrees. Case A, B and C are  $\Delta$ 's due to the 2nd plus 4th order TMO or BW potentials outside  $x=1.0$ , 0.6 and 0.3 respectively, the inside potentials being cut to be zero.  $\Delta_{\text{exp}}=-0.109 \pm 0.020^\circ$ ,

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## A Note on Superexchange Interaction

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In the previous paper<sup>1)</sup> we have developed a theory of the superexchange interaction of MnO by the method of Heitler, London and Löwdin. The energy difference between the ordered antiparallel spin state and the parallel spin state was found to be

$$E(f) - E(a) = 6NH_{ga}^2 / (H_{gg} - H_{aa}) \cdot Q, \quad (1)$$

where  $N$  is the number of ion pairs in the unit volume,  $H_{gg}$  is the energy of the totally ionic state (ground state),  $H_{aa}$  is the energy of the excited state, where one pair of  $\text{Mn}^{2+}-\text{O}^-$  ions exists,  $H_{ga}$  is the transition matrix element between the ground state and the excited state and  $Q$  is the net energy difference between the two spin arrangements in excited state. We see from eq. (1) that, if  $Q$  is positive, the ordered antiparallel spin state is more stable than the parallel spin state at low temperature. Thus, in our theory the decision of the sign of  $Q$  is very important. Using Löwdin's<sup>2)</sup> notation  $Q$  is approximately expressed as

$$Q = B(2f0|3d0) + C(2f0|3d), \quad (2)$$

where  $B$  is the S-energy between a  $(2f0)$  electron of an  $\text{O}^-$  ion and a  $(3d0)$  electron of a adjacent  $\text{Mn}^{2+}$  ion and  $C$  is the exchange energy between a  $(2f0)$

electron of an  $\text{O}^-$  ion and  $(3d)^5$  electrons of a adjacent  $\text{Mn}^{2+}$  ion. In the previous paper, however, we left quantitative consideration of the quantities  $Q$ ,  $H_{ga}$  and  $(H_{gg} - H_{aa})$  untouched. In this note we shall compute them approximately, in order to get the more or less quantitative information about the mechanism of the superexchange interaction developed by Kramers<sup>3)</sup> and Anderson.<sup>4)</sup>

In fact, the rigorous computation of  $Q$  seems to be a rather complicated problem, because we must have many electron configurations in order to get a good representation of the state of an  $\text{O}^-$  ion within a MnO crystal. We, however, want to know only the order of magnitude of  $Q$  at present. Thus for our semi-quantitative purpose it may be allowed to use the Hartree-Fock wave function of  $(2f)^5$  state of the  $\text{O}^-$  ion and an approximate  $(3d)^5$  wave function of  $\text{Mn}^{2+}$  ion, which is determined by the interpolation from the Hartree-Fock wave functions of  $\text{Cu}^+$  and  $\text{Cr}^{2+}$ .

$$P_{3d}(r) = \{57.44 \exp(-4.835r) + 2.366 \exp(-1.871r)\} r^3. \quad (3)$$

This function gives the better agreement with the experimental charge distribution<sup>5)</sup> than Dancoff's exact H-F wave function of free ion. When the wave functions are given, the computation of  $B$  and  $C$  is performed straightforwardly by Löwdin's method and the results are as follows (we use the atomic unit):

$$S(2f0|3d0) = 0.11, \quad B = 0.021,$$

$$C = -0.016 \quad \text{and} \quad Q = 0.005.$$

Next,  $H_{ga}$  is approximately given by

$$H_{ga} = S^{-2} \int \psi_{2p}(R : \text{O}^-) \left( V_{\text{Mn}}(r) + \frac{2\alpha_M - 1}{a} + V_0'(r) \right) \psi_{3d0}(r : \text{Mn}) d\tau, \quad (4)$$

where  $V_{\text{Mn}}(r)$  is the potential of a  $\text{Mn}^{2+}$  ion,  $\alpha_M$  is the Madelung constant and  $2a$  is the lattice constant and the last term gives the correction due to the finite size of the oxygen ion, which is expressed as:

$$V_0'(r) = 6 \times \frac{1}{2\pi r} \int_{a-r}^{a+r} \left\{ V_0(R) - \left( -\frac{2}{R} \right) \right\} R dR. \quad (5)$$

( $V_0(R)$  = the Hartree field of the oxygen ion.)

Using the above mentioned wave functions we find  $H_{ga} \sim 0.0075$ . Finally  $(H_{gg} - H_{aa})$  is roughly

estimated by the following expression :

$$\Delta E = (H_{gg} - H_{aa}) = \frac{(4\alpha_M - 1)e^2}{a} + E - I + E_P, \quad (6)$$

where  $I$  is the ionization energy of  $\text{Mn}^{++}$  ion (about 16ev.?) and  $E$  is the affinity of the  $\text{O}^{--}$  ion (about -9ev.?) and  $E_P$  is the polarization energy, which is estimated by Klemm's<sup>5)</sup> method (about -5ev.). Using these values we get the value 9ev for  $\Delta E$ . In the case of alkali-halide crystals the values of  $\Delta E$  estimated from (6) are nearly equal to the observed values (the difference is about one or two ev.). However, in the case of  $\text{MnO}$  the value of  $\Delta E$  is supposed to be two or three ev. Thus we see that eq. (6) fails in giving the right order of  $\Delta E$ . Using the value  $\Delta E = 3\text{ev.}$  we find :

$$\{E(f) - E(n)\} / N = 6 \times (0.0006) \text{ev.}$$

Thus we find that our computation gives the right order of magnitude for the superexchange interaction of  $\text{MnO}$  crystals. Although our computation has only a semi-quantitative nature owing to many approximations, it may be allowed to say that our computation gives some further support to the Kramers-Anderson's theory. Here we notice that for quantitative consideration our method is much more convenient than the usual spin operator method. Finally the author wishes to express his thanks to Prof. T. Muto and Prof. R. Kubo and Dr. P. W. Anderson for helpful discussions.

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## A Remark on Convergence Factor\*

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We want to use the following convergence factor

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including the idea of the cohesive meson theory<sup>1)</sup> in order to calculate the finite values of the Dirac particles' self energy:

$$C = C((P-k)^2 - m^2) - g \cdot C_0(k^2), \quad (1)$$

where

$$C'((P-k)^2 - m^2) = -\lambda^2((P-k)^2 - m^2 - \lambda^2)^{-1},$$

$$C_0(k^2) = -\lambda^2 \exp(-(\mu/\lambda)^2) (k^2 - \lambda^2 \exp(-(\mu/\lambda)^2))^{-1},$$

$m$ : mass of Dirac particle,  $\mu$ : mass of  $C$ -meson,  
 $g$ : constant, (2)

and

$$(1-g)^{-1}[C]_{\lambda \rightarrow \infty} = 1. \quad (3)$$

The 2nd order self energy integrals of an electron in the energy momentum space representation is given by

$$\delta m = e^2 / \pi i \int \gamma_\mu (P-k-m)^{-1} \gamma_\mu k^{-2} d^4 k (1-g)^{-1} \cdot [C]_{\lambda \rightarrow \infty} \quad (4)$$

$$= (\delta m_1 - \delta m_2) (1-g)^{-1},$$

where

$$\delta m_1 = [e^2 / \pi i] \int \gamma_\mu (P-k-m)^{-1} \gamma_\mu k^{-2} d^4 k$$

$$C((P-k)^2 - m^2) |_{\lambda \rightarrow \infty},$$

$$\delta m_2 = [e^2 / \pi i] \int \gamma_\mu (P-k-m)^{-1} \gamma_\mu k^{-2} d^4 k \cdot C_0(k^2) |_{\lambda \rightarrow \infty}. \quad (5)$$

Owing to the formulae

$$\left. \begin{aligned} & ((P-k)^2 - m^2)^{-1} C((P-k)^2 - m^2) \\ &= - \int_0^{\lambda^2} ((P-k)^2 - m^2 - L)^{-2} dL, \\ & k^{-2} C_0(k^2) = - \int_0^{\lambda^2 \exp(-(\mu/\lambda)^2)} (k^2 - L)^{-2} dL, \\ & a^{-2} b^{-1} = \int_0^1 2x dx (ax + b(1-x))^{-2}, \end{aligned} \right\} \quad (6)$$

we have

$$\delta m_1 = [e^2 / \pi i] \int \gamma_\mu (P-k+m) \gamma_\mu d^4 k ((P-k)^2 - m^2) \cdot C((P-k)^2 - m^2) k^{-2} |_{\lambda \rightarrow \infty} \quad (7)$$

$$= [e^2 / \pi i] \int \gamma_\mu (P-k+m) \gamma_\mu d^4 k (k^2 - 2Pk - L)^{-2} \cdot (k^2)^{-1} |_{\lambda \rightarrow \infty},$$



On the other hand,

$$(8i) \int (1 : k_\sigma) d^4k (k^2 - 2Pk - L)^{-2} (k^2)^{-1} \\ = \int_0^1 (1 : xP_\sigma) 2dx \log(\lambda^2/m^2x) \quad (8) \\ \text{where } \lambda \gg m$$

$$\delta m_1 = 3e^2/2\pi [\log \lambda/m] \cdot m + 3e^2/2\pi \cdot m [5/12]. \quad (9) \\ (\text{log-term}) \quad (\text{finite-term})$$

The value of  $\delta m_1$  is the same as that of V. Weisskopf<sup>2)</sup> and J. Schwinger,<sup>3)</sup> that is to say,

$$V. Weisskopf: 3e^2/2\pi [\log \lambda/m] \cdot m \\ + 3e^2/2\pi \cdot m [\log(1 + \sqrt{1 + (\mu/\lambda)^2}) - (1/6)]_{\lambda \rightarrow \infty} \quad (10)$$

$$J. Schwinger: 3e^2/2\pi [\log \lambda/m] \cdot m \\ + 3e^2/2\pi \cdot m [5/6 - \log \gamma], \quad \gamma = 1.781. \quad (11)$$

We calculate  $\delta m_2$  as well and get the following result:

$$\delta m_2 = 3e^2/2\pi [\log \lambda/m - \frac{1}{2} \cdot (\mu/\lambda)^2] \cdot m \cdot g \\ + 3e^2/2\pi \cdot m \cdot g \cdot (1/4). \quad (12)$$

If  $g=1$ , this is entirely the same expression as the result obtained by Feynman.<sup>4)</sup> According to the calculation by M. Hamaguchi,<sup>5)</sup>  $\delta m$  includes the logarithmically divergent part and the 2nd finite term does not appear. In my calculation  $\delta m$  is as follows,

$$\delta m = (\delta m_1 - \delta m_2) (1-g)^{-1} = 3e^2/2\pi \cdot m \cdot (5-3g)/12 \\ \cdot (1-g)^{-1}. \quad (13)$$

$3e^2/2\pi \cdot m \cdot \frac{5-3g}{12(1-g)}$  gives for the change in mass of electron. If  $g=1$ ,  $g=0$ ,  $g=2$ ,  $g=3$ , etc. we get the values of the change in mass:  $e^2/2\pi \cdot m$ ,  $e^2/8\pi \cdot 5m$ ,  $e^2/8\pi \cdot m$ ,  $e^2/4\pi \cdot m$ , respectively.

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## On the Hydrodynamical Representation of Non-Relativistic Spinor Equation

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The hydrodynamical representation of Schrödinger equation formerly suggested by Madelung<sup>1)</sup> has been developed in our previous paper,<sup>2)</sup> considering its bearings from various points of view. The similar method was taken up by Schönberg in a recent paper<sup>3)</sup> where he briefly quoted an unpublished investigation of Bohm, Schiller, and Tiomno which attempts to represent a quantum-mechanical motion by a *rotational* hydrodynamical field in the case of non-relativistic spinning particle. We have incidentally touched upon the latter problem in a short note<sup>4)</sup> which was intended to give some remarks to Schönberg's article. In the present note we would like to state briefly our treatment of the problem of spin separately in a more sufficient form, though it would be expected to be quite similar to the work of Bohm et al.

The quantum-mechanical motion of a non-relativistic spinning particle is described by the Schrödinger equation

$$((\hbar/i)\partial/\partial t + H)\psi = 0, \quad (1)$$

with the two-component spinor function  $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ , and the Hamiltonian

$$H = 1/2m \cdot (\mathbf{p} - e\mathbf{A}/c)^2 + eA_0 - (e/mc)\mathbf{s} \cdot \mathbf{H}, \quad (2)$$

as we consider, for simplicity, in the lowest approximation, where  $(\mathbf{A}, A_0)$  is the external electromagnetic potential,  $\mathbf{H} = \text{curl} \mathbf{A}$  the magnetic field strength, and  $\mathbf{s} = (\hbar/2)\boldsymbol{\sigma}$ ,  $\boldsymbol{\sigma}$  being the Pauli matrices.

First we introduce a scalar

$$P = \psi^* \psi \quad (3)$$

and a vector

$$\mathbf{v} = \frac{\hbar}{2mi} \frac{\psi^* \nabla \psi - \nabla \psi^* \cdot \psi}{\psi^* \psi} - \frac{e}{mc} \mathbf{A}, \quad (4)$$

which satisfy the continuity equation

$$\dot{P} + \text{div}(P\mathbf{v}) = 0, \quad (5)$$



and so can be regarded respectively as a density and a velocity functions, the latter being a general rotational field due to  $\psi$  having two components. Further, we introduce a vector  $\mathbf{S}$  by

$$\mathbf{S} = \hbar/2 \cdot \psi^* \boldsymbol{\sigma} \psi / \psi^* \psi \quad (6)$$

which has a constant length:

$$S^2 = \hbar^2/4, \quad (7)$$

and can be regarded, in our picture, as the 'spin field', i.e. the intrinsic angular momentum distributed in space. It is not independent of the flow velocity  $\mathbf{v}$  but is connected with the rotational part of the latter by

$$\begin{aligned} \text{curl}_{j \times k} \mathbf{v} &\equiv \partial_j v_k - \partial_k v_j \\ &= (1/m S_1) (\partial_j S_2 \partial_k S_3 - \partial_k S_2 \partial_j S_3) - (e/mc) H_{j \times k}, \end{aligned} \quad (8)$$

with  $\partial_j \equiv \partial/\partial x_j$ . The first term in the right side of (8) can be rewritten in a more symmetrical form

$$\frac{4}{m \hbar^2} \begin{vmatrix} S_1 & S_2 & S_3 \\ \partial_j S_1 & \partial_j S_2 & \partial_j S_3 \\ \partial_k S_1 & \partial_k S_2 & \partial_k S_3 \end{vmatrix},$$

whose divergence vanishes identically on account of (7), as it should.

We can now find, after some calculations using (1), the Euler equation of our velocity function to be

$$\begin{aligned} m dv_i/dt &= K_i + e/mc \cdot \sum_k \partial_i H_k \cdot S_k \\ &+ (\hbar^2/2m) \cdot \partial_i (\Delta \sqrt{P}/\sqrt{P}) \\ &- (1/mP) \sum_{k,l} \partial_k (P \partial_i S_l \cdot \partial_k S_l), \end{aligned} \quad (9)$$

where  $d/dt \equiv \partial/\partial t + (\mathbf{v} \cdot \nabla)$  means the substantial derivative. In the right side of (9),  $K_i = e E_i + e/c \cdot [\mathbf{v} \times \mathbf{H}]_i$  is the Lorentz force, and the third term is the well-known quantum force, but there appear, in addition, the second and the fourth terms dependent on the spin field. The former is taken as the reaction on the orbital motion of the element of the fluid due to the inhomogeneity of the magnetic field, as the result of the spin being associated with the magnetic moment  $(e/mc)\mathbf{S}$ , while the latter is the new stress which appears due to the inhomogeneity of the spin distribution.

Next we calculate the equation of motion for  $\mathbf{S}$  to obtain

$$\begin{aligned} d\mathbf{S}/dt &= (e/mc) [\mathbf{S} \times \mathbf{H}] + 1/m \\ &\cdot [\mathbf{S} \times (\Delta \mathbf{S} + (1/P) \sum_k \partial_k P \partial_k \mathbf{S})], \end{aligned} \quad (10)$$

where the first term in the right side is the torque acted on the spin by the magnetic field due to the magnetic moment associated to the spin, while the second term means that the spin is also subjected to the additional torque dependent on the gradient of the spin field itself.

We could thus transform the spinor equation (1) into simultaneous equations of motion, (5), (9), and (10), for a scalar  $P$  and two vectors  $\mathbf{v}$  and  $\mathbf{S}$ , together with the subsidiary conditions (7) and (8) compatible with the equations of motion.<sup>(5)</sup> These equations are all real and *explicitly gauge-invariant* and could be associated with the extended hydrodynamical picture. The more detailed properties of it can be investigated in a similar manner as in the original hydrodynamical picture<sup>(6)</sup>; for instance, the energy density is given by

$$\begin{aligned} \mathcal{H} = P \left\{ \frac{m}{2} v^2 + e A_0 + \frac{\hbar^2}{8m} \frac{(\nabla P)^2}{P^2} \right. \\ \left. - \frac{e}{mc} \mathbf{H} \mathbf{S} + \frac{1}{2m} (\nabla \mathbf{S})^2 \right\}. \end{aligned}$$

Since the above formulation presents a new picture and a method for the treatment of a spinning particle or of a spinor field, it is expected to be useful for various purposes, though the picture itself should not be taken too realistically.<sup>(7)</sup> For instance it gives immediately the 'classical or W.K.B.-like approximation' for the quantum mechanics of a spinning particle: If we neglect the quantities of the order  $\hbar^2$ , taking into account that  $\mathbf{S}$  is of the order  $\hbar$ , eqs. (9) and (10) are reduced to

$$m dv_i/dt = K_i + (e/mc) \sum_k \partial_i H_k \cdot S_k, \quad (9')$$

$$d\mathbf{S}/dt = e/mc \cdot [\mathbf{S} \times \mathbf{H}] \quad (10')$$

respectively, while eqs. (5), (7), and (8) remain the same. This shows that the picture is reduced, in this approximation, to such an ensemble of the numerous and mutually independent classical motions of a *classical particle with spin*, as to satisfy the condition (8), besides (5).

At this point it would be useful to add an explanation on the classical spin. It is to be defined formally as the fourth intrinsic degree of freedom such as to bring about to the Hamiltonian (2), taken as a classical quantity, its last term  $H_s = -(e/mc)\mathbf{s} \cdot \mathbf{H}$ , which is to be considered as a function of the conjugate canonical variables<sup>(8)</sup>,

$$q_s = \tan^{-1}(s_2/s_1), \quad p_s = s_3,$$

under the constraint  $s^2 = \hbar^2/4$ . The equations of



we assume that the meson field is unquantized, and regard spin and isotopic spin of the nucleon as classical unit vectors. Then the equations of motion of the system under the influence of the incident plane electromagnetic wave can be solved without any approximations other than the expansions in powers of elementary charge  $e$ . Only the  $e^2$ -parts of charge and current densities are necessary for our purpose. We denote them by  $\rho_2(\mathbf{x}, t)$  and  $\mathbf{j}_2(\mathbf{x}, t)$ , and write them as

$$\left. \begin{aligned} \rho_2(\mathbf{x}, t) \\ \mathbf{j}_2(\mathbf{x}, t) \end{aligned} \right\} = (2\pi)^{-3} e^{-i\omega t} \int d^3\ell e^{i\ell \cdot \mathbf{x}} \left\{ \begin{aligned} \rho_w(\ell) \\ \mathbf{j}_w(\ell) \end{aligned} \right\}, \quad (1)$$

where  $\omega$  is frequency of the incident wave. The scattering takes place through the three processes represented schematically as above. Corresponding to this,  $\mathbf{j}_w(\ell)$  can be written as

$$\mathbf{j}_w(\ell) = \mathbf{j}_I(\ell) + \mathbf{j}_{II}(\ell) + \mathbf{j}_{III}(\ell), \quad (2)$$

where  $\mathbf{j}_I$ ,  $\mathbf{j}_{II}$  and  $\mathbf{j}_{III}$  correspond to the processes  $I$ ,  $II$ , and  $III$  respectively. In  $\mathbf{j}_I$  and  $\mathbf{j}_{II}$ , therefore, the resonance occurs at the excitation energy of the isobar.  $\mathbf{j}_{II}$  and  $\mathbf{j}_{III}$  are expressed by integrals over Fourier components of meson field around the nucleon. The integral for  $\mathbf{j}_{III}$  is linearly divergent, while the one for  $\mathbf{j}_{II}$  is convergent. We cut off this divergent integral. Then a difficulty arises. Namely, the equation of continuity for the charge and current densities

$$-i\omega\rho_w(\ell) + i\ell \cdot \mathbf{j}_w(\ell) = 0 \quad (3)$$

does not hold. As a consequence of this, the cross section does not vanish at  $\omega=0$ , even if the nucleon is assumed to be at rest. We find that this difficulty is due to certain terms appearing in  $\rho_w$  and  $\mathbf{j}_{III}$ . These terms are the so-called surface terms, and are quite different in the form from the other terms. We drop these terms entirely. Then the terms containing the cut off momentum disappear from  $\mathbf{j}_{III}$ , and only  $\mathbf{j}_I$  and  $\mathbf{j}_{II}$  depend on the cut off momentum through the  $e^2$ -parts of spin and isotopic spin. We have verified that the charge and current densities outside the nucleon as well as the total charge are not affected by dropping the terms injuring the equation of continuity. These terms contain a term proportional to the cut off momentum in spite of the fact that the cut off is the very cause of the breakdown of the equation of continuity. This is due to the strong singularity of the derivative coupling between nucleon and meson. In the quantum mechanical treatment, all the divergences cancel, so

cut off is not necessary, therefore the above difficulty does not arise<sup>2)</sup>.

Now we add to  $\mathbf{j}_2(\mathbf{x}, t)$  the term giving rise to the ordinary Compton scattering

$$-(e^2/2M)(1-\tau_3)\mathbf{A}_0(\mathbf{x}_0, t)\delta(\mathbf{x}-\mathbf{x}_0), \quad (4)$$

where  $\mathbf{A}_0$  is vector potential of the incident wave, and  $\mathbf{x}_0$  denotes position of the nucleon. Then we have instead of (2)

$$\mathbf{j}_w(\ell) = \mathbf{j}_0(\ell) + \mathbf{j}_I(\ell) + \mathbf{j}_{II}(\ell) + \mathbf{j}_{III}(\ell),$$

where  $\mathbf{j}_0(\ell)$  denotes the Fourier transform of (4). Owing to  $\mathbf{j}_0$ ,  $\mathbf{j}_w$  does not satisfy the equation of continuity (3) again, and at  $\omega=0$  the cross section takes the Thomson value instead of zero.

The differential cross section per unit solid angle in the direction of the unit vector  $\mathbf{n}$  is given by

$$d\sigma/d\Omega = |\mathbf{j}_w(\mathbf{n}\omega)|^2 - |\mathbf{n} \cdot \mathbf{j}_w(\mathbf{n}\omega)|^2.$$

We average this over all directions of the spin and the isotopic spin. Therefore our calculation gives only the average of the cross sections for proton and neutron. If we do not take the average over the isotopic spin directions, we are led to an unaccountable result that  $\mathbf{j}_{III}$  does not contribute to the cross sections. This is because the isotopic spin has been treated as a classical unit vector.

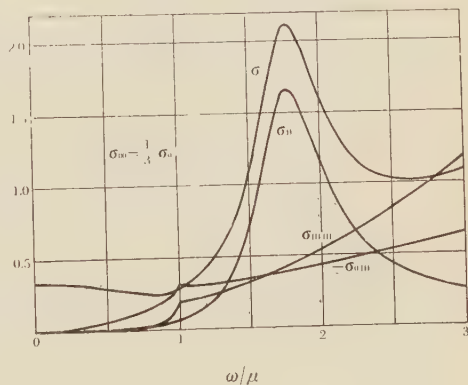


Fig. 1

In Fig. 1 we show the total cross section  $\sigma$  and the main contributions to it as functions of  $\omega$ .  $\sigma_{00}$ ,  $\sigma_{II}$ , and  $\sigma_{III}$  are the contributions of  $\mathbf{j}_0$ ,  $\mathbf{j}_I$  and  $\mathbf{j}_{III}$  respectively, and  $\sigma_{III-III}$  is the interference term between  $\mathbf{j}_0$  and  $\mathbf{j}_{III}$ . The other terms contribute to  $\sigma$  at most 7% for  $\omega < 3\mu$ . Unit of the ordinate has been taken to be the Thomson cross section

$$\sigma_0 = (8\pi/3)(e^2/M)^2.$$

Values of the coupling constant and the cut off momentum have been taken from reference 1):

$$f^2/4\pi = 1/6, \quad k_{\max} = (3\pi/8)M.$$

In virtue of  $\sigma_{II}$ ,  $\sigma$  has a maximum at  $\omega = 1.8\mu$ .  $\sigma_{II}$  depends strongly on the magnetic moments of nucleon, for which we have taken the static values.  $\sigma_{00}$ ,  $\sigma_{III\ III}$  and  $\sigma_{0III}$  are independent of the nucleon magnetic moments and the cut off momentum.

Sachs and Foldy<sup>2)</sup> have calculated only the terms corresponding to  $\sigma_{00} + \sigma_{III\ III} + \sigma_{0III}$ , and have shown that these terms have maxima at  $\omega = \mu$ . In our case, however,  $\sigma_{00} + \sigma_{III\ III} + \sigma_{0III}$  does not show

any remarkable anomaly at  $\omega = \mu$ . Sachs and Foldy have taken a larger value for the coupling constant:  $f^2/4\pi = 0.252$ . Even for this value of the coupling constant, our values of  $\sigma_{00} + \sigma_{III\ III} + \sigma_{0III}$  are much smaller than those of Sachs and Foldy. This is partly due to our classical averaging over the spin variables.

- 1) K. A. Brueckner and K. M. Case, Phys. Rev. **83** (1951), 1141.
- 2) R. G. Sachs and L. L. Foldy, Phys. Rev. **80** (1950), 824.

## ERRATA

### Kinematical Investigations of Meson-Nucleon Reactions

Satio HAYAKAWA, Masaaki KAWAGUCHI and Shigeo MINAMI

Prog. Theor. Phys. **12** (1954), 355

p. 358	10 lines from the bottom	for	They	read	$\alpha$ and $\beta$
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